

ABSTRACT

THOMPSON, KYLE ANDREW. Commuting Involutions of $SL(n, k)$. (Under the direction of Dr. Aloysius Helminck).

The classification of commuting involutions of a connected, reductive algebraic group over an algebraically closed field of characteristic not two was completed by Helminck in [Hel88]. In this thesis, the isomorphism classes of commuting pairs of involutions of $SL(n, k)$ for arbitrary fields of characteristic not two are determined. This is done in three main parts: commuting pairs of inner involutions, commuting pairs of involutions with one inner involution and one outer involution, and commuting pairs of outer involutions. The classification is done up to (almost) inner automorphism of $SL(n, k)$ as in the case of a single involution, completed in [HWD06]. For commuting pairs of involutions, the classification is done in several pieces, each of which uses an isomorphism class representative of a single involution of $SL(n, k)$ as the first entry. First, it is shown that if a mapping of a certain form is invertible, it will partially zero-out the matrix representative of one of the entries in the pair of commuting involutions while fixing the other pair, and will be an inner automorphism of $SL(n, k)$. Then, it is shown that, indeed, the given mapping must be invertible for one of a few given forms. This leads to an induction argument that gives a partial classification of the commuting inner pairs. To finish the classification, whether or not the remaining pairs are isomorphic is determined. Next, using results on Quadratic forms, a 'nice' form for commuting pairs with one inner and one outer involution is found. To finish the classification of such pairs, work is completed on a field (of characteristic not two) by field basis and results are explicitly obtained for algebraically closed fields, the real numbers, finite fields, and the p -adic numbers. Lastly, equivalence between the classification of commuting outer pairs and commuting pairs with one inner involution and one outer involution is shown.

Commuting Involutions of $SL(n, k)$

by
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BIOGRAPHY

Kyle was born and grew up in the Midwest. He attended Rollins College where he swam distance freestyle, earning three conference titles his senior year as Captain of the team. He met his wife, Carolyn, a few lanes down the pool; they were married in 2007. Kyle graduated Summa Cum Laude from Rollins before attending graduate school at North Carolina State University. There, he earned the Armstrong Maltbie Teaching Award given by the Mathematics Department. While at NC State, Kyle worked under the direction of Dr. Aloysius Helminck.

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Chapter 1

Introduction

For any field k of characteristic not two we can define a symmetric k -variety as the homogeneous space G_k/H_k where G is a reductive algebraic group defined over k and H is the fixed point group of a k -involution with G_k and H_k the k -rational points of G and H , respectively. The study of these symmetric k -varieties occur in problems in representation theory, geometry, singularity theory, the study of character sheaves and the study of cohomology of arithmetic subgroups (as in [BB81], [Vog83] and [Vog82] for representation theory, [DCP83], [DCP85] and [Abe88] for geometry, [LV83] [HS90] for singularity theory, [Lus90] and [Gro92] for the study of character sheaves, and [TW89] for the study of cohomology of arithmetic subgroups).

Determining various orbit decompositions of symmetric k -varieties play a vital role in the study of these symmetric spaces, especially in representation theory (see [OM84], [Vog83] and [Vog82], [HH05, HH08], [vdBS97, BD92, Del98, FJ80, OS80], [Kir93], and [Gro92]). The orbits of minimal parabolic k -subgroups on symmetric varieties and the orbits of a θ -stable symmetric k -subgroup acting on the symmetric k -variety for θ a k -involution are important in the study of the representation theory of symmetric varieties and harmonic analysis, respectively. The related double cosets play an important role in their study, including the double cosets of a pair of commuting involutions (see [Hel88] [Mat02]).

Although the double cosets of a pair of commuting involutions are important in several areas of mathematics, there still is not a complete classification of the

isomorphism classes of commuting involutions for general base fields. For algebraically closed fields, this was completed by Helminck in [Hel88] and some partial results were obtained by Matsuki [Mat02] over the real numbers for compact groups. This thesis is on the classification of commuting involutions of $\mathrm{SL}(n, k)$, for k an arbitrary field of characteristic not two. The classification of single involutions of $\mathrm{SL}(n, k)$ was completed by Helminck, Wu, and Dometrius up to conjugacy by (almost) inner automorphisms of $\mathrm{SL}(n, k)$ in [HWD06]. I used their results to classify commuting pairs of involutions up to conjugacy by (almost) inner automorphisms of $\mathrm{SL}(n, k)$ in the following manner: a pair of commuting involutions (σ, θ) of $\mathrm{SL}(n, k)$ is isomorphic to any commuting pair of the form $(\mathrm{Inn}_G^{-1} \sigma \mathrm{Inn}_G, \mathrm{Inn}_G^{-1} \theta \mathrm{Inn}_G)$, for Inn_G an (almost) inner automorphism of $\mathrm{SL}(n, k)$. This can lead to the determination of the maximal (σ, θ, k) -split tori for the isomorphism classes given by (σ, θ) . Furthermore, this can lead to a generalization as described below.

The classification of commuting involutions of an algebraic group over an algebraically closed field was completed by Helminck [Hel88] using commuting involutions of the root system (that can be lifted to commuting involutions of the group) and quadratic elements in a (σ, θ) -split torus. Also, work on the classification of one involution of an algebraic group over an arbitrary field of characteristic not two was done by Helminck [Hel00] using the root system of a maximal k -split torus and something similar to quadratic elements. From the classification of commuting involutions of $\mathrm{SL}(n, k)$, one can work to generalize to commuting involutions of an arbitrary reductive algebraic group by using the root system of a maximal k -split torus built from a maximal (σ, θ, k) -split torus and some form of quadratic elements. In doing this, one would want to synthesize the work on commuting involutions over an algebraically closed field and single involutions over an arbitrary field of characteristic not two. To begin this process one might, in order to simplify things, first focus on the case of split reductive algebraic groups where maximal k -split tori are maximal tori. Along the way, one might want to determine the classification of (σ, θ, k) -split tori over the intersection of fixed point groups of σ and θ , the relationship between commuting involutions of the root system and commuting involutions of the group, and an appropriate classification of quadratic elements in a (σ, θ, k) -split torus or a

torus containing one.

In this dissertation, the classification of commuting involutions of $\mathrm{SL}(n, k)$ is split in 3 parts: commuting pairs of inner involutions, commuting pairs of involutions with one inner involution and one outer involution, and commuting pairs of outer involutions. The classification is done up to (almost) inner automorphisms of $\mathrm{SL}(n, k)$ as in the case of a single involution, completed in [HWD06], i.e. inner automorphisms of $\mathrm{SL}(n, \bar{k})$ that keep $\mathrm{SL}(n, k)$ invariant. For commuting pairs of inner involutions, the classification is done in several pieces, each of which uses an isomorphism class representative of a single involution of $\mathrm{SL}(n, k)$ as the first entry. First, it is shown that if a mapping of a certain form is invertible, it will partially zero-out the matrix representative of one of the entries in the pair of commuting involutions while fixing the other pair, and will be an inner automorphism of $\mathrm{SL}(n, k)$. This is done in lemma 4.1.2. Then it is shown in lemma 4.1.5 that, indeed, the given mapping must be invertible for one of a few given forms. This leads to an induction argument that gives a partial classification of the commuting inner pairs. To finish the classification of inner pairs, the remaining ones are determined to be isomorphic or not on a case-by-case basis. This leads to the isomorphism classes, appearing as theorems 4.3.14, 4.3.16, and 4.3.17 in this dissertation:

Theorem 1.0.1. *For n divisible by 4, p, q representatives of elements in k^*/k^{*2} , with $p, q \not\equiv 1 \pmod{k^{*2}}$, the isomorphism classes of commuting inner involutions of $\mathrm{SL}(n, k)$ are as follows:*

$$\begin{aligned}
& (\mathrm{Inn}_{L'_{n,q}}, \mathrm{Inn}_Y), (\mathrm{Inn}_{L_{n,q}}, \mathrm{Inn}_{Y'}), (\mathrm{Inn}_{I_{n-i,i}}, \mathrm{Inn}_{Y''}), \left(\mathrm{Inn}_{I_{\frac{n}{2}, \frac{n}{2}}}, \mathrm{Inn}_{Y'''} \right) \text{ for} \\
Y &= L_{n,p} (p \not\equiv q \pmod{k^{*2}}), I_{\frac{n}{2}, \frac{n}{2}}, I_{\frac{n}{2}, \frac{n}{2}} L_{n,p}, \text{ or } \begin{pmatrix} I_{\frac{n}{2}-i,i} & 0_{\frac{n}{2}} \\ 0_{\frac{n}{2}} & I_{\frac{n}{2}-i,i} \end{pmatrix}, i \in \left\{ 1, 2, \dots, \frac{n}{4} \right\}, \\
& Y' = I_{n-2i, 2i} L_{n,q}, i \in \left\{ 1, 2, \dots, \frac{n}{4} \right\}, \\
Y'' &= L_{n,p}, \text{ or } \begin{pmatrix} I_{n-i-j,j} & 0_{(n-i) \times i} \\ 0_{i \times (n-i)} & I_{i-k,k} \end{pmatrix}, i \in \left\{ 1, 2, \dots, \frac{n}{2} \right\}, \\
Y''' &= L'_{n,p}, \text{ or } \begin{pmatrix} I_{\frac{n}{2}-j,j} & 0_{\frac{n}{2} \times \frac{n}{2}} \\ 0_{\frac{n}{2} \times \frac{n}{2}} & I_{\frac{n}{2}-k,k} \end{pmatrix}, j \in \left\{ 1, 2, \dots, \frac{n}{4} \right\}.
\end{aligned}$$

For Y'' , we have: $n - i \neq i$, and $j \in \{1, 2, \dots, \frac{n-i}{2}\}$, if $n - i - j \neq j$, $i - k \neq k$ then $k \in \{1, 2, \dots, i\}$ or else if $n - i - j = j$, or $i - k = k$, then $k \in \{1, 2, \dots, \frac{i}{2}\}$. Also, for Y''' we have: if $\frac{n}{2} - j \neq j$, $\frac{n}{2} - k \neq k$, then for $k \in \{1, 2, \dots, \frac{n}{2}\}$ we get distinct isomorphism classes for distinct j, k such that as a pair (j, k) is distinct, independent of order, or if $\frac{n}{2} - j = j$, or $\frac{n}{2} - k = k$, then $k \in \{1, 2, \dots, \frac{n}{4}\}$,

Theorem 1.0.2. For n even but not divisible by 4, p, q representatives of elements k^*/k^{*2} , $p, q \not\equiv 1 \pmod{k^{*2}}$, $a^2 - qb^2 = p$ the isomorphism classes of commuting inner involutions of $\text{SL}(n, k)$ are as follows:

$(\text{Inn}_{L_{n,q}}, \text{Inn}_{I_{n-2i, 2i} L_{n,q}})$, $(\text{Inn}_{I_{n-i, i}}, \text{Inn}_{Y''})$ for $i \in \{1, 2, \dots, \frac{n}{2}\}$, and

$(\text{Inn}_{\begin{pmatrix} L_{2,q} & 0 \\ 0 & L'_{n-2,q} \end{pmatrix}}, \text{Inn}_Y)$, $(\text{Inn}_{L'_{n,q}}, \text{Inn}_{Y'})$, $(\text{Inn}_{I_{\frac{n}{2}, \frac{n}{2}}}, \text{Inn}_{Y'''})$ for

$$Y = \begin{pmatrix} \begin{pmatrix} a & b \\ -qb & -a \end{pmatrix} & 0 \\ 0 & I_{\frac{n-2}{2}, \frac{n-2}{2}} L_{n-2,p} \end{pmatrix}, \begin{pmatrix} I_{1,1} & 0 \\ 0 & I_{\frac{n-2}{2}, \frac{n-2}{2}} \end{pmatrix},$$

$$Y' = \begin{pmatrix} I_{\frac{n}{2}-i, i} & 0 \\ 0 & I_{\frac{n}{2}-i, i} \end{pmatrix}, i \in \left\{1, 2, \dots, \left\lfloor \frac{n-2}{4} \right\rfloor\right\},$$

$$Y'' = L_{n,p}, \text{ or } \begin{pmatrix} I_{n-i-j, j} & 0_{(n-i) \times i} \\ 0_{i \times (n-i)} & I_{i-k, k} \end{pmatrix}, i \in \left\{1, 2, \dots, \frac{n}{2}\right\},$$

$$Y''' = L'_{n,p}, \text{ or } \begin{pmatrix} I_{\frac{n}{2}-j, j} & 0_{\frac{n}{2} \times \frac{n}{2}} \\ 0_{\frac{n}{2} \times \frac{n}{2}} & I_{\frac{n}{2}-k, k} \end{pmatrix}, j \in \left\{1, 2, \dots, \frac{n}{4}\right\}$$

Where, for Y'' we have: for $n - i \neq i$, and $j \in \{1, 2, \dots, \frac{n-i}{2}\}$, if $n - i - j \neq j$, $i - k \neq k$ then $k \in \{1, 2, \dots, i\}$ or else if $n - i - j = j$, or $i - k = k$, then $k \in \{1, 2, \dots, \frac{i}{2}\}$, and for Y''' , we have: if $\frac{n}{2} - j \neq j$, $\frac{n}{2} - k \neq k$, then for $k \in \{1, 2, \dots, \frac{n}{2}\}$ we get distinct isomorphism classes for distinct j, k such that as a pair (j, k) is distinct, independent of order, or if $\frac{n}{2} - j = j$, or $\frac{n}{2} - k = k$, then $k \in \{1, 2, \dots, \frac{n}{4}\}$,

Theorem 1.0.3. For n odd, the isomorphism classes of commuting pairs of involutions of $\text{SL}(n, k)$ are: $(\text{Inn}_{I_{n-i, i}}, \text{Inn}_Y)$ for

$$Y = \begin{pmatrix} I_{n-i-j, j} & 0_{(n-i) \times i} \\ 0_{i \times (n-i)} & I_{i-k, k} \end{pmatrix}, i \in \left\{1, 2, \dots, \frac{n}{2}\right\},$$

for $n - i \neq i$, and $j \in \{1, 2, \dots, \frac{n-i}{2}\}$, if $n - i - j \neq j$, $i - k \neq k$ then $k \in \{1, 2, \dots, i\}$ or else if $n - i - j = j$, or $i - k = k$, then $k \in \{1, 2, \dots, \frac{i}{2}\}$.

Next, using results on Quadratic forms, a 'nice' form for commuting pairs with one inner and one outer involution is obtained in lemmas 5.1.4 and 5.2.4 corresponding to symmetric and skew-symmetric forms (that give rise to outer involutions), respectively. To finish the classification of such pairs, work is done on a field (of characteristic not two) by field basis and results are explicitly obtained for algebraically closed fields, the real numbers, finite fields, and the p -adic numbers. This is summarized in the tables herein and in theorem 5.3.11. Lastly, the equivalence between the classification of commuting outer pairs and commuting pairs with one inner involution and one outer involution is shown. In the last section, a summary of the isomorphism classes of pairs of commuting involutions of $SL(n, k)$ is given.

Chapter 2

Background Material

In this section, an overview of the material on which most of this thesis relies is presented. This will include the theory and results from [HWD06], [Hel88], and a few selections from [Sch85], [Jon50], [Dom03], [Jac05], and [Hel00], as well as a short introduction to symmetric k -varieties.

2.1 Involutions of $\mathrm{SL}(n, k)$

This thesis is about commuting pairs of involutions of $\mathrm{SL}(n, k)$ for k a field of characteristic not two, and their isomorphism classes. Thus, a thorough grasp of single involutions and their isomorphism classes is needed, as it is used throughout this work. Throughout this section k is a field of characteristic not two, k_1 is an extension field of k and \bar{k} is the algebraic closure of k .

Definition 2.1.1. An involution defined over a field k of a connected, reductive algebraic group, G , defined over k is an automorphism (over k) of order two, i.e. an automorphism ϕ such that $\phi^2 = \mathrm{Id}$ and $\phi \neq \mathrm{Id}$.

Before defining what it is meant by isomorphic involutions, some notation is needed:

Notation 2.1.2. Let G be a connected, reductive algebraic group defined over a field k . Then let $\mathrm{Aut}(G)$ denote the set of all automorphisms of G . Let $\mathrm{Inn}_k(G)$ denote

the set of all inner (k)-automorphisms of G , i.e. $\text{Inn}_k(G) = \{\text{Inn}_A | A \in G\}$, where for $X \in G$, $\text{Inn}_A(X) = A^{-1}XA$ for $A \in G$, and any automorphism of G that is not an inner automorphism is referred to as an outer automorphism. Also, let $\text{Inn}(G)$ denote the set of automorphisms of G that are 'almost' an inner automorphism, i.e. $\text{Inn}(G) = \{\text{Inn}_A | A \in \bar{G}, \text{Inn}_A(G) = G\}$ where \bar{G} is the group G over \bar{k} , i.e. for $G = \text{SL}(n, k)$, $\bar{G} = \text{SL}(n, \bar{k})$.

Definition 2.1.3. Given two involutions τ and σ of G , τ is isomorphic or $\text{Inn}(G)$ -isomorphic to σ if and only if there is a $\phi \in \text{Inn}(G)$ such that $\tau = \phi^{-1}\sigma\phi$, often denoted $\tau \approx \sigma$.

Note that in the above definition, we take the definition of isomorphic involutions to be that of $\text{Inn}(G)$ -isomorphism. There is also a similar notion for $\text{Aut}(G)$ and $\text{Inn}_k(G)$ -isomorphism that are not considered in this thesis.

Beginning with the classification of involutions of $\text{SL}(2, k)$, as in [HW02], an important lemma that holds for $n = 2$ is the following [Bor91]:

Lemma 2.1.4. *For $\phi \in \text{Aut}(\text{SL}(2, k))$ and k not algebraically closed, there exists an extension field k_1 of k and a $\tau \in \text{Inn}(\text{SL}(2, k))$ such that $\tau|_{\text{SL}(2, k)} = \phi$, i.e. there is always a 2×2 matrix $A \in \text{SL}(2, k_1)$ such that $\phi = \text{Inn}_A|_{\text{SL}(2, k)}$. Furthermore, for k algebraically closed, $\text{Aut}(\text{SL}(2, k)) = \text{Inn}(\text{SL}(2, k))$.*

Unfortunately, as soon described, the above lemma does not hold for $n > 2$; however, it turns out that for $n > 2$ there are only two types of involutions to classify: inner and outer. Considering the automorphisms of $\text{SL}(2, k)$ of the form Inn_A for $A \in \text{SL}(2, k_1)$, by the above lemma this covers all of them, the following lemma from [HW02], lead to a 'nice' form for the matrix A :

Lemma 2.1.5. *The automorphism $\text{Inn}_A \in \text{Inn}(\text{SL}(2, k_1))$ keeps $\text{SL}(2, k)$ invariant if and only if $A = pB$ for some $p \in k_1$ and $B \in \text{GL}(2, k)$.*

In other words, since $\text{Inn}_{pB} = \text{Inn}_B$ for any constant p , every automorphism of $\text{SL}(2, k)$ can be written as $\text{Inn}_B|_{\text{SL}(2, k)}$ for $B \in \text{GL}(2, k)$. Knowing this, and applying some simple matrix algebra, the form of B can be further simplified [HW02]:

Lemma 2.1.6. *Suppose $\phi \in \text{Aut}(\text{SL}(2, k))$ is an involution. Then there exists a matrix $A \in \text{GL}(2, k)$ such that $\phi = \text{Inn}_A|_{\text{SL}(2, k)}$ and Inn_A is isomorphic to $\text{Inn}\begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}$.*

Therefore, all isomorphism classes of involutions of $\text{SL}(2, k)$ can be represented by 2×2 matrices of the form $\begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix} \in \text{GL}(2, k)$. The problem of determining the isomorphism classes of involutions of $\text{SL}(2, k)$, then, has been reduced to determining which q 's give distinct isomorphism classes, i.e. the classification has been reduced to the field k . In fact, distinct isomorphism classes of involutions of $\text{SL}(2, k)$ arise from distinct square classes, i.e. distinct elements of k^*/k^{*2} [HW02]:

Theorem 2.1.7. *Suppose $\phi, \sigma \in \text{Aut}(\text{SL}(2, k))$ are involutions of which the corresponding matrices in G are $A = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}$. Then ϕ is conjugate (isomorphic) to σ if and only if q/p is a square in k^* .*

Corollary 2.1.8. *The number of isomorphism classes of involutions of $\text{SL}(2, k)$ equals the order of k^*/k^{*2} , and distinct isomorphism classes of involutions have matrix representatives $\begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}$ for distinct $q \in k^*/k^{*2}$.*

Since the groups k^*/k^{*2} for k algebraically closed fields, $\mathbb{R}, \mathbb{F}_p\mathbb{Q}_2$, and \mathbb{Q}_p for $p \neq 2$ are considered in this thesis, their groups are briefly listed (as in [HW02] and [HWD06]):

1. $k = \bar{k}$. In this case $|k^*/k^{*2}| = 1$
2. $k = \mathbb{R}$. In this case $k^*/k^{*2} \approx \mathbb{Z}_2$, and thus $|k^*/k^{*2}| = 2$.
3. $k = \mathbb{Q}$. In this case $|k^*/k^{*2}| = \infty$.
4. $k = \mathbb{F}_p, p \neq 2$. Then, $k^*/k^{*2} \approx \mathbb{Z}_2$ and thus $|k^*/k^{*2}| = 2$. For convenience (and to refer to later in this thesis) the 'smallest' non-square in \mathbb{F}_p will be referred to as s_p , as in [HWD06], and then take $\{1, s_p\}$ as the elements in k^*/k^{*2} .
5. $k = \mathbb{Q}_p, p \neq 2$. In this case, $k^*/k^{*2} \approx \mathbb{Z}_2 \times \mathbb{Z}_2$, hence $|k^*/k^{*2}| = 4$. Again letting s_p refer to the 'smallest' non-square in the field \mathbb{F}_p , the elements of k^*/k^{*2} can be taken in the following set: $\{1, p, s_p, ps_p\}$, as in [HW02].

6. $k = \mathbb{Q}_2$. In this case, $|k^*/k^{*2}| = 8$. Also, the elements of k^*/k^{*2} can be taken in the following set: $\{1, -1, 2, -2, 3, -3, 6, -6\}$, as in [HW02].

Although the classification of involutions of $\mathrm{SL}(2, k)$ can be reduced to the structure of the field k , namely to the number of square classes, the same does not hold for $\mathrm{SL}(n, k)$, $n > 2$; however, for involutions of $\mathrm{SL}(n, k)$, $n > 2$, of inner type, i.e. in $\mathrm{Inn}(\mathrm{SL}(n, k))$, the structure of k^*/k^{*2} plays an important role.

The classification (up to $\mathrm{Inn}(G)$ -isomorphism) of involutions of $\mathrm{SL}(n, k)$, $n > 2$ is broken up into two parts: classifying inner involutions (inner automorphisms that are involutions) and classifying outer involutions (outer automorphisms that are involutions). First, inner involutions of $\mathrm{SL}(n, k)$ are discussed for k a field of characteristic not two. Note that for $A \in \mathrm{GL}(n, \bar{k})$, $\mathrm{Inn}_A = \mathrm{Inn}_{A'}$ for $A' \in \mathrm{SL}(n, \bar{k})$, because $A' = \left(\frac{1}{\det(A)}\right)^{1/n} A$ and $\mathrm{Inn}_A = \mathrm{Inn}_{\alpha A}$ for any constant α . Thus, one can consider an automorphism of the following type: $\mathrm{Inn}_Y|_{\mathrm{SL}(n, k)}$ for $Y \in \mathrm{GL}(n, \bar{k})$. From [HWD06], we have the following two lemmas:

Lemma 2.1.9. *Let $Y \in \mathrm{GL}(n, \bar{k})$. If $\mathrm{Inn}_Y|_{\mathrm{SL}(n, k)} = \mathrm{Id}$, then $Y = pI$ for some $p \in \bar{k}$.*

Lemma 2.1.10. *Let $\phi \in \mathrm{Inn}(\mathrm{SL}(n, k)) = \{\mathrm{Inn}_A | A \in \mathrm{SL}(n, \bar{k}), \mathrm{Inn}_A(\mathrm{SL}(n, k)) = \mathrm{SL}(n, k)\} = \{\mathrm{Inn}_A | A \in \mathrm{GL}(n, \bar{k}), \mathrm{Inn}_A(\mathrm{SL}(n, k)) = \mathrm{SL}(n, k)\}$ (by the remark above), and let $Y \in \mathrm{GL}(n, \bar{k})$. Then, for $\phi = \mathrm{Inn}_Y \in \mathrm{Inn}(\mathrm{SL}(n, \bar{k}))$ keeps $\mathrm{SL}(n, k)$ invariant if and only if $Y = pB$ for $p \in \bar{k}$ and $B \in \mathrm{GL}(n, k)$.*

The above lemma implies that any automorphism in $\mathrm{Inn}(\mathrm{SL}(n, k))$ of $\mathrm{SL}(n, k)$ can be written as Inn_B for $B \in \mathrm{GL}(n, k)$. Therefore, any involution of $\mathrm{SL}(n, k)$ of inner type can be written as $\mathrm{Inn}_Y|_{\mathrm{SL}(n, k)}$ for $Y \in \mathrm{GL}(n, k)$. Furthermore, isomorphism of involutions can be written as conjugation by Inn_Y for $Y \in \mathrm{GL}(n, k)$. In fact, we get:

Lemma 2.1.11. *The automorphisms of inner type $\mathrm{Inn}_{Y_1}, \mathrm{Inn}_{Y_2} \in \mathrm{Inn}(\mathrm{SL}(n, k))$ are conjugate (isomorphic) if and only if the matrix Y_1 is conjugate to cY_2 for some $c \in k$.*

Proof. Inn_{Y_1} is conjugate to Inn_{Y_2} if and only if there is $\phi \in \mathrm{Inn}(\mathrm{SL}(n, k))$ such that $\phi^{-1} \mathrm{Inn}_{Y_1} \phi = \mathrm{Inn}_{Y_2}$ if and only if there is an $A \in \mathrm{GL}(n, \bar{k})$ such that $\phi =$

$\text{Inn}_A |_{\text{SL}(n,k)}, \text{Inn}_A(\text{SL}(n,k)) = \text{SL}(n,k)$ and $\phi \in \text{Inn}(\text{SL}(n,k))$ such that $\text{Inn}_{Y_2} = \text{Inn}_A^{-1} \text{Inn}_{Y_1} \text{Inn}_A$ if and only if there is a $B \in \text{GL}(n,k)$ such that $\text{Inn}_B^{-1} \text{Inn}_{Y_1} \text{Inn}_B = \text{Inn}_{Y_2}$. This holds if and only if $\text{Inn}_{BY_1B^{-1}Y_2^{-1}} = \text{Id}$ if and only if $BY_1B^{-1} = cY_2$ for $c \in \bar{k}$. But by lemma 2.1.10 $Y_1, Y_2 \in \text{GL}(n,k)$ and since $B \in \text{GL}(n,k)$ this holds if and only if $c \in k$. \square

Returning to involutions, the matrix representatives of an involution of inner type of $\text{SL}(n,k)$, they have the form Inn_Y for $Y \in \text{GL}(n,k)$ such that $\text{Inn}_Y^2 = \text{Id}$. This implies that $Y^2 = pI_n$ for $p \in k$. By determining whether or not p is a square in k^* , the isomorphism class of Inn_Y can be narrowed down:

Lemma 2.1.12. *Suppose $Y \in \text{GL}(n,k)$ with $Y^2 = pI_n$. Then if $p = c^2 \in k^{*2}$ then Y is conjugate to $cI_{n-i,i}$ for some $i = 0, 1, \dots, n$. If $p \notin k^{*2}$ then n is even and Y is conjugate to $L_{n,p}$.*

In fact, the distinctness of involutions of $\text{SL}(n,k)$ up to $\text{Inn}(\text{SL}(n,k))$ -isomorphism depends in part on the structure of the field k , because distinct isomorphism classes of involutions $\text{Inn}_{L_{n,p}}$ come from distinct $p \in k^*/k^{*2}$. The complete classification (of involutions of inner type of $\text{SL}(n,k)$ is used in theorem 3.3.1), and is as follows:

Theorem 2.1.13. *Suppose $\phi \in \text{Aut}(\text{SL}(n,k))$ is of inner type. Then up to isomorphism ($\text{Inn}(\text{SL}(n,k))$ -isomorphism), ϕ is one of the following:*

1. $\text{Inn}_Y |_{\text{SL}(n,k)}$, where $Y = I_{n-i,i} \in \text{GL}(n,k)$, and $i \in \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$
2. $\text{Inn}_Y |_{\text{SL}(n,k)}$, where $Y = L_{n,p} \in \text{GL}(n,k)$ and $p \in k^*/k^{*2}, p \not\equiv 1 \pmod{k^{*2}}$

and these are distinct classes, thus ϕ is only one of the above.

As there exists a 'nice' relationship between involutions of $\text{SL}(n,k)$ of inner type and the structure of the field k given by the group k^*/k^{*2} , outer involutions of $\text{SL}(n,k)$ are related to bilinear forms on the vector space k^n . Before describing the relationship between bilinear forms on k^n and outer involutions of $\text{SL}(n,k)$ an important result from [Bor91] is vital:

Lemma 2.1.14. *Any outer automorphism can be written as $\text{Inn}_M \theta$ for θ a fixed outer automorphism.*

Thus, taking the outer automorphism of $\text{SL}(n, k)$ to be $\theta(A) = A^{T-1}$ for $A \in \text{SL}(n, k)$, any outer automorphism of $\text{SL}(n, k)$ can be written as $\text{Inn}_M \theta$. Knowing this, one needs to classify the automorphism of inner type Inn_M for $\text{Inn}_M \theta$ an involution of $\text{SL}(n, k)$. It turns out [HWD06] that M must be either symmetric or skew-symmetric (hence a matrix representative of a symmetric or skew-symmetric bilinear form on k^n):

Lemma 2.1.15. *For θ the outer involution given by $\theta(A) = A^{T-1}$, $\text{Inn}_M \theta$ is an involution of $\text{SL}(n, k)$ if and only if M is symmetric or skew-symmetric and M is only skew-symmetric if n is even.*

This lemma allows one to describe the outer involutions of $\text{SL}(n, k)$ but not classify them up to $\text{Inn}(\text{SL}(n, k))$ -isomorphism. To do that, first recall the definition of congruence:

Definition 2.1.16. Two bilinear forms with matrices M_1 and M_2 represent the same bilinear form on k^n with respect to different bases if and only if they are congruent, i.e. if and only if there exists a $Q \in \text{GL}(n, k)$ such that $Q^T M_1 Q = M_2$.

Unfortunately, there is not a one-to-one correspondence between the congruence classes of symmetric/skew-symmetric matrices M (and hence of symmetric/skew-symmetric bilinear forms) and isomorphism classes of outer involutions of $\text{SL}(n, k)$ of the form $\text{Inn}_M \theta$; however, there is a close relationship between the two [HWD06]. A quick definition is needed first:

Definition 2.1.17. Two bilinear forms on k^n with associated matrices M_1 and M_2 are semi-congruent over k if there exists a $Q \in \text{GL}(n, k)$ and an $\alpha \in k$ such that $Q^T M_1 Q = \alpha M_2$.

The notion of semi-congruence is what is needed to tie together the classification of symmetric/skew-symmetric bilinear forms on k^n and isomorphism classes of outer involutions of $\text{SL}(n, k)$ [HWD06]:

Theorem 2.1.18. *If $\text{Inn}_{M_1} \theta$ and $\text{Inn}_{M_2} \theta$ are two outer involutions of $\text{SL}(n, k)$, then they come from (nondegenerate) bilinear symmetric or skew-symmetric forms represented by M_1 and M_2 , respectively, and $\text{Inn}_{M_1} \theta$ is isomorphic to $\text{Inn}_{M_2} \theta$ if and only if M_1 is semi-congruent to M_2 . Furthermore, if the matrix representative is skew-symmetric then n is even.*

Therefore, to complete the classification of involutions of $\text{SL}(n, k)$ one needs to determine the semi-congruence classes of symmetric and skew-symmetric bilinear forms on k^n . This depends on the structure of the field k and has been determined [HWD06] for algebraically closed fields, $\mathbb{R}, \mathbb{F}_p, \mathbb{Q}_p, \mathbb{Q}_2$ for $p \neq 2$.

2.2 Bilinear Forms

Symmetric and skew-symmetric bilinear forms on k^n play an important role in the construction and to some degree, the classification of outer involutions of $\text{SL}(n, k)$ as discussed above and in [HWD06]. In this section we briefly review some results of bilinear forms appearing in [Jon50] and [Sch85]. Recall the following definition from [Sch85]:

Definition 2.2.1. A bilinear form on a vector space $V = k^n$ is a map

$$\beta : V \times V \rightarrow k$$

such that for all $x, x', y, y' \in V$ and $\alpha \in k$ the following hold:

$$\beta(x + x', y) = \beta(x, y) + \beta(x', y)$$

$$\beta(x, y + y') = \beta(x, y) + \beta(x, y')$$

$$\beta(\alpha x, y) = \alpha \beta(x, y) = \beta(x, \alpha y).$$

Furthermore, the bilinear form β is said to be symmetric if and only if $\beta(x, y) = \beta(y, x)$ for all $x, y \in V$ and is said to be skew-symmetric if and only if $\beta(x, y) = -\beta(y, x)$ for all $x, y \in V$.

Given a matrix representative M of a bilinear form on k^n (matrix representatives of bilinear forms are congruent if and only if they represent the 'same' bilinear form with respect to a new basis), then from [Sch85] we get:

Lemma 2.2.2. *Every symmetric matrix B is congruent to a diagonal matrix.*

Hence, congruence of A and B being defined by $Q \in \text{GL}(n, k)$ such that $Q^T B Q = A$, then for each B there exists a $Q \in \text{GL}(n, k)$ such that $Q^T B Q$ is diagonal, say $Q^T B Q = \sum_{i=1}^n q_i * x_i^2 E_{i,i}$ where q_i is a representative of k^*/k^{*2} and $x \in k$. Then using $Q' = \sum_{i=1}^n \frac{1}{x_i} E_{i,i}$, the following holds:

Corollary 2.2.3. *Every symmetric matrix B is congruent to a diagonal matrix whose entries are representatives of the square-class group k^*/k^{*2} .*

Also, a 'nice' form exists for each nonsingular skew-symmetric matrix [Sch85]:

Lemma 2.2.4. *Every skew-symmetric matrix in $\text{GL}(n, k)$ is congruent to the matrix:*

$$\begin{pmatrix} 0 & I_{\frac{n}{2}} \\ -I_{\frac{n}{2}} & 0 \end{pmatrix},$$

hence n must be even.

Considering bilinear forms over \mathbb{Q}_p , some results from [Jon50] are useful. First, a few definitions are needed:

Definition 2.2.5. The Hilbert Symbol, $(\cdot, \cdot)_p$ of two nonzero p -adic numbers α and β is defined as follows: $(\alpha, \beta)_p = 1$ if $\alpha x^2 + \beta y^2 = 1$ has a solution over the p -adics, and equals -1 otherwise.

Definition 2.2.6. The Hasse symbol, $c_p(M)$ of M the matrix representative of a bilinear form on k^n is defined by:

$$c_p(M) = (-1, -D_n)_p \prod_{i=1}^{n-1} (D_i, -D_{i+1})_p$$

for D_i the determinant of the upper left $i \times i$ square submatrix of M .

Using the above notions, and the two results below becomes useful when determining semi-congruence. This is because the first lemma relates the Hasse symbol of a matrix M to the Hasse symbol of αM for $\alpha \in k$ and the second result gives necessary and sufficient conditions for congruence.

Lemma 2.2.7. *If M is an $n \times n$ matrix (with entries in \mathbb{Q}_p) and $\alpha \in \mathbb{Q}_p$ then:*

$$c_p(\alpha M) = \begin{cases} (\alpha, -1)_p^{n/2} (\alpha, D_n)_p c_p(M) & : n \text{ even} \\ (\alpha, -1)_p^{(n+1)/2} c_p(M) & : n \text{ odd} \end{cases}$$

Theorem 2.2.8. *Two symmetric matrices M_1 and M_2 with entries in \mathbb{Q}_p are congruent if and only if: $\det(M_1) = \alpha^2 \det(M_2)$ and $c_p(M_1) = c_p(M_2)$.*

2.3 Automorphisms fixing $\mathrm{Sp}(2n, k)$ or $\mathrm{SO}(n, k, \beta)$

As the characterization theorems of automorphisms Inn_A for $A \in \mathrm{GL}(n, \bar{k})$ that keep $\mathrm{Sp}(2n, \bar{k})$ invariant or keep $\mathrm{SO}(n, \bar{k}, \beta)$ invariant in [Jac05] and [Dom03], respectively, is used in the classification of commuting involutions of $\mathrm{SL}(n, k)$ with an outer involution, a brief overview of the main results of those theses is included here. First, consider automorphisms of inner type, having the form Inn_A for $A \in \mathrm{GL}(2n, \bar{k})$, that keep $\mathrm{Sp}(2n, \bar{k})$ invariant. Then, it is shown in [Jac05] that:

Theorem 2.3.1. *Inn_A keeps $\mathrm{Sp}(2n, \bar{k})$ invariant if and only if $A = pM$ for $p \in \bar{k}$ and $M \in \mathrm{Sp}(2n, \bar{k})$*

In other words, every inner automorphism that keeps $\mathrm{Sp}(2n, \bar{k})$ invariant can be written as Inn_A for $A \in \mathrm{Sp}(2n, \bar{k})$. This can be taken one step further:

Theorem 2.3.2. *If $A \in \mathrm{Sp}(2n, \bar{k})$, then Inn_A keeps $\mathrm{Sp}(2n, k)$ invariant if and only if $A = pM$ for some $p \in \bar{k}$ and $M \in \mathrm{Sp}(2n, k)$.*

Thus, every inner automorphism that keeps $\mathrm{Sp}(2n, \bar{k})$ and $\mathrm{Sp}(2n, k)$ invariant, can be written as Inn_M for $M \in \mathrm{Sp}(2n, k)$. Also, inner automorphisms that keep $\mathrm{SO}(n, k, \beta)$ invariant for β a bilinear symmetric (non-degenerate) form on k^n have a nice form [Dom03].

Theorem 2.3.3. *Suppose β is a non-degenerate symmetric bilinear form with matrix M over k^n where the characteristic of k is not two. Then for $A \in \text{GL}(n, \bar{k})$, if Inn_A keeps $\text{SO}(n, \bar{k}, \beta)$ invariant if and only if $A = \alpha \hat{A}$ for $\alpha \in \bar{k}$ and:*

1. $\hat{A} \in \text{SO}(n, \bar{k}, \beta)$ if n is odd
2. $\hat{A} \in \text{SO}(n, \bar{k}, \beta)$ or $\hat{A} \in \text{O}(n, \bar{k}, \beta)$ and $\det(\hat{A}) = -1$, if n is even.

Taking this one step further, [Dom03] showed:

Theorem 2.3.4. *For $A \in \text{SO}(n, \bar{k}, \beta)$ or n even and $A \in \text{O}(n, \bar{k}, \beta)$, the inner automorphism Inn_A keeps $\text{SO}(n, k, \beta)$ invariant if and only if $A = \alpha \hat{A}$ for some $\alpha \in \bar{k}$ and:*

1. $\hat{A} \in \text{SO}(n, k, \beta)$ if n is odd
2. $\hat{A} \in \text{SO}(n, k, \beta)$ or $\hat{A} \in \text{O}(n, k, \beta)$ and $\det(\hat{A}) = -1$ if n is even.

Thus, every inner automorphism that keeps $\text{SO}(n, \bar{k}, \beta)$ and $\text{SO}(n, k, \beta)$ invariant, can be written as Inn_M for $M \in \text{SO}(n, k, \beta)$ or $M \in \text{O}(n, k, \beta)$ and $\det(M) = -1$, n even.

2.4 Commuting Pairs of Involutions over Algebraically Closed Fields

A classification of commuting pairs of involutions for G a connected, reductive algebraic group defined over an algebraically closed field was determined in [Hel88] and is briefly outlined below.

The main result on commuting involutions in this paper can be summarized as follows: for a root system of a certain torus, T , in G , the classification of commuting involutions of G can be reduced to the classification of commuting (admissible) involutions of the root system of T up to a certain 'quadratic element' of the torus T .

To begin, the following definition is needed:

Definition 2.4.1. Let T be a maximal torus of G , a connected reductive algebraic group over an algebraically closed field. An automorphism ϕ of G of order ≤ 2 is normally related to T if and only if $\phi(T) = T$ and $T_\phi^- = \{t \in T \mid \phi(t) = t^{-1}\}^0$ is a maximal ϕ -split torus of G .

With this, a correspondence between isomorphism at the group level and Weyl-group conjugacy at the associated root level can be given [Hel88]:

Theorem 2.4.2. *Let $\phi_1, \phi_2 \in \text{Aut}(G)$ be such that $\phi_1^2 = \phi_2^2 = \text{Id}$ and assume ϕ_1, ϕ_2 are normally related to T , for T a maximal torus of G . Then ϕ_1 and ϕ_2 are conjugate under $\text{Inn}(G)$ if and only if $\phi_1|_T$ and $\phi_2|_T$ are conjugate under $W(T)$, the Weyl group of T .*

Furthermore, as shown in the proof of the above theorem in [Hel88], if two involutions ϕ_1 and ϕ_2 , restricted to T are the same (on the torus) then, on the group G , $\phi_1 = \phi_2 \text{Inn}_t$ for some $t \in T_{\phi_2}^-$.

Before developing a full correspondence between involutions on the root level and involutions on the group level, one needs to determine which involutions on the root level can be 'lifted' to the group G .

Definition 2.4.3. An involution $\sigma \in \text{Aut}(X, \Phi)$ can be lifted if there is an involutorial automorphism $\phi \in \text{Aut}(G, T)$ inducing σ on (X, Φ) , i.e. $\phi|_T = \sigma$. Where Φ is the root system of T and $\text{Aut}(V, W)$ is the set of automorphisms, α , of V such that $\alpha(W) = W$.

Now, for T a maximal torus of G , involutions which can be lifted are given by the following [Hel88]

Proposition 2.4.4. *Let $\phi \in \text{Aut}(X, \Phi)$ be an involution and Δ a basis of Φ . Then the following are equivalent:*

1. ϕ can be lifted
2. There is a $t \in T$ such that $\phi_\Delta \text{Inn}_t$ is an involution, for ϕ_Δ defined by

$$\phi_\Delta(\chi_\alpha(\xi)) = \chi_{\phi(\alpha)}(\xi)$$

for all $\alpha \in \Delta, \xi \in k$, and χ_α the corresponding one-parameter additive subgroup of G defined by α .

3. There is a $t \in T$ such that $c_{\phi(\alpha), \phi_\Delta} = \alpha(\phi(t)t)$ for all $\alpha \in \Delta$, where $c_{\phi(\alpha), \phi_\Delta}$ are structure constants of G relative to $\{\chi_\alpha\}$.
4. There is a $t \in T_\phi^+$ such that $c_{\phi(\alpha), \phi_\Delta} = \alpha(t)$ for all $\alpha \in \Delta$.

A concept related to lifting involutions from the root system of a given connected, reductive algebraic group is determining whether or not an involution, ϕ of some root datum can be realized as a lifted involution, $\hat{\phi}$ of some reductive algebraic group G with maximal torus T with T_ϕ^- is a maximal $\hat{\phi}$ -split torus of G . In the case that this holds, the datum is said to be admissible. These indices are given in [Hel88].

Continuing to commuting involutions, a definition is needed first:

Definition 2.4.5. A torus A of G is called (σ, θ) -split if A is σ - and θ -split. Also, a torus T of G which is both σ and θ stable is called (σ, θ) -stable and define $T_{\sigma, \theta}^- = \{t \in T | \sigma(t) = \theta(t) = t\}^0$.

Given this definition, two results that are useful in the classification of commuting involutions are the following [Hel88]:

Lemma 2.4.6. 1. There exists a maximal torus of G , which is (σ, θ) -stable.

2. All maximal (σ, θ) -split tori of G are conjugate under $(G_\sigma \cap G_\theta)^0$, where $G_\phi = \{g \in G | \phi(g) = g\}^0$.

A notion of normally related exists for commuting pairs of involutions as well:

Definition 2.4.7. If (σ, θ) is a pair of commuting involutorial automorphisms of G and T is a maximal torus of G , then (σ, θ) is normally related to T if and only if $\phi(T) = \theta(T) = T$ and $T_{\sigma, \theta}^-, T_\sigma^-, T_\theta^-$ are maximal (σ, θ) -split, σ -split, and θ -split, respectively.

Also, it was shown in [Hel88] that such tori always exists, and they are all conjugate under $(G_\sigma \cap G_\theta)^0$.

The above leads to one of the main results of [Hel88], namely the relationship between classifying commuting involutions of a reductive algebraic group G and classifying commuting involutions of the associated root system:

Theorem 2.4.8. *Let (σ_1, θ_1) and (σ_2, θ_2) be pairs of commuting involutorial automorphisms of G , normally related to T . Then $(\sigma_1, \theta_1)|_T$ and $(\sigma_2, \theta_2)|_T$ are conjugate under $W(T)$ if and only if there exists $\epsilon \in T_{\sigma_1, \theta_1}^-$ with $\epsilon^2 \in Z(G)$ such that (σ_2, θ_2) is isomorphic to $(\sigma_1, \theta_1 \text{Inn}_\epsilon)$.*

The ϵ in the above theorem is called a quadratic element of the torus T . These elements are vital to the classification of commuting involutions. Before proceeding, let $C(T)$ denote the set of $W(T)$ conjugacy classes of ordered pairs of commuting involutions of the root system of a torus T . Since every pair of commuting involutorial automorphisms of G is isomorphic to one normally related to T , there is a map [Hel88] that takes pairs of commuting involutions of G to $C(T)$. Let $C(\sigma, \theta)$ denote the fiber above the image of the map on a pair (σ, θ) . With this, and letting F denote the family of all pairs of commuting involutorial automorphisms of a G , let $F(\sigma, \theta)$ denote the subset of F consisting of pairs of commuting involutions of G whose isomorphism classes are in $C(\sigma, \theta)$ and A a maximal (σ, θ) -split torus of G , as in [Hel88]. Then for $F_A(\sigma, \theta) = \{(\sigma, \theta \text{Inn}_\epsilon) | \epsilon \in A, \epsilon^2 \in Z(G)\}$. Thus, as shown in [Hel88], any pair in $F(\sigma, \theta)$ is isomorphic to a pair $(\sigma, \theta \text{Inn}_a) \in F_A(\sigma, \theta)$, thus, the isomorphism classes in $C(\sigma, \theta)$ can be represented by a set of quadratic elements of A . Finally, determining which quadratic elements a_i give isomorphic pairs $(\sigma, \theta \text{Inn}_{a_i})$ is given in [Hel88]:

Proposition 2.4.9. *Let (σ, θ) be a pair of commuting involutions of G , A a maximal (σ, θ) -split torus of G and $T \supset A$ a (σ, θ) -stable maximal torus of G such that T_σ^- is a maximal σ -split torus of G and T_θ^- is a maximal θ -split torus of G . Then two pairs $(\sigma, \theta \text{Inn}_{a_1})$ and $(\sigma, \theta \text{Inn}_{a_2})$ in $F_A(\sigma, \theta)$ are isomorphic under $Z_G(A)$ if and only if there exists a $t \in T$ such that $\sigma(t) = t$ and $a_1 a_2 = \theta(t) t^{-1}$.*

Thus, to classify commuting involutions of a connected, reductive algebraic group G over an algebraically closed field, one needs to determine Weyl group conjugacy classes of (admissible) commuting involutions of a root system (of some torus) and

then classify quadratic elements corresponding to pairs in $F_A(\sigma, \theta)$, for $A \subset T$ as above.

2.5 k -Involutions over Fields of Characteristic Not Two

As a classification of commuting involutions of an arbitrary connected, reductive algebraic group over algebraically closed fields (of characteristic not two) is given in [Hel88], a classification of single k -involutions of a connected, reductive algebraic group (defined over k) for k an arbitrary field of characteristic not two is discussed in [Hel00]. A few of the results of this paper are discussed below. First, a few definitions are needed [Hel00]

Definition 2.5.1. A mapping $\phi : G \rightarrow G$ is called a k -automorphism of G if ϕ is a bijective rational k -homomorphism whose inverse is a rational k -homomorphism, for G a connected, reductive algebraic k -group. Furthermore, for G and ϕ defined over k with ϕ an automorphism of G of order 2, ϕ is called a k -involution.

Definition 2.5.2. A k -torus A of G is called (θ, k) -split if it is both θ -split and k -split.

Because the classification of k -involutions of a group G defined over k is related to k -tori of G , some results on these tori are needed from [Hel00]:

Proposition 2.5.3. *Let A be a maximal (θ, k) -split torus of G . Then A is the unique maximal (θ, k) -split torus of $Z_G(A)$.*

Furthermore, for $G_\theta = \{g \in G \mid \theta(g) = g\}$, G reductive, and H a k -open subgroup of G_θ , the following is shown in [Hel00]:

Proposition 2.5.4. *Let A_1 and A_2 be maximal (θ, k) -split tori of G and \tilde{A}_1 , and \tilde{A}_2 be maximal k -split tori of G containing A_1 and A_2 , respectively. Then there exists a $g \in (Z_G(\tilde{A}_1)H^0)_k$ such that $g^{-1}A_1g = A_2$ and $g^{-1}\tilde{A}_1g = \tilde{A}_2$.*

As with the action of an involution on a group G over an algebraically closed field being given by an index (related to the associated root system of G), the actions of the Galois group Γ of K/k where for G a reductive k -group, T a maximal k -torus of G , K is a finite Galois extension of k which splits T [Hel00], and the involution θ can be described by an index [Hel00]. In doing this, one could hope for a characterization of k -involutions of G by characterizing admissible involutions of the root system of a certain torus; however, more is needed [Hel00]. It will be described later that there are three [Hel00] invariants needed to classify k -involutions. Working toward classifying k -involutions on a torus T in G , a few preliminaries from [Hel00] are needed:

Definition 2.5.5. Let A be a maximal k -split torus of G . A k -involution θ of G is normally related to A if $\theta(A) = A$ and A_{θ}^{-} is a maximal (θ, k) -split torus of G .

Lemma 2.5.6. *Let A be a maximal k -split torus of G . Every k -involution is G_k -isomorphic with one normally related to A .*

A result analogous to the relationship between involutions of a connected, reductive algebraic group and involutions of the associated torus (and hence root system) for k -groups G and k -involutions is the following [Hel00]:

Theorem 2.5.7. *Let G be a connected reductive semi-simple algebraic group defined over k , A a maximal k -split torus of G and θ_1, θ_2 be k -involutions of G , normally related to A . Then, θ_1 is G_k -isomorphic to $\theta_2 \text{Inn}_a$ for $a \in A_{\theta_2}^{-1}$ if and only if $\theta_1|_{Z_G(A)}$ and $\theta_2|_{Z_G(A)}$ are isomorphic under G_k .*

Note how this differs from the result for groups over algebraically closed fields: for k groups one needs to restrict to the centralizer of a maximal k -split torus (for split groups this is no different, since maximal k -split tori are maximal tori) and furthermore, one needs isomorphism over G_k on the restriction, unlike isomorphism over the Weyl group (of a torus T) on the restriction for the algebraically closed case. By results on [Hel00], one can do slightly better than isomorphism over G_k on the restriction. In fact, one only needs to use $N(A, \theta_2) = N_{G_k}(A).Z(A, \theta_2)$ for $Z(A, \theta_2)$ the set of $z_i \in Z_G(A)$ for $i \in I$ for I a certain index set stemming from a minimal parabolic k -subgroup of G acting on G_k/H_k [Hel00].

The above relates isomorphism of k -involutions of G on the group level to isomorphism on the restriction to the centralizer of a maximal k -split torus. As stated above, this is not enough for a full classification of k -involutions of G . Three invariants are needed [Hel00]:

1. classification of admissible k -involutions of $(X^*(T), \Phi(T), \Phi(A))$ for T a maximal torus of G containing a maximal k -split torus A of G .
2. classification of the G_k -isomorphism classes of k -involutions of the k -anisotropic kernel of G , i.e. the product of the derived group $Z_G(A)$ and the maximal anisotropic k -torus in the center of $Z_G(A)$ [Hum75]
3. classification of G_k -isomorphy classes of k -inner elements $a \in I_k(A_\theta^{-1})$

By synthesizing the results in [Hel00] and [Hel88] one could work towards a generalized result of commuting k -involutions of an arbitrary connected, reductive algebraic group defined over a field k of characteristic not two.

2.6 Symmetric k -Varieties

As the study of commuting involutions is motivated by the study of symmetric k -varieties and associated double coset decompositions, a very brief discussion is included. Following the work in [HP94], let G be a connected k -group and θ a k -automorphism of G of order two. The fixed point group of the k -involution θ is given by $G_\theta = \{g \in G \mid \theta(g) = g\}$ and it is a k -group that is also reductive (for G reductive). Also, as stated in [HP94], if G is semi-simple and simply connected, then G_θ is connected. A criterion for determining whether an involution of G is a k -involution is the following [HP94]:

Proposition 2.6.1. *Let G be a connected semi-simple algebraic k -group and θ an involution of G . Then θ is defined over k if and only if G_θ^0 is defined over k .*

Let H be a k -open subgroup of G_θ . Then for G reductive:

Definition 2.6.2. The variety G/H is called a symmetric variety and the variety G_k/H_k is called a symmetric k -variety.

There is a 'natural' root system associated to the symmetric space G/H [HP94]. Namely, for a torus T of G that is θ -stable (where H is defined as a k -open subgroup of the fixed point group of θ in G) and θ -split (recall a torus, A , is θ -split if and only if $\theta(a) = a^{-1}$ for all $a \in A$), if T is maximal θ -split it gives the 'natural' root system of G/H . Similarly, for T a maximal (θ, k) -split k -torus of G , the root system given by T is the 'natural' root system of the symmetric k -variety, G_k/H_k [HP94].

As the study of orbits of minimal parabolic k -subgroups acting on symmetric k -varieties play an important role in the study of such varieties [HP94], a result on the decomposition of the orbits is included as in [HP94]:

Theorem 2.6.3. *Let G be a reductive algebraic k -group, θ a k -involution of G , H a k -open subgroup of G_θ , and P a minimal parabolic k -subgroup of G . Then, for $\{A_i | i \in I\}$ be the set of representatives of the H_k -conjugacy classes of θ -stable maximal k -split tori in G , the following holds:*

$$H_k \backslash G_k / P_k \approx \bigcup_{i \in I} W_{H_k}(A_i) \backslash W_{G_k}(A_i)$$

for $W_{G_k}(A_i) = N_{G_k}(A_i) / Z_{G_k}(A_i)$ and $W_{H_k}(A_i)$ defined similarly.

Chapter 3

Preliminaries

Our basic references for reductive groups, Borel subgroups, and maximal tori will be the books [Bor91], [Hum75], and [Spr98]. For background on the classification of pairs of commuting involutions over algebraically closed fields, our basic reference will be the paper [Hel88]. We will follow their notations and terminology.

3.1 Some notation

Throughout this thesis, k will be a field of characteristic not equal to 2, and G is a reductive algebraic group defined over k . Let T be a maximal k -split torus in G , and P be a minimal parabolic subgroup of G . Let θ be an involution of G , and $K = G^\theta$ be the fixed point group of θ . Let G_k , K_k and T_k be the k -rational points of G , K and T , respectively. $W_{G_k}(T) = N_{G_k}(T)/Z_{G_k}(T)$ is the Weyl group of T in G_k , where $N_{G_k}(T)$ is the normalizer of T in G_k and $Z_{G_k}(T)$ is the centralizer of T in G_k . For a closed subgroup U of G we write U° for the connected component containing the identity. The following is some notation we will use throughout the rest of this paper.

Notation 3.1.1. 1. We will use \bar{q} to denote the entire square class of q .

2. By abuse of notation, we will use $q \in k^*/(k^*)^2$ to denote that q is the representative of the square class \bar{q} of k .

Notation 3.1.2. Let $E_{i,j}$ be the appropriate (clear from the context) size matrix with a 1 in the (i,j) th entry and zeros elsewhere.

Notation 3.1.3. 1. Let I_j be the $j \times j$ identity matrix. We will sometimes refer to the identity matrix as Id , where its size is apparent from the context, and with a slight abuse of notation also refer to the identity operator by Id .

2. Let $I_{n-i,i} \in \text{GL}(n, k)$ be the matrix given by:
$$\begin{pmatrix} I_{n-i} & \\ & -I_i \end{pmatrix}.$$

3. Let $L_{n,q}$ be the matrix given by:
$$\begin{pmatrix} 0 & 1 & & \\ q & 0 & & \\ & & \ddots & \\ & & & 0 & 1 \\ & & & q & 0 \end{pmatrix}.$$

4. Let $L'_{n,q}$ be the matrix given by: $\begin{pmatrix} 0 & I_{\frac{n}{2}} \\ qI_{\frac{n}{2}} & 0 \end{pmatrix}$ for n even.

5. Let $M_{r,s,t} \in \text{GL}(n, k)$ be the matrix given by:
$$\begin{pmatrix} I_{n-3} & & & \\ & r & & \\ & & s & \\ & & & t \end{pmatrix}.$$

6. Let J_{2m} be the matrix:
$$\begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}.$$

7. Let P_n be the permutation matrix: $\sum_{i=1}^{\frac{k}{2}} (E_{i,2i-1} + E_{\frac{k}{2}+i,2i})$

Notation 3.1.4. Let $M(n, k)$ denote $n \times n$ matrices over the field k , and $M(n \times m, k)$ denote $n \times m$ matrices over the field k .

Notation 3.1.5. Let Inn_A be the mapping defined by $\text{Inn}_A(X) = A^{-1}XA$. We will sometimes refer this mapping by its matrix representative, A .

Notation 3.1.6. A commuting pair of involutions of $\mathrm{SL}(n, k)$ will sometimes be referred to as such, in addition to being referred to as: a pair, a commuting pair, or a commuting pair of involutions.

3.2 θ -stable Tori

T is θ -stable if $\theta(T) = T$. Then, $T = T^+T^-$, where

1. $T^+ = \{t \in T \mid \theta(t) = t\}^\circ$
2. $T^- = \{t \in T \mid \theta(t) = t^{-1}\}^\circ$. The torus T^- is called θ -split.

Moreover, $T^+ \cap T^-$ is finite.

Definition 3.2.1. A torus T is called (θ, k) -split if T is k -split and θ -split. We will let \mathcal{A}_θ denote the set of maximal (θ, k) -split tori.

All maximal k -split tori of G are G -conjugate. If T is a maximal torus, then $Z_G(T) = T$. By a result in [Ste68], any Borel subgroup contains a θ -stable maximal torus.

3.3 Useful Results

Lemma 3.3.1. *Every pair of commuting inner involutions is isomorphic to the pair $(\mathrm{Inn}_Y, \mathrm{Inn}_A)$ for either:*

1. $Y = I_{n-i, i} \in \mathrm{GL}(n, k)$ where $i \in \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$
2. $Y = L_{n, q} \in \mathrm{GL}(n, k)$ where $q \in k^*/k^{*2}$ with $q \neq 1 \pmod{k^{*2}}$, n even.

and these are distinct for distinct $i \in \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$ and distinct $q \in k^*/k^{*2}$ with $q \neq 1 \pmod{k^{*2}}$.

Proof. This follows immediately from the definition of isomorphic pairs and the classification of inner involutions of $\mathrm{SL}(n, k)$ found in [HWD06]. \square

We will need another lemma from [Sil00].

Lemma 3.3.2. *For $A, B, C, D \in M(\frac{n}{2}, k)$, $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BC)$ when $CD = DC$*

Chapter 4

Classification of Commuting Pairs of Inner Involutions

Using the results of [HWD06], we classify the pairs of commuting inner involutions of $\mathrm{SL}(n, k)$, for k a field of characteristic not 2. A pair of commuting inner involutions will have the form $(\mathrm{Inn}_A, \mathrm{Inn}_B)$ for $A, B \in \mathrm{GL}(n, k)$, where $\mathrm{Inn}_A(X) = A^{-1}XA$, $\forall X \in \mathrm{SL}(n, k)$.

Dropping the Inn_A notation, we will sometimes refer to the matrix $A \in \mathrm{GL}(n, k)$ for the corresponding inner automorphism, Inn_A , and (A, B) for the pair $(\mathrm{Inn}_A, \mathrm{Inn}_B)$. Note that Inn_A is an automorphism when $A \in \mathrm{GL}(n, k)$ and is an involution when $A^2 = p\mathrm{Id}$. Also note that $\mathrm{Inn}_A \approx \mathrm{Inn}_B$ if and only if there is a $C \in \mathrm{GL}(n, k)$ such that $C^{-1}AC = cB$ for $c \in \bar{k}$, as in [HWD06]. We will first focus on pairs of the form $(\mathrm{Inn}_Y, \mathrm{Inn}_A)$ for $Y = L_{n,q}$ and then for $Y = I_{n-i,i}$, because every pair of involutions of $\mathrm{SL}(n, k)$ has the first entry in the pair isomorphic to one of these [HWD06]. In each case, the form of the matrix A in the pair $(\mathrm{Inn}_Y, \mathrm{Inn}_A)$ is determined first. Then, the fact that Inn_A is an involution of $\mathrm{SL}(n, k)$ is used next to 'simplify' the form of A while keeping the form of Y fixed. After this is completed for each Y , there are only a few possible isomorphism classes of commuting inner involutions of $\mathrm{SL}(n, k)$. The last step, then, is to determine the distinctness of the remaining pairs.

4.1 Commuting Inner Involutions with the First Pair Isomorphic to $\text{Inn}_{L_{n,q}}$

As stated above, we first consider pairs of commuting inner involutions of the form $(\text{Inn}_{L_{n,q}}, \text{Inn}_A)$, for $q \not\equiv 1 \pmod{k^*2}$.

Lemma 4.1.1. $(\text{Inn}_{L_{n,q}}, \text{Inn}_A)$, is a pair of commuting involutions only if $A = (A_{i,j})$ and one of the following:

$$A_{i,j} = \begin{pmatrix} a_{i,j} & b_{i,j} \\ qb_{i,j} & a_{i,j} \end{pmatrix} \text{ or } A_{i,j} = \begin{pmatrix} a_{i,j} & b_{i,j} \\ -qb_{i,j} & -a_{i,j} \end{pmatrix}$$

Proof. Let $A = (A_{i,j})_{i,j=1}^{n/2}$ and $Y = L_{\frac{n}{2},q} = (Y_{i,j})_{i,j=1}^{n/2}$, where $A_{i,j} = \begin{pmatrix} a_{i,j} & b_{i,j} \\ c_{i,j} & d_{i,j} \end{pmatrix}$ and $Y_{i,j} = \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}$ for $i = j$ and $Y_{i,j} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for $i \neq j$, then $\text{Inn}_Y \text{Inn}_A = \text{Inn}_A \text{Inn}_Y$
 $\leftrightarrow (\text{Inn}_Y)^{-1} (\text{Inn}_A)^{-1} \text{Inn}_Y \text{Inn}_A = \text{Id} \leftrightarrow \text{Inn}_{AYA^{-1}Y^{-1}} = \text{Id} \leftrightarrow AY A^{-1}Y^{-1} = cI, c \in \bar{k} - \{0\} \leftrightarrow AY = cYA \leftrightarrow (A_{i,j})_{i,j=1}^{n/2} (Y_{i,j})_{i,j=1}^{n/2} = c (Y_{i,j})_{i,j=1}^{n/2} (A_{i,j})_{i,j=1}^{n/2} \leftrightarrow (Y_{i,i} A_{i,j})_{i,j=1}^{n/2} = c (A_{i,j} Y_{i,i})_{i,j=1}^{n/2} \leftrightarrow$ For each fixed $i, j \in \{1, 2, \dots, n/2\}$,

$$\begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix} \begin{pmatrix} a_{i,j} & b_{i,j} \\ c_{i,j} & d_{i,j} \end{pmatrix} = c \begin{pmatrix} a_{i,j} & b_{i,j} \\ c_{i,j} & d_{i,j} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix} \\ \leftrightarrow \begin{pmatrix} c_{i,j} & d_{i,j} \\ qa_{i,j} & qb_{i,j} \end{pmatrix} = c \begin{pmatrix} qb_{i,j} & a_{i,j} \\ qd_{i,j} & c_{i,j} \end{pmatrix},$$

if and only if $c_{i,j} = cqb_{i,j}$, $d_{i,j} = ca_{i,j}$, $qa_{i,j} = cq d_{i,j}$, and $qb_{i,j} = cc_{i,j}$. This implies that $c_{i,j} = c^2 c_{i,j}$, and $d_{i,j} = c^2 d_{i,j}$, giving us $c^2 = 1$ or $c_{i,j} = 0 = d_{i,j}$. But if $c_{i,j} = 0 = d_{i,j}$ then $a_{i,j} = 0 = b_{i,j}$ since $q, c \neq 0 \rightarrow A$ singular, a contradiction since Inn_A is an automorphism $\rightarrow c^2 = 1 \rightarrow c = \pm 1 \rightarrow c_{i,j} = qb_{i,j}$ and $d_{i,j} = a_{i,j}$ or $c_{i,j} = -qb_{i,j}$ and $d_{i,j} = -a_{i,j}$. \square

Therefore, for $q \in k^*/k^{*2}, q \not\equiv 1 \pmod{k^*2}$ and suppressing the Inn notation, com-

muting pairs:

$$\left(\begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ q & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & q & 0 \end{pmatrix}, \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \dots & a_{n,n-1} & a_{n,n} \end{pmatrix} \right)$$

are one of the following forms:

$$\left(\begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ q & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & q & 0 \end{pmatrix}, \begin{pmatrix} a_{1,1} & b_{1,1} & \dots & a_{1,n/2} & b_{1,n/2} \\ qb_{1,1} & a_{1,1} & \dots & qb_{1,n/2} & a_{1,n/2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n/2,1} & b_{n/2,1} & \dots & a_{n/2,n/2} & b_{n/2,n/2} \\ qb_{n/2,1} & a_{n/2,1} & \dots & qb_{n/2,n/2} & a_{n/2,n/2} \end{pmatrix} \right) \text{ or} \quad (4.1)$$

$$\left(\begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ q & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & q & 0 \end{pmatrix}, \begin{pmatrix} a_{1,1} & b_{1,1} & \dots & a_{1,n/2} & b_{1,n/2} \\ -qb_{1,1} & -a_{1,1} & \dots & -qb_{1,n/2} & -a_{1,n/2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n/2,1} & b_{n/2,1} & \dots & a_{n/2,n/2} & b_{n/2,n/2} \\ -qb_{n/2,1} & -a_{n/2,1} & \dots & -qb_{n/2,n/2} & -a_{n/2,n/2} \end{pmatrix} \right). \quad (4.2)$$

We now prove a lemma about pairs of the above two forms that will aid in their classification. Partition the matrix A above as follows:

$$\begin{pmatrix} A_1 & U \\ V & C \end{pmatrix}$$

for $A_1 \in M(n-4, k)$, $U \in M(n-4 \times 4, k)$, $V \in M(4 \times n-4, k)$, $C \in M(4, k)$. The following lemma gives a sufficient condition to 'zero-out' blocks in the matrix A , which we will see makes the classification of pairs much easier.

Lemma 4.1.2. *If for some $M_i \in \text{GL}(4, k)$, with $M_i^2 = pI_4$, we have $\begin{pmatrix} A_1 & U \\ V & C+M_i \end{pmatrix}$ is invertible and commutes (up to a scalar multiple) with $L_{n,q}$, then we get the following isomorphism if $A^2 = pI$:*

$$\left(L_{n,q}, \begin{pmatrix} A_1 & U \\ V & C \end{pmatrix} \right) \approx \left(L_{n,q}, \begin{pmatrix} A'_1 & 0_{(n-4) \times 4} \\ 0_{4 \times (n-4)} & M_i \end{pmatrix} \right).$$

Proof. Suppose that $A^2 = pI$, then:

$$\begin{pmatrix} A_1^2 + UV & A_1U + UC \\ VA_1 + CV & VU + C^2 \end{pmatrix} = \begin{pmatrix} pI_{n-4} & 0 \\ 0 & pI_4 \end{pmatrix}. \quad (4.3)$$

This gives the following relations:

$$A_1^2 + UV = pI_{n-4} \quad (4.4)$$

$$A_1U + UC = 0 \quad (4.5)$$

$$VA_1 + CV = 0 \quad (4.6)$$

$$VU + C^2 = pI_4 \quad (4.7)$$

But then,

$$\begin{aligned} & \begin{pmatrix} A_1 & U \\ V & C + M_i \end{pmatrix} \begin{pmatrix} A_1 & U \\ V & C \end{pmatrix} \begin{pmatrix} A_1 & U \\ V & C + M_i \end{pmatrix}^{-1} \\ &= \begin{pmatrix} A_1^2 + UV & A_1U + UC \\ VA_1 + CV + M_iV & VU + C^2 + M_iC \end{pmatrix} \begin{pmatrix} A_1 & U \\ V & C + M_i \end{pmatrix}^{-1} \\ &= \begin{pmatrix} pI_{n-4} & 0 \\ M_iV & pI_4 + M_iC \end{pmatrix} \begin{pmatrix} A_1 & U \\ V & C + M_i \end{pmatrix}^{-1} \\ &= \begin{pmatrix} I_{n-4} & 0 \\ 0 & M_i \end{pmatrix} \begin{pmatrix} pI_{n-4} & 0 \\ V & C + M_i \end{pmatrix} \begin{pmatrix} A_1 & U \\ V & C + M_i \end{pmatrix}^{-1}, \end{aligned}$$

since, from above, we know $M_i^2 = pI_4$. Also, because, by supposition, $\begin{pmatrix} A_1 & U \\ V & C + M_i \end{pmatrix}$

is invertible, there exists a matrix: $\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$ such that

$$\begin{pmatrix} A_1 & U \\ V & C + M_i \end{pmatrix} \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} = I_n,$$

i.e.

$$\begin{pmatrix} A_1X_1 + UX_3 & A_1X_2 + UX_4 \\ VX_1 + (C + M_i)X_3 & VX_2 + (C + M_i)X_4 \end{pmatrix} = \begin{pmatrix} I_{n-4} & 0 \\ 0 & I_4 \end{pmatrix}.$$

This results in the following relations:

$$VX_1 + (C + M_i)X_3 = 0 \quad (4.8)$$

$$VX_2 + (C + M_i)X_4 = I_4 \quad (4.9)$$

This gives:

$$\begin{aligned} & \begin{pmatrix} I_{n-4} & 0 \\ 0 & M_i \end{pmatrix} \begin{pmatrix} pI_{n-4} & 0 \\ V & C + M_i \end{pmatrix} \begin{pmatrix} A_1 & U \\ V & C + M_i \end{pmatrix}^{-1} \\ = & \begin{pmatrix} I_{n-4} & 0 \\ 0 & M_i \end{pmatrix} \begin{pmatrix} pI_{n-4} & 0 \\ V & C + M_i \end{pmatrix} \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \\ = & \begin{pmatrix} I_{n-4} & 0 \\ 0 & M_i \end{pmatrix} \begin{pmatrix} pX_1 & pX_2 \\ VX_1 + (C + M_i)X_3 & VX_2 + (C + M_i)X_4 \end{pmatrix} \\ = & \begin{pmatrix} I_{n-4} & 0 \\ 0 & M_i \end{pmatrix} \begin{pmatrix} pX_1 & pX_2 \\ 0 & I_4 \end{pmatrix} \\ = & \begin{pmatrix} pX_1 & pX_2 \\ 0 & M_i \end{pmatrix}. \end{aligned}$$

This implies that,

$$\begin{pmatrix} pX_1 & pX_2 \\ 0 & M_i \end{pmatrix}^2 = \begin{pmatrix} pI_{n-4} & 0 \\ 0 & pI_4 \end{pmatrix},$$

because the corresponding involution is isomorphic to Inn_A , but then

$$\begin{aligned} (pX_1)^2 &= pI_{n-4}, \\ pX_1pX_2 + pX_2M_i &= 0. \end{aligned}$$

Also, since k has characteristic not 2,

$$\frac{1}{p} \begin{pmatrix} 2pX_1 & pX_2 \\ 0 & M_i \end{pmatrix} \begin{pmatrix} \frac{1}{2}pX_1 & \frac{1}{2}pX_2 \\ 0 & M_i \end{pmatrix} = \begin{pmatrix} I_{n-4} & 0 \\ 0 & I_4 \end{pmatrix},$$

i.e. $\begin{pmatrix} 2pX_1 & pX_2 \\ 0 & M_i \end{pmatrix}$ is invertible (assuming $\begin{pmatrix} A_1 & U \\ V & C + M_i \end{pmatrix}$ is invertible) so that,

$$\begin{aligned} & \begin{pmatrix} 2pX_1 & pX_2 \\ 0 & M_i \end{pmatrix} \begin{pmatrix} pX_1 & pX_2 \\ 0 & M_i \end{pmatrix} \begin{pmatrix} 2pX_1 & pX_2 \\ 0 & M_i \end{pmatrix}^{-1} = \\ & \begin{pmatrix} 2(pX_1)^2 & 2pX_1pX_2 + pX_2M_i \\ 0 & M_i^2 \end{pmatrix} \begin{pmatrix} 2pX_1 & pX_2 \\ 0 & M_i \end{pmatrix}^{-1} = \\ & \begin{pmatrix} 2pI_{n-4} & pX_1pX_2 \\ 0 & pI_4 \end{pmatrix} \begin{pmatrix} 2pX_1 & pX_2 \\ 0 & M_i \end{pmatrix}^{-1} = \\ & \begin{pmatrix} pX_1 & 0 \\ 0 & M_i \end{pmatrix} \begin{pmatrix} 2pX_1 & pX_2 \\ 0 & M_i \end{pmatrix} \begin{pmatrix} 2pX_1 & pX_2 \\ 0 & M_i \end{pmatrix}^{-1} = \\ & \begin{pmatrix} pX_1 & 0 \\ 0 & M_i \end{pmatrix}, \end{aligned}$$

by the above relations and that $M_i^2 = pI_4$.

It remains to show that $\begin{pmatrix} 2pX_1 & pX_2 \\ 0 & M_i \end{pmatrix}$ commutes (up to a scalar multiple) with $L_{n,q}$. It was shown above that $\begin{pmatrix} pX_1 & pX_2 \\ 0 & M_i \end{pmatrix}$ commutes with $L_{n,q}$:

$$\left(L_{n,q}, \begin{pmatrix} A_1 & U \\ V & C \end{pmatrix} \right)$$

is a commuting pair (with Inn notation suppressed) isomorphic to

$$\left(L_{n,q}, \begin{pmatrix} pX_1 & pX_2 \\ 0 & M_i \end{pmatrix} \right)$$

because, by supposition, $L_{n,q}$ and $\begin{pmatrix} A_1 & U \\ V & C + M_i \end{pmatrix}$ commute (as involutions). Parti-

tion the matrix $L_{n,q}$ into blocks: $\begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}$, with $L_1 \in \text{GL}(n-4, k)$ and $L_2 \in \text{GL}(4, k)$

which is possible because we are assuming n is even. Then,

$$\begin{aligned} \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix} \begin{pmatrix} pX_1 & pX_2 \\ 0 & M_i \end{pmatrix} &= c \begin{pmatrix} pX_1 & pX_2 \\ 0 & M_i \end{pmatrix} \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix} \\ \rightarrow \begin{pmatrix} pL_1X_1 & pL_1X_2 \\ 0 & L_2M_i \end{pmatrix} &= c \begin{pmatrix} pX_1L_1 & pX_2L_2 \\ 0 & M_iL_2 \end{pmatrix} \end{aligned}$$

for some $c \in \bar{k}$. This gives the relations:

$$pL_1X_1 = cpX_1L_1 \quad (4.10)$$

$$pL_1X_2 = cpX_2L_2 \quad (4.11)$$

$$L_2M_i = cM_iL_2 \quad (4.12)$$

And because the characteristic of k is not two, equation (4.10) implies that $2pL_1X_1 = 2cpX_1L_1$, so that

$$\begin{aligned} \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix} \begin{pmatrix} 2pX_1 & pX_2 \\ 0 & M_i \end{pmatrix} &= \begin{pmatrix} 2pL_1X_1 & pL_1X_2 \\ 0 & L_2M_i \end{pmatrix} \\ &= \begin{pmatrix} 2cpX_1L_1 & cpX_2L_2 \\ 0 & cM_iL_2 \end{pmatrix} \\ &= c \begin{pmatrix} 2pX_1 & pX_2 \\ 0 & M_i \end{pmatrix} \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}. \end{aligned}$$

□

Before proceeding to the next theorem, some small results are needed.

Lemma 4.1.3. *For a fixed $q \in k$, the set $\left\{ \begin{pmatrix} x & y \\ qy & x \end{pmatrix} \mid x, y \in k \right\}$ is commutative.*

Proof.

$$\begin{pmatrix} x & y \\ qy & x \end{pmatrix} \begin{pmatrix} w & z \\ qz & w \end{pmatrix} = \begin{pmatrix} xw + qyz & xz + yw \\ qyw + qxz & qyz + xw \end{pmatrix} = \begin{pmatrix} w & z \\ qz & w \end{pmatrix} \begin{pmatrix} x & y \\ qy & x \end{pmatrix}$$

□

Lemma 4.1.4. $\forall x, y \in k, \epsilon = \pm 1$, if $\det \begin{pmatrix} x & y \\ \epsilon qy & \epsilon x \end{pmatrix} = 0$, then $\begin{pmatrix} x & y \\ \epsilon qy & \epsilon x \end{pmatrix} = 0$, when $q \not\equiv 1 \pmod{k^{*2}}$.

Proof. $\det \begin{pmatrix} x & y \\ \epsilon qy & \epsilon x \end{pmatrix} = \epsilon(x^2 - qy^2)$ which is zero if and only if $x^2 = qy^2$. If $y = 0$, then $x = 0$ and $\begin{pmatrix} x & y \\ \epsilon qy & \epsilon x \end{pmatrix} = 0$, or $y \neq 0$ which implies $q = \frac{x^2}{y^2} = \left(\frac{x}{y}\right)^2 \equiv 1 \pmod{k^{*2}}$, a contradiction. \square

Let $M_1, M_2, M_3, \tilde{M}_1$, and \tilde{M}_2 be the matrices:

$$\begin{aligned} & \pm \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & q & 0 \\ 0 & \frac{p}{q} & 0 & 0 \\ p & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ p-1 & 0 & -1 & 0 \\ 0 & p-1 & 0 & -1 \end{pmatrix}, \\ & \pm \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ p & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -q & -1 \\ \frac{p}{1-q} & \frac{p}{1-q} & 0 & 0 \\ -\frac{pq}{1-q} & -\frac{p}{1-q} & 0 & 0 \end{pmatrix}, \end{aligned}$$

respectively, and note that $M_1^2 = M_2^2 = M_3^2 = pI_{4 \times 4}$, and $\tilde{M}_1^2 = \tilde{M}_2^2 = pI_{4 \times 4}$.

Theorem 4.1.5. For $A = (A_{i,j})_{i,j=1}^n = \begin{pmatrix} A_1 & U \\ V & C \end{pmatrix}$, $M_1, M_2, M_3, \tilde{M}_1, \tilde{M}_2$ as above, we have one of the following:

1. $A_{i,j} = \begin{pmatrix} a_{i,j} & b_{i,j} \\ qb_{i,j} & a_{i,j} \end{pmatrix}$ and $\begin{pmatrix} A_1 & U \\ V & C + M \end{pmatrix}$ is invertible for $M = M_1, M_2$, or M_3 , or $\exists a, b \in k$ such that $a^2 + qb^2 = p$, $2ab = 0$.
2. $A_{i,j} = \begin{pmatrix} a_{i,j} & b_{i,j} \\ -qb_{i,j} & -a_{i,j} \end{pmatrix}$ and $\begin{pmatrix} A_1 & U \\ V & C + M \end{pmatrix}$ is invertible for $M = \tilde{M}_1$ or \tilde{M}_2 .

Proof. Since $\begin{pmatrix} A_1 & U \\ V & C \end{pmatrix}^2 = pI_n$, we know $\begin{pmatrix} A_1 & U \\ V & C \end{pmatrix}$ is invertible. Therefore, the matrix $\begin{pmatrix} A_1 & U \\ V & C + M_i \end{pmatrix}$ is invertible if and only if

$$\begin{aligned} \begin{pmatrix} A_1 & U \\ V & C + M_i \end{pmatrix} \begin{pmatrix} A_1 & U \\ V & C \end{pmatrix} &= \begin{pmatrix} A_1^2 + UV & A_1U + UC \\ VA_1 + CV + M_iV & VU + C^2 + M_iC \end{pmatrix} \\ &= \begin{pmatrix} pI_{n-4} & 0 \\ M_iV & pI_4 + M_iC \end{pmatrix} \text{ (by equations (4.4) - (4.7))} \end{aligned}$$

is invertible, which is invertible if and only if $pI_4 + MC$ is invertible, by lemma 3.3.2.

We show first that if $pI_4 + MC$ is singular for $M = M_i$, $i = 1, 2, 3$ then there exists a solution $a^2 + qb^2 = p, 2ab = 0$, for $a, b \in k$, and second we show that $pI_4 + MC$ is nonsingular for some $M = \tilde{M}_i$, $i = 1, 2$, corresponding to parts (1) and (2) of the theorem, respectively.

Considering the first part, suppose that $pI_4 + M_iC$ is singular for $i = 1, 2, 3$. We know, by supposition, that:

$$C = \begin{pmatrix} a & b & c & d \\ qb & a & qd & c \\ e & f & g & h \\ qf & e & qh & g \end{pmatrix},$$

for $a, b, c, d, e, f, g, h \in k, q \not\equiv 1 \pmod{k^*2}$. Now, $\det(pI_4 + M_1C) = 0$, implies that:

$$\det \begin{pmatrix} p \pm e & \pm f & \pm g & \pm h \\ \pm qf & p \pm e & \pm qh & \pm g \\ \pm pa & \pm pb & p \pm pc & \pm pd \\ \pm qpb & \pm pa & \pm qpd & p \pm pc \end{pmatrix} = 0$$

along with lemmas 3.3.2 and 4.1.3, imply that

$$0 = \det \left(\begin{pmatrix} p \pm e & \pm f \\ \pm qf & p \pm e \end{pmatrix} \begin{pmatrix} p \pm pc & \pm pd \\ \pm qpd & p \pm pc \end{pmatrix} - p \begin{pmatrix} g & h \\ qh & g \end{pmatrix} \begin{pmatrix} a & b \\ qb & a \end{pmatrix} \right)$$

and thus

$$\begin{aligned} 0 &= \det \left(\begin{pmatrix} p+e & f \\ qf & p+e \end{pmatrix} \begin{pmatrix} p+pc & pd \\ qpd & p+pc \end{pmatrix} - p \begin{pmatrix} g & h \\ qh & g \end{pmatrix} \begin{pmatrix} a & b \\ qb & a \end{pmatrix} \right) \\ &= \det \left(\begin{pmatrix} p-e & -f \\ -qf & p-e \end{pmatrix} \begin{pmatrix} p-pc & -pd \\ -qpd & p-pc \end{pmatrix} - p \begin{pmatrix} g & h \\ qh & g \end{pmatrix} \begin{pmatrix} a & b \\ qb & a \end{pmatrix} \right). \end{aligned}$$

By lemma 4.1.4 this gives:

$$\begin{aligned} 0 &= \left(\begin{pmatrix} p-e & -f \\ -qf & p-e \end{pmatrix} \begin{pmatrix} p-pc & -pd \\ -qpd & p-pc \end{pmatrix} - p \begin{pmatrix} g & h \\ qh & g \end{pmatrix} \begin{pmatrix} a & b \\ qb & a \end{pmatrix} \right) \\ &= \left(\begin{pmatrix} p-e & -f \\ -qf & p-e \end{pmatrix} \begin{pmatrix} p-pc & -pd \\ -qpd & p-pc \end{pmatrix} - p \begin{pmatrix} g & h \\ qh & g \end{pmatrix} \begin{pmatrix} a & b \\ qb & a \end{pmatrix} \right), \end{aligned}$$

and thus

$$\begin{aligned} p^2 \begin{pmatrix} c & d \\ qd & c \end{pmatrix} + p \begin{pmatrix} e & f \\ qf & e \end{pmatrix} &= -p^2 \begin{pmatrix} c & d \\ qd & c \end{pmatrix} - p \begin{pmatrix} e & f \\ qf & e \end{pmatrix} \\ &\rightarrow 2p \left(p \begin{pmatrix} c & d \\ qd & c \end{pmatrix} + \begin{pmatrix} e & f \\ qf & e \end{pmatrix} \right) = 0 \\ &\rightarrow p \begin{pmatrix} c & d \\ qd & c \end{pmatrix} + \begin{pmatrix} e & f \\ qf & e \end{pmatrix} = 0, \end{aligned}$$

since the characteristic of k is not 2 and $p \neq 0$. Therefore,

$$\begin{pmatrix} e & f \\ qf & e \end{pmatrix} = -p \begin{pmatrix} c & d \\ qd & c \end{pmatrix}. \quad (4.13)$$

Also, $\det(pI_4 + M_2C) = 0$ implies that

$$\det \begin{pmatrix} p \pm qf & \pm e & \pm qh & \pm g \\ \pm qe & p \pm qf & \pm qg & \pm qh \\ \pm pb & \pm \frac{p}{q}a & p \pm pd & p \pm \frac{p}{q}c \\ \pm pa & \pm pb & p \pm pc & p \pm pd \end{pmatrix} = 0,$$

along with lemmas 3.3.2 and 4.1.3 gives:

$$\det \left(\begin{pmatrix} p \pm qf & \pm e \\ \pm qe & p \pm qf \end{pmatrix} \begin{pmatrix} p \pm pd & \pm \frac{p}{q}c \\ \pm pc & p \pm pd \end{pmatrix} - \frac{p}{q} \begin{pmatrix} qh & g \\ qg & qh \end{pmatrix} \begin{pmatrix} qb & a \\ qa & qb \end{pmatrix} \right) = 0$$

and thus

$$\begin{aligned} 0 &= \det \left(\begin{pmatrix} p+qf & e \\ qe & p+qf \end{pmatrix} \begin{pmatrix} p+pd & \frac{p}{q}c \\ pc & p+pd \end{pmatrix} - \frac{p}{q} \begin{pmatrix} qh & g \\ qg & qh \end{pmatrix} \begin{pmatrix} qb & a \\ qa & qb \end{pmatrix} \right) \\ &= \det \left(\begin{pmatrix} p- qf & -e \\ -qe & p- qf \end{pmatrix} \begin{pmatrix} p- pd & -\frac{p}{q}c \\ -pc & p- pd \end{pmatrix} - \frac{p}{q} \begin{pmatrix} qh & g \\ qg & qh \end{pmatrix} \begin{pmatrix} qb & a \\ qa & qb \end{pmatrix} \right). \end{aligned}$$

By lemma 4.1.4,

$$\begin{aligned} 0 &= \left(\begin{pmatrix} p+qf & e \\ qe & p+qf \end{pmatrix} \begin{pmatrix} p+pd & \frac{p}{q}c \\ pc & p+pd \end{pmatrix} - \frac{p}{q} \begin{pmatrix} qh & g \\ qg & qh \end{pmatrix} \begin{pmatrix} qb & a \\ qa & qb \end{pmatrix} \right) \\ &= \left(\begin{pmatrix} p- qf & -e \\ -qe & p- qf \end{pmatrix} \begin{pmatrix} p- pd & -\frac{p}{q}c \\ -pc & p- pd \end{pmatrix} - \frac{p}{q} \begin{pmatrix} qh & g \\ qg & qh \end{pmatrix} \begin{pmatrix} qb & a \\ qa & qb \end{pmatrix} \right), \end{aligned}$$

which implies

$$\begin{aligned} \frac{p^2}{q} \begin{pmatrix} qd & c \\ qc & qd \end{pmatrix} + p \begin{pmatrix} qf & e \\ qe & qf \end{pmatrix} &= -\frac{p^2}{q} \begin{pmatrix} qd & c \\ qc & qd \end{pmatrix} - p \begin{pmatrix} qf & e \\ qe & qf \end{pmatrix} \\ \rightarrow 2p \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix} \left(\frac{p}{q} \begin{pmatrix} c & d \\ qd & c \end{pmatrix} + \begin{pmatrix} e & f \\ qf & e \end{pmatrix} \right) &= 0 \\ \rightarrow \frac{p}{q} \begin{pmatrix} c & d \\ qd & c \end{pmatrix} + \begin{pmatrix} e & f \\ qf & e \end{pmatrix} &= 0, \end{aligned}$$

since the characteristic of k is not 2, $p \neq 0$, and $\begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}$ is invertible. Therefore,

$$\begin{pmatrix} e & f \\ qf & e \end{pmatrix} = -\frac{p}{q} \begin{pmatrix} c & d \\ qd & c \end{pmatrix}. \quad (4.14)$$

Putting (4.13) and (4.14) together,

$$-p \begin{pmatrix} c & d \\ qd & c \end{pmatrix} = \begin{pmatrix} e & f \\ qf & e \end{pmatrix} = -\frac{p}{q} \begin{pmatrix} c & d \\ qd & c \end{pmatrix}$$

so that $c, d = 0$, since if $-p = -\frac{p}{q}$ then $q = \frac{p}{p} = 1$, a contradiction to the fact that $q \not\equiv 1 \pmod{k^{*2}}$ and hence $e, f = 0$. Plugging back into $pI_4 + M_1C$, (which is singular by supposition) we get

$$\begin{aligned} \begin{pmatrix} p^2 & 0 \\ 0 & p^2 \end{pmatrix} - p \begin{pmatrix} g & h \\ qh & g \end{pmatrix} \begin{pmatrix} a & b \\ qb & a \end{pmatrix} &= 0 \\ \rightarrow \begin{pmatrix} g & h \\ qh & g \end{pmatrix} &= p \begin{pmatrix} a & b \\ qb & a \end{pmatrix}^{-1} \end{aligned} \quad (4.15)$$

(we know that $\begin{pmatrix} a & b \\ qb & a \end{pmatrix}$ is invertible since it multiplies with $\begin{pmatrix} g & h \\ qh & g \end{pmatrix}$ to get pI_4). Using that $pI_4 + M_3C$ is also singular, $\det(pI_4 + M_3C) = 0$, i.e.

$$\det \begin{pmatrix} \begin{pmatrix} p+a & b \\ qb & p+a \end{pmatrix} & p \begin{pmatrix} a & b \\ qb & a \end{pmatrix}^{-1} \\ (p-1) \begin{pmatrix} a & b \\ qb & a \end{pmatrix} & \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} - p \begin{pmatrix} a & b \\ qb & a \end{pmatrix}^{-1} \end{pmatrix} = 0$$

which implies, by lemmas 3.3.2 and 4.1.3, p not zero, and $q \not\equiv 1 \pmod{k^{*2}}$, that

$$\det \left(\begin{pmatrix} p+a & b \\ qb & p+a \end{pmatrix} \left(\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} - p \begin{pmatrix} a & b \\ qb & a \end{pmatrix}^{-1} \right) - p(p-1)I_2 \right) = 0,$$

so that by lemma 4.1.4,

$$\begin{aligned} \begin{pmatrix} p^2 & 0 \\ 0 & p^2 \end{pmatrix} - p^2 \begin{pmatrix} a & b \\ qb & a \end{pmatrix}^{-1} + p \begin{pmatrix} a & b \\ qb & a \end{pmatrix} - pI_2 - p(p-1)I_2 &= 0 \\ \rightarrow pI_2 - p \begin{pmatrix} a & b \\ qb & a \end{pmatrix}^{-1} + \begin{pmatrix} a & b \\ qb & a \end{pmatrix} - I_2 - (p-1)I_2 &= 0 \\ \rightarrow (p-1)I_2 - (p-1)I_2 + \left(\begin{pmatrix} a & b \\ qb & a \end{pmatrix} - p \begin{pmatrix} a & b \\ qb & a \end{pmatrix}^{-1} \right) &= 0 \end{aligned}$$

$$\begin{aligned} &\rightarrow \begin{pmatrix} a & b \\ qb & a \end{pmatrix} = p \begin{pmatrix} a & b \\ qb & a \end{pmatrix}^{-1} \\ &\rightarrow \begin{pmatrix} a^2 + qb^2 & 2ab \\ 2qab & a^2 + qb^2 \end{pmatrix} = \begin{pmatrix} a & b \\ qb & a \end{pmatrix}^2 = pI_2 \end{aligned}$$

$\rightarrow \exists a, b \in k$ such that $a^2 + qb^2 = p, 2ab = 0$. Therefore, either one of $pI_4 + M_i C$ is nonsingular for $i = 1, 2$, or 3, or $\exists a, b \in k$ such that $a^2 + qb^2 = p, 2ab = 0$. This completes the first part of the theorem.

We now prove the second part of the theorem. Suppose, by way of contradiction, that $pI_4 + \tilde{M}_i C$ for $i = 1, 2$ is singular. We know, in this case, that the matrix C has the form:

$$C = \begin{pmatrix} a & b & c & d \\ -qb & -a & -qd & -c \\ e & f & g & h \\ -qf & -e & -qh & -g \end{pmatrix},$$

for some $a, b, c, d, e, f, g, h \in k, q \not\equiv 1 \pmod{k^{*2}}$. Now, $\det(pI_4 + \tilde{M}_1 C) = 0$, gives:

$$\det \begin{pmatrix} p \pm e & \pm f & \pm g & \pm h \\ \pm qf & p \pm e & \pm qh & \pm g \\ \pm pa & \pm pb & p \pm pc & \pm pd \\ \pm qpb & \pm pa & \pm qpd & p \pm pc \end{pmatrix} = 0.$$

This implies, by lemmas 3.3.2 and 4.1.3, that

$$\det \left(\begin{pmatrix} p \pm e & \pm f \\ \pm qf & p \pm e \end{pmatrix} \begin{pmatrix} p \pm pc & \pm pd \\ \pm qpd & p \pm pc \end{pmatrix} - p \begin{pmatrix} g & h \\ qh & g \end{pmatrix} \begin{pmatrix} a & b \\ qb & a \end{pmatrix} \right) = 0$$

and thus

$$\begin{aligned} 0 &= \det \left(\begin{pmatrix} p + e & f \\ qf & p + e \end{pmatrix} \begin{pmatrix} p + pc & pd \\ qpd & p + pc \end{pmatrix} - p \begin{pmatrix} g & h \\ qh & g \end{pmatrix} \begin{pmatrix} a & b \\ qb & a \end{pmatrix} \right) \\ &= \det \left(\begin{pmatrix} p - e & -f \\ -qf & p - e \end{pmatrix} \begin{pmatrix} p - pc & -pd \\ -qpd & p - pc \end{pmatrix} - p \begin{pmatrix} g & h \\ qh & g \end{pmatrix} \begin{pmatrix} a & b \\ qb & a \end{pmatrix} \right). \end{aligned}$$

Lemma 4.1.4 gives:

$$\begin{aligned} 0 &= \left(\begin{pmatrix} p+e & f \\ qf & p+e \end{pmatrix} \begin{pmatrix} p+pc & pd \\ qpd & p+pc \end{pmatrix} - p \begin{pmatrix} g & h \\ qh & g \end{pmatrix} \begin{pmatrix} a & b \\ qb & a \end{pmatrix} \right) \\ &= \left(\begin{pmatrix} p-e & -f \\ -qf & p-e \end{pmatrix} \begin{pmatrix} p-pc & -pd \\ -qpd & p-pc \end{pmatrix} - p \begin{pmatrix} g & h \\ qh & g \end{pmatrix} \begin{pmatrix} a & b \\ qb & a \end{pmatrix} \right) \end{aligned}$$

which implies

$$\begin{aligned} p^2 \begin{pmatrix} c & d \\ qd & c \end{pmatrix} + p \begin{pmatrix} e & f \\ qf & e \end{pmatrix} &= -p^2 \begin{pmatrix} c & d \\ qd & c \end{pmatrix} - p \begin{pmatrix} e & f \\ qf & e \end{pmatrix} \\ &\rightarrow 2p \left(p \begin{pmatrix} c & d \\ qd & c \end{pmatrix} + \begin{pmatrix} e & f \\ qf & e \end{pmatrix} \right) = 0 \\ &\rightarrow p \begin{pmatrix} c & d \\ qd & c \end{pmatrix} + \begin{pmatrix} e & f \\ qf & e \end{pmatrix} = 0 \\ &\rightarrow -p \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c & d \\ qd & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e & f \\ qf & e \end{pmatrix} \\ &\rightarrow \begin{pmatrix} e & f \\ -qf & -e \end{pmatrix} = -p \begin{pmatrix} c & d \\ -qd & -c \end{pmatrix}, \end{aligned} \tag{4.16}$$

since the characteristic of k is not 2 and $p \neq 0$. Similarly, $\det(pI_4 + \tilde{M}_2C) = 0$, gives:

$$\det \begin{pmatrix} p \pm (e - qf) & \pm(f - e) & \pm(g - qh) & \pm(h - g) \\ \pm(qf - qe) & p \pm (e - qf) & \pm(qh - qg) & \pm(g - qh) \\ \pm \frac{p}{1-q}(a - qb) & \pm \frac{p}{1-q}(b - a) & p \pm \frac{p}{1-q}(c - qd) & \pm \frac{p}{1-q}(d - c) \\ \pm \frac{p}{1-q}(qb - qa) & \pm \frac{p}{1-q}(a - qb) & \pm \frac{p}{1-q}(qd - qc) & p \pm \frac{p}{1-q}(c - qd) \end{pmatrix} = 0.$$

By lemmas 3.3.2 and 4.1.3,

$$\det(C_1 - C_2) = 0$$

for

$$C_1 = \begin{pmatrix} p \pm (e - qf) & \pm(f - e) \\ \pm(qf - qe) & p \pm (e - qf) \end{pmatrix} \begin{pmatrix} p \pm \frac{p}{1-q}(c - qd) & \pm \frac{p}{1-q}(d - c) \\ \pm \frac{p}{1-q}(qd - qc) & p \pm \frac{p}{1-q}(c - qd) \end{pmatrix} \text{ and}$$

$$C_2 = \begin{pmatrix} \pm(g - qh) & \pm(h - g) \\ \pm(qh - qg) & \pm(g - qh) \end{pmatrix} \begin{pmatrix} \pm \frac{p}{1-q}(a - qb) & \pm \frac{p}{1-q}(b - a) \\ \pm \frac{p}{1-q}(qb - qa) & \pm \frac{p}{1-q}(a - qb) \end{pmatrix}$$

This implies that both $\det(C'_1 + C'_2 - C'_3)$ and $\det(C''_1 + C'_2 - C'_3)$ are zero for:

$$C'_1 = \begin{pmatrix} p^2 + \frac{p^2}{1-q}(c - qd) + p(e - qf) & \frac{p^2}{1-q}(d - c) + p(f - e) \\ \frac{p^2}{1-q}(qd - qc) + p(qf - qe) & p^2 + \frac{p^2}{1-q}(c - qd) + p(e - qf) \end{pmatrix}$$

$$C''_1 = \begin{pmatrix} p^2 - \frac{p^2}{1-q}(c - qd) - p(e - qf) & -\frac{p^2}{1-q}(d - c) - p(f - e) \\ -\frac{p^2}{1-q}(qd - qc) - p(qf - qe) & p^2 - \frac{p^2}{1-q}(c - qd) - p(e - qf) \end{pmatrix}$$

$$C'_2 = \frac{p}{1-q} \begin{pmatrix} e - qf & f - e \\ qf - qe & e - qf \end{pmatrix} \begin{pmatrix} c - qd & d - c \\ qd - qc & c - qd \end{pmatrix}$$

$$C'_3 = \frac{p}{1-q} \begin{pmatrix} g - qh & h - g \\ qh - qg & g - qh \end{pmatrix} \begin{pmatrix} a - qb & b - a \\ qb - qa & a - qb \end{pmatrix}.$$

Lemma 4.1.4 implies $C'_1 + C'_2 - C'_3 = 0$ and $C''_1 + C'_2 - C'_3 = 0$. Therefore,

$$2p \left(\frac{p}{1-q} \begin{pmatrix} c - qd & d - c \\ qd - qc & c - qd \end{pmatrix} + \begin{pmatrix} e - qf & f - e \\ qf - qe & e - qf \end{pmatrix} \right) = 0$$

$$\rightarrow -\frac{p}{1-q} \begin{pmatrix} c - qd & d - c \\ qd - qc & c - qd \end{pmatrix} = \begin{pmatrix} e - qf & f - e \\ qf - qe & e - qf \end{pmatrix}$$

since the characteristic of k is not 2, and

$$-\frac{p}{1-q} \begin{pmatrix} 1 & 1 \\ -q & -1 \end{pmatrix} \begin{pmatrix} c & d \\ -qd & -c \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -q & -1 \end{pmatrix} \begin{pmatrix} e & f \\ -qf & -e \end{pmatrix}$$

$$\rightarrow -\frac{p}{1-q} \begin{pmatrix} c & d \\ -qd & -c \end{pmatrix} = \begin{pmatrix} e & f \\ -qf & -e \end{pmatrix}, \quad (4.17)$$

because $q \not\equiv 1 \pmod{k^{*2}}$. Putting (4.16) and (4.17) together:

$$-p \begin{pmatrix} c & d \\ qd & c \end{pmatrix} = -\frac{p}{1-q} \begin{pmatrix} c & d \\ qd & c \end{pmatrix}$$

$\rightarrow \begin{pmatrix} c & d \\ -qd & -c \end{pmatrix} = 0$, since if $-p = -\frac{p}{1-q} \rightarrow 1 - q = \frac{p}{p} = 1$, since $p \neq 0 \rightarrow q = 0$,

a contradiction. Therefore, $\begin{pmatrix} e & f \\ -qf & -e \end{pmatrix}$ is also the zero matrix. Using this, and

using $(pI_4 + \tilde{M}_i C)$ is singular for $i = 1, 2$,

$$\begin{aligned} 0 &= \det \begin{pmatrix} p & 0 & \pm g & \pm h \\ 0 & p & \pm qh & \pm g \\ \pm pa & \pm pb & p & 0 \\ \pm qpb & \pm pa & 0 & p \end{pmatrix} \\ &= \det \begin{pmatrix} p & 0 & \pm(g - qh) & \pm(h - g) \\ 0 & p & \pm(qh - qg) & \pm(g - qh) \\ \pm\frac{p}{1-q}(a - qb) & \pm\frac{p}{1-q}(b - a) & p & 0 \\ \pm\frac{p}{1-q}(qb - qa) & \pm\frac{p}{1-q}(a - qb) & 0 & p \end{pmatrix}, \end{aligned}$$

thus,

$$\begin{aligned} 0 &= \det \left(\begin{pmatrix} p^2 & 0 \\ 0 & p^2 \end{pmatrix} - p \begin{pmatrix} g & h \\ qh & g \end{pmatrix} \begin{pmatrix} a & b \\ qb & a \end{pmatrix} \right) \\ &= \det \left(\begin{pmatrix} p^2 & 0 \\ 0 & p^2 \end{pmatrix} - \frac{p}{1-q} \begin{pmatrix} g - qh & h - g \\ qh - qg & g - qh \end{pmatrix} \begin{pmatrix} a - qb & b - a \\ qb - qa & a - qb \end{pmatrix} \right), \end{aligned}$$

by lemmas 3.3.2 and 4.1.3. Therefore, by lemma 4.1.4,

$$\begin{aligned} p \begin{pmatrix} g & h \\ qh & g \end{pmatrix} \begin{pmatrix} a & b \\ qb & a \end{pmatrix} &= \begin{pmatrix} p^2 & 0 \\ 0 & p^2 \end{pmatrix} \\ &= \frac{p}{1-q} \begin{pmatrix} g - qh & h - g \\ qh - qg & g - qh \end{pmatrix} \begin{pmatrix} a - qb & b - a \\ qb - qa & a - qb \end{pmatrix} \end{aligned}$$

so that

$$\begin{pmatrix} g & h \\ qh & g \end{pmatrix} \begin{pmatrix} a & b \\ qb & a \end{pmatrix} = pI_2 = \frac{1}{1-q} \begin{pmatrix} g - qh & h - g \\ qh - qg & g - qh \end{pmatrix} \begin{pmatrix} a - qb & b - a \\ qb - qa & a - qb \end{pmatrix}.$$

This results in the following relations:

$$\begin{aligned} ag + qbh &= p \\ ah + bg &= 0 \\ \frac{1}{1-q}((a-qb)(g-gh) + q(b-a)(h-g)) &= p \end{aligned}$$

which imply

$$\begin{aligned} ag - qah - qbg + q^2bh + qbh - qbg - qah - qah + qag &= p(1-q) \\ \rightarrow (ag + qbh) + q(ag + qbh) - q(ah + bg) - q(ah + bg) &= p(1-q) \\ \rightarrow p + qp = p(1+q) = p(1-q) & \\ \rightarrow 1 + q = 1 - q & \\ \rightarrow 2q = 0 & \\ \rightarrow q = 0 & \end{aligned}$$

since the characteristic of k is not 2, which gives us a contradiction since q was assumed to be nonzero. Therefore, one of $pI_4 + \tilde{M}_i\tilde{C}, i = 1, 2$ is invertible. \square

We are now at a point where we can use the results above and induction to further simplify the form of a commuting pair of involutions of $SL(n, k)$ where the first entry is isomorphic to $L_{n,q}$, for $q \not\equiv 1 \pmod{k^2}$. Namely, we get the following theorem by repeatedly applying the techniques above, and using the fact that a matrix A in $GL(n, k)$ defines an inner automorphism of $SL(n, k)$, via the map Inn_A . Before proceeding, observe that the invertible matrices from theorem 4.1.5 commute (up to scalar multiple) with $L_{n,q}$.

4.1.6 n divisible by 4

Corollary 4.1.7. *Given that n is divisible by 4 and q is not equivalent to $1 \pmod{k^2}$, then the pair: $(\text{Inn}_{L_{n,q}}, \text{Inn}_A)$ is isomorphic to the pair $(\text{Inn}_{L'_{n,q}}, \text{Inn}_B)$, for any A such that the above is a pair of commuting involutions, and*

$$B = \begin{pmatrix} Y & 0 \\ 0 & \pm Y \end{pmatrix}, L'_{n,q} = \begin{pmatrix} 0 & I_{\frac{n}{2}} \\ qI_{\frac{n}{2}} & 0 \end{pmatrix},$$

where Y is the matrix representative of an isomorphism class of inner involutions of $\text{SL}\left(\frac{n}{2}, k\right)$, or $(\text{Inn}_{L_{n,q}}, \text{Inn}_A)$ is isomorphic to $(\text{Inn}_{L_{n,q}}, \text{Inn}_{I_{n-2i, 2i}L_{n,q}})$.

Proof. Without loss of generality, suppose $A^2 = pI_n$. Then, by lemma 4.1.2 and theorem 4.1.5 we know if there is not a solution to $a^2 + qb^2 = p$ with $2ab = 0$ over k that

$$(\text{Inn}_{L_{n,q}}, \text{Inn}_A) \approx (\text{Inn}_{L_{n,q}}, \text{Inn}_{A'}) , A' = \begin{pmatrix} A'_1 & 0 \\ 0 & M \end{pmatrix}$$

with $M = M_i, i = 1, 2, 3$ or $M = \tilde{M}_i, i = 1, 2$. In fact, using the automorphism defined by $\begin{pmatrix} I_{n-4} & 0 \\ 0 & R \end{pmatrix}$ for R :

$$\begin{pmatrix} 0 & 0 & 1 & \pm 1 \\ 0 & 0 & \pm q & 1 \\ p & \pm \frac{p}{q} & 0 & 0 \\ \pm p & p & 0 & 0 \end{pmatrix}, \begin{pmatrix} I_2 & 0 \\ -I_2 & -I_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & q & 0 \\ \frac{pq}{1-q} & \frac{p}{1-q} & 0 & 0 \\ \frac{-qp}{1-q} & \frac{-pq}{1-q} & 0 & 0 \end{pmatrix},$$

the above isomorphism becomes:

$$(\text{Inn}_{L_{n,q}}, \text{Inn}_A) \approx (\text{Inn}_{L_{n,q}}, \text{Inn}_{A'}) , A' = \begin{pmatrix} A'_1 & 0 \\ 0 & M \end{pmatrix},$$

for M either M_1 or \tilde{M}_1 , depending on which form from lemma 4.1.1 is the matrix A . Namely, if $M = M_1$ or \tilde{M}_1 , nothing needs to be done. If $M = M_2$ then set

$$R = \begin{pmatrix} 0 & 0 & 1 & \pm 1 \\ 0 & 0 & \pm q & 1 \\ p & \pm \frac{p}{q} & 0 & 0 \\ \pm p & p & 0 & 0 \end{pmatrix}$$

(keeping the signs of M_2 and R the same). Then, R is invertible because $q \not\equiv 1 \pmod{k^{*2}}$ and

$$\begin{pmatrix} I_{n-4} & 0 \\ 0 & R \end{pmatrix} L_{n,q} \begin{pmatrix} I_{n-4} & 0 \\ 0 & R \end{pmatrix}^{-1} = L_{n,q},$$

$$\begin{pmatrix} I_{n-4} & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} A'_1 & 0 \\ 0 & M_2 \end{pmatrix} \begin{pmatrix} I_{n-4} & 0 \\ 0 & R \end{pmatrix}^{-1} = \begin{pmatrix} A'_1 & 0 \\ 0 & M_1 \end{pmatrix}.$$

If $M = M_3$ set

$$R = \begin{pmatrix} I_2 & 0 \\ -I_2 & -I_2 \end{pmatrix}.$$

Then R is invertible and

$$\begin{aligned} \begin{pmatrix} I_{n-4} & 0 \\ 0 & R \end{pmatrix} L_{n,q} \begin{pmatrix} I_{n-4} & 0 \\ 0 & R \end{pmatrix}^{-1} &= L_{n,q}, \\ \begin{pmatrix} I_{n-4} & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} A'_1 & 0 \\ 0 & M_3 \end{pmatrix} \begin{pmatrix} I_{n-4} & 0 \\ 0 & R \end{pmatrix}^{-1} &= \begin{pmatrix} A'_1 & 0 \\ 0 & M_1 \end{pmatrix}. \end{aligned}$$

If $M = \tilde{M}_2$ set

$$R = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & q & 0 \\ \frac{pq}{1-q} & \frac{p}{1-q} & 0 & 0 \\ \frac{-qp}{1-q} & \frac{-pq}{1-q} & 0 & 0 \end{pmatrix}.$$

Then R is invertible because q is neither 0 nor 1, and

$$\begin{aligned} \begin{pmatrix} I_{n-4} & 0 \\ 0 & R \end{pmatrix} L_{n,q} \begin{pmatrix} I_{n-4} & 0 \\ 0 & R \end{pmatrix}^{-1} &= L_{n,q}, \\ \begin{pmatrix} I_{n-4} & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} A'_1 & 0 \\ 0 & \tilde{M}_2 \end{pmatrix} \begin{pmatrix} I_{n-4} & 0 \\ 0 & R \end{pmatrix}^{-1} &= \begin{pmatrix} A'_1 & 0 \\ 0 & \tilde{M}_1 \end{pmatrix}. \end{aligned}$$

But $A^2 = pI_n$ implies that $A'^2 = (T^{-1}AT)^2 = pI_n$, for $T = \begin{pmatrix} A_1 & U \\ V & C + M \end{pmatrix}$, thus $A_1'^2 = pI_{n-4}$. And since we are assuming that there is no solution to $a^2 + qb^2 = p$ with $2ab = 0$ over k , by lemmas 4.1.2 and 4.1.5, and the above argument, there is an invertible matrix S such that:

$$\begin{aligned} S^{-1}L_{n-4,q}S &= aL_{n-4,q} \\ S^{-1}A'_1S &= b \begin{pmatrix} A''_1 & 0 \\ 0 & M' \end{pmatrix} \end{aligned}$$

for $M' = M_1$, or $M' = \tilde{M}_1$. This implies that the commuting pair $(\text{Inn}_{L_{n,q}}, \text{Inn}_A)$ is isomorphic to the commuting pair:

$$\begin{aligned} & \left(\text{Inn} \begin{pmatrix} S^{-1}L_{n-4,q}S & 0 \\ 0 & L_{4,q} \end{pmatrix}, \text{Inn} \begin{pmatrix} S^{-1}A_1'S & 0 \\ 0 & M \end{pmatrix} \right) \\ &= \left(\text{Inn} \begin{pmatrix} aL_{n-4,q} & 0 \\ 0 & L_{4,q} \end{pmatrix}, \text{Inn} \begin{pmatrix} b \begin{pmatrix} A_1'' & 0 \\ 0 & M' \end{pmatrix} & 0 \\ 0 & M \end{pmatrix} \right) \end{aligned}$$

via the inner automorphism defined by $\begin{pmatrix} S & 0 \\ 0 & I_4 \end{pmatrix}$. This forces $a = \pm 1$ because $L_{n-4,q}^2 = qI_{n-4} = (S^{-1}L_{n-4,q}S)^2 = a^2qI_{n-4}$. If $a = -1$, the inner automorphism defined by the matrix:

$$T' = \begin{pmatrix} I_{n-4} & 0 \\ 0 & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{pmatrix}$$

gives the required isomorphism to the pair:

$$\left(\text{Inn} \begin{pmatrix} L_{n-4,q} & 0 \\ 0 & L_{4,q} \end{pmatrix}, \text{Inn} \begin{pmatrix} b \begin{pmatrix} A_1'' & 0 \\ 0 & M' \end{pmatrix} & 0 \\ 0 & M \end{pmatrix} \right),$$

i.e. to the pair with $a = 1$ by factoring out the constant -1 (after conjugation by the above inner automorphism defined by T') because

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -q & 0 \end{pmatrix}$$

and because (since we know $M = M_1$ or \tilde{M}_1),

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

commutes with M . Also, $b = \pm 1$, because

$$A_1'^2 = pI_{n-4} = (S^{-1}A_1'S)^2 = b^2 \begin{pmatrix} A_1'' & 0 \\ 0 & M' \end{pmatrix}^2 = \begin{pmatrix} b^2 A_1''^2 & 0 \\ 0 & b^2 p I_4 \end{pmatrix},$$

since $M' = M_1$ or \tilde{M}_1 . But by definition of M_1 and \tilde{M}_1 we can ignore a constant of ± 1 . Therefore, if there is no solution over k to $a^2 + qb^2 = p$ with $2ab = 0$ then the pair $(\text{Inn}_{L_{n,q}}, \text{Inn}_A)$ is isomorphic to the pair

$$\left(\text{Inn} \begin{pmatrix} L_{n-4,q} & 0 \\ 0 & L_{4,q} \end{pmatrix}, \text{Inn} \begin{pmatrix} \pm A_1'' & 0 \\ 0 & M' \\ & & M \end{pmatrix} \right),$$

for M, M' either (both) M_1 or (both) \tilde{M}_1 . Using this as an induction step, by repeatedly applying lemma 4.1.2 and theorem 4.1.5 to the block A_1'' , this gives us, by induction, the isomorphism:

$$(\text{Inn}_{L_{n,q}}, \text{Inn}_A) \approx (\text{Inn}_{L_{n,q}}, \text{Inn}_{A''}), A'' = \begin{pmatrix} X_1 & & 0 \\ & \ddots & \\ 0 & & X_{\frac{n}{4}} \end{pmatrix},$$

for $X_j = M_1$, or $X_j = \tilde{M}_1$.

Consider Inn_P for the permutation matrix:

$$P_n = \sum_{i=1}^{\frac{n}{2}} (E_{i,2i-1} + E_{\frac{n}{2}+i,2i}).$$

For the cases that A has the form $(A_{i,j})$ with $A_{i,j} = \begin{pmatrix} a_{i,j} & b_{i,j} \\ qb_{i,j} & a_{i,j} \end{pmatrix}$ and no solution over k of $a^2 + qb^2 = p$ with $2ab = 0$ or A has the form $(A_{i,j})$ with $A_{i,j} = \begin{pmatrix} a_{i,j} & b_{i,j} \\ -qb_{i,j} & -a_{i,j} \end{pmatrix}$ it remains to show that the commuting pair $(\text{Inn}_{L_{n,q}}, \text{Inn}_{A''})$, with

$$A'' = \begin{pmatrix} X_1 & & 0 \\ & \ddots & \\ 0 & & X_{\frac{n}{4}} \end{pmatrix} \text{ is isomorphic to } (\text{Inn}_{L'_{n,q}}, \text{Inn}_B), \text{ with } B = \begin{pmatrix} Y & 0 \\ 0 & \pm Y \end{pmatrix} \text{ for}$$

Y the matrix representative of an isomorphy class of inner involutions of $\text{SL}(\frac{n}{2}, k)$.

First, it can be shown that $P_n L_{n,q} P_n^{-1} = L'_{n,q}$. Note that, because P_n is a permutation matrix,

$$P_n^{-1} = P_n^T = \left(\sum_{i=1}^{\frac{n}{2}} (E_{2i-1,i} + E_{2i,\frac{n}{2}+i}) \right)$$

and note

$$L_{n,q} = \sum_{i=1}^{\frac{n}{2}} (E_{2i-1,2i} + qE_{2i,2i-1}), L'_{n,q} = \sum_{i=1}^{\frac{n}{2}} (E_{i,\frac{n}{2}+i} + qE_{\frac{n}{2}+i,i}),$$

so that $P_n L_{n,q} P_n^{-1}$ is equal to:

$$\begin{aligned} & \left(\sum_{i=1}^{\frac{n}{2}} (E_{i,2i-1} + E_{\frac{n}{2}+i,2i}) \right) \left(\sum_{i=1}^{\frac{n}{2}} (E_{2i-1,2i} + qE_{2i,2i-1}) \right) \left(\sum_{i=1}^{\frac{n}{2}} (E_{2i-1,i} + E_{2i,\frac{n}{2}+i}) \right) \\ &= \left(\sum_{i=1}^{\frac{n}{2}} E_{i,2i-1} \sum_{j=1}^{\frac{n}{2}} E_{2j-1,2j} + \sum_{i=1}^{\frac{n}{2}} E_{i,2i-1} \sum_{j=1}^{\frac{n}{2}} qE_{2j,2j-1} + \sum_{i=1}^{\frac{n}{2}} E_{\frac{n}{2}+i,2i} \sum_{j=1}^{\frac{n}{2}} E_{2j-1,2j} \right. \\ & \quad \left. + \sum_{i=1}^{\frac{n}{2}} E_{\frac{n}{2}+i,2i} \sum_{j=1}^{\frac{n}{2}} qE_{2j,2j-1} \right) \left(\sum_{i=1}^{\frac{n}{2}} (E_{2i-1,i} + E_{2i,\frac{n}{2}+i}) \right) \\ &= \left(\sum_{i=1}^{\frac{n}{2}} E_{i,2i} + 0 + 0 + \sum_{i=1}^{\frac{n}{2}} qE_{\frac{n}{2}+i,2i-1} \right) \left(\sum_{i=1}^{\frac{n}{2}} (E_{2i-1,i} + E_{2i,\frac{n}{2}+i}) \right) \\ &= \sum_{i=1}^{\frac{n}{2}} E_{i,2i} \sum_{j=1}^{\frac{n}{2}} E_{2j-1,j} + \sum_{i=1}^{\frac{n}{2}} E_{i,2i} \sum_{j=1}^{\frac{n}{2}} E_{2j,\frac{n}{2}+j} + \sum_{i=1}^{\frac{n}{2}} qE_{\frac{n}{2}+i,2i-1} \sum_{j=1}^{\frac{n}{2}} E_{2j-1,j} \\ & \quad + \sum_{i=1}^{\frac{n}{2}} qE_{\frac{n}{2}+i,2i-1} \sum_{j=1}^{\frac{n}{2}} E_{2j,\frac{n}{2}+j} \\ &= 0 + \sum_{i=1}^{\frac{n}{2}} E_{i,\frac{n}{2}+i} + q \sum_{i=1}^{\frac{n}{2}} E_{\frac{n}{2}+i,i} + 0 \\ &= \sum_{i=1}^{\frac{n}{2}} (E_{i,\frac{n}{2}+i} + qE_{\frac{n}{2}+i,i}) = L'_{n,q}, \end{aligned}$$

with the zeros following from the fact that $2j-1 \neq 2i$ for any i or j . Next, it is shown that $P_n A'' P_n^{-1}$, for A'' as above, has the block form: $\begin{pmatrix} Y' & 0 \\ 0 & \pm Y' \end{pmatrix}$ with $Y' \in \text{GL}(\frac{n}{2}, k)$.

By the argument above, it is known that A'' is one of two forms:

$$\begin{pmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_1 \end{pmatrix} \text{ or } \begin{pmatrix} \tilde{M}_1 & & 0 \\ & \ddots & \\ 0 & & \tilde{M}_1 \end{pmatrix},$$

i.e. A'' is one of the following forms:

$$\begin{aligned} & \sum_{i=0}^{\frac{n}{4}-1} \epsilon_i (E_{1+4i,3+4i} + E_{2+4i,4+4i} + pE_{3+4i,1+4i} + pE_{4+4i,2+4i}) \\ & \sum_{i=0}^{\frac{n}{4}-1} \epsilon_i (E_{1+4i,3+4i} - E_{2+4i,4+4i} + pE_{3+4i,1+4i} - pE_{4+4i,2+4i}) \end{aligned}$$

for $\epsilon_i = \pm 1$. Therefore, $P_n A'' P_n^{-1}$ is equal to:

$$\begin{aligned} & \left(\sum_{i=1}^{\frac{n}{2}} E_{i,2i-1} \sum_{j=1}^{\frac{n}{4}} \epsilon_j E_{4j-3,4j-1} + \sum_{i=1}^{\frac{n}{2}} E_{i,2i-1} \sum_{j=1}^{\frac{n}{4}} \epsilon_j p E_{4j-1,4j-3} \right. \\ & \left. + \sum_{i=1}^{\frac{n}{2}} E_{\frac{n}{2}+i,2i} \sum_{j=1}^{\frac{n}{4}} \epsilon_j E_{4j-2,4j} + \sum_{i=1}^{\frac{n}{2}} E_{\frac{n}{2}+i,2i} \sum_{j=1}^{\frac{n}{4}} \epsilon_j p E_{4j,4j-2} \right) P_n^{-1} \text{ or} \\ & \left(\sum_{i=1}^{\frac{n}{2}} E_{i,2i-1} \sum_{j=1}^{\frac{n}{4}} \epsilon_j E_{4j-3,4j-1} + \sum_{i=1}^{\frac{n}{2}} E_{i,2i-1} \sum_{j=1}^{\frac{n}{4}} \epsilon_j p E_{4j-1,4j-3} \right. \\ & \left. - \sum_{i=1}^{\frac{n}{2}} E_{\frac{n}{2}+i,2i} \sum_{j=1}^{\frac{n}{4}} \epsilon_j E_{4j-2,4j} - \sum_{i=1}^{\frac{n}{2}} E_{\frac{n}{2}+i,2i} \sum_{j=1}^{\frac{n}{4}} \epsilon_j p E_{4j,4j-2} \right) P_n^{-1} \end{aligned}$$

because $2i - 1 \neq 4j - 2$, $2i - 1 \neq 4j$, $2i \neq 4j - 1$, $2i \neq 4j - 3$ for any integers i, j .

These are equal to:

$$\begin{aligned} & \left(\sum_{j=1}^{\frac{n}{4}} \epsilon_j E_{2j-1,4j-1} + \sum_{j=1}^{\frac{n}{4}} p \epsilon_j E_{2j,4j-3} + \sum_{j=1}^{\frac{n}{4}} \epsilon_j E_{\frac{n}{2}+2j-1,4j} + \sum_{j=1}^{\frac{n}{4}} p \epsilon_j E_{\frac{n}{2}+2j,4j-2} \right) P_n^{-1} \text{ or} \\ & \left(\sum_{j=1}^{\frac{n}{4}} \epsilon_j E_{2j-1,4j-1} + \sum_{j=1}^{\frac{n}{4}} p \epsilon_j E_{2j,4j-3} - \sum_{j=1}^{\frac{n}{4}} \epsilon_j E_{\frac{n}{2}+2j-1,4j} - \sum_{j=1}^{\frac{n}{4}} p \epsilon_j E_{\frac{n}{2}+2j,4j-2} \right) P_n^{-1}, \end{aligned}$$

respectively, because $2i = 4j - 2$ when $i = 2j - 1$ and $2i = 4j$ when $i = 2j$. Similarly solving for i in terms of j and using that for integers i, j , $2i - 1 \neq 4j$, $4j - 2$ and

$2i \neq 4j - 1, 4j - 3$, this results in:

$$\begin{aligned}
& \sum_{j=1}^{\frac{n}{4}} \epsilon_j E_{2j-1, 2j} + \sum_{j=1}^{\frac{n}{4}} p \epsilon_j E_{2j, 2j-1} + \sum_{j=1}^{\frac{n}{4}} \epsilon_j E_{\frac{n}{2}+2j-1, \frac{n}{2}+2j} + \sum_{j=1}^{\frac{n}{4}} p \epsilon_j E_{\frac{n}{2}+2j, \frac{n}{2}+2j-1} \quad \text{or} \\
& \sum_{j=1}^{\frac{n}{4}} \epsilon_j E_{2j-1, 2j} + \sum_{j=1}^{\frac{n}{4}} p \epsilon_j E_{2j, 2j-1} - \sum_{j=1}^{\frac{n}{4}} \epsilon_j E_{\frac{n}{2}+2j-1, \frac{n}{2}+2j} - \sum_{j=1}^{\frac{n}{4}} p \epsilon_j E_{\frac{n}{2}+2j, \frac{n}{2}+2j-1} \\
& = \sum_{j=1}^{\frac{n}{4}} \epsilon_j E_{2j-1, 2j} + p \epsilon_j E_{2j, 2j-1} + \epsilon_j E_{\frac{n}{2}+2j-1, \frac{n}{2}+2j} + p \epsilon_j E_{\frac{n}{2}+2j, \frac{n}{2}+2j-1} \quad \text{or} \\
& \sum_{j=1}^{\frac{n}{4}} \epsilon_j E_{2j-1, 2j} + p \epsilon_j E_{2j, 2j-1} - \epsilon_j E_{\frac{n}{2}+2j-1, \frac{n}{2}+2j} - p \epsilon_j E_{\frac{n}{2}+2j, \frac{n}{2}+2j-1}
\end{aligned}$$

respectively. Note that, in both cases, these end up as matrices with two-by-two blocks on the diagonal. Because n is divisible by 4, we get the desired block form. Thus, the commuting pair $(\text{Inn}_{L_{n,q}}, \text{Inn}_A)$ for A one of the forms in lemma 4.1.1 is isomorphic to a commuting pair of the form $(\text{Inn}_{L'_{n,q}}, \text{Inn}_{\begin{pmatrix} Y' & 0 \\ 0 & \pm Y' \end{pmatrix}})$ for $Y' \in \text{GL}(\frac{n}{2}, k)$, when there is no solution over k to $a^2 + qb^2 = p$ with $2ab = 0$. Because Inn_A is an involution isomorphic to $\text{Inn}_{\begin{pmatrix} Y' & 0 \\ 0 & \pm Y' \end{pmatrix}}$, there exists an invertible C' such that $C'^{-1}AC' = \alpha Y'$ so that $pI_n = A^2 = (C'^{-1}AC')^2 = \alpha^2 Y'^2$, i.e. Y' is the matrix representative of an involution of $\text{SL}(\frac{n}{2}, k)$. This implies that there is a matrix $T' \in \text{GL}(\frac{n}{2}, k)$ such that $T'^{-1}Y'T' = \beta Y$ for Y the matrix representative of an isomorphism class of involution of $\text{SL}(\frac{n}{2}, k)$. Therefore, $(\text{Inn}_{L_{n,q}}, \text{Inn}_A)$ is isomorphic to

$$\begin{aligned}
& \left(\text{Inn}_{L'_{n,q}}, \text{Inn}_{\begin{pmatrix} Y' & 0 \\ 0 & \pm Y' \end{pmatrix}} \right) \\
& \approx \left(\text{Inn}_{\begin{pmatrix} T' & 0 \\ 0 & T' \end{pmatrix}^{-1} L'_{n,q} \begin{pmatrix} T' & 0 \\ 0 & T' \end{pmatrix}}, \text{Inn}_{\begin{pmatrix} T' & 0 \\ 0 & T' \end{pmatrix}^{-1} \begin{pmatrix} Y' & 0 \\ 0 & \pm Y' \end{pmatrix} \begin{pmatrix} T' & 0 \\ 0 & T' \end{pmatrix}} \right) \\
& = \left(\text{Inn}_{\begin{pmatrix} T' & 0 \\ 0 & T' \end{pmatrix}^{-1} \begin{pmatrix} 0 & I_{\frac{n}{2}} \\ pI_{\frac{n}{2}} & 0 \end{pmatrix} \begin{pmatrix} T' & 0 \\ 0 & T' \end{pmatrix}}, \text{Inn}_{\begin{pmatrix} Y & 0 \\ 0 & \pm Y \end{pmatrix}} \right) \\
& = \left(\text{Inn}_{L'_{n,q}}, \text{Inn}_{\begin{pmatrix} Y & 0 \\ 0 & \pm Y \end{pmatrix}} \right),
\end{aligned}$$

whenever A is of the form $(A_{i,j})$ with $A_{i,j} = \begin{pmatrix} a_{i,j} & b_{i,j} \\ qb_{i,j} & a_{i,j} \end{pmatrix}$ and no solution over k of

$a^2 + qb^2 = p$ with $2ab = 0$ or A has the form $(A_{i,j})$ with $A_{i,j} = \begin{pmatrix} a_{i,j} & b_{i,j} \\ -qb_{i,j} & -a_{i,j} \end{pmatrix}$, for Y the matrix representative of an isomorphy class of inner involution of $\text{SL}(\frac{n}{2}, k)$.

Lastly, if A has the form $(A_{i,j})$ with $A_{i,j} = \begin{pmatrix} a_{i,j} & b_{i,j} \\ qb_{i,j} & a_{i,j} \end{pmatrix}$ and each of M_i for $i = 1, 2, 3$ is singular, it was shown in theorem 4.1.5 that there is $a, b \in k$ such that $a^2 + qb^2 = p, 2ab = 0$. Let M_4 be the two by two matrix $\begin{pmatrix} a & b \\ qb & a \end{pmatrix}$ for a, b the solution over k to $x^2 + qy^2 = p, 2xy = 0$. By lemma 4.1.1 and lemma 4.1.3, the matrix

$$A \pm \begin{pmatrix} 0 & 0 \\ 0 & M_4 \end{pmatrix} = \begin{pmatrix} A_1 & U \\ V & C \pm \begin{pmatrix} 0 & 0 \\ 0 & M_4 \end{pmatrix} \end{pmatrix}$$

commutes with $L_{n,q}$. Also, one of them is invertible:

$$\begin{aligned} & \begin{pmatrix} A_1 & U \\ V & C \pm \begin{pmatrix} 0 & 0 \\ 0 & M_4 \end{pmatrix} \end{pmatrix} \text{ is invertible} \\ \Leftrightarrow & \begin{pmatrix} A_1 & U \\ V & C \pm \begin{pmatrix} 0 & 0 \\ 0 & M_4 \end{pmatrix} \end{pmatrix} \begin{pmatrix} A_1 & U \\ V & C \end{pmatrix} \text{ is invertible} \\ \Leftrightarrow & \begin{pmatrix} pI_{n-4} & 0 \\ \pm \begin{pmatrix} 0 & 0 \\ 0 & M_4 \end{pmatrix} V & pI_4 \pm \begin{pmatrix} 0 & 0 \\ 0 & M_4 \end{pmatrix} C \end{pmatrix} \\ \Leftrightarrow & pI_4 \pm \begin{pmatrix} 0 & 0 \\ 0 & M_4 \end{pmatrix} C \text{ is invertible} \\ \Leftrightarrow & pI_2 \pm M_4 C_2 \text{ is invertible} \end{aligned}$$

for C_2 the lower right two-by-two block of C , i.e. $C_2 = \begin{pmatrix} g & h \\ qh & g \end{pmatrix}$ for $g, h \in k$. If both

$pI_2 + M_4C_2$ and $pI_2 - M_4C_2$ are singular, then

$$\begin{aligned} 0 &= \det \left(\begin{pmatrix} p + ag + qbh & ah + bg \\ q(ah + bg) & p + aq + qbh \end{pmatrix} \right) \\ &= \det \left(\begin{pmatrix} p - (ag + qbh) & -(ah + bg) \\ -q(ah + bg) & p - (aq + qbh) \end{pmatrix} \right) \end{aligned}$$

and thus

$$\begin{aligned} q(ah + bg)^2 &= (p + ag + qbh)^2 \quad \text{and} \\ q(ah + bg)^2 &= (p - (ag + qbh))^2. \end{aligned}$$

If $(ah + bg) \neq 0$ then $(p + ag + qbh) \neq 0$ and $q = (p + ag + qbh)^2 / (ah + bg)^2$, a contradiction to the fact that $q \not\equiv 1 \pmod{k^{*2}}$. Therefore, $(ah + bg) = 0$ which implies $(p + ag + qbh) = 0 = (p - (ag + qbh))$. Thus, $p = ag + qbh = -p$ and $2p = 0$. This gives $p = 0$ because the characteristic of k is not two, but this is a contradiction to the fact that Inn_A is an automorphism. Therefore one of

$$\begin{pmatrix} A_1 & U \\ V & C \pm \begin{pmatrix} 0 & 0 \\ 0 & M_4 \end{pmatrix} \end{pmatrix}$$

is invertible. Therefore, by the proof of lemma 4.1.2 and the above induction argument, the pair $(\text{Inn}_{L_{n,q}}, \text{Inn}_A)$ is isomorphic to the pair $(\text{Inn}_{L_{n,q}}, \text{Inn}_{B'})$, for B' the matrix of 2×2 blocks of the form $\epsilon_i \begin{pmatrix} a & b \\ qb & a \end{pmatrix}$ on the diagonal and zeros elsewhere, with $\epsilon_i = \pm 1$. But since the characteristic of the field k is not two and $2ab = 0$, either a or b is 0. If b is zero, then the pair $(\text{Inn}_{L_{n,q}}, \text{Inn}_{B'})$ for B' as described directly above is equal to:

$$\begin{aligned} \left(\text{Inn}_{L_{n,q}}, \text{Inn} \begin{pmatrix} \epsilon_1 a I_2 & & \\ & \ddots & \\ & & \epsilon_{\frac{n}{2}} a I_2 \end{pmatrix} \right) &= \left(\text{Inn}_{L_{n,q}}, \text{Inn} \begin{pmatrix} \epsilon_1 I_2 & & \\ & a & \\ & & \ddots & \\ & & & \epsilon_{\frac{n}{2}} I_2 \end{pmatrix} \right) \\ &= \left(\text{Inn}_{L_{n,q}}, \text{Inn} \begin{pmatrix} \epsilon_1 I_2 & & \\ & \ddots & \\ & & \epsilon_{\frac{n}{2}} I_2 \end{pmatrix} \right). \end{aligned}$$

After conjugation by Inn_{P_n} for P_n the permutation matrix: $\sum_{i=1}^{\frac{n}{2}} (E_{i,2i-1} + E_{\frac{n}{2}+i,2i})$, this gives:

$$\left(\text{Inn}_{L'_{n,q}}, \text{Inn}_{P_n} \left(\begin{array}{cccc} \epsilon_1 I_2 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \epsilon_{\frac{n}{2}} I_2 \end{array} \right) P_n^T \right) = \left(\text{Inn}_{L'_{n,q}}, \text{Inn} \begin{pmatrix} Y & 0 \\ 0 & \pm Y \end{pmatrix} \right),$$

for Y the matrix representative of an isomorphy class of inner involutions of $\text{SL}(\frac{n}{2}, k)$, because, $P_n B' P_n^T$ is equal to:

$$\begin{aligned} & \left(\sum_{i=1}^{\frac{n}{2}} (E_{i,2i-1} + E_{\frac{n}{2}+i,2i}) \right) \left(\sum_{j=1}^{\frac{n}{2}} \epsilon_j (E_{2j-1,2j-1} + E_{2j,2j}) \right) P_n^T \\ &= \left(\sum_{i=1}^{\frac{n}{2}} \epsilon_i E_{i,2i-1} + \sum_{i=1}^{\frac{n}{2}} \epsilon_i E_{\frac{n}{2}+i,2i} \right) \left(\sum_{j=1}^{\frac{n}{2}} (E_{2j-1,j} + E_{2j,\frac{n}{2}+j}) \right) \\ &= \sum_{i=1}^{\frac{n}{2}} \epsilon_i E_{i,i} + \sum_{i=1}^{\frac{n}{2}} \epsilon_i E_{\frac{n}{2}+i,\frac{n}{2}+i}, \end{aligned}$$

where the multiplications follow from the fact that $2i - 1 \neq 2j$ for integers i, j and $2j - 1 = 2i - 1$ or $\frac{n}{2} + i = \frac{n}{2} + j$ implies $i = j$. Note that this gives B' the desired form (possibly after factoring out a -1). If $a = 0$, then $(\text{Inn}_{L_{n,q}}, \text{Inn}_A)$ is isomorphic to

$$\begin{aligned} & \left(\text{Inn}_{L_{n,q}}, \text{Inn} \left(\begin{array}{cccc} \begin{pmatrix} 0 & b \\ qb & 0 \end{pmatrix} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \begin{pmatrix} 0 & b \\ qb & 0 \end{pmatrix} \end{array} \right) \right) \\ &= \left(\text{Inn}_{L_{n,q}}, \text{Inn} \left(\begin{array}{cccc} \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix} \end{array} \right) \right) \end{aligned}$$

This implies that $p \equiv q$ because, by supposition $B'^2 = pI_n$. Therefore, the pair $(\text{Inn}_{L_{n,q}}, \text{Inn}_A)$ is isomorphic to the pair $(\text{Inn}_{L_{n,q}}, \text{Inn}_{I_{n-2i, 2iL_{n,q}}})$, for some $i \in \{0, 1, \dots, \frac{n}{4}\}$. \square

4.1.8 n even, not divisible by 4

When n is even but not divisible by 4, we can do something similar.

Corollary 4.1.9. *Given that n is even but not divisible by 4 and q is not equivalent to 1 mod k^{*2} , and $A^2 = pI_n$, then $(\text{Inn}_{L_{n,q}}, \text{Inn}_A) \approx$*

$$(\text{Inn}_{L_{n,q}}, \text{Inn}_{I_{n-2i, 2iL_{n,q}}}), \text{ or } \left(\text{Inn}_{L'_{n,q}}, \text{Inn} \begin{pmatrix} I_{\frac{n}{2}-i} & 0 \\ 0 & I_{\frac{n}{2}-i} \end{pmatrix} \right),$$

if there is a solution over k to $a^2 + qb^2 = p, 2ab = 0$, otherwise it is isomorphic to the pair, $P(a, b) =$

$$\left(\text{Inn} \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix} & 0 \\ 0 & L'_{n-2,q} \end{pmatrix}, \text{Inn} \begin{pmatrix} \begin{pmatrix} a & b \\ -qb & -a \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} Y & 0 \\ 0 & -Y \end{pmatrix} \end{pmatrix} \right),$$

for Y the matrix representative of an isomorphism class of inner involution of the group $\text{SL}(\frac{n-2}{2}, k)$. Furthermore, for fixed Y , all such pairs are isomorphic.

Proof. The first statements follow from the proof of corollary 4.1.7. Fix a representative, Y , of an inner isomorphism class of involution of $\text{SL}(\frac{n-2}{2}, k)$ and suppose that we have pairs $P(a, b)$ and $P(c, d)$, as defined above. Then, $P(a, b)$ is isomorphic to $P(c, d)$ if there is an $x, y \in k$ such that:

$$\begin{pmatrix} x & y \\ -qy & -x \end{pmatrix} \begin{pmatrix} a & b \\ -qb & -a \end{pmatrix} \begin{pmatrix} x & y \\ -qy & -x \end{pmatrix}^{-1} = \begin{pmatrix} c & d \\ -qd & -c \end{pmatrix},$$

because, then, for

$$\begin{aligned}
S &= \begin{pmatrix} \begin{pmatrix} x & y \\ -qy & -x \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} I_{\frac{n}{2}-1} & 0 \\ 0 & -I_{\frac{n}{2}-1} \end{pmatrix} \end{pmatrix}, \\
&\begin{pmatrix} \text{Inn}_S^{-1} \text{Inn} \begin{pmatrix} L_{2,q} & 0 \\ 0 & L'_{\frac{n}{2}-1,q} \end{pmatrix} \text{Inn}_S, \text{Inn}_S^{-1} \text{Inn} \begin{pmatrix} \begin{pmatrix} a & b \\ -qb & -a \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} Y & 0 \\ 0 & -Y \end{pmatrix} \end{pmatrix} \text{Inn}_S \end{pmatrix} \\
&= \begin{pmatrix} \text{Inn}_S \begin{pmatrix} L_{2,q} & 0 \\ 0 & L'_{\frac{n}{2}-1,q} \end{pmatrix} \text{Inn}_S^{-1}, \text{Inn}_S \begin{pmatrix} \begin{pmatrix} a & b \\ -qb & -a \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} Y & 0 \\ 0 & -Y \end{pmatrix} \end{pmatrix} \text{Inn}_S^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \text{Inn} \begin{pmatrix} -L_{2,q} & 0 \\ 0 & -L'_{\frac{n}{2}-1,q} \end{pmatrix}, \text{Inn} \begin{pmatrix} \begin{pmatrix} c & d \\ -qd & -c \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} Y & 0 \\ 0 & -Y \end{pmatrix} \end{pmatrix} \end{pmatrix} \\
&= P(c, d).
\end{aligned}$$

For any $x, y \in k$, not both zero, $\det \begin{pmatrix} x & y \\ -qy & -x \end{pmatrix} = qy^2 - x^2$, and since q is not a square in k , the determinant is nonzero. This implies that the matrix $\begin{pmatrix} x & y \\ -qy & -x \end{pmatrix}$

is either the zero matrix or is invertible. Therefore, for x, y not both zero,

$$\begin{aligned} & \begin{pmatrix} x & y \\ -qy & -x \end{pmatrix} \begin{pmatrix} a & b \\ -qb & -a \end{pmatrix} \begin{pmatrix} x & y \\ -qy & -x \end{pmatrix}^{-1} \\ &= \frac{1}{qy^2 - x^2} \begin{pmatrix} ax - qby & bx - ay \\ q(bx - ay) & ax - qby \end{pmatrix} \begin{pmatrix} -x & -y \\ qy & x \end{pmatrix} \\ &= \frac{1}{qy^2 - x^2} \begin{pmatrix} -ax^2 + 2qboxy - qay^2 & bx^2 - 2axy + qby^2 \\ -q(bx^2 - 2axy + qby^2) & -(-ax^2 + 2qboxy - qay^2) \end{pmatrix} \end{aligned}$$

must equal $\begin{pmatrix} c & d \\ -qd & -c \end{pmatrix}$. But this happens if and only if the following equations are satisfied:

$$\begin{aligned} -ax^2 + 2qboxy - qay^2 &= (qy^2 - x^2)c \\ bx^2 - 2axy + qby^2 &= (qy^2 - x^2)d. \end{aligned}$$

The first of the above equations holds if and only if there is an $x, y \in k$ not both zero such that $(c-a)x^2 + 2qboxy - (c+a)qy^2 = 0$. If $(c-a) = 0$ then we can take $x = 1$ and $y = 0$. If $(c-a) \neq 0$ then the above holds if $(2qby)^2 + 4(c-a)(c+a)qy^2$ is a square in k (by the quadratic formula). But that is square $\leftrightarrow 4y^2q(qb^2 + c^2 - a^2)$ is a square $\leftrightarrow 4y^2q(c^2 - p)$ is square (since $a^2 - qb^2 = p$ by construction). But $4y^2q(c^2 - p)$ is square $\leftrightarrow q(c^2 - p)$ is square. Also, we know $c^2 - qd^2 = p$, by construction $\rightarrow c^2 - p = qd^2$ and thus $q(c^2 - p) = q(qd^2) = q^2d^2$, a square. Therefore, we know there is $x, y \in k$ not both zero (the above shows that $x = \frac{-2qby \pm \sqrt{(2qby)^2 + 4(c-a)(c+a)qy^2}}{2(c-a)}$ is a solution to the first equation and thus we can let $y = 1 \neq 0$) such that we get $P(a, b)$ isomorphic to $P(c, d')$ but then $c^2 - qd'^2 = p$, and we know $c^2 - qd^2 = p$, so $-qd'^2 = -qd^2 \rightarrow d' = \pm d$. If $d' = -d$ the theorem follows from using the the inner automorphism defined by:

$$S_1 = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 \\ 0 & I_{\frac{n}{2}-1, \frac{n}{2}-1} \end{pmatrix}$$

□

Note that by viewing the Y 's in the pairs above as trivial, the classification of commuting involutions of $\mathrm{SL}(2, k)$ follows.

4.2 Commuting Inner Involutions with the First Pair Isomorphic to $\mathrm{Inn}_{I_{n-i,i}}$

We now consider commuting pairs of inner involutions of $\mathrm{SL}(n, k)$ with the first involution isomorphic to $\mathrm{Inn}_{I_{n-i,i}}$, and proceed in the same manner as when the first entry is isomorphic to $\mathrm{Inn}_{L_{n,q}}$.

Corollary 4.2.1. *A commuting pair of involutions with the first entry isomorphic to $\mathrm{Inn}_{I_{n-i,i}}$ for $i \in \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$ is isomorphic to either $(\mathrm{Inn}_{I_{n-i,i}}, \mathrm{Inn}_B)$ with $B = \begin{pmatrix} 0 & I_i \\ pI_i & 0 \end{pmatrix}$, for $p \in k$ or $(\mathrm{Inn}_{I_{n-i,i}}, \mathrm{Inn}_B)$ with $B = \begin{pmatrix} Y_1 & 0 \\ 0 & \pm Y_2 \end{pmatrix}$, with Y_1 the matrix representative of an isomorphy class of inner involutions of $\mathrm{SL}(n-i, k)$ and Y_2 the matrix representative of an isomorphy class of inner involutions of $\mathrm{SL}(i, k)$.*

Proof. For

$$B' = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix},$$

for $B_1 \in M(n-i \times n-i, k)$, $B_2 \in M(n-i \times i, k)$, $B_3 \in M(i \times n-i, k)$, and $B_4 \in M(i \times i, k)$, the commuting condition gives:

$$\begin{aligned} B'^{-1}I_{n-i,i}B' &= cI_{n-i,i} \\ \Leftrightarrow I_{n-i,i}B' &= cB'I_{n-i,i} \\ \Leftrightarrow \begin{pmatrix} B_1 & B_2 \\ -B_3 & -B_4 \end{pmatrix} &= c \begin{pmatrix} B_1 & -B_2 \\ B_3 & -B_4 \end{pmatrix} \\ \Leftrightarrow c = 1 \text{ and } B_2 = 0, B_3 = 0 \\ \text{or } c = -1 \text{ and } B_1 = 0, B_4 = 0, \end{aligned}$$

because the characteristic of k is not two. If $c = 1$, then the commuting pair $(\mathrm{Inn}_{I_{n-i,i}}, \mathrm{Inn}_{B'})$ is equal to the pair $(\mathrm{Inn}_{I_{n-i,i}}, \mathrm{Inn}_{B'})$ for $B' = \begin{pmatrix} B_1 & 0 \\ 0 & -B_4 \end{pmatrix}$. If

$A^2 = pI_n$ then $B_1^2 = pI_{n-i}$ and $B_4^2 = pI_i$, and thus there is a $C_1 \in \text{GL}(n-i, k)$ and $C_4 \in \text{GL}(i, k)$ such that:

$$\begin{aligned} C_1^{-1}B_1C_1 &= \alpha Y_1 \quad \text{and} \\ C_4^{-1}B_4C_4 &= \beta Y_2 \end{aligned}$$

for Y_1, Y_2 matrix representatives of isomorphy classes of inner involutions of $\text{SL}(n-i, k)$ and $\text{SL}(i, k)$, respectively. This gives isomorphism with the pair

$$\left(\text{Inn}_{I_{n-i,i}}, \text{Inn} \begin{pmatrix} Y_1 & 0 \\ 0 & \gamma Y_2 \end{pmatrix} \right)$$

by factoring out an α . But $B'^2 = pI_n$ implies that $(\gamma Y_2)^2 = \gamma^2 Y_2^2 = \gamma^2 \sigma I_i$ because, by construction, Y_2 represents an inner involution. This implies that σ and p are in the same square class, so we can take Y_2 such that $Y_2^2 = pI_i$, so that $\gamma = \pm 1$.

Next, if $c = -1$ then the commuting pair $(\text{Inn}_{I_{n-i,i}}, \text{Inn}_A)$ is equal to the pair $(\text{Inn}_{I_{n-i,i}}, \text{Inn}_{B'})$ for $B' = \begin{pmatrix} 0 & B_2 \\ B_3 & 0 \end{pmatrix}$. But because $\text{Inn}_{B'}$ is an automorphism and

$$\det(B'^2) = \det(B')^2 = \det \left(\begin{pmatrix} 0 & B_2 \\ B_3 & 0 \end{pmatrix} \begin{pmatrix} 0 & B_2 \\ B_3 & 0 \end{pmatrix} \right) = \det \left(\begin{pmatrix} B_2B_3 & \\ 0 & B_3B_2 \end{pmatrix} \right),$$

$B_2B_3 \in \text{GL}(n-i, k)$ and $B_3B_2 \in \text{GL}(i, k)$ and thus B_2, B_3 are nonsingular. This implies that $n-i = i$, because the rank of the product is at most two times the minimum of $n-i, i$. This gives, since $B'^2 = pI_n$, that $B_2B_3 = pI_i$ and thus $B_3 = pB_2^{-1}$ so that the pair is equal to $(\text{Inn}_{I_{n-i,i}}, \text{Inn}_{B'})$ for $B' = \begin{pmatrix} 0 & B_2 \\ pB_2^{-1} & 0 \end{pmatrix}$. But then

using the inner automorphism defined by the matrix $\begin{pmatrix} I_i & 0 \\ 0 & B_2 \end{pmatrix}$ gives the desired

isomorphism:

$$\begin{aligned} \begin{pmatrix} I_i & 0 \\ 0 & B_2 \end{pmatrix} I_{i,i} \begin{pmatrix} I_i & 0 \\ 0 & B_2 \end{pmatrix}^{-1} &= I_{i,i} \quad \text{and} \\ \begin{pmatrix} I_i & 0 \\ 0 & B_2 \end{pmatrix} \begin{pmatrix} 0 & B_2 \\ pB_2^{-1} & 0 \end{pmatrix} \begin{pmatrix} I_i & 0 \\ 0 & B_2 \end{pmatrix}^{-1} &= \begin{pmatrix} 0 & I_i \\ pI_i & 0 \end{pmatrix} \end{aligned}$$

□

4.3 Classification of Remaining Pairs

The isomorphism classes of commuting inner involutions of $\text{SL}(n, k)$, now depend, in part, on the isomorphism classes of the blocks Y , Y_1 , and Y_2 (viewed as a matrix representative of an inner involution).

Lemma 4.3.1. *Let A and B be a matrix representatives of inner involutions such that $A^2 = p_1 I_n$ and $B^2 = p_2 I_n$. If Inn_A is isomorphic to Inn_B , as inner involutions, then $p_1 \equiv p_2 \pmod{k^{*2}}$.*

Proof. Inn_A isomorphic to Inn_B implies there is a $C \in \text{GL}(n, k)$ such that $C^{-1}AC = cB \rightarrow (cB)^2 = (C^{-1}AC)^2 = C^{-1}ACC^{-1}AC = C^{-1}p_1 I_n C = p_1 I_n \rightarrow p_2 I_n = B^2 = \frac{1}{c^2} p_1 I_n \rightarrow p_2 = \left(\frac{1}{c}\right)^2 p_1 \rightarrow p_1 \equiv p_2 \pmod{k^{*2}}$. □

This lemma implies that we need to determine the distinctness only of those remaining commuting pairs where the matrix representatives of the inner involutions (corresponding to pairs with isomorphic first entries) square to the same square class multiple of the identity.

Lemma 4.3.2. *Suppressing the Inn notation, we have that*

$$\left(L'_{n,q}, \begin{pmatrix} L_{\frac{n}{2},p} & 0 \\ 0 & L_{\frac{n}{2},p} \end{pmatrix} \right) \not\cong \left(L'_{n,q}, \begin{pmatrix} L_{\frac{n}{2},p} & 0 \\ 0 & -L_{\frac{n}{2},p} \end{pmatrix} \right)$$

Proof. If the above pairs are isomorphic then there is a matrix in $\text{GL}(n, k)$ of the

form $\begin{pmatrix} A & B \\ qB & A \end{pmatrix}$ or $\begin{pmatrix} A & B \\ -qB & -A \end{pmatrix}$ such that

$$\begin{aligned} & \begin{pmatrix} A & B \\ \pm qB & \pm A \end{pmatrix} \begin{pmatrix} L_{\frac{n}{2},p} & 0 \\ 0 & L_{\frac{n}{2},p} \end{pmatrix} \begin{pmatrix} A & B \\ \pm qB & \pm A \end{pmatrix}^{-1} = c \begin{pmatrix} L_{\frac{n}{2},p} & 0 \\ 0 & -L_{\frac{n}{2},p} \end{pmatrix} \\ \Leftrightarrow & \begin{pmatrix} A & B \\ \pm qB & \pm A \end{pmatrix} \begin{pmatrix} L_{\frac{n}{2},p} & 0 \\ 0 & L_{\frac{n}{2},p} \end{pmatrix} = c \begin{pmatrix} L_{\frac{n}{2},p} & 0 \\ 0 & -L_{\frac{n}{2},p} \end{pmatrix} \begin{pmatrix} A & B \\ \pm qB & \pm A \end{pmatrix} \\ \Leftrightarrow & \begin{pmatrix} AL_{\frac{n}{2},p} & BL_{\frac{n}{2},p} \\ \pm qBL_{\frac{n}{2},p} & \pm AL_{\frac{n}{2},p} \end{pmatrix} = c \begin{pmatrix} L_{\frac{n}{2},p}A & L_{\frac{n}{2},p}B \\ \mp qL_{\frac{n}{2},p}B & \mp L_{\frac{n}{2},p}A \end{pmatrix} \end{aligned}$$

$$cA = L_{\frac{n}{2},p}^{-1}AL_{\frac{n}{2},p} = -cA \text{ and } cB = L_{\frac{n}{2},p}^{-1}BL_{\frac{n}{2},p} = -cB \quad (4.18)$$

which implies that A and B are both zero since the characteristic of k is not two, or that c is zero. But both of these contradict the fact that the matrix representative of the inner automorphism defined by the matrix $\begin{pmatrix} A & B \\ \pm qB & \pm A \end{pmatrix}$ is invertible. \square

Lemma 4.3.3. *Suppressing the Inn notation we have that, for*

$i, j \in \{0, 1, \dots, \lfloor \frac{n}{4} \rfloor\}$, $i \neq j$,

$$\left(L'_{n,q}, \begin{pmatrix} I_{\frac{n}{2}-i,i} & 0 \\ 0 & I_{\frac{n}{2}-i,i} \end{pmatrix} \right) \not\approx \left(L'_{n,q}, \begin{pmatrix} I_{\frac{n}{2}-j,j} & 0 \\ 0 & I_{\frac{n}{2}-j,j} \end{pmatrix} \right).$$

Proof. This follows from the fact that $I_{n-2j,2j}$ is not isomorphic to $I_{n-2i,2i}$ for $i, j \in \{0, 1, \dots, \lfloor \frac{n}{4} \rfloor\}$, $i \neq j$, and

$$\text{Inn} \begin{pmatrix} I_{\frac{n}{2}-k,k} & 0 \\ 0 & I_{\frac{n}{2}-k,k} \end{pmatrix} \approx \text{Inn}_{I_{n-2k,2k}}$$

for $k \in \{0, 1, \dots, \lfloor \frac{n}{4} \rfloor\}$. To show the above isomorphism, consider the matrix

$$P = \sum_{i=1}^{\frac{n}{2}-k} E_{i,i} + E_{\frac{n}{2}-k+i, \frac{n}{2}+i} + \sum_{i=1}^k E_{n-2k+i, \frac{n}{2}-k+i} + E_{n-k+i, n-k+i}.$$

Then,

$$\begin{aligned}
PP^T &= P \left(\sum_{i=1}^{\frac{n}{2}-k} E_{i,i} + E_{\frac{n}{2}+i, \frac{n}{2}-k+i} + \sum_{i=1}^k E_{\frac{n}{2}-k+i, n-2k+i} + E_{n-k+i, n-k+i} \right) \\
&= \sum_{i=1}^{\frac{n}{2}-k} E_{i,i} + E_{\frac{n}{2}-k+i, \frac{n}{2}-k+i} + \sum_{i=1}^k E_{n-2k+i, n-2k+i} + E_{n-k+i, n-k+i} \\
&= \sum_{i=1}^{n-2k} E_{i,i} + \sum_{i=1}^{2k} E_{n-2k+i, n-2k+i} \\
&= \sum_{i=1}^n E_{i,i} \\
&= I_n
\end{aligned}$$

because of the following: $i = \frac{n}{2} + i, i = \frac{n}{2} - k + i, i = n - k + i, \frac{n}{2} + i = \frac{n}{2} - k + i, \frac{n}{2} + i = n - k + i, \frac{n}{2} - k + i = n - k + i$ all contradict either that fact that k is no larger than $\frac{n}{4}$ or that $n \neq 0$. Note that for $k = 0$ the isomorphism is trivial and the matrix P is not needed. Furthermore,

$$\begin{aligned}
&P \begin{pmatrix} I_{\frac{n}{2}-k, k} & 0 \\ 0 & I_{\frac{n}{2}-k, k} \end{pmatrix} P^T \\
&= P \left(\sum_{j=1}^{\frac{n}{2}-k} E_{j,j} + E_{\frac{n}{2}+j, \frac{n}{2}+j} - \sum_{j=1}^k E_{\frac{n}{2}-k+j, \frac{n}{2}-k+j} + E_{n-k+j, n-k+j} \right) P^T \\
&= \left(\sum_{j=1}^{\frac{n}{2}-k} E_{j,j} + \sum_{j=1}^{\frac{n}{2}-k} E_{\frac{n}{2}-k+j, \frac{n}{2}+j} - \sum_{j=1}^k E_{n-2k+j, \frac{n}{2}-k+j} - \sum_{j=1}^k E_{n-k+j, n-k+j} \right) P^T \\
&= \sum_{j=1}^{\frac{n}{2}-k} E_{j,j} + \sum_{j=1}^{\frac{n}{2}-k} E_{\frac{n}{2}-k+j, \frac{n}{2}-k+j} - \sum_{j=1}^k E_{n-2k+j, n-2k+j} - \sum_{j=1}^k E_{n-k+j, n-k+j} \\
&= \sum_{j=1}^{n-2k} E_{j,j} - \sum_{j=1}^{2k} E_{n-2k+j, n-2k+j} \\
&= I_{n-2k, 2k}
\end{aligned}$$

□

Having distinct isomorphism class representatives of inner involutions of $\text{SL}(\frac{n}{2}, k)$ as blocks, does not always guarantee distinct pairs, as the following lemma shows.

Lemma 4.3.4. *Suppressing the Inn notation, for $i \in \{0, 1, \dots, \lfloor \frac{n}{4} \rfloor\}$,*

$$\left(L'_{n,q}, \begin{pmatrix} I_{\frac{n}{2}-i,i} & 0 \\ 0 & -I_{\frac{n}{2}-i,i} \end{pmatrix} \right) \approx \left(L'_{n,q}, \begin{pmatrix} I_{\frac{n}{2}} & 0 \\ 0 & -I_{\frac{n}{2}} \end{pmatrix} \right).$$

Proof. This isomorphism follows from considering the inner involution defined by:

$$P = \sum_{j=1}^{\frac{n}{2}-i} E_{j,j} + E_{\frac{n}{2}+j, \frac{n}{2}+j} + \sum_{j=1}^i E_{\frac{n}{2}-i+j, n-i+j} + qE_{n-i+j, \frac{n}{2}-i+j}.$$

Note that for

$$Q = \sum_{j=1}^{\frac{n}{2}-i} E_{j,j} + E_{\frac{n}{2}+j, \frac{n}{2}+j} + \sum_{j=1}^i \frac{1}{q} E_{\frac{n}{2}-i+j, n-i+j} + E_{n-i+j, \frac{n}{2}-i+j},$$

it follows that $PQ = I_n$, and $Q = P^{-1}$. Then,

$$\begin{aligned} & P^{-1} \begin{pmatrix} I_{\frac{n}{2}} & 0 \\ 0 & -I_{\frac{n}{2}} \end{pmatrix} P \\ &= P^{-1} \left(\sum_{j=1}^{\frac{n}{2}-i} E_{j,j} - E_{\frac{n}{2}+j, \frac{n}{2}+j} + \sum_{j=1}^i E_{\frac{n}{2}-i+j, n-i+j} - qE_{n-i+j, \frac{n}{2}-i+j} \right) \\ &= \sum_{j=1}^{\frac{n}{2}-i} E_{j,j} - \sum_{j=1}^{\frac{n}{2}-i} E_{\frac{n}{2}+j, \frac{n}{2}+j} - \sum_{j=1}^i E_{\frac{n}{2}-i+j, \frac{n}{2}-i+j} + \sum_{j=1}^i E_{n-i+j, n-i+j} \\ &= \sum_{j=1}^{\frac{n}{2}-i} E_{j,j} - E_{\frac{n}{2}+j, \frac{n}{2}+j} + \sum_{j=1}^i -E_{\frac{n}{2}-i+j, \frac{n}{2}-i+j} + E_{n-i+j, n-i+j} \\ &= \begin{pmatrix} I_{\frac{n}{2}-i,i} & 0 \\ 0 & -I_{\frac{n}{2}-i,i} \end{pmatrix}. \end{aligned}$$

The proof follows from the fact that P commutes with $L'_{n,q}$. □

Lemma 4.3.5. *The pairs $(\text{Inn}_{L_{n,q}}, \text{Inn}_{I_{n-2i, 2i}L_{n,q}})$ for $i \in \{0, 1, \dots, \lfloor \frac{n}{4} \rfloor\}$ are distinct for distinct i .*

Proof. Let $i \neq j$ with $i, j \in \{0, 1, \dots, \lfloor \frac{n}{4} \rfloor\}$. Then if $(L_{n,q}, I_{n-2i, 2i}L_{n,q})$ is isomorphic to $(L_{n,q}, I_{n-2j, 2j}L_{n,q})$ then there exists an $A \in \text{GL}(n, k)$ such that

$$AL_{n,q}A^{-1} = cL_{n,q} \text{ and } AI_{n-2i, 2i}L_{n,q}A^{-1} = \kappa I_{n-2j, 2j}L_{n,q}$$

this implies that $A^{-1}L_{n,q} = \frac{1}{c}L_{n,q}A^{-1}$ so that

$$\begin{aligned} AI_{n-2i,2i}A^{-1}L_{n,q} &= \frac{1}{c}AI_{n-2i,2i}L_{n,q}A^{-1} = \frac{\kappa}{c}I_{n-2j,2j}L_{n,q} \\ \rightarrow AI_{n-2i,2i}A^{-1} &= \frac{\kappa}{c}I_{n-2j,2j}, \end{aligned}$$

a contradiction to the fact that the inner involutions defined by the matrices $I_{n-2i,2i}$ and $I_{n-2j,2j}$ are in distinct isomorphism classes for $i \neq j$, $i, j \in \{0, 1, \dots, \lfloor \frac{n}{4} \rfloor\}$. \square

Lemma 4.3.6. *Suppressing the Inn notation we have the following: the commuting pair $(L_{n,q}, I_{n-2i,2i}L_{n,q})$ is not isomorphic to $\left(L'_{n,q}, \begin{pmatrix} L_{\frac{n}{2},q} & 0 \\ 0 & -L_{\frac{n}{2},q} \end{pmatrix}\right)$, for $i \in \{0, 1, \dots, \lfloor \frac{n}{4} \rfloor\}$ (n divisible by 4), and the pair $\left(L'_{n,q}, \begin{pmatrix} L_{\frac{n}{2},q} & 0 \\ 0 & L_{\frac{n}{2},q} \end{pmatrix}\right)$ is isomorphic to $(L_{n,q}, I_{\frac{n}{2},\frac{n}{2}}L_{n,q})$.*

Proof. First, note that $(L_{n,q}, I_{n-2i,2i}L_{n,q}) \approx \left(L'_{n,q}, \begin{pmatrix} 0 & I_{\frac{n}{2}-i,i} \\ qI_{\frac{n}{2}-i,i} & 0 \end{pmatrix}\right)$, via the inner automorphism defined by the permutation matrix P_n , as defined above. Considering first the case when we use the negative block, for the pair to be isomorphic it must be that for some invertible matrix of the form $\begin{pmatrix} A & B \\ \pm qB & \pm A \end{pmatrix}$ for $A, B \in M(\frac{n}{2}, k)$ (it commutes up to a scalar multiple with $L'_{n,q}$) that

$$\begin{aligned} &\begin{pmatrix} A & B \\ \pm qB & \pm A \end{pmatrix} \begin{pmatrix} L_{\frac{n}{2},q} & 0 \\ 0 & -L_{\frac{n}{2},q} \end{pmatrix} = c \begin{pmatrix} 0 & I_{\frac{n}{2}-i,i} \\ qI_{\frac{n}{2}-i,i} & 0 \end{pmatrix} \begin{pmatrix} A & B \\ \pm qB & \pm A \end{pmatrix} \\ \rightarrow &\begin{pmatrix} AL_{\frac{n}{2},q} & -BL_{\frac{n}{2},q} \\ \pm qBL_{\frac{n}{2},q} & \mp AL_{\frac{n}{2},q} \end{pmatrix} = c \begin{pmatrix} \pm qI_{\frac{n}{2}-i,i}B & \pm I_{\frac{n}{2}-i,i}A \\ qI_{\frac{n}{2}-i,i}A & qI_{\frac{n}{2}-i,i}B \end{pmatrix} \\ \rightarrow &AL_{\frac{n}{2},q} = \pm cqI_{\frac{n}{2}-i,i}B, BL_{\frac{n}{2},q} = \pm cI_{\frac{n}{2}-i,i}A \text{ and} \\ &AL_{\frac{n}{2},q} = \mp cqI_{\frac{n}{2}-i,i}B, BL_{\frac{n}{2},q} = \mp cI_{\frac{n}{2}-i,i}A, \text{ signs taken respectively} \\ \rightarrow &A = qcI_{\frac{n}{2}-i,i}BL_{\frac{n}{2},q}^{-1} = -A \text{ and } B = cI_{\frac{n}{2}-i,i}AL_{\frac{n}{2},q}^{-1} = -B \end{aligned}$$

which implies both A and B are the zero matrix, since the characteristic of k is not two. This is a contradiction to the invertibility of an inner automorphism defined by the matrix $\begin{pmatrix} A & B \\ \pm qB & \pm A \end{pmatrix}$, as above.

Considering the case when the positive block is used, i.e.

$$\left(L'_{n,q}, \begin{pmatrix} L_{\frac{n}{2},q} & 0 \\ 0 & +L_{\frac{n}{2},q} \end{pmatrix} \right) = (L'_{n,q}, L_{n,q}).$$

Let

$$S = \begin{pmatrix} L_{\frac{n}{2},-q} & I_{\frac{n}{2}} \\ qI_{\frac{n}{2}} & L_{\frac{n}{2},-q} \end{pmatrix},$$

and note that S commutes with $L'_{n,q}$. Furthermore, $\det(S) = \det\left(L_{\frac{n}{2},-q} - qI_{\frac{n}{2}}\right)$ by lemma 3.3.2, which is equal to $\det(-qI_{\frac{n}{2}} - qI_{\frac{n}{2}}) = \det(-2qI_{\frac{n}{2}}) \neq 0$, because the characteristic of k is not two. Therefore, S is invertible and thus so is the inner automorphism defined by S . Now, for $X = \sum_{i=1}^{\frac{n}{2}} E_{2i-1,2i-1} - E_{2i,2i}$:

$$\begin{aligned} SL_{n,q}S^{-1} &= \begin{pmatrix} L_{\frac{n}{2},-q} & I_{\frac{n}{2}} \\ qI_{\frac{n}{2}} & L_{\frac{n}{2},-q} \end{pmatrix} \begin{pmatrix} L_{\frac{n}{2},q} & 0 \\ 0 & L_{\frac{n}{2},q} \end{pmatrix} \begin{pmatrix} L_{\frac{n}{2},-q} & I_{\frac{n}{2}} \\ qI_{\frac{n}{2}} & L_{\frac{n}{2},-q} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} L_{\frac{n}{2},-q}L_{\frac{n}{2},q} & L_{\frac{n}{2},q} \\ qL_{\frac{n}{2},q} & L_{\frac{n}{2},-q}L_{\frac{n}{2},q} \end{pmatrix} \begin{pmatrix} L_{\frac{n}{2},-q} & I_{\frac{n}{2}} \\ qI_{\frac{n}{2}} & L_{\frac{n}{2},-q} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} qX & XL_{\frac{n}{2},-q} \\ qXL_{\frac{n}{2},-q} & qX \end{pmatrix} \begin{pmatrix} L_{\frac{n}{2},-q} & I_{\frac{n}{2}} \\ qI_{\frac{n}{2}} & L_{\frac{n}{2},-q} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 0 & X \\ qX & 0 \end{pmatrix} \begin{pmatrix} L_{\frac{n}{2},-q} & I_{\frac{n}{2}} \\ qI_{\frac{n}{2}} & L_{\frac{n}{2},-q} \end{pmatrix} \begin{pmatrix} L_{\frac{n}{2},-q} & I_{\frac{n}{2}} \\ qI_{\frac{n}{2}} & L_{\frac{n}{2},-q} \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 0 & X \\ qX & 0 \end{pmatrix}. \end{aligned}$$

Furthermore, $\begin{pmatrix} P_{\frac{n}{2}} & 0 \\ 0 & P_{\frac{n}{2}} \end{pmatrix}$ commutes with $L'_{n,q}$ and

$$\begin{pmatrix} P_{\frac{n}{2}} & 0 \\ 0 & P_{\frac{n}{2}} \end{pmatrix} \begin{pmatrix} 0 & X \\ qX & 0 \end{pmatrix} \begin{pmatrix} P_{\frac{n}{2}} & 0 \\ 0 & P_{\frac{n}{2}} \end{pmatrix}^T = \begin{pmatrix} 0 & I_{\frac{n}{4},\frac{n}{4}} \\ qI_{\frac{n}{4},\frac{n}{4}} & 0 \end{pmatrix}$$

because

$$\begin{aligned}
& P_{\frac{n}{2}} X P_{\frac{n}{2}}^T \\
&= \left(\sum_{i=1}^{\frac{n}{4}} E_{i,2i-1} + E_{\frac{n}{4}+i,2i} \right) \left(\sum_{i=1}^{\frac{n}{2}} E_{2i-1,2i-1} - E_{2i,2i} \right) \left(\sum_{i=1}^{\frac{n}{4}} E_{2i-1,i} + E_{2i,\frac{n}{4}+i} \right) \\
&= \left(\sum_{i=1}^{\frac{n}{4}} E_{i,2i-1} - E_{\frac{n}{4}+i,2i} \right) \left(\sum_{i=1}^{\frac{n}{4}} E_{2i-1,i} + E_{2i,\frac{n}{4}+i} \right) \\
&= \left(\sum_{i=1}^{\frac{n}{4}} E_{i,i} - E_{\frac{n}{4}+i,\frac{n}{4}+i} \right) \\
&= I_{\frac{n}{4},\frac{n}{4}}.
\end{aligned}$$

Therefore,

$$(L'_{n,q}, L_{n,q}) \approx \left(L'_{n,q}, \begin{pmatrix} 0 & I_{\frac{n}{4},\frac{n}{4}} \\ qI_{\frac{n}{4},\frac{n}{4}} & 0 \end{pmatrix} \right) \approx (L_{n,q}, I_{\frac{n}{2},\frac{n}{2}} L_{n,q}).$$

Hence, by lemma 4.3.5, the pair $\left(L'_{n,q}, \begin{pmatrix} L_{\frac{n}{2},q} & 0 \\ 0 & L_{\frac{n}{2},q} \end{pmatrix} \right)$ is not isomorphic to any pair $(L_{n,q}, I_{n-2i,2i} L_{n,q})$ unless $i = \frac{n}{4}$. \square

Lemma 4.3.7. *Suppressing the Inn notation, the following isomorphism holds:*

$$\left(I_{n-i,i}, \begin{pmatrix} L_{n-i,q} & 0 \\ 0 & L_{i,q} \end{pmatrix} \right) \approx \left(I_{n-i,i}, \begin{pmatrix} L_{n-i,q} & 0 \\ 0 & -L_{i,q} \end{pmatrix} \right).$$

Proof. Let

$$S = \sum_{k=1}^{n-i} E_{k,k} + \sum_{k=n-i+1}^n E_{2k-1,2k-1} - E_{2k,2k},$$

then S is invertible and because S is diagonal, it commutes with the diagonal matrix

$I_{n-i,i}$. Furthermore,

$$\begin{aligned}
& S \begin{pmatrix} L_{n-i,q} & 0 \\ 0 & L_{i,q} \end{pmatrix} S^{-1} \\
&= \begin{pmatrix} L_{n-i,q} & 0 \\ 0 & \left(\sum_{k=1}^i E_{2k-1,2k-1} - E_{2k,2k} \right) L_{i,q} \left(\sum_{k=1}^i E_{2k-1,2k-1} - E_{2k,2k} \right)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} L_{n-i,q} & 0 \\ 0 & \left(\sum_{k=1}^i E_{2k-1,2k} - qE_{2k,2k-1} \right) \left(\sum_{k=1}^i E_{2k-1,2k-1} - E_{2k,2k} \right) \end{pmatrix} \\
&= \begin{pmatrix} L_{n-i,q} & 0 \\ 0 & \left(\sum_{k=1}^i -E_{2k-1,2k} - qE_{2k,2k-1} \right) \end{pmatrix} \\
&= \begin{pmatrix} L_{n-i,q} & 0 \\ 0 & - \left(\sum_{k=1}^i E_{2k-1,2k} + qE_{2k,2k-1} \right) \end{pmatrix} \\
&= \begin{pmatrix} L_{n-i,q} & 0 \\ 0 & -L_{i,q} \end{pmatrix}
\end{aligned}$$

□

Lemma 4.3.8. For $n - i \neq i$, and distinct $j \in \{1, 2, \dots, \lfloor \frac{n-i}{2} \rfloor\}$ or distinct $k \in \{1, 2, \dots, \lfloor \frac{i}{2} \rfloor\}$, pairs of the following form are distinct:

$$\left(\text{Inn}_{I_{n-i,i}}, \text{Inn} \begin{pmatrix} I_{n-i-j,j} & 0 \\ 0 & I_{i-k,k} \end{pmatrix} \right).$$

Proof. If not, then we would have the isomorphism, for $A \in \text{GL}((n-i), k)$ and $B \in \text{GL}(i, k)$:

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I_{n-i-j,j} & 0 \\ 0 & I_{i-k,k} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}^{-1} = c \begin{pmatrix} I_{n-i-l,l} & 0 \\ 0 & I_{i-m,m} \end{pmatrix}$$

$$\rightarrow AI_{n-i-j,j}A^{-1} = cI_{n-i-l,l} \text{ and } BI_{i-k,k}B^{-1} = cI_{i-m,m},$$

at least one of which leads to a contradiction, since for $j \neq l$, $I_{n-i-j,j}$ and $I_{n-i-l,l}$ are in distinct isomorphism classes for $j, l \in \{1, 2, \dots, \lfloor \frac{n-i}{2} \rfloor\}$, and $k \neq m$, $I_{i-k,k}$ and $I_{i-m,m}$ are in distinct isomorphism classes for $k, m \in \{1, 2, \dots, \lfloor \frac{i}{2} \rfloor\}$. □

Lemma 4.3.9. For $n - i \neq i$, $j \in \{1, 2, \dots, \lfloor \frac{n-i}{2} \rfloor\}$, and $k \in \{1, 2, \dots, \lfloor \frac{i}{2} \rfloor\}$, and suppressing the Inn notation,

$$\left(I_{n-i,i}, \begin{pmatrix} I_{n-i-j,j} & 0 \\ 0 & I_{i-k,k} \end{pmatrix} \right) \approx \left(I_{n-i,i}, \begin{pmatrix} I_{n-i-j,j} & 0 \\ 0 & -I_{i-k,k} \end{pmatrix} \right)$$

if and only if either $i - k = k$ or $n - i - j = j$.

Proof. (\Leftarrow) First, suppose that $i - k = k$, then $i = 2k$, and let

$$S = \begin{pmatrix} I_{n-i} & 0 \\ 0 & \begin{pmatrix} 0_k & I_k \\ I_k & 0_k \end{pmatrix} \end{pmatrix}.$$

Then, $SI_{n-i,i}S^{-1} = I_{n-i,i}$ and

$$\begin{aligned} & S \begin{pmatrix} I_{n-i-j,j} & 0 \\ 0 & I_{i-k,k} \end{pmatrix} S^{-1} \\ &= \begin{pmatrix} I_{n-i-j,j} & 0 \\ 0 & \begin{pmatrix} 0_k & I_k \\ I_k & 0_k \end{pmatrix} I_{i-k,k} \begin{pmatrix} 0_k & I_k \\ I_k & 0_k \end{pmatrix}^{-1} \end{pmatrix} \\ &= \begin{pmatrix} I_{n-i-j,j} & 0 \\ 0 & \begin{pmatrix} 0_k & -I_k \\ I_k & 0_k \end{pmatrix} \begin{pmatrix} 0_k & I_k \\ I_k & 0_k \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} I_{n-i-j,j} & 0 \\ 0 & -I_{i-k,k} \end{pmatrix} \end{aligned}$$

Next, if $n - i - j = j$ then $n - i = 2j$, and let

$$S' = \begin{pmatrix} \begin{pmatrix} 0_j & I_j \\ I_j & 0_j \end{pmatrix} & 0 \\ 0 & I_i \end{pmatrix}.$$

Then, $S'I_{n-i,i}S'^{-1} = I_{n-i,i}$ and

$$\begin{aligned}
& S' \begin{pmatrix} I_{n-i-j,j} & 0 \\ 0 & I_{i-k,k} \end{pmatrix} S'^{-1} \\
&= \begin{pmatrix} \begin{pmatrix} 0_j & I_j \\ I_j & 0_j \end{pmatrix} I_{n-i-j,j} \begin{pmatrix} 0_j & I_j \\ I_j & 0_j \end{pmatrix}^{-1} & 0 \\ 0 & I_{i-k,k} \end{pmatrix} \\
&= \begin{pmatrix} \begin{pmatrix} 0_j & -I_j \\ I_j & 0_j \end{pmatrix} \begin{pmatrix} 0_j & I_j \\ I_j & 0_j \end{pmatrix} & 0 \\ 0 & I_{i-k,k} \end{pmatrix} \\
&= - \begin{pmatrix} I_{n-i-j,j} & 0 \\ 0 & -I_{i-k,k} \end{pmatrix}.
\end{aligned}$$

Thus, the first part follows from the fact that $\text{Inn}_{\alpha A} = \text{Inn}_A$ for constant α , invertible A .

(\Rightarrow) If the above pair is isomorphic, then there is an $S \in \text{GL}(n, k)$ such that S has the block form, for $A \in \text{GL}(n-i, k)$ and $B \in \text{GL}(i, k)$, $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, and

$$\begin{aligned}
AI_{n-i-j,j}A^{-1} &= cI_{n-i-j,j} \\
BI_{i-k,k}B^{-1} &= -cI_{i-k,k}.
\end{aligned}$$

If $AI_{n-i-j,j}A^{-1} = cI_{n-i-j,j}$, then either $c = -1$ or $c = 1$, because

$$I_{n-i} = (AI_{n-i-j,j}A^{-1})^2 = c^2I_{n-i-j,j}^2 = c^2I_{n-i}.$$

If $c = -1$ then for $A = \begin{pmatrix} Q & R \\ S & T \end{pmatrix}$, with $Q \in M(n-i-j, k)$, $T \in M(j, k)$, $R \in M(n-i-j \times j, k)$, $S \in M(j \times n-i-j, k)$,

$$\begin{aligned}
\begin{pmatrix} Q & R \\ S & T \end{pmatrix} \begin{pmatrix} I_{n-i-j} & 0 \\ 0 & -I_j \end{pmatrix} &= - \begin{pmatrix} I_{n-i-j} & 0 \\ 0 & -I_j \end{pmatrix} \begin{pmatrix} Q & R \\ S & T \end{pmatrix} \\
&\rightarrow \begin{pmatrix} Q & -R \\ S & -T \end{pmatrix} = \begin{pmatrix} -Q & -R \\ S & T \end{pmatrix},
\end{aligned}$$

which implies that Q is the zero $(n - i - j \times n - i - j)$ matrix and T is the $(j \times j)$ zero matrix because the characteristic of the field k is not two. Since A is invertible this implies that $n - i - j = j$, because if not then the rank of A would be at most two times the minimum of $\{n - i - j, j\}$. If $c = 1$ then $BI_{i-k,k}B^{-1} = -I_{i-k,k}$ and by the same reasoning as above, $i - k = k$. \square

Lemma 4.3.10. *For $n - i = i$, and $j, k, l, m \in \{1, 2, \dots, \lfloor \frac{i}{2} \rfloor\}$, suppressing the Inn notation,*

$$\left(I_{i,i}, \begin{pmatrix} I_{i-j,j} & 0 \\ 0 & I_{i-k,k} \end{pmatrix} \right) \approx \left(I_{i,i}, \begin{pmatrix} I_{i-l,l} & 0 \\ 0 & I_{i-m,m} \end{pmatrix} \right).$$

if and only if either $j = l$ and $k = m$ or $j = m$ and $k = l$.

Proof. The isomorphism holds if and only if either

$$\begin{aligned} & \begin{pmatrix} A_{i \times i} I_{i-k,k} A_{i \times i}^{-1} & 0 \\ 0 & B_{i \times i} I_{i-j,j} B_{i \times i}^{-1} \end{pmatrix} \\ &= \begin{pmatrix} A_{i \times i} I_{i-k,k} & 0 \\ 0 & B_{i \times i} I_{i-j,j} \end{pmatrix} \begin{pmatrix} 0 & B_{i \times i}^{-1} \\ A_{i \times i}^{-1} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & A_{i \times i} \\ B_{i \times i} & 0 \end{pmatrix} \begin{pmatrix} I_{i-j,j} & 0 \\ 0 & I_{i-k,k} \end{pmatrix} \begin{pmatrix} 0 & A_{i \times i} \\ B_{i \times i} & 0 \end{pmatrix}^{-1} = c \begin{pmatrix} I_{i-l,l} & 0 \\ 0 & I_{i-m,m} \end{pmatrix} \end{aligned}$$

for $A, B \in GL(i, k)$, if and only if $k = l$ and $j = m$, by the classification of involutions of $SL(n, k)$ (in [HWD06]), or for $A, B \in GL(i, k)$,

$$\begin{pmatrix} A_{i \times i} & 0 \\ 0 & B_{i \times i} \end{pmatrix} \begin{pmatrix} I_{i-j,j} & 0 \\ 0 & I_{i-k,k} \end{pmatrix} \begin{pmatrix} A_{i \times i} & 0 \\ 0 & B_{i \times i} \end{pmatrix}^{-1} = c \begin{pmatrix} I_{i-l,l} & 0 \\ 0 & I_{i-m,m} \end{pmatrix}$$

if and only if $j = l$ and $k = m$, by similar reasoning. \square

Lemma 4.3.11. *For $j = l$ and $k = m$, as in the statement of lemma 4.3.10, $i - j = j$ or $i - k = k$ if and only if the following isomorphism holds, suppressing the Inn notation:*

$$\left(I_{i,i}, \begin{pmatrix} I_{i-j,j} & 0 \\ 0 & I_{i-k,k} \end{pmatrix} \right) \approx \left(I_{i,i}, \begin{pmatrix} I_{i-j,j} & 0 \\ 0 & -I_{i-k,k} \end{pmatrix} \right)$$

Proof. (\Rightarrow) If $i - j = j$ or $i - k = k$ then the isomorphism holds by the proof of lemma 4.3.9.

(\Leftarrow) If the isomorphism holds then either there is an invertible $A, B \in \text{GL}(i, k)$ such that $AI_{i-j,j}A^{-1} = cI_{i-j,j}$ and $BI_{i-k,k}B^{-1} = -cI_{i-k,k}$ or $A, B \in \text{GL}(i, k)$ such that $AI_{i-k,k}A^{-1} = cI_{i-j,j}$ and $BI_{i-j,j}B^{-1} = -cI_{i-k,k}$, as in the proof of lemma 4.3.10. The first of these cases gives $c^2I_i = (cI_{i-j,j})^2 = (AI_{i-j,j}A^{-1})^2 = I_i$ which implies that $c = \pm 1$. This, in turn, implies that either $i - j = j$ or $i - k = k$, as in the proof of lemma 4.3.9. The second case forces $k = j$, because k, j are such that if they are distinct then $I_{i-k,k}$ and $I_{i-j,j}$ are in distinct isomorphism classes of involutions of $\text{SL}(n, k)$, but if $k = j$, then this reduces to the first case and thus $i - j = j$ or $i - k = k$. \square

Lemma 4.3.12. *For $k = l$ and $j = m$, as in the statement of lemma 4.3.10, $i - j = j$ or $i - k = k$ if and only if the following isomorphism holds, suppressing the Inn notation:*

$$\left(I_{i,i}, \begin{pmatrix} I_{i-j,j} & 0 \\ 0 & I_{i-k,k} \end{pmatrix} \right) \approx \left(I_{i,i}, \begin{pmatrix} I_{i-k,k} & 0 \\ 0 & -I_{i-j,j} \end{pmatrix} \right)$$

Proof. (\Rightarrow) If $i - j = j$ or $i - k = k$ then the isomorphism holds by the proof of lemma 4.3.9 and by using the matrix:

$$S' = \begin{pmatrix} 0 & I_i \\ \begin{pmatrix} 0_k & I_k \\ I_k & 0_k \end{pmatrix} & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & \begin{pmatrix} 0_k & I_k \\ I_k & 0_k \end{pmatrix} \\ I_i & 0 \end{pmatrix},$$

similar to the use of S in the proof of lemma 4.3.9.

(\Leftarrow) If the isomorphism holds then either there is an invertible $A, B \in \text{GL}(i, k)$ such that $AI_{i-j,j}A^{-1} = cI_{i-k,k}$ and $BI_{i-k,k}B^{-1} = -cI_{i-j,j}$ or $A, B \in \text{GL}(i, k)$ such that $AI_{i-j,j}A^{-1} = cI_{i-j,j}$ and $BI_{i-k,k}B^{-1} = -cI_{i-k,k}$. The second of these cases forces $c = \pm 1$, which in turn forces either $i - j = j$ or $i - k = k$. The first case forces $k = j$, because k, j are such that if they are distinct then $I_{i-k,k}$ and $I_{i-j,j}$ are in distinct isomorphism classes of involutions of $\text{SL}(n, k)$, but if $k = j$, then by the above reasoning, we still have $i - j = j$ or $i - k = k$, as in the proof of lemma 4.3.11. \square

Lemma 4.3.13. $\left(I_{i,i}, \begin{pmatrix} 0 & I_i \\ pI_i & 0 \end{pmatrix} \right) \not\approx \left(I_{i,i}, \begin{pmatrix} L_{\frac{n}{2},p} & 0 \\ 0 & L_{\frac{n}{2},p} \end{pmatrix} \right)$

Proof. $\begin{pmatrix} A_{i \times i} & 0 \\ 0 & B_{i \times i} \end{pmatrix}$ and $\begin{pmatrix} 0 & A_{i \times i} \\ B_{i \times i} & 0 \end{pmatrix}$ are the two forms of matrices that commute with $I_{i,i}$ up to a multiple, but

$$\begin{pmatrix} A_{i \times i} & 0 \\ 0 & B_{i \times i} \end{pmatrix} \begin{pmatrix} 0 & I_i \\ pI_i & 0 \end{pmatrix} \begin{pmatrix} A_{i \times i} & 0 \\ 0 & B_{i \times i} \end{pmatrix}^{-1} = \begin{pmatrix} 0 & A_{i \times i} B_{i \times i}^{-1} \\ pB_{i \times i} A_{i \times i}^{-1} & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & A_{i \times i} \\ B_{i \times i} & 0 \end{pmatrix} \begin{pmatrix} 0 & I_i \\ pI_i & 0 \end{pmatrix} \begin{pmatrix} 0 & A_{i \times i} \\ B_{i \times i} & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & pA_{i \times i} B_{i \times i}^{-1} \\ B_{i \times i} A_{i \times i}^{-1} & 0 \end{pmatrix},$$

both of which do not equal a scalar multiple of $L_{n,p}$. \square

Theorem 4.3.14. *For n divisible by 4, p, q representatives of elements in k^*/k^{*2} , with $p, q \not\equiv 1 \pmod{k^{*2}}$, the isomorphism classes of commuting inner involutions of $SL(n, k)$ are as follows:*

$$\begin{aligned} & (\text{Inn}_{L'_{n,q}}, \text{Inn}_Y), (\text{Inn}_{L_{n,q}}, \text{Inn}_{Y'}), (\text{Inn}_{I_{n-i,i}}, \text{Inn}_{Y''}), \left(\text{Inn}_{I_{\frac{n}{2}, \frac{n}{2}}}, \text{Inn}_{Y'''} \right) \text{ for} \\ Y &= L_{n,p} (p \not\equiv q \pmod{k^{*2}}), I_{\frac{n}{2}, \frac{n}{2}}, I_{\frac{n}{2}, \frac{n}{2}} L_{n,p}, \text{ or } \begin{pmatrix} I_{\frac{n}{2}-i,i} & 0_{\frac{n}{2}} \\ 0_{\frac{n}{2}} & I_{\frac{n}{2}-i,i} \end{pmatrix}, i \in \left\{ 1, 2, \dots, \frac{n}{4} \right\}, \\ & Y' = I_{n-2i, 2i} L_{n,q}, i \in \left\{ 1, 2, \dots, \frac{n}{4} \right\}, \\ Y'' &= L_{n,p}, \text{ or } \begin{pmatrix} I_{n-i-j} & 0_{(n-i) \times i} \\ 0_{i \times (n-i)} & I_{i-k,k} \end{pmatrix}, i \in \left\{ 1, 2, \dots, \frac{n}{2} \right\}, \\ Y''' &= L'_{n,p}, \text{ or } \begin{pmatrix} I_{\frac{n}{2}-j} & 0_{\frac{n}{2} \times \frac{n}{2}} \\ 0_{\frac{n}{2} \times \frac{n}{2}} & I_{\frac{n}{2}-k,k} \end{pmatrix}, j \in \left\{ 1, 2, \dots, \frac{n}{4} \right\}. \end{aligned}$$

For Y'' , we have: $n-i \neq i$, and $j \in \{1, 2, \dots, \frac{n-i}{2}\}$, if $n-i-j \neq j$, $i-k \neq k$ then $k \in \{1, 2, \dots, i\}$ or else if $n-i-j = j$, or $i-k = k$, then $k \in \{1, 2, \dots, \frac{i}{2}\}$. Also, for Y''' we have: if $\frac{n}{2}-j \neq j$, $\frac{n}{2}-k \neq k$, then for $k \in \{1, 2, \dots, \frac{n}{2}\}$ we get distinct isomorphism classes for distinct j, k such that as a pair (j, k) is distinct, independent of order, or if $\frac{n}{2}-j = j$, or $\frac{n}{2}-k = k$, then $k \in \{1, 2, \dots, \frac{n}{4}\}$.

Proof. This follows directly from above. \square

Lemma 4.3.15. *For n even, not divisible by 4, $a^2 - qb^2 = q$, $i \in \{1, 2, \dots, \frac{n-2}{4}\}$ the pair $P(a, b)$ as defined in corollary 4.1.9, with $Y = L_{\frac{n-2}{2}, q}$ is not isomorphic to the pair $(L_{n, q}, I_{n-2i, 2i} L_{n, q})$.*

Proof. Let $P(a, b)_1$ be the first involution in the (ordered) pair $P(a, b)$ and $P(a, b)_2$ be the second involution in the pair $P(a, b)$, (as defined in corollary 4.1.9) for $Y = L_{\frac{n-2}{2}, q}$. If the pairs are isomorphic for some i , there would be an $A \in \text{GL}(n, k)$ such that, for some i , $AI_{n-2i, 2i} L_{n, q} A^{-1} = \tau P(a, b)_2$ and $AL_{n, q} A^{-1} = \alpha P(a, b)_1$. This implies that $\tau P(a, b)_2 = AI_{n-2i, 2i} L_{n, q} A^{-1} = \alpha AI_{n-2i, 2i} A^{-1} P(a, b)_1$. This implies that $\alpha AI_{n-2i, 2i} A^{-1} = \frac{\tau}{\alpha} P(a, b)_2 P(a, b)_1$.

If -1 is not a square in k then the involution represented by:

$\frac{\tau}{\alpha} P(a, b)_2 P(a, b)_1$ is isomorphic to the involution defined by the matrix $L_{n, -1}$, which would contradict $I_{n-2i, 2i}$ and $L_{n, -1}$ being in separate isomorphism classes. In this case the pairs would not be isomorphic.

If -1 is a square, then the involution represented by $\frac{\tau}{\alpha} P(a, b)_2 P(a, b)_1$ is isomorphic to the involution represented by $\frac{\tau}{\alpha} P(a, b)'$, for $Y = I_{\frac{n-2}{2} - i, i}$, where the lower block has an even number of -1 's. Also, since -1 is a square $\tau \begin{pmatrix} b & \frac{a}{q} \\ -a & -b \end{pmatrix}$, is isomorphic to $\tau I_{1, 1}$, i.e. $\begin{pmatrix} x & y \\ -qy & -x \end{pmatrix} \begin{pmatrix} b & \frac{a}{q} \\ -a & -b \end{pmatrix} \begin{pmatrix} x & y \\ -qy & -x \end{pmatrix}^{-1}$ is equal to $\begin{pmatrix} -bx^2 + 2axy - qby^2 & ay^2 - 2bxy + \frac{a}{q}x^2 \\ -q(ay^2 - 2bxy + \frac{a}{q}x^2) & -(-bx^2 + 2axy - qby^2) \end{pmatrix}$, and $ay^2 - 2bxy + \frac{a}{q}x^2 = 0$ has a nonzero solution (pick an x nonzero) because $4x^2(b^2 - \frac{a^2}{q}) = -4x^2$, a square if -1 is a square. But this gives an odd number of -1 's whereas $I_{n-2i, 2i}$ has an even number of -1 's and neither have more than $\lfloor \frac{n}{2} \rfloor$ -1 's. Therefore they cannot be isomorphic, and thus the corresponding pairs cannot be isomorphic. \square

This gives us the isomorphy classes of commuting involutions of $\text{SL}(n, k)$ for n even and not divisible by 4.

Theorem 4.3.16. *For n even but not divisible by 4, p, q representatives of elements k^*/k^{*2} , $p, q \not\equiv 1 \pmod{k^{*2}}$, $a^2 - qb^2 = p$ the isomorphism classes of commuting inner involutions of $\text{SL}(n, k)$ are as follows:*

$(\text{Inn}_{L_{n,q}}, \text{Inn}_{I_{n-2i,2i}L_{n,q}}), (\text{Inn}_{I_{n-i,i}}, \text{Inn}_{Y''})$ for $i \in \{1, 2, \dots, \frac{n}{2}\}$, and

$(\text{Inn}_{\begin{pmatrix} L_{2,q} & 0 \\ 0 & L'_{n-2,q} \end{pmatrix}}, \text{Inn}_Y), (\text{Inn}_{L'_{n,q}}, \text{Inn}_{Y'}), (\text{Inn}_{I_{\frac{n}{2}, \frac{n}{2}}}, \text{Inn}_{Y'''})$ for

$$Y = \begin{pmatrix} \begin{pmatrix} a & b \\ -qb & -a \end{pmatrix} & 0 \\ 0 & I_{\frac{n-2}{2}, \frac{n-2}{2}} L_{n-2,p} \end{pmatrix}, \begin{pmatrix} I_{1,1} & 0 \\ 0 & I_{\frac{n-2}{2}, \frac{n-2}{2}} \end{pmatrix},$$

$$Y' = \begin{pmatrix} I_{\frac{n}{2}-i, i} & 0 \\ 0 & I_{\frac{n}{2}-i, i} \end{pmatrix}, i \in \left\{1, 2, \dots, \left\lfloor \frac{n-2}{4} \right\rfloor\right\},$$

$$Y'' = L_{n,p}, \text{ or } \begin{pmatrix} I_{n-i-j, j} & 0_{(n-i) \times i} \\ 0_{i \times (n-i)} & I_{i-k, k} \end{pmatrix}, i \in \left\{1, 2, \dots, \frac{n}{2}\right\},$$

$$Y''' = L'_{n,p}, \text{ or } \begin{pmatrix} I_{\frac{n}{2}-j, j} & 0_{\frac{n}{2} \times \frac{n}{2}} \\ 0_{\frac{n}{2} \times \frac{n}{2}} & I_{\frac{n}{2}-k, k} \end{pmatrix}, j \in \left\{1, 2, \dots, \frac{n}{4}\right\}$$

Where, for Y'' we have: for $n-i \neq i$, and $j \in \{1, 2, \dots, \frac{n-i}{2}\}$, if $n-i-j \neq j$, $i-k \neq k$ then $k \in \{1, 2, \dots, i\}$ or else if $n-i-j = j$, or $i-k = k$, then $k \in \{1, 2, \dots, \frac{i}{2}\}$, and for Y''' , we have: if $\frac{n}{2} - j \neq j$, $\frac{n}{2} - k \neq k$, then for $k \in \{1, 2, \dots, \frac{n}{2}\}$ we get distinct isomorphism classes for distinct j, k such that as a pair (j, k) is distinct, independent of order, or if $\frac{n}{2} - j = j$, or $\frac{n}{2} - k = k$, then $k \in \{1, 2, \dots, \frac{n}{4}\}$.

Proof. This follows from the preceding lemma, corollary 4.1.9, lemma 4.3.4 and the classification of involutions for n divisible by 4. \square

We can now complete the classification by consider when n is odd:

Theorem 4.3.17. *For n odd, the isomorphism classes of commuting pairs of involutions of $\text{SL}(n, k)$ are: $(\text{Inn}_{I_{n-i,i}}, \text{Inn}_Y)$ for*

$$Y = \begin{pmatrix} I_{n-i-j, j} & 0_{(n-i) \times i} \\ 0_{i \times (n-i)} & I_{i-k, k} \end{pmatrix}, i \in \left\{1, 2, \dots, \frac{n}{2}\right\},$$

for $n-i \neq i$, and $j \in \{1, 2, \dots, \frac{n-i}{2}\}$, if $n-i-j \neq j$, $i-k \neq k$ then $k \in \{1, 2, \dots, i\}$ or else if $n-i-j = j$, or $i-k = k$, then $k \in \{1, 2, \dots, \frac{i}{2}\}$

Proof. This follows from above. \square

Chapter 5

Classification of Commuting Pairs of Involutions with one Outer Involution

In this chapter, we consider the classification of commuting pairs of involutions, with one involution an outer involution, the other an inner involution. Throughout this section, let θ be the (outer) involution of $\mathrm{SL}(n, k)$ defined by $\theta(A) = A^{T^{-1}}$. From [HWD06], we know that such a commuting pair has the form $(\mathrm{Inn}_M \theta, \mathrm{Inn}_A)$ for M the matrix of a symmetric or skew-symmetric bilinear form over k^n , where M is skew-symmetric only if n is even.

Lemma 5.0.18. *For an inner automorphism, Inn_A , of $\mathrm{SL}(n, k)$: $\mathrm{Inn}_A \theta = \theta \mathrm{Inn}_{A^{T^{-1}}}$.*

Proof. Let $X \in \mathrm{SL}(n, k)$, then

$$\begin{aligned} \mathrm{Inn}_A \theta(X) &= \mathrm{Inn}_A (X^{T^{-1}}) = A^{-1} X^{T^{-1}} A = \theta \theta (A^{-1} X^{T^{-1}} A) = \theta \left((A^{-1} X^{T^{-1}} A)^{T^{-1}} \right) \\ &= \theta (A^T X A^{T^{-1}}) = \theta \mathrm{Inn}_{A^{T^{-1}}} (X). \end{aligned}$$

□

Lemma 5.0.19. *$(\mathrm{Inn}_M \theta, \mathrm{Inn}_A)$ is a commuting pair of involutions if and only if $A^2 = pI_n$, $M = \pm M^T$, and $A^T M A = \alpha M$.*

Proof. That the pair is comprised of involutions $\text{Inn}_M \theta$ and Inn_A if and only if $A^2 = pI_n$, $M = \pm M^T$, follows from [HWD06]. Thus, it suffices to show that the above pair of involutions is a commuting pair if and only if $A^T M A = \alpha M$. This follows because $(\text{Inn}_M \theta, \text{Inn}_A)$ is a commuting pair only if $\text{Inn}_M \text{Inn}_{A^{T-1}} \theta = \text{Inn}_A \text{Inn}_M \theta$ if and only if $\text{Inn}_{AMA^T M^{-1}} = \text{Id}$. \square

There are two cases to consider: when M is symmetric and when M is skew-symmetric, and they will be considered in that order.

5.1 Forms of Pairs when M is Symmetric

Lemma 5.1.1. *If $M = M^T$, then $(\text{Inn}_M \theta, \text{Inn}_A)$ is a commuting pair of involutions if and only if $(\text{Inn}_M \theta, \text{Inn}_A) = (\text{Inn}_M \theta, \text{Inn}_{A'})$ $A'^2 = pI_n$, and $A'^T M A' = M$.*

Proof. The if statement follows from lemma 5.0.19, and by lemma 5.0.19 to show the only if, it suffices to show that $\text{Inn}_A = \text{Inn}_{A'}$, $A'^T M A' = M$. We know that $A \in \text{GL}(n, k)$ from [HWD06] and that for $X \in \text{SO}(n, \bar{k}, \beta)$ (for β a non-degenerate symmetric bilinear form over k^n with matrix M , as in [Dom03]), $X^T M X = M$. Therefore, $(\text{Inn}_A(X))^T M \text{Inn}_A(X) = (A^{-1} X A)^T M A^{-1} X A = A^T X^T (A^{T-1} M A^{-1}) X A$, but since $A^T M A = \alpha M$ then $\frac{1}{\alpha} M = A^{T-1} M A^{-1}$. Thus, $A^T X^T (A^{T-1} M A^{-1}) X A = A^T X^T (\frac{1}{\alpha} M) X A = \frac{1}{\alpha} A^T M A = \frac{1}{\alpha} (\alpha M) = M$. Therefore, Inn_A keeps $\text{SO}(n, \bar{k}, \beta)$ invariant, which implies (by theorem 5.2 in [Dom03]) that $A = c_1 \hat{A}$ for $c_1 \in \bar{k}$ and $\hat{A} \in \text{SO}(n, \bar{k}, \beta)$ or $\hat{A} \in \text{O}(n, \bar{k}, \beta)$, and $\text{Inn}_A = \text{Inn}_{c_1 \hat{A}} = \text{Inn}_{\hat{A}} \rightarrow \hat{A}^T M \hat{A} = \alpha' M$. By the argument above, $\text{Inn}_{\hat{A}}$ keeps $\text{SO}(n, \bar{k}, \beta)$ invariant, but $\text{Inn}_A = \text{Inn}_{\hat{A}} \rightarrow \text{Inn}_{\hat{A}}$ is an involution of $\text{SL}(n, k)$, thus $\text{Inn}_{\hat{A}}$ keeps $\text{SL}(n, k)$ invariant and for $X \in \text{SO}(n, k, \beta)$, $\text{Inn}_{\hat{A}}(X) \in \text{SO}(n, \bar{k}, \beta) \rightarrow \text{Inn}_{\hat{A}} \in \text{SO}(n, k, \beta)$ since X is in $\text{SL}(n, k)$. Therefore, $\text{Inn}_{\hat{A}}$ keeps $\text{SO}(n, k, \beta)$ invariant, which implies by theorem 5.2 in [Dom03], $\hat{A} = c_2 A'$ for $A' \in \text{SO}(n, k, \beta)$ or $A' \in \text{O}(n, k, \beta)$. Therefore, $\text{Inn}_A = \text{Inn}_{\hat{A}} = \text{Inn}_{c_2 A'} = \text{Inn}_{A'}$. \square

Corollary 5.1.2. *If $M = M^T$, then $(\text{Inn}_M \theta, \text{Inn}_A)$ is a commuting pair of involutions if and only if it equals $(\text{Inn}_M \theta, \text{Inn}_{A'})$ with $A'^2 = \pm I_n$, $A'^T M A' = M$.*

Proof. By lemma 5.1.1, it suffices to show that $A'^2 = \pm I_n$. Since the above pair is a pair of involutions, $A'^2 = pI_n \rightarrow A' = pA'^{-1}$, and from lemma 5.1.1, $A'^T M A' = M \rightarrow I_n = A'^T M A' M^{-1} = A'^T M^T (pA'^{-1} M^{-1}) = p(MA')^T M A'$ since $M = M^T$. Therefore, $p(MA')^T = M A'$, and thus $\left(p(MA')^T\right)^T = (MA')^T \rightarrow M A' = \frac{1}{p}(MA')^T$. This implies $p(MA')^T = M A' = \frac{1}{p}(MA')^T$ and thus $p^2(MA')^T = (MA')^T$, and since $\text{Inn}_{A'}$ is an automorphism and M is the matrix of a non-degenerate bilinear form, $p^2 = 1 \rightarrow p = \pm 1$. \square

Therefore by the results above, every pair of commuting involutions of $\text{SL}(n, k)$ with one involution an inner involution and one involution an outer involution (corresponding to a symmetric bilinear form on k^n) is isomorphic to: $(\text{Inn}_M \theta, \text{Inn}_A)$, for $A^2 = \pm I_n$, M symmetric.

Theorem 5.1.3. *The pair, $(\text{Inn}_M \theta, \text{Inn}_A)$, for $A^2 = I_n$, M symmetric, is isomorphic to $(\text{Inn}_D \theta, \text{Inn}_{I_{n-i,i}})$, for D , a diagonal matrix with entries in k^*/k^{*2} , $i \in \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$. Also, if -1 is not a square in k^* , $(\text{Inn}_M \theta, \text{Inn}_A)$, for $A^2 = -I_n$ (this forces n to be even), M symmetric, is isomorphic to $(\text{Inn}_C \theta, \text{Inn}_{L_{n,-1}})$, for $C = (C_{j,k})_{j,k=1}^{\frac{n}{2}}$, $C_{j,k} = \begin{pmatrix} a_{j,k} & b_{j,k} \\ -b_{j,k} & a_{j,k} \end{pmatrix}$, C symmetric.*

Proof. First, we prove that $(\text{Inn}_M \theta, \text{Inn}_A)$, for $A^2 = I_n$, M symmetric, is isomorphic to $(\text{Inn}_D \theta, \text{Inn}_{I_{n-i,i}})$, for D , a diagonal matrix with entries in k^*/k^{*2} . Since $A^2 = I_n$ we know there is an invertible matrix S such that $S^{-1} A S = c_1 I_{n-i,i}$, $c_1 \in k^* \rightarrow$ the above pair is isomorphic to $(\text{Inn}_{M'} \theta, \text{Inn}_{I_{n-i,i}})$, with M' symmetric since $M'^T = (S^T M S)^T = S^T M^T S = S^T M S = M'$. Also, by lemma 5.1.1, we can take M' such that $I_{n-i,i}^T M' I_{n-i,i} = M'$ (if it does equal M' then there is some constant $\alpha \in \bar{k}^*$ and invertible B such that $B^T M' B = M'$ and $\text{Inn}_{I_{n-i,i}} = \text{Inn}_{\alpha I_{n-i,i}} = \text{Inn}_B$.) But, if $I_{n-i,i}^T M' I_{n-i,i} = M' \rightarrow$ for $M1, M3$ symmetric,

$$M' = \begin{pmatrix} M1_{(n-i) \times (n-i)} & M2_{(n-i) \times i} \\ M2_{i \times (n-i)}^T & M3_{i \times i} \end{pmatrix},$$

then

$$\begin{aligned} I_{n-i,i}^T M' I_{n-i,i} &= \begin{pmatrix} M1_{(n-i) \times (n-i)} & M2_{(n-i) \times i} \\ -M2_{i \times (n-i)}^T & -M3_{i \times i} \end{pmatrix} I_{n-i,i} \\ &= \begin{pmatrix} M1_{(n-i) \times (n-i)} & -M2_{(n-i) \times i} \\ -M2_{i \times (n-i)}^T & M3_{i \times i} \end{pmatrix} = M' \end{aligned}$$

and since the characteristic of k is not two this gives $M2 = 0$. Therefore, because M' is symmetric, $M1$ and $M3$ are symmetric. From [Sch85], we have that there is an invertible S_1 , and an invertible S_2 such that $S_1^T M1 S_1 = D1$ and $S_2^T M3 S_2 = D2$, for $D1, D2$ diagonal matrices with elements in k^*/k^{*2} . This implies that $(\text{Inn}_M \theta, \text{Inn}_A)$ is isomorphic to $(\text{Inn}_D \theta, \text{Inn}_{I_{n-i,i}})$ for D , diagonal with elements in k^*/k^{*2} , because the blocks S_1 and S_2 commute with the blocks I_{n-i} and I_i , respectively. This proves the first statement, because the inner automorphism defined by the matrix $\begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}$ commutes with the inner automorphism defined by $I_{n-i,i}$.

We now show the second statement, that if -1 is not a square (if it is then this reduces to the first statement) in k^* , $(\text{Inn}_M \theta, \text{Inn}_A)$, for $A^2 = -I_n$, M symmetric, is isomorphic to $(\text{Inn}_C \theta, \text{Inn}_{L_{n,-1}})$, for $C = (C_{j,k})_{j,k=1}^{\frac{n}{2}}$, $C_{j,k} = \begin{pmatrix} a_{j,k} & b_{j,k} \\ -b_{j,k} & a_{j,k} \end{pmatrix}$, C symmetric. From [HWD06] we know that there is an invertible T such that $T^{-1}AT = L_{n,-1}$ so that $(\text{Inn}_M \theta, \text{Inn}_A)$ is isomorphic to $(\text{Inn}_C \theta, \text{Inn}_{L_{n,-1}})$, with C symmetric. Furthermore, as in the proof of the first statement we can take C such that $L_{n,-1}^T C L_{n,-1} = C$ considering 2×2 blocks, which implies for $C = (C_{j,k})_{j,k=1}^{\frac{n}{2}}$, $C_{j,k} = \begin{pmatrix} a_{j,k} & b_{j,k} \\ c_{j,k} & d_{j,k} \end{pmatrix}$,

$$\begin{aligned} C_{j,k} &= (L_{n,-1}^T C L_{n,-1})_{j,k} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{j,k} & b_{j,k} \\ c_{j,k} & d_{j,k} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -c_{j,k} & -d_{j,k} \\ a_{j,k} & b_{j,k} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} d_{j,k} & -c_{j,k} \\ -b_{j,k} & a_{j,k} \end{pmatrix} \end{aligned}$$

for $j, k \in \{1, 2, \dots, \frac{n}{2}\}$. This implies that for each fixed $j, k \in \{1, 2, \dots, \frac{n}{2}\}$, $a_{j,k} = d_{j,k}$ and $c_{j,k} = -b_{j,k}$, so that C has the desired form. \square

Lemma 5.1.4. *The commuting pair of involutions (for -1 not a square in k^*) $(\text{Inn}_C \theta, \text{Inn}_{L_{n,-1}})$, for $C = (C_{j,k})_{j,k=1}^{\frac{n}{2}}$, $C_{j,k} = \begin{pmatrix} a_{j,k} & b_{j,k} \\ -b_{j,k} & a_{j,k} \end{pmatrix}$, C symmetric, is isomorphic to the pair $(\text{Inn}_{C'} \theta, \text{Inn}_{L_{n,-1}})$, for $C' = (C'_{j,k})_{j,k}^{\frac{n}{2}}$, with $C'_{j,k} = d_j I_2$ for $j = k$, $d_i \in k^*/k^{*2}$, and $C'_{j,k}$ is the 2×2 zero matrix when $j \neq k$.*

Proof. First, observe that $a_{j,k}^2 + b_{j,k}^2 = 0$ if and only if either $b_{j,k} = 0 = a_{j,k}$ or $-1 = \left(\frac{b_{j,k}}{a_{j,k}}\right)^2$, a contradiction to the fact that -1 is not a square in k^* . This implies that the matrix $C_{j,k}$ is either the 2×2 zero matrix or is invertible. Observe also that because C is symmetric, $C_{j,j} = a_{j,j} I_2$ and $C_{j,k} = C_{k,j}$. Second, for $n = 4$ if $C_{1,2} = 0$, we're done. Suppose $C_{1,2} \neq 0$, and consider the matrix:

$$S = \begin{pmatrix} a_{1,2} & b_{1,2} & 0 & 0 \\ -b_{1,2} & a_{1,2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $S^{-1}L_{n,-1}S = L_{n,-1}$ and

$$\begin{aligned}
& S^T C S \\
&= \begin{pmatrix} a_{1,2} & b_{1,2} & 0 & 0 \\ -b_{1,2} & a_{1,2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^T \begin{pmatrix} a_{1,1} & 0 & a_{1,2} & b_{1,2} \\ 0 & a_{1,1} & -b_{1,2} & a_{1,2} \\ a_{1,2} & -b_{1,2} & a_{2,2} & 0 \\ b_{1,2} & a_{1,2} & 0 & a_{2,2} \end{pmatrix} \begin{pmatrix} a_{1,2} & b_{1,2} & 0 & 0 \\ -b_{1,2} & a_{1,2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} a_{1,2} & -b_{1,2} & 0 & 0 \\ b_{1,2} & a_{1,2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{1,1}a_{1,2} & a_{1,1}b_{1,2} & a_{1,2} & b_{1,2} \\ -a_{1,1}b_{1,2} & a_{1,1}a_{1,2} & -b_{1,2} & a_{1,2} \\ a_{1,2}^2 + b_{1,2}^2 & 0 & a_{2,2} & 0 \\ 0 & a_{1,2}^2 + b_{1,2}^2 & 0 & a_{2,2} \end{pmatrix} \\
&= \begin{pmatrix} a_{1,1}(a_{1,2}^2 + b_{1,2}^2) & 0 & a_{1,2}^2 + b_{1,2}^2 & 0 \\ 0 & a_{1,1}(a_{1,2}^2 + b_{1,2}^2) & 0 & a_{1,2}^2 + b_{1,2}^2 \\ a_{1,2}^2 + b_{1,2}^2 & 0 & a_{2,2} & 0 \\ 0 & a_{1,2}^2 + b_{1,2}^2 & 0 & a_{2,2} \end{pmatrix}.
\end{aligned}$$

Also, for P_4 as above, we have, from the proof of lemma 4.1.7, that $P_4^T = P_4^{-1}$, $P_4^{-1}L_{4,-1}P_4 = J_4$. Then,

$$P_4^T S^T C S P_4 = \begin{pmatrix} a_{1,1}(b_{1,2}^2 + a_{1,2}^2) & a_{1,2}^2 + b_{1,2}^2 & 0 & 0 \\ a_{1,2}^2 + b_{1,2}^2 & a_{2,2} & 0 & 0 \\ 0 & 0 & a_{1,1}(a_{1,2}^2 + b_{1,2}^2) & a_{1,2}^2 + b_{1,2}^2 \\ 0 & 0 & a_{1,2}^2 + b_{1,2}^2 & a_{2,2} \end{pmatrix}.$$

But this gives us identical symmetric 2×2 blocks on the diagonal, thus we know from [Sch85] there is an invertible 2×2 matrix, Q , such that for

$$\bar{Q} = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} \rightarrow \bar{Q}^T P_4^T S^T C S P_4 \bar{Q} = \bar{D} = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix},$$

for D a 2×2 diagonal matrix. Also, the form of \bar{Q} guarantees that it commutes with J_4 . Finally, 'undoing' the permutation matrix P_4 , we get $P_4 \bar{D} P_4^T = D'$, a diagonal matrix, and $P_4 J_4 P_4^{-1} = L_{4,-1}$. Thus, for $n = 4$ the isomorphism holds. Proceeding by induction, assume the result holds for some n , we aim to show it holds for $n + 2$.

Let $(\text{Inn}_C \theta, \text{Inn}_{L_{n+2,-1}})$ be a commuting pair of involutions of $\text{SL}(n+2, k)$. Partition C into block form in the following way:

$$C = (C_{j,k})_{j,k=1}^{\frac{n+2}{2}} = \begin{pmatrix} C_1 & U \\ U^T & C_{\frac{n+2}{2}, \frac{n+2}{2}} \end{pmatrix},$$

for $C_1 \in M(n, k)$, $U \in M(n \times 2, k)$, and the form of C forces $C_1 = (C_{j,k})_{j,k=1}^{\frac{n}{2}}$. From our induction hypothesis, there is an invertible B such that $B^T C_1 B$ has 2×2 blocks on the diagonal of the form $d_j I_2$ with zeros elsewhere, and $B^{-1} L_{n,-1} B = L_{n,-1}$. This implies that

$$\begin{pmatrix} B^T & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} C_1 & U \\ U^T & C_{\frac{n+2}{2}, \frac{n+2}{2}} \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & I_2 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} d_1 I_2 & 0 \\ \vdots & \vdots \\ 0 & d_{\frac{n}{2}} I_2 \end{pmatrix} & U' \\ U'^T & C_{\frac{n+2}{2}, \frac{n+2}{2}} \end{pmatrix},$$

for $U' = B^T U B$. For $\bar{B} = \begin{pmatrix} B^T & 0 \\ 0 & I_2 \end{pmatrix}$, by the induction hypothesis and the form of \bar{B} , $\bar{B}^{-1} L_{n+2,-1} \bar{B} = L_{n+2,-1}$. By the form of C and thus of $\bar{B}^T C \bar{B}$ (because the involution $\text{Inn}_{\bar{B}^T C \bar{B}} \theta$ commutes with the involution $\text{Inn}_{L_{n+2,-1}}$ and by the proof of lemma 5.2.3), $U' = (C'_{j, \frac{n+2}{2}})_{j=1}^{\frac{n}{2}}$ for

$$C'_{j, \frac{n+2}{2}} = \begin{pmatrix} a_{j, \frac{n+2}{2}} & b_{j, \frac{n+2}{2}} \\ -b_{j, \frac{n+2}{2}} & a_{j, \frac{n+2}{2}} \end{pmatrix}. \text{ Let } S' = \begin{pmatrix} C''_{1, \frac{n+2}{2}} & & 0 \\ & \ddots & \\ & & C''_{\frac{n}{2}, \frac{n+2}{2}} \\ 0 & & & I_2 \end{pmatrix},$$

where $C''_{j, \frac{n+2}{2}} = C'_{j, \frac{n+2}{2}}$ if it is nonzero, and I_2 if it is zero. Note that S' is such that

$S'^{-1}L_{n+2,-1}S' = L_{n+2,-1}$, and $S'^T \bar{B}^T C \bar{B} S' =$

$$\begin{aligned}
& \begin{pmatrix} C''_{1, \frac{n+2}{2}} & & & 0 \\ & \ddots & & \\ & & C''_{\frac{n}{2}, \frac{n+2}{2}} & \\ 0 & & & I_2 \end{pmatrix}^T \begin{pmatrix} \begin{pmatrix} d_1 I_2 & & 0 \\ & \ddots & \\ 0 & & d_{\frac{n}{2}} I_2 \end{pmatrix} & U' \\ & U'^T & & C_{\frac{n+2}{2}, \frac{n+2}{2}} \end{pmatrix} S' \\
&= \begin{pmatrix} \begin{pmatrix} d_1 C''_{1, \frac{n+2}{2}} & & 0 \\ & \ddots & \\ 0 & & d_{\frac{n}{2}} C''_{\frac{n}{2}, \frac{n+2}{2}} \end{pmatrix} & \begin{pmatrix} C''_{1, \frac{n+2}{2}} C_{1, \frac{n+2}{2}} \\ \vdots \\ C''_{\frac{n}{2}, \frac{n+2}{2}} C_{\frac{n}{2}, \frac{n+2}{2}} \end{pmatrix} \\ C_{1, \frac{n+2}{2}} \cdots C_{\frac{n}{2}, \frac{n+2}{2}} & C_{\frac{n+2}{2}, \frac{n+2}{2}} \end{pmatrix} S' \\
&= \begin{pmatrix} \begin{pmatrix} d_1 C''_{1, \frac{n+2}{2}} C''_{1, \frac{n+2}{2}} & & 0 \\ & \ddots & \\ 0 & & d_{\frac{n}{2}} C''_{\frac{n}{2}, \frac{n+2}{2}} C''_{\frac{n}{2}, \frac{n+2}{2}} \end{pmatrix} & \begin{pmatrix} C''_{1, \frac{n+2}{2}} C_{1, \frac{n+2}{2}} \\ \vdots \\ C''_{\frac{n}{2}, \frac{n+2}{2}} C_{\frac{n}{2}, \frac{n+2}{2}} \end{pmatrix} \\ C_{1, \frac{n+2}{2}} C''_{1, \frac{n+2}{2}} \cdots C_{\frac{n}{2}, \frac{n+2}{2}} C''_{\frac{n}{2}, \frac{n+2}{2}} & C_{\frac{n+2}{2}, \frac{n+2}{2}} \end{pmatrix} \\
&= \begin{pmatrix} \begin{pmatrix} d_1 d'_1 I_2 & & 0 \\ & \ddots & \\ 0 & & d_{\frac{n}{2}} d'_{\frac{n}{2}} I_2 \end{pmatrix} & \begin{pmatrix} d'_1 I_2 \\ \vdots \\ d'_{\frac{n}{2}} I_2 \end{pmatrix} \\ d'_1 I_2 \cdots d'_{\frac{n}{2}} I_2 & C_{\frac{n+2}{2}, \frac{n+2}{2}} \end{pmatrix} \\
&= \hat{C},
\end{aligned}$$

by the construction of $C''_{j, \frac{n+2}{2}}$. Therefore, $(\text{Inn}_C \theta, \text{Inn}_{L_{n+2,-1}})$ is isomorphic to the pair $(\text{Inn}_{\hat{C}} \theta, \text{Inn}_{L_{n+2,-1}})$.

Claim 5.1.5. $(\text{Inn}_{\hat{C}} \theta, \text{Inn}_{L_{n+2,-1}})$ is isomorphic to $(\text{Inn}_{\hat{M}} \theta, \text{Inn}_{J_{n+2}})$, for \hat{M} the matrix $\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$, with M a symmetric matrix.

Proof. The pairs are Inn_P isomorphic for $P = \sum_{i=1}^{\frac{n}{2}+1} (E_{i, 2i-1} + E_{\frac{n}{2}+1+i, 2i}) = P_{n+2}$. This is true because it was shown in the proof of corollary 4.1.7 for positive integers

n that $P_n L_{n,q} P_n^{-1} = L'_{n,q} = J_n$ (for n even). Also,

$$\begin{aligned} \text{Inn}_{P_{n+2}^{-1}} \text{Inn}_{\hat{C}} \theta \text{Inn}_{P_{n+2}} &= \text{Inn}_{P_{n+2}^{-1}} \text{Inn}_{\hat{C}} \text{Inn}_{P_{n+2}^{T^{-1}}} \theta \\ &= \text{Inn}_{P_{n+2} \hat{C} P_{n+2}^T} \theta, \end{aligned}$$

since P_{n+2} is a permutation matrix. Furthermore, $P_{n+2} \hat{C} P_{n+2}^T$ is equal to:

$$\begin{aligned}
& P_{n+2} \sum_{i=1}^{\frac{n}{2}} d_i d'_i (E_{2i-1,2i-1} + E_{2i,2i}) P_{n+2}^T \\
& + P_{n+2} \sum_{i=1}^{\frac{n}{2}} d'_i (E_{2i-1,n+1} + E_{2i,n+2} + E_{n+1,2i-1} + E_{n+2,2i}) P_{n+2}^T \\
& + P_{n+2} a_{\frac{n+2}{2}, \frac{n+2}{2}} (E_{n+1,n+1} + E_{n+2,n+2}) P_{n+2}^T \\
& + P_{n+2} b_{\frac{n+2}{2}, \frac{n+2}{2}} (E_{n+1,n+2} - E_{n+2,n+1}) P_{n+2}^T \\
= & \sum_{i=1}^{\frac{n}{2}+1} (E_{i,2i-1} + E_{\frac{n}{2}+1+i,2i}) \sum_{i=1}^{\frac{n}{2}} d_i d'_i (E_{2i-1,2i-1} + E_{2i,2i}) P_{n+2}^T \\
& + \sum_{i=1}^{\frac{n}{2}+1} (E_{i,2i-1} + E_{\frac{n}{2}+1+i,2i}) \sum_{i=1}^{\frac{n}{2}} d'_i (E_{2i-1,n+1} + E_{2i,n+2} + E_{n+1,2i-1} + E_{n+2,2i}) P_{n+2}^T \\
& + \sum_{i=1}^{\frac{n}{2}+1} (E_{i,2i-1} + E_{\frac{n}{2}+1+i,2i}) a_{\frac{n+2}{2}, \frac{n+2}{2}} (E_{n+1,n+1} + E_{n+2,n+2}) P_{n+2}^T \\
& + \sum_{i=1}^{\frac{n}{2}+1} (E_{i,2i-1} + E_{\frac{n}{2}+1+i,2i}) b_{\frac{n+2}{2}, \frac{n+2}{2}} (E_{n+1,n+2} - E_{n+2,n+1}) P_{n+2}^T \\
= & \sum_{i=1}^{\frac{n}{2}} d_i d'_i (E_{i,2i-1} + E_{\frac{n}{2}+1+i,2i}) \sum_{i=1}^{\frac{n}{2}+1} (E_{2i-1,i} + E_{2i,\frac{n}{2}+1+i}) \\
& + \sum_{i=1}^{\frac{n}{2}} d'_i (E_{i,n+1} + E_{\frac{n}{2}+1+i,n+2} + E_{\frac{n}{2}+1,2i-1} + E_{n+2,2i}) \sum_{i=1}^{\frac{n}{2}+1} (E_{2i-1,i} + E_{2i,\frac{n}{2}+1+i}) \\
& + a_{\frac{n+2}{2}, \frac{n+2}{2}} (E_{\frac{n}{2}+1,n+1} + E_{n+2,n+2}) \sum_{i=1}^{\frac{n}{2}+1} (E_{2i-1,i} + E_{2i,\frac{n}{2}+1+i}) \\
& + b_{\frac{n+2}{2}, \frac{n+2}{2}} (E_{\frac{n}{2}+1,n+2} - E_{n+2,n+1}) \sum_{i=1}^{\frac{n}{2}+1} (E_{2i-1,i} + E_{2i,\frac{n}{2}+1+i}) \\
= & \sum_{i=1}^{\frac{n}{2}} d_i d'_i (E_{i,i} + E_{\frac{n}{2}+1+i,\frac{n}{2}+1+i}) \\
& + \sum_{i=1}^{\frac{n}{2}} d'_i (E_{i,\frac{n}{2}+1} + E_{\frac{n}{2}+1+i,n+2} + E_{\frac{n}{2}+1,i} + E_{n+2,\frac{n}{2}+1+i}) \\
& + a_{\frac{n+2}{2}, \frac{n+2}{2}} (E_{\frac{n}{2}+1,\frac{n}{2}+1} + E_{n+2,n+2}) + b_{\frac{n+2}{2}, \frac{n+2}{2}} (E_{\frac{n}{2}+1,n+2} - E_{n+2,\frac{n}{2}+1})
\end{aligned}$$

and this is equal to $\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$ for $M = \sum_{i=1}^{\frac{n}{2}} d_i d'_i E_{i,i} + \sum_{i=1}^{\frac{n}{2}} d'_i (E_{i, \frac{n}{2}+1} + E_{\frac{n}{2}+1, i}) + a_{\frac{n+2}{2}, \frac{n+2}{2}} E_{\frac{n}{2}+1, \frac{n}{2}+1}$, because, since \hat{C} is symmetric, $b_{\frac{n+2}{2}, \frac{n+2}{2}} = 0$ (as shown above), and $n+1 = 2i-1$ implies that $i = \frac{n}{2} + 1$, as well as $n+2 = 2i$ implies that $i = \frac{n}{2} + 1$. Also, since n is even, $n+1 \neq 2i$ for integers i . \square

By results of [Sch85], there exists an invertible Q such that $Q^T M Q$ is diagonal, therefore, because

$$\begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}^T \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} = \begin{pmatrix} Q^T M Q & 0 \\ 0 & Q^T M Q \end{pmatrix}$$

is diagonal and

$$\begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}^{-1} \begin{pmatrix} 0 & I_{\frac{n+2}{2}} \\ -I_{\frac{n+2}{2}} & 0 \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} = \begin{pmatrix} 0 & I_{\frac{n+2}{2}} \\ -I_{\frac{n+2}{2}} & 0 \end{pmatrix},$$

then $(\text{Inn}_{\hat{C}} \theta, \text{Inn}_{L_{n+2, -1}})$ is isomorphic to $(\text{Inn}_D \theta, \text{Inn}_{J_{n+2}})$, for D diagonal. But since P_{n+2} is a permutation matrix, $(\text{Inn}_D \theta, \text{Inn}_{J_{n+2}})$, for D diagonal is isomorphic to $(\text{Inn}_{C'} \theta, \text{Inn}_{L_{n+2, -1}})$, for C' diagonal, and since C' commutes with $L_{n+2, -1}$, it has the desired form. \square

5.2 Forms of Pairs when M is Skew-Symmetric

Next, consider the case when M is skew-symmetric (and thus n is even, $n = 2m$.)

Let

$$J_{2m} = \begin{pmatrix} 0 & I_{\frac{n}{2}} \\ -I_{\frac{n}{2}} & 0 \end{pmatrix}$$

Lemma 5.2.1. *If $M = -M^T$ then the pair $(\text{Inn}_M \theta, \text{Inn}_A)$ is isomorphic to the pair $(\text{Inn}_{J_{2m}} \theta, \text{Inn}_{A'})$ for $A'^T J_{2m} A' = J_{2m}$*

Proof. From [Sch85], we have that M is congruent to $J_{2m} \rightarrow \exists Q \in \text{GL}(n, k)$ such

that $Q^T M Q = J_{2m}$, then

$$\begin{aligned} (\text{Inn}_M \theta, \text{Inn}_A) &\approx (\text{Inn}_Q \text{Inn}_M \theta \text{Inn}_{Q^{-1}}, \text{Inn}_Q \text{Inn}_A \text{Inn}_{Q^{-1}}) \\ &= (\text{Inn}_Q \text{Inn}_M \text{Inn}_{Q^T} \theta, \text{Inn}_{Q^{-1} A Q}) \\ &= (\text{Inn}_{J_{2m}} \theta, \text{Inn}_{\hat{A}}). \end{aligned}$$

Also, $(\text{Inn}_{J_{2m}} \theta, \text{Inn}_{\hat{A}})$ is a commuting pair of involutions if and only if $\hat{A}^2 = pI_n$ and $\hat{A}^T J_{2m} \hat{A} = \alpha J_{2m}$. We now show $(\text{Inn}_{J_{2m}} \theta, \text{Inn}_{\hat{A}}) = (\text{Inn}_{J_{2m}} \theta, \text{Inn}_{A'})$. for A' such that $A'^T J_{2m} A' = J_{2m}$. Let $X \in \text{Sp}(2m, \bar{k})$. Then,

$$\begin{aligned} (\text{Inn}_{\hat{A}}(X))^T J_{2m} \text{Inn}_{\hat{A}}(X) &= (\hat{A}^{-1} X \hat{A})^T J_{2m} \hat{A}^{-1} X \hat{A} \\ &= \hat{A}^T X^T (\hat{A}^{-1} J_{2m} \hat{A}^{-1}) X \hat{A} \\ &= \hat{A}^T X^T \left(\frac{1}{\alpha} J_{2m} \right) X \hat{A} \\ &= \frac{1}{\alpha} \hat{A}^T (X^T J_{2m} X) \hat{A} \\ &= \frac{1}{\alpha} \hat{A}^T J_{2m} \hat{A} \\ &= \frac{1}{\alpha} (\alpha J_{2m}) \\ &= J_{2m}. \end{aligned}$$

Therefore, $\text{Inn}_{\hat{A}}$ keeps $\text{Sp}(2m, \bar{k})$ invariant, and thus by theorem 5.5 in [Jac05], $\hat{A} = c_1 B$ for $B \in \text{Sp}(2m, \bar{k})$, $c_1 \in \bar{k}$. Furthermore, $\text{Inn}_{\hat{A}} = \text{Inn}_{c_1 B} = \text{Inn}_B$, and Inn_B keeps $\text{Sp}(2m, k)$ invariant because Inn_B is also an involution of $\text{SL}(n, k)$, thus for $X \in \text{Sp}(2m, k)$, $\text{Inn}_B(X)$ is in $\text{Sp}(2m, \bar{k})$ and in $\text{SL}(n, k) \rightarrow \text{Inn}_B(X)$ is in $\text{Sp}(2m, k)$. Because Inn_B keeps $\text{Sp}(2n, k)$ invariant, $B = c_2 A'$ for $c_2 \in \bar{k}$, $A' \in \text{Sp}(2m, k)$, by theorem 5.5 in [Jac05]. Therefore, $\text{Inn}_{\hat{A}} = \text{Inn}_B = \text{Inn}_{c_2 A'} = \text{Inn}_{A'}$. \square

Corollary 5.2.2. *If $M = -M^T$ then the commuting pair of involutions, $n = 2m$, $(\text{Inn}_M \theta, \text{Inn}_A)$, is isomorphic to the pair $(\text{Inn}_{J_{2m}} \theta, \text{Inn}_{A'})$ for $A'^T J_{2m} A' = J_{2m}$, and $A'^2 = \pm I_n$.*

Proof. The first statement is a restatement of lemma 5.2.1, so it remains to show that $A'^2 = \pm I_n$. Suppose $A'^2 = pI_n$, then $A'^T J_{2m} A' = J_{2m}$ which implies that

$$I_n = A'^T J_{2m} A' J_{2m}^{-1} = -A'^T J_{2m}^T A' J_{2m}^{-1} = -p (J_{2m} A')^T (J_{2m} A')^{-1}$$

giving $-p \left((J_{2m}A')^T \right)^T = (J_{2m}A')^T$ and thus $-p(J_{2m}A') = (J_{2m}A')^T$. Therefore, $-p(J_{2m}A')^T = (J_{2m}A') = -\frac{1}{p}(J_{2m}A')^T \rightarrow (J_{2m}A')^T = p^2(J_{2m}A')^T \rightarrow p^2 = 1 \rightarrow p = \pm 1$. \square

Therefore by the results above, every pair of commuting involutions of $\text{SL}(n, k)$ with one involution an inner involution and one involution an outer involution (corresponding to a skew-symmetric bilinear form on k^n) is isomorphic to: $(\text{Inn}_{J_{2m}}\theta, \text{Inn}_A)$, for $A^2 = \pm I_n$.

Theorem 5.2.3. *The pair, $(\text{Inn}_M\theta, \text{Inn}_A)$, for $A^2 = I_n$, M skew-symmetric (hence n is even), is isomorphic to $(\text{Inn}_{J_{n-i,i}}\theta, \text{Inn}_{I_{n-i,i}})$, for*

$$J_{n-i,i} = \begin{pmatrix} J_{n-i} & 0_{(n-i) \times i} \\ 0_{i \times (n-i)} & J_i \end{pmatrix},$$

$i \in \{2, 4, \dots, \frac{n}{2}\}$. Also, if -1 is not a square in k^* , $(\text{Inn}_M\theta, \text{Inn}_A)$, for $A^2 = -I_n$, M skew-symmetric, is isomorphic to $(\text{Inn}_C\theta, \text{Inn}_{L_{n,-1}})$, for C skew-symmetric, $C = (C_{j,k})_{j,k=1}^{\frac{n}{2}}$, $C_{j,k} = \begin{pmatrix} a_{j,k} & b_{j,k} \\ -b_{j,k} & a_{j,k} \end{pmatrix}$.

Proof. First, we prove that $(\text{Inn}_M\theta, \text{Inn}_A)$, for $A^2 = I_n$, the matrix, M , skew-symmetric, is isomorphic to $(\text{Inn}_{J_{n-i,i}}\theta, \text{Inn}_{I_{n-i,i}})$. Since $A^2 = I_n$ we know there is an invertible matrix S such that $S^{-1}AS = c_1 I_{n-i,i}$, $c_1 \in k^* \rightarrow$ the above pair is isomorphic to $(\text{Inn}_{M'}\theta, \text{Inn}_{I_{n-i,i}})$, with M' skew-symmetric since $M'^T = (S^TMS)^T = S^T M^T S = -S^TMS = -M'$. Also, by the proof of lemma 5.2.1, we can take M' such that $I_{n-i,i}^T M' I_{n-i,i} = M'$ (if equality does not hold then, by results in [Jac05] and as in the proof of lemma 5.2.1, there is some constant $\alpha \in \bar{k}^*$ and invertible B such that $B^T M' B = M'$ and $\text{Inn}_{I_{n-i,i}} = \text{Inn}_{\alpha I_{n-i,i}} = \text{Inn}_B$.) So, taking $I_{n-i,i}^T M' I_{n-i,i} = M'$ this gives for $M1, M3$ skew-symmetric,

$$M' = \begin{pmatrix} M1_{(n-i) \times (n-i)} & M2_{(n-i) \times i} \\ -M2_{i \times (n-i)}^T & M3_{i \times i} \end{pmatrix},$$

$$\begin{aligned}
M' &= I_{n-i,i}^T M' I_{n-i,i} \\
&= \begin{pmatrix} M1_{(n-i) \times (n-i)} & M2_{(n-i) \times i} \\ M2_{i \times (n-i)}^T & -M3_{i \times i} \end{pmatrix} I_{n-i,i} \\
&= \begin{pmatrix} M1_{(n-i) \times (n-i)} & -M2_{(n-i) \times i} \\ M2_{i \times (n-i)}^T & M3_{i \times i} \end{pmatrix}
\end{aligned}$$

and since the characteristic of k is not two, this gives $M2 = 0$. Therefore, because M' skew-symmetric, $M1$ and $M3$ are skew-symmetric. From [Sch85], we have that there is an invertible S_1 , and an invertible S_2 such that $S_1^T M1 S_1 = J_{n-i}$ and $S_2^T M3 S_2 = J_i$. This implies that $(\text{Inn}_M \theta, \text{Inn}_A)$ is isomorphic to $(\text{Inn}_{J_{n-i,i}} \theta, \text{Inn}_{I_{n-i,i}})$, because

$$\begin{pmatrix} S_1 & 0_{(n-i) \times i} \\ 0_{i \times (n-i)} & S_2 \end{pmatrix}^{-1} I_{n-i,i} \begin{pmatrix} S_1 & 0_{(n-i) \times i} \\ 0_{i \times (n-i)} & S_2 \end{pmatrix} = I_{n-i,i}.$$

This proves the first statement. We now show the second statement, that if -1 is not a square (if it is then this reduces to the first statement) in k^* , $(\text{Inn}_M \theta, \text{Inn}_A)$, for $A^2 = -I_n$, M skew-symmetric, is isomorphic to $(\text{Inn}_C \theta, \text{Inn}_{L_{n,-1}})$, for $C = (C_{j,k})_{j,k=1}^{\frac{n}{2}}$, $C_{j,k} = \begin{pmatrix} a_{j,k} & b_{j,k} \\ -b_{j,k} & a_{j,k} \end{pmatrix}$, C skew-symmetric. From [HWD06] we know that there is an invertible T such that $T^{-1}AT = L_{n,-1}$ so that $(\text{Inn}_M \theta, \text{Inn}_A)$ is isomorphic to $(\text{Inn}_C \theta, \text{Inn}_{L_{n,-1}})$, with C skew-symmetric. Furthermore, as in the proof of the first statement we can take C such that $L_{n,-1}^T C L_{n,-1} = C$ considering 2×2 blocks, \rightarrow for

$$C = (C_{j,k})_{j,k=1}^{\frac{n}{2}}, C_{j,k} = \begin{pmatrix} a_{j,k} & b_{j,k} \\ c_{j,k} & d_{j,k} \end{pmatrix},$$

$$C_{j,k} = (L_{n,-1}^T C L_{n,-1})_{j,k} = \begin{pmatrix} -c_{j,k} & -d_{j,k} \\ a_{j,k} & b_{j,k} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} d_{j,k} & -c_{j,k} \\ -b_{j,k} & a_{j,k} \end{pmatrix},$$

for $j, k \in \{1, 2, \dots, \frac{n}{2}\}$. This implies that for each fixed $j, k \in \{1, 2, \dots, \frac{n}{2}\}$, $a_{j,k} = d_{j,k}$ and $c_{j,k} = -b_{j,k}$, so that $C_{j,k}$ has the desired form and $C = (C_{j,k})_{j,k=1}^{\frac{n}{2}}$ is skew-symmetric. \square

Lemma 5.2.4. *The commuting pair of involutions (for -1 not a square in k^*)*

$(\text{Inn}_C \theta, \text{Inn}_{L_{n,-1}})$, for $C = (C_{j,k})_{j,k=1}^{\frac{n}{2}}$, $C_{j,k} = \begin{pmatrix} a_{j,k} & b_{j,k} \\ -b_{j,k} & a_{j,k} \end{pmatrix}$, C skew-symmetric, is

isomorphic to the pair $(\text{Inn}_{C'} \theta, \text{Inn}_{L_{n,-1}})$, for $C' = (C'_{j,k})_{j,k=1}^{\frac{n}{2}}$, $C'_{j,k} = d_j \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for $d_j \in k^*/k^{*2}$ when $j = k$, and the 2×2 zero matrix when $j \neq k$.

Proof. As above, observe that $a_{j,k}^2 + b_{j,k}^2 = 0$ if and only if either $b_{j,k} = 0 = a_{j,k}$ or $-1 = \left(\frac{b_{j,k}}{a_{j,k}}\right)^2$, a contradiction to the fact that -1 is not a square in k^* . This implies that the matrix $C_{j,k}$ is either the 2×2 zero matrix or is invertible. Observe also that because C is skew-symmetric,

$$C_{j,j} = \begin{pmatrix} 0 & b_{j,j} \\ -b_{j,j} & 0 \end{pmatrix}$$

and $C_{j,k} = -C_{k,j}$. Second, for $n = 4$ if $C_{1,2} = 0$, we're done. Suppose $C_{1,2} \neq 0$, and consider the matrix:

$$S = \begin{pmatrix} b_{1,2} & -a_{1,2} & 0 & 0 \\ a_{1,2} & b_{1,2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $S^{-1}L_{n,-1}S = L_{n,-1}$ and

$$S^T C S$$

$$\begin{aligned} &= \begin{pmatrix} b_{1,2} & -a_{1,2} & 0 & 0 \\ a_{1,2} & b_{1,2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^T \begin{pmatrix} 0 & b_{1,1} & a_{1,2} & b_{1,2} \\ -b_{1,1} & 0 & -b_{1,2} & a_{1,2} \\ -a_{1,2} & b_{1,2} & 0 & b_{2,2} \\ -b_{1,2} & -a_{1,2} & -b_{2,2} & 0 \end{pmatrix} \begin{pmatrix} b_{1,2} & -a_{1,2} & 0 & 0 \\ a_{1,2} & b_{1,2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} b_{1,2} & a_{1,2} & 0 & 0 \\ -a_{1,2} & b_{1,2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_{1,1}a_{1,2} & b_{1,1}b_{1,2} & a_{1,2} & b_{1,2} \\ -b_{1,1}b_{1,2} & b_{1,1}a_{1,2} & -b_{1,2} & a_{1,2} \\ 0 & a_{1,2}^2 + b_{1,2}^2 & 0 & b_{2,2} \\ -a_{1,2}^2 - b_{1,2}^2 & 0 & -b_{2,2} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & b_{1,1}(a_{1,2}^2 + b_{1,2}^2) & 0 & a_{1,2}^2 + b_{1,2}^2 \\ -b_{1,1}(a_{1,2}^2 + b_{1,2}^2) & 0 & -a_{1,2}^2 - b_{1,2}^2 & 0 \\ 0 & a_{1,2}^2 + b_{1,2}^2 & 0 & b_{2,2} \\ -a_{1,2}^2 - b_{1,2}^2 & 0 & -b_{2,2} & 0 \end{pmatrix}. \end{aligned}$$

Also, for P_4 , as above, it was shown in lemma 4.1.7 that $P_4^T = P_4^{-1}$, $P_4^{-1}L_{4,-1}P_4 = J_4$. Also, $P_4^T S^T C S P_4 =$

$$\begin{pmatrix} 0 & 0 & b_{1,1}(a_{1,2}^2 + b_{1,2}^2) & a_{1,2}^2 + b_{1,2}^2 \\ 0 & 0 & a_{1,2}^2 + b_{1,2}^2 & b_{2,2} \\ -b_{1,1}(a_{1,2}^2 - b_{1,2}^2) & -a_{1,2}^2 - b_{1,2}^2 & 0 & 0 \\ -a_{1,2}^2 - b_{1,2}^2 & -b_{2,2} & 0 & 0 \end{pmatrix}.$$

But this gives us symmetric 2×2 blocks on the off diagonal that are negatives of each other, thus we know from [Sch85] there is an invertible 2×2 matrix, Q , such that for

$$\bar{Q} = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}, \bar{Q}^T P_4^T S^T C S P_4 \bar{Q} = \bar{D} = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix},$$

for D a 2×2 diagonal matrix. Also, the form of \bar{Q} guarantees that it commutes with J_4 . Finally, 'undoing' the permutation matrix P_4 , we get $P_4 \bar{D} P_4^T = D'$, a block diagonal matrix of the form:

$$\begin{pmatrix} 0 & d_1 & 0 & 0 \\ -d_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_2 \\ 0 & 0 & -d_2 & 0 \end{pmatrix}$$

and $P_4 J_4 P_4^{-1} = L_{4,-1}$. Thus, for $n = 4$ the isomorphism holds. Proceeding by induction, assume the result holds for some n , we aim to show it holds for $n + 2$. Let $(\text{Inn}_C \theta, \text{Inn}_{L_{n+2,-1}})$ be a commuting pair of involutions of $\text{SL}(n + 2, k)$. Partition C into block form in the following way:

$$C = (C_{j,k})_{j,k=1}^{\frac{n+2}{2}} \begin{pmatrix} C_1 & U \\ -U^T & C_{\frac{n+2}{2}, \frac{n+2}{2}} \end{pmatrix},$$

for $C_1 \in M(n, k)$, $U \in M(n \times 2, k)$. Note that the form of C forces $C_1 = (C_{j,k})_{j,k=1}^{\frac{n}{2}}$, for $(C_{j,k})_{j,k=1}^{\frac{n}{2}}$ as described in the statement of this lemma. From our induction hypothesis, there is an invertible B such that $B^T C_1 B =$

$$\begin{pmatrix} d_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & & 0 \\ & \ddots & & \\ & & & \\ 0 & & & d_{\frac{n}{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix},$$

and $B^{-1}L_{n,-1}B = L_{n,-1}$. This implies that $\bar{B} = \begin{pmatrix} B & 0 \\ 0 & I_2 \end{pmatrix}$ is invertible and

$$\begin{aligned} & \bar{B}^T C \bar{B} \\ &= \begin{pmatrix} B^T & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} C_1 & U \\ -U^T & C_{\frac{n+2}{2}, \frac{n+2}{2}} \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & I_2 \end{pmatrix} \\ &= \begin{pmatrix} B^T C_1 B & U' \\ -U'^T & C_{\frac{n+2}{2}, \frac{n+2}{2}} \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} d_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & d_{\frac{n}{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix} & U' \\ -U'^T & & C_{\frac{n+2}{2}, \frac{n+2}{2}} \end{pmatrix}. \end{aligned}$$

Also, $\bar{B}^{-1}L_{n+2,-1}\bar{B} = L_{n+2,-1}$. By the form of C and thus of $\bar{B}^T C \bar{B}$, it must be that $U' = \left(C'_{j, \frac{n+2}{2}} \right)_{j=1}^{\frac{n}{2}}$ for

$$C'_{j, \frac{n+2}{2}} = \begin{pmatrix} b_{j, \frac{n+2}{2}} & -a_{j, \frac{n+2}{2}} \\ a_{j, \frac{n+2}{2}} & b_{j, \frac{n+2}{2}} \end{pmatrix}.$$

Let

$$S' = \begin{pmatrix} C''_{1, \frac{n+2}{2}} & & 0 \\ & \ddots & \\ & & C''_{\frac{n}{2}, \frac{n+2}{2}} \\ 0 & & & I_2 \end{pmatrix},$$

where $C''_{j, \frac{n+2}{2}} = C'_{j, \frac{n+2}{2}}$ if it is nonzero, and I_2 if it is zero. Note that S' is such that

$S'^{-1}L_{n+2,-1}S' = L_{n+2,-1}$, and

$$\begin{aligned}
& S'^T \bar{B}^T C \bar{B} S' \\
&= \begin{pmatrix} C''_{1, \frac{n+2}{2}} & & 0 \\ & \ddots & \\ & & C''_{\frac{n}{2}, \frac{n+2}{2}} \\ 0 & & & I_2 \end{pmatrix}^T \begin{pmatrix} \begin{pmatrix} d_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & d_{\frac{n}{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix} & U' \\ & & & -U'^T & C_{\frac{n+2}{2}, \frac{n+2}{2}} \end{pmatrix} S' \\
&= \begin{pmatrix} \begin{pmatrix} d_1 C''^T_{1, \frac{n+2}{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & d_{\frac{n}{2}} C''^T_{\frac{n}{2}, \frac{n+2}{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix} & C''^T_{1, \frac{n+2}{2}} C_{1, \frac{n+2}{2}} \\ & & & \vdots \\ -C^T_{1, \frac{n+2}{2}} & \cdots & -C^T_{\frac{n}{2}, \frac{n+2}{2}} & C_{\frac{n+2}{2}, \frac{n+2}{2}} \end{pmatrix} S' \\
&= \begin{pmatrix} \begin{pmatrix} d_1 C''^T_{1, \frac{n+2}{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} C''_{1, \frac{n+2}{2}} & & 0 \\ & \ddots & \\ 0 & & d_{\frac{n}{2}} C''^T_{\frac{n}{2}, \frac{n+2}{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} C''_{\frac{n}{2}, \frac{n+2}{2}} \end{pmatrix} & C''^T_{1, \frac{n+2}{2}} C_{1, \frac{n+2}{2}} \\ & & & \vdots \\ -C^T_{1, \frac{n+2}{2}} C''_{1, \frac{n+2}{2}} & \cdots & -C^T_{\frac{n}{2}, \frac{n+2}{2}} C''_{\frac{n}{2}, \frac{n+2}{2}} & C_{\frac{n+2}{2}, \frac{n+2}{2}} \end{pmatrix} \\
&= \begin{pmatrix} \begin{pmatrix} d_1 d'_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & d_{\frac{n}{2}} d'_{\frac{n}{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix} & d'_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ & & & \vdots \\ d'_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \cdots & d'_{\frac{n}{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & C_{\frac{n+2}{2}, \frac{n+2}{2}} \end{pmatrix} \\
&= \hat{C},
\end{aligned}$$

by the construction of $C''_{j, \frac{n+2}{2}}$. Therefore, $(\text{Inn}_C \theta, \text{Inn}_{L_{n+2,-1}})$ is isomorphic to the pair $(\text{Inn}_{\hat{C}} \theta, \text{Inn}_{L_{n+2,-1}})$.

Claim 5.2.5. $(\text{Inn}_{\hat{C}} \theta, \text{Inn}_{L_{n+2,-1}})$ is isomorphic to $(\text{Inn}_{\hat{M}} \theta, \text{Inn}_{J_{n+2}})$, for \hat{M} the skew-symmetric matrix $\begin{pmatrix} 0 & M \\ -M & 0 \end{pmatrix}$, and M a symmetric matrix.

Proof. The pairs are Inn_P isomorphic for $P = \sum_{i=1}^{\frac{n}{2}+1} (E_{i,2i-1} + E_{\frac{n}{2}+1+i,2i}) = P_{n+2}$. Then $P_{n+2}^T = P_{n+2}^{-1}$, and $P_{n+2} L_{n+2,-1} P_{n+2}^{-1} = P_{n+2} L_{n+2,-1} P_{n+2}^T = L'_{n+2,-1} = J_{n+2}$, as

shown in the proof of lemma 4.1.7. Also,

$$\begin{aligned} \text{Inn}_{P_{n+2}^{-1}} \text{Inn}_{\hat{C}} \theta \text{Inn}_{P_{n+2}} &= \text{Inn}_{P_{n+2}^{-1}} \text{Inn}_{\hat{C}} \text{Inn}_{P_{n+2}^{T-1}} \theta \\ &= \text{Inn}_{P_{n+2} \hat{C} P_{n+2}^T} \theta, \end{aligned}$$

since P_{n+2} is a permutation matrix. Furthermore,

$$\begin{aligned} &P_{n+2} \hat{C} P_{n+2}^T \\ &= P_{n+2} \sum_{i=1}^{\frac{n}{2}} d_i d'_i (E_{2i-1,2i} - E_{2i,2i-1}) P_{n+2}^T \\ &\quad + P_{n+2} \sum_{i=1}^{\frac{n}{2}} d'_i (E_{2i-1,n+2} - E_{2i,n+1} + E_{n+1,2i} - E_{n+2,2i-1}) P_{n+2}^T \\ &\quad + b_{\frac{n+2}{2}, \frac{n+2}{2}} P_{n+2} (E_{n+1,n+2} - E_{n+2,n+1}) P_{n+2}^T \\ &= \sum_{i=1}^{\frac{n}{2}+1} (E_{i,2i-1} + E_{\frac{n}{2}+1+i,2i}) \sum_{i=1}^{\frac{n}{2}} d_i d'_i (E_{2i-1,2i} - E_{2i,2i-1}) P_{n+2}^T \\ &\quad + \sum_{i=1}^{\frac{n}{2}+1} (E_{i,2i-1} + E_{\frac{n}{2}+1+i,2i}) \sum_{i=1}^{\frac{n}{2}} d'_i (E_{2i-1,n+2} - E_{2i,n+1} + E_{n+1,2i} - E_{n+2,2i-1}) P_{n+2}^T \\ &\quad + b_{\frac{n+2}{2}, \frac{n+2}{2}} \sum_{i=1}^{\frac{n}{2}+1} (E_{i,2i-1} + E_{\frac{n}{2}+1+i,2i}) (E_{n+1,n+2} - E_{n+2,n+1}) P_{n+2}^T \\ &= \sum_{i=1}^{\frac{n}{2}} d_i d'_i (E_{i,2i} - E_{\frac{n}{2}+1+i,2i-1}) \sum_{i=1}^{\frac{n}{2}+1} (E_{2i-1,i} + E_{2i,\frac{n}{2}+1+i}) \\ &\quad + \sum_{i=1}^{\frac{n}{2}} d'_i (E_{i,n+2} - E_{\frac{n}{2}+1+i,n+1} + E_{\frac{n}{2}+1,2i} - E_{n+2,2i-1}) \sum_{i=1}^{\frac{n}{2}+1} (E_{2i-1,i} + E_{2i,\frac{n}{2}+1+i}) \\ &\quad + b_{\frac{n+2}{2}, \frac{n+2}{2}} (E_{\frac{n}{2}+1,n+2} - E_{n+2,n+1}) \sum_{i=1}^{\frac{n}{2}+1} (E_{2i-1,i} + E_{2i,\frac{n}{2}+1+i}) \\ &= \sum_{i=1}^{\frac{n}{2}} d_i d'_i (E_{i,\frac{n}{2}+1+i} - E_{\frac{n}{2}+1+i,i}) \\ &\quad + \sum_{i=1}^{\frac{n}{2}} d'_i (E_{i,n+2} - E_{\frac{n}{2}+1+i,\frac{n}{2}+1} + E_{\frac{n}{2}+1,\frac{n}{2}+1+i} - E_{n+2,i}) \\ &\quad + b_{\frac{n+2}{2}, \frac{n+2}{2}} (E_{\frac{n}{2}+1,n+2} - E_{n+2,\frac{n}{2}+1}). \end{aligned}$$

This is equal to $\begin{pmatrix} 0 & M \\ -M & 0 \end{pmatrix}$ for $M = \sum_{i=1}^{\frac{n}{2}} d_i d'_i E_{i,i} + \sum_{i=1}^{\frac{n}{2}} d'_i (E_{i, \frac{n}{2}+1} + E_{\frac{n}{2}+1, i}) + b_{\frac{n+2}{2}, \frac{n+2}{2}} E_{\frac{n}{2}+1, \frac{n}{2}+1}$, and note that M is symmetric. \square

By results of [Sch85], there exists an invertible Q such that $Q^T M Q$ is diagonal, therefore, because

$$\begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}^T \begin{pmatrix} 0 & M \\ -M & 0 \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} = \begin{pmatrix} 0 & Q^T M Q \\ -Q^T M Q & 0 \end{pmatrix}$$

has diagonal blocks and

$$\begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}^{-1} \begin{pmatrix} 0 & I_{\frac{n+2}{2}} \\ -I_{\frac{n+2}{2}} & 0 \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} = \begin{pmatrix} 0 & I_{\frac{n+2}{2}} \\ -I_{\frac{n+2}{2}} & 0 \end{pmatrix},$$

then $(\text{Inn}_{\hat{C}} \theta, \text{Inn}_{L_{n+2, -1}})$ is isomorphic to $(\text{Inn}_{\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}} \theta, \text{Inn}_{J_{n+2}})$, for D diagonal. But by the form of P_{n+2} , the pair is isomorphic to $(\text{Inn}_{C'} \theta, \text{Inn}_{L_{n+2, -1}})$, for

$$C' = \begin{pmatrix} d_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & d_{\frac{n}{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix}, d_i \in k^*/k^{*2},$$

and since C' commutes with $L_{n+2, -1}$, it has the desired form. \square

5.3 Classification of the Remaining Four Types

The above results imply that there are 4 possible types of commuting pairs (with one involution an inner involution and one involution an outer involution) of involutions of $\text{SL}(n, k)$:

Corollary 5.3.1. *Every commuting pair of involutions, one inner and one outer, of*

$SL(n, k)$ is isomorphic to one of the following pairs:

$$(1) \left(\text{Inn} \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix} \theta, \text{Inn}_{I_{n-i, i}} \right), \quad (2) \left(\text{Inn} \begin{pmatrix} J_{n-i} & \\ & J_i \end{pmatrix} \theta, \text{Inn}_{I_{n-i, i}} \right),$$

$$(3) \left(\text{Inn} \begin{pmatrix} d_1 I_2 & & \\ & \ddots & \\ & & d_{\frac{n}{2}} I_2 \end{pmatrix} \theta, \text{Inn}_{L_{n, -1}} \right), \quad (4) \left(\text{Inn} \begin{pmatrix} d_1 J_2 & & \\ & \ddots & \\ & & d_{\frac{n}{2}} J_2 \end{pmatrix} \theta, \text{Inn}_{L_{n, -1}} \right)$$

and note that pairs of type (3) and (4) only need to be considered when $-1 \notin k^{*2}$, and pairs of type (2) can occur only when $n-i$ and i are even.

Proof. This is a direct result of the work above. \square

Note that pairs of type (2), above, are distinct for distinct $i \in \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$ with $i, n-i$ even.

We first consider pairs of type (1) and their distinct isomorphism classes. Note that these pairs are equal to pairs of the form $\left(\text{Inn} \begin{pmatrix} M & 0 \\ 0 & \alpha N \end{pmatrix} \theta, \text{Inn}_{I_{n-i, i}} \right)$, for M a $(n-i) \times (n-i)$ representative of a semi-congruence class of symmetric matrices and N an $i \times i$ representative of a semi-congruence class of symmetric matrices. If $n-i \neq i$ then different semi-congruence classes (for M and N) give different isomorphism classes, but different values of α might not give isomorphic pairs for fixed M and N .

Lemma 5.3.2. *Let M and N , $n-i \neq i$, be fixed matrix representatives of semi-congruence classes of symmetric bilinear forms on k^{n-i} and k^i , respectively. Then*

$$\left(\text{Inn} \begin{pmatrix} M & \\ & \alpha_1 N \end{pmatrix} \theta, \text{Inn}_{I_{n-i, i}} \right) \approx \left(\text{Inn} \begin{pmatrix} M & \\ & \alpha_2 N \end{pmatrix} \theta, \text{Inn}_{I_{n-i, i}} \right)$$

*if and only if either $\alpha_1 \equiv \alpha_2 \pmod{k^{*2}}$ or there exists a $\beta \in k^*$ such that M_{n-i} is congruent to βM_{n-i} and $\alpha_1 M_i$ is congruent to $\beta \alpha_2 M_i$.*

Proof. This follows directly from the definition of isomorphic pairs and the fact that $n-i \neq i$. \square

Furthermore, complete results on the existence of such a β have been determined for algebraically closed fields, \mathbb{R} , \mathbb{F}_p , and the p -adic numbers and are given in the tables below. Additionally, results for $n-i = i$ exist and are also given in tables below. Before giving the tables, we need some results given in [Jon50]:

Definition 5.3.3. The Hilbert symbol $(\alpha, \beta)_p$ is ± 1 depending on whether or not the equation $\alpha x^2 + \beta y^2 = 1$ has a solution of the p -adic numbers, it is 1 if it has a solution over the p -adics and is -1 otherwise.

Definition 5.3.4. The Hasse symbol, $c_p(f)$, of the matrix, M , of a bilinear form, f , is defined in terms of the Hilbert Symbol:

$$c_p(f) = (-1, -D_n)_p \prod_{i=1}^{n-1} (D_i, -D_{i+1})_p,$$

where D_i is the $i \times i$ determinant in the upper left hand corner of the matrix, M , of the bilinear form, f .

The following two lemmas come from [Jon50]:

Lemma 5.3.5. *For a bilinear form, f , we have:*

- $c_p(\omega f) = (\omega, (-1)^{(n+1)/2})_p c_p(f)$ if n is odd and
- $c_p(\omega f) = (\omega, (-1)^{n/2} \delta)_p c_p(f)$ if n is even

where δ is the determinant of the matrix, M , of the bilinear form, f .

Lemma 5.3.6. *Two symmetric matrices M_1 and M_2 with entries in \mathbb{Q}_p are congruent if and only if*

$$\det(M_1) = \alpha^2 \det(M_2) \text{ and } c_p(M_1) = c_p(M_2)$$

First, we consider when $n - i \neq i$. For algebraically closed fields there is only one element in k^*/k^{*2} and thus we only need to consider distinct pairs of semi-congruence class matrix representatives of size $n - i$ and i on the diagonal. For $k = \mathbb{R}$ no such β exists unless $n - i - j = j$ or $i - k = k$ for $M = I_{n-i-j, j}$, $N = I_{i-k, k}$, which follows from Sylvester's law regarding the signature of a symmetric real bilinear form found in [Art91]. We know such a β exists for $n - i - j = j$ or $i - k = k$ since:

$$\begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}^T \begin{pmatrix} I_m & 0 \\ 0 & -I_m \end{pmatrix} \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix} = \begin{pmatrix} -I_m & 0 \\ 0 & I_m \end{pmatrix}$$

For finite fields (of characteristic not 2,) we first need an important lemma from [Sch85]:

Table 5.1: Distinct isomorphism classes given by the multiple $\alpha \in \mathbb{F}_p^*/\mathbb{F}_p^{*2}$ for fixed semi-congruence class representatives M and N

		i			
		odd		even	
		I_i	I_i	M_{i,s_p}	
$n - i$	odd	I_{n-i}	$1, s_p$	1	1
	even	I_{n-i}	1	1	1
		M_{n-i,s_p}	1	1	1

Lemma 5.3.7. *Every element of \mathbb{F}_p can be written as the sum of 2 squares in \mathbb{F}_p .*

Note that this implies that there exists a symmetric matrix $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ such that for s_p a representative of $\mathbb{F}_p^*/\mathbb{F}_p^{*2} = \{1, s_p\}$ that is not a square, $A^T A = \begin{pmatrix} s_p & 0 \\ 0 & s_p \end{pmatrix}$, for $a, b \in \mathbb{F}_p$, and this implies that for even m , I_m is congruent to $s_p I_m$ over $\text{GL}(\mathbb{F}_p, m)$. This also implies that for $M_{m,s_p} = \begin{pmatrix} I_{m-1} & 0 \\ 0 & s_p \end{pmatrix}$ and m odd, M_{m,s_p} is semi-congruent to I_m . From this we can determine isomorphism classes:

Lemma 5.3.8. *The distinct isomorphism classes of commuting pairs of involutions of $\text{SL}(\mathbb{F}_p, n)$ of the above type (1), are given by $\left(\text{Inn}_{\begin{pmatrix} M & 0 \\ 0 & \alpha N \end{pmatrix}} \theta, \text{Inn}_{I_{n-i,i}} \right)$ for $i \in \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$. For each fixed pair, M and N , of semi-congruence class representatives of symmetric bilinear forms of k^{n-i}, k^i , respectively, the distinct isomorphism classes for $\alpha \in \mathbb{F}_p^*/\mathbb{F}_p^{*2}$ are given in table I.*

Proof. This follows from lemmas 5.3.6 and 5.3.7, and the following (for $n - i$ even):

$$\begin{aligned}
\begin{pmatrix} M_{1,1,s_p} & 0 \\ 0 & s_p N \end{pmatrix} &= \begin{pmatrix} \begin{pmatrix} I_{n-i-2} & 0 \\ & 1 \end{pmatrix} & 0 \\ 0 & s_p N \end{pmatrix} \\
&\approx \begin{pmatrix} \begin{pmatrix} S & 0 \\ 0 & I_{i+2} \end{pmatrix}^T & \begin{pmatrix} I_{n-i-2} & 0 \\ & 1 \\ 0 & s_p \\ & 0 \end{pmatrix} & 0 \\ & & s_p N \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & I_{i+2} \end{pmatrix} \\
&= \begin{pmatrix} \begin{pmatrix} S^T S & 0 \\ & 1 \end{pmatrix} & 0 \\ 0 & s_p N \end{pmatrix}
\end{aligned}$$

for $S = \sum_{k=1}^{n-i-2} (aE_{2i-1,2i-1} + aE_{2i,2i} + bE_{2i-1,2i} - bE_{2i,2i-1})$, where $a, b \in k$ with

$a^2 + b^2 = s_p$ as in lemma 5.3.7. Furthermore,

$$\begin{aligned}
& S^T S \\
&= \sum_{k=1}^{\frac{n-i-2}{2}} (aE_{2i-1,2i-1} + aE_{2i,2i} + bE_{2i-1,2i} - bE_{2i,2i-1})^T \times \\
&\quad \sum_{k=1}^{\frac{n-i-2}{2}} (aE_{2i-1,2i-1} + aE_{2i,2i} + bE_{2i-1,2i} - bE_{2i,2i-1}) \\
&= \sum_{k=1}^{\frac{n-i-2}{2}} (aE_{2i-1,2i-1} + aE_{2i,2i} - bE_{2i-1,2i} + bE_{2i,2i-1}) \times \\
&\quad \sum_{k=1}^{\frac{n-i-2}{2}} (aE_{2i-1,2i-1} + aE_{2i,2i} + bE_{2i-1,2i} - bE_{2i,2i-1}) \\
&= \sum_{k=1}^{\frac{n-i-2}{2}} (a^2 E_{2i-1,2i-1} + abE_{2i-1,2i} + a^2 E_{2i,2i} - abE_{2i,2i-1}) + \\
&\quad \sum_{k=1}^{\frac{n-i-2}{2}} (-abE_{2i-1,2i} + b^2 E_{2i-1,2i-1} + abE_{2i,2i-1} + b^2 E_{2i,2i}) \\
&= \sum_{k=1}^{\frac{n-i-2}{2}} ((a^2 + b^2)(E_{2i-1,2i-1} + E_{2i,2i})) \\
&= s_p I_{n-i-2}
\end{aligned}$$

so that the above matrix is equal to:

$$\begin{aligned}
\left(\begin{array}{c} \left(\begin{array}{cc} s_p I_{n-i-2} & 0 \\ & 1 \end{array} \right) & 0 \\ \left(\begin{array}{cc} 0 & s_p \end{array} \right) & \\ 0 & s_p N \end{array} \right) &= s_p \left(\begin{array}{c} \left(\begin{array}{cc} I_{n-i-2} & 0 \\ & \frac{1}{s_p} \end{array} \right) & 0 \\ \left(\begin{array}{cc} 0 & 1 \end{array} \right) & \\ 0 & N \end{array} \right) \\
&\approx s_p \left(\begin{array}{c} \left(\begin{array}{cc} I_{n-i-2} & 0 \\ & 1 \end{array} \right) & 0 \\ \left(\begin{array}{cc} 0 & \frac{1}{s_p} \end{array} \right) & \\ 0 & N \end{array} \right)
\end{aligned}$$

via the permutatoin matrix, P , that commutes with $I_{n-i,i}$,

$$P = \sum_{k=1}^{n-i-2} E_{k,k} + \sum_{k=1}^i E_{n-i+k,n-i+k} + E_{n-i-1,n-i} + E_{n-i,n-i-1}.$$

The proof follows, then, from the fact that $\text{Inn}_{\alpha A} = \text{Inn}_A$ for any constant α and the fact that

$$M_{1,1,s_p}^T M_{1,1,\frac{1}{s_p}} M_{1,1,s_p} = M_{1,1,s_p}$$

for any size M (in this case $M \in \text{GL}(n-i, k)$). \square

Lemma 5.3.9. *For the p -adic numbers, we have the following isomorphism classes given in table format for commuting pairs of the form: $\left(\text{Inn}_{\begin{pmatrix} M & 0 \\ 0 & \alpha N \end{pmatrix}} \theta, \text{Inn}_{I_{n-i,i}} \right)$ for $i \in \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$ and for each fixed pair of semi-congruence class representatives M and N , the distinct isomorphism classes for $\alpha \in \mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$, are given in the tables below, first for $p \neq 2$, $\alpha \in \mathbb{Q}_p^*/\mathbb{Q}_p^{*2} = \{1, p, s_p, ps_p\}$, then for $p = 2$, $\alpha \in \mathbb{Q}_2^*/\mathbb{Q}_2^{*2} = \{1, -1, 2, -2, 3, -3, 6, -6\}$. Furthermore, for $n-i \neq i$ distinct semi-congruence class elements along with distinct α 's listed in the tables below give distinct isomorphism classes, and for $n-i = i$ distinct semi-congruence class elements along with distinct α 's listed in the tables below give distinct isomorphism classes except for the fact that*

$$\left(\text{Inn}_{\begin{pmatrix} M & 0 \\ 0 & \alpha N \end{pmatrix}} \theta, \text{Inn}_{I_{n-i,i}} \right) \approx \left(\text{Inn}_{\begin{pmatrix} N & 0 \\ 0 & \alpha M \end{pmatrix}} \theta, \text{Inn}_{I_{n-i,i}} \right)$$

Lastly, for $i = 1$ only the column for $i = 4l + 1$ applies, and for $i = 2$ only the column $i = 4l + 2$ applies except for the entry $M_{2,3,-6}$.

Proof. This follows from a direct computation of the Hilbert symbol and from lemma 5.3.5. \square

Notation 5.3.10. Let $M_{r,s,t}$ be the appropriate size matrix representative of a semi-congruence class of symmetric bilinear form (either $n-i \times n-i$ or $i \times i$) of the form:

$$\begin{pmatrix} I_{j-3} & & 0 \\ & r & \\ & & s \\ 0 & & & t \end{pmatrix},$$

for $j = n-i$ or i , depending on the location in the tables below.

Table 5.2: Isomorphism classes given by $\alpha \in \mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$, $p \neq 2$, part I

		$i = 4l$					
		I_i	M_{p,s_p,ps_p}	$M_{1,p,p}$	$M_{1,1,p}$	$M_{1,1,s_p}$	$M_{1,1,ps_p}$
$-1 \notin \mathbb{Q}_p^{*2}$							
$4k$	I_{n-i}	1	1	x	1	1	1
	M_{p,s_p,ps_p}	1	1	x	1	1	1
	$M_{1,1,p}$	1	1	x	$1,p$	1	1
	$M_{1,1,s_p}$	1	1	x	1	$1,p$	1
	$M_{1,1,ps_p}$	1	1	x	1	1	$1,s_p$
$4k+1$	I_{n-i}	1	1	x	$1,p$	$1,p$	$1,s_p$
$4k+2$	I_{n-i}	1	1	x	1	$1,p$	1
	M_{1,p,ps_p}	1	1	x	1	1	1
	$M_{1,1,p}$	1	1	x	1	1	$1,s_p$
	$M_{1,1,s_p}$	1	1	x	1	1	1
	$M_{1,1,ps_p}$	1	1	x	$1,p$	1	1
$4k+3$	I_{n-i}	1	1	x	$1,p$	$1,p$	$1,s_p$
	M_{p,s_p,ps_p}	1	1	x	$1,p$	$1,p$	$1,s_p$
$-1 \in \mathbb{Q}_p^{*2}$							
$4k$	I_{n-i}	1	x	1	1	1	1
	$M_{1,p,p}$	1	x	1	1	1	1
	$M_{1,1,p}$	1	x	1	$1,s_p$	1	1
	$M_{1,1,s_p}$	1	x	1	1	$1,p$	1
	$M_{1,1,ps_p}$	1	x	1	1	1	$1,p$
odd	I_{n-i}	1	x	1	$1,s_p$	$1,p$	$1,p$
	M_{p,s_p,ps_p}	1	x	1	$1,s_p$	$1,p$	$1,p$
$4k+2$	I_{n-i}	1	x	1	1	1	1
	M_{p,s_p,ps_p}	1	x	1	1	1	1
	$M_{1,1,p}$	1	x	1	$1,s_p$	1	1
	$M_{1,1,s_p}$	1	x	1	1	$1,p$	1
	$M_{1,1,ps_p}$	1	x	1	1	1	$1,p$

Table 5.3: Isomorphism classes given by $\alpha \in \mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$, $p \neq 2$, part II

		$i = 4l + 2$					
		I_i	M_{1,p,ps_p}	M_{1,p,s_p,ps_p}	$M_{1,1,p}$	$M_{1,1,s_p}$	$M_{1,1,ps_p}$
$-1 \notin \mathbb{Q}_p^{*2}$							
$4k$	I_{n-i}	1	1	x	1	1	1
	M_{p,s_p,ps_p}	1	1	x	1	1	1
	$M_{1,1,p}$	1	1	x	1	1	$1,p$
	$M_{1,1,s_p}$	$1,p$	1	x	1	1	1
	$M_{1,1,ps_p}$	1	1	x	$1,s_p$	1	1
$4k + 1$	I_{n-i}	$1,p$	1	x	$1,s_p$	1	$1,p$
$4k + 2$	I_{n-i}	$1,p$	1	x	1	1	1
	M_{1,p,ps_p}	1	1	x	1	1	1
	$M_{1,1,p}$	1	1	x	$1,s_p$	1	1
	$M_{1,1,s_p}$	1	1	x	1	1	1
	$M_{1,1,ps_p}$	1	1	x	1	1	$1,p$
$4k + 3$	I_{n-i}	$1,p$	1	x	$1,s_p$	1	$1,p$
	M_{p,s_p,ps_p}	$1,p$	1	x	$1,s_p$	1	$1,p$
$-1 \in \mathbb{Q}_p^{*2}$							
$4k$	I_{n-i}	1	x	1	1	1	1
	$M_{1,p,p}$	1	x	1	1	1	1
	$M_{1,1,p}$	1	x	1	$1,s_p$	1	1
	$M_{1,1,s_p}$	1	x	1	1	$1,p$	1
	$M_{1,1,ps_p}$	1	x	1	1	1	$1,p$
odd	I_{n-i}	1	x	1	$1,s_p$	$1,p$	$1,p$
	M_{p,s_p,ps_p}	1	x	1	$1,s_p$	$1,p$	$1,p$
$4k + 2$	I_{n-i}	1	x	1	1	1	1
	M_{p,s_p,ps_p}	1	x	1	1	1	1
	$M_{1,1,p}$	1	x	1	$1,s_p$	1	1
	$M_{1,1,s_p}$	1	x	1	1	$1,p$	1
	$M_{1,1,ps_p}$	1	x	1	1	1	$1,p$

Table 5.4: Isomorphism classes given by $\alpha \in \mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$, $p \neq 2$, part III

		$i = 4l + 1$		$i = 4l + 3$	
		I_i	M_{1,p,s_p,ps_p}	I_i	M_{1,p,s_p,ps_p}
$-1 \notin \mathbb{Q}_p^{*2}$					
$4k$	I_{n-i}	1	x	1	1
	M_{p,s_p,ps_p}	1	x	1	1
	$M_{1,1,p}$	$1,p$	x	$1,p$	$1,p$
	$M_{1,1,s_p}$	$1,p$	x	$1,p$	$1,p$
	$M_{1,1,ps_p}$	$1,s_p$	x	$1,s_p$	$1,s_p$
$4k + 1$	I_{n-i}	$1,p,s_p,ps_p$	x	$1,p,s_p,ps_p$	$1,p,s_p,ps_p$
$4k + 2$	I_{n-i}	$1,p$	x	$1,p$	$1,p$
	M_{1,p,ps_p}	1	x	1	1
	$M_{1,1,p}$	$1,s_p$	x	$1,s_p$	$1,s_p$
	$M_{1,1,s_p}$	1	x	1	1
	$M_{1,1,ps_p}$	$1,p$	x	$1,p$	$1,p$
$4k + 3$	I_{n-i}	$1,p,s_p,ps_p$	x	$1,p,s_p,ps_p$	$1,p,s_p,ps_p$
	M_{p,s_p,ps_p}	$1,p,s_p,ps_p$	x	$1,p,s_p,ps_p$	$1,p,s_p,ps_p$
$-1 \in \mathbb{Q}_p^{*2}$					
$4k$	I_{n-i}	1	1	1	1
	$M_{1,p,p}$	1	1	1	1
	$M_{1,1,p}$	$1,s_p$	$1,s_p$	$1,s_p$	$1,s_p$
	$M_{1,1,s_p}$	$1,p$	$1,p$	$1,p$	$1,p$
	$M_{1,1,ps_p}$	$1,p$	$1,p$	$1,p$	$1,p$
odd	I_{n-i}	$1,p,s_p,ps_p$	$1,p,s_p,ps_p$	$1,p,s_p,ps_p$	$1,p,s_p,ps_p$
	M_{p,s_p,ps_p}	$1,p,s_p,ps_p$	$1,p,s_p,ps_p$	$1,p,s_p,ps_p$	$1,p,s_p,ps_p$
$4k + 2$	I_{n-i}	1	1	1	1
	M_{p,s_p,ps_p}	1	1	1	1
	$M_{1,1,p}$	$1,s_p$	$1,s_p$	$1,s_p$	$1,s_p$
	$M_{1,1,s_p}$	$1,p$	$1,p$	$1,p$	$1,p$
	$M_{1,1,ps_p}$	$1,p$	$1,p$	$1,p$	$1,p$

Table 5.5: Isomorphism classes given by $\alpha \in \mathbb{Q}_2^*/\mathbb{Q}_2^{*2}$, part Ia

		$i = 4l$				
		I_i	$M_{2,3,6}$	$M_{1,1,-1}$	$M_{1,1,2}$	$M_{1,1,-2}$
$4k$	I_{n-i}	1	1	1	1	1
	$M_{2,3,6}$	1	1	1	1	1
	$M_{1,1,-1}$	1	1	1,-1	1	1
	$M_{1,1,2}$	1	1	1	1,3	1
	$M_{1,1,-2}$	1	1	1	1	1,-1
	$M_{1,1,3}$	1	1	1	1	1
	$M_{1,1,-3}$	1	1	1	1	1
	$M_{1,1,6}$	1	1	1	1	1
	$M_{1,1,-6}$	1	1	1	1	1
$4k + 1$	I_{n-i}	1	1	1,-1	1,3	1,-1
$4k + 2$	I_{n-i}	1	1	1,-1	1	1
	$M_{2,3,-6}$	1	1	1	1	1
	$M_{1,1,-1}$	1	1	1	1	1
	$M_{1,1,2}$	1	1	1	1	1,-1
	$M_{1,1,-2}$	1	1	1	1,3	1
	$M_{1,1,3}$	1	1	1	1	1
	$M_{1,1,-3}$	1	1	1	1	1
	$M_{1,1,6}$	1	1	1	1	1
	$M_{1,1,-6}$	1	1	1	1	1
$4k + 3$	I_{n-i}	1	1	1,-1	1,3	1,-1
	$M_{2,3,6}$	1	1	1,-1	1,3	1,-1

Table 5.6: Isomorphism classes given by $\alpha \in \mathbb{Q}_2^*/\mathbb{Q}_2^{*2}$, part Ib

		$i = 4l$			
		$M_{1,1,3}$	$M_{1,1,-3}$	$M_{1,1,6}$	$M_{1,1,-6}$
$4k$	I_{n-i}	1	1	1	1
	$M_{2,3,6}$	1	1	1	1
	$M_{1,1,-1}$	1	1	1	1
	$M_{1,1,2}$	1	1	1	1
	$M_{1,1,-2}$	1	1	1	1
	$M_{1,1,3}$	1,-1	1	1	1
	$M_{1,1,-3}$	1	1,3	1	1
	$M_{1,1,6}$	1	1	1,-1	1
	$M_{1,1,-6}$	1	1	1	1,3
$4k + 1$	I_{n-i}	1,-1	1,2	1,-1	1,2
$4k + 2$	I_{n-i}	1	1	1	1
	$M_{2,3,-6}$	1	1	1	1
	$M_{1,1,-1}$	1	1	1	1
	$M_{1,1,2}$	1	1	1	1
	$M_{1,1,-2}$	1	1	1	1
	$M_{1,1,3}$	1	1,2	1	1
	$M_{1,1,-3}$	1,-1	1	1	1
	$M_{1,1,6}$	1	1	1	1,3
	$M_{1,1,-6}$	1	1	1,-1	1
$4k + 3$	I_{n-i}	1,-1	1,2	1,-1	1,2
	$M_{2,3,6}$	1,-1	1,2	1,-1	1,2

Table 5.7: Isomorphism classes given by $\alpha \in \mathbb{Q}_2^*/\mathbb{Q}_2^{*2}$, part IIa

		$i = 4l + 2$				
		I_i	$M_{2,3,6}$	$M_{1,1,-1}$	$M_{1,1,2}$	$M_{1,1,-2}$
$4k$	I_{n-i}	1	1	1	1	1
	$M_{2,3,6}$	1	1	1	1	1
	$M_{1,1,-1}$	1,-1	1	1	1	1
	$M_{1,1,2}$	1	1	1	1	1,3
	$M_{1,1,-2}$	1	1	1	1,-1	1
	$M_{1,1,3}$	1	1	1	1	1
	$M_{1,1,-3}$	1	1	1	1	1
	$M_{1,1,6}$	1	1	1	1	1
	$M_{1,1,-6}$	1	1	1	1	1
$4k + 1$	I_{n-i}	1,-1	1	1	1,-2	1,3
$4k + 2$	I_{n-i}	1,-1	1	1	1	1
	$M_{2,3,-6}$	1	1	1	1	1
	$M_{1,1,-1}$	1	1	1	1	1
	$M_{1,1,2}$	1	1	1	1,-1	1
	$M_{1,1,-2}$	1	1	1	1	1,3
	$M_{1,1,3}$	1	1	1	1	1
	$M_{1,1,-3}$	1	1	1	1	1
	$M_{1,1,6}$	1	1	1	1	1
	$M_{1,1,-6}$	1	1	1	1	1
$4k + 3$	I_{n-i}	1,-1	1	1	1,-2	1,3
	$M_{2,3,6}$	1,-1	1	1	1,-2	1,3

Table 5.8: Isomorphism classes given by $\alpha \in \mathbb{Q}_2^*/\mathbb{Q}_2^{*2}$, part IIb

		$i = 4l + 2$			
		$M_{1,1,3}$	$M_{1,1,-3}$	$M_{1,1,6}$	$M_{1,1,-6}$
$4k$	I_{n-i}	1	1	1	1
	$M_{2,3,6}$	1	1	1	1
	$M_{1,1,-1}$	1	1	1	1
	$M_{1,1,2}$	1	1	1	1
	$M_{1,1,-2}$	1	1	1	1
	$M_{1,1,3}$	1	1,-1	1	1
	$M_{1,1,-3}$	1,2	1	1	1
	$M_{1,1,6}$	1	1	1	1,-1
	$M_{1,1,-6}$	1	1	1,3	1
$4k + 1$	I_{n-i}	1,-2	1,3	1,3	1,-3
$4k + 2$	I_{n-i}	1	1	1	1
	$M_{2,3,-6}$	1	1	1	1
	$M_{1,1,-1}$	1	1	1	1
	$M_{1,1,2}$	1	1	1	1
	$M_{1,1,-2}$	1	1	1	1
	$M_{1,1,3}$	1,2	1	1	1
	$M_{1,1,-3}$	1	1,-1	1	1
	$M_{1,1,6}$	1	1	1,3	1
	$M_{1,1,-6}$	1	1	1	1,-1
$4k + 3$	I_{n-i}	1,-2	1,3	1,3	1,-3
	$M_{2,3,6}$	1,-2	1,3	1,3	1,-3

Table 5.9: Isomorphism classes given by $\alpha \in \mathbb{Q}_2^*/\mathbb{Q}_2^{*2}$, part III

		$i = 4l + 1$	$i = 4l + 3$	
		I_i	I_i	$M_{2,3,6}$
$4k$	I_{n-i}	1	1	1
	$M_{2,3,6}$	1	1	1
	$M_{1,1,-1}$	1,-1	1,-1	1,-1
	$M_{1,1,2}$	1,3	1,3	1,3
	$M_{1,1,-2}$	1,-1	1,-1	1,-1
	$M_{1,1,3}$	1,-1	1,-1	1,-1
	$M_{1,1,-3}$	1,2	1,2	1,2
	$M_{1,1,6}$	1,-1	1,-1	1,-1
	$M_{1,1,-6}$	1,2	1,2	1,2
$4k + 1$	I_{n-i}	1,-1,2,-2,3,-3,6,-6	1,-1,2,-2,3,-3,6,-6	1,-1,2,-2,3,-3,6,-6
$4k + 2$	I_{n-i}	1,-1	1,-1	1,-1
	$M_{2,3,-6}$	1	1	1
	$M_{1,1,-1}$	1	1	1
	$M_{1,1,2}$	1,-2	1,-2	1,-2
	$M_{1,1,-2}$	1,3	1,3	1,3
	$M_{1,1,3}$	1,-2	1,-2	1,-2
	$M_{1,1,-3}$	1,3	1,3	1,3
	$M_{1,1,6}$	1,3	1,3	1,3
	$M_{1,1,-6}$	1,-3	1,-3	1,-3
$4k + 3$	I_{n-i}	1,-1,2,-2,3,-3,6,-6	1,-1,2,-2,3,-3,6,-6	1,-1,2,-2,3,-3,6,-6
	$M_{2,3,6}$	1,-1,2,-2,3,-3,6,-6	1,-1,2,-2,3,-3,6,-6	1,-1,2,-2,3,-3,6,-6

Continuing on from Corollary 5.3.1, we now consider commuting pairs of involutions of $\mathrm{SL}(n, k)$ of type (3), when $-1 \notin k^{*2}$, which are isomorphic to pairs of the form

$$\left(\mathrm{Inn}_{\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}} \theta, \mathrm{Inn}_{J_n} \right),$$

M a representative of a semi-congruence class of symmetric bilinear form of $k^{n/2}$. Note, however, that distinct semi-congruence class elements M do not necessarily yield distinct isomorphism classes of commuting involutions of the above type, since we can transform (the matrix representatives of the involutions of) the pair by nonsingular matrices of the block form $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$, while preserving J_n . We give results of the isomorphism classes of the above type for specific fields:

Theorem 5.3.11. *For:*

- $k = \bar{k}$, -1 is a square and thus we do not need to consider pairs of the above type.
- $k = \mathbb{R}$, we get distinct pairs for distinct $M_i = I_{\frac{n}{2}-i, i}$, $i \in \{1, 2, \dots, \lfloor \frac{n}{4} \rfloor\}$.
- $k = \mathbb{F}_p$, with $-1 \notin k^{*2}$, there is only one isomorphism class, $(\theta, \mathrm{Inn}_{L_{n,-1}})$
- $k = \mathbb{Q}_p$, with $-1 \notin k^{*2}$, $p \neq 2$
 - $\frac{n}{2} = 4l + 1$ There is one isomorphism class with commuting pair representative: $(\theta, \mathrm{Inn}_{L_{n,-1}})$.
 - $\frac{n}{2} = 4l + 3$ There is one isomorphism class with commuting pair representative: $(\theta, \mathrm{Inn}_{L_{n,-1}})$.
 - $\frac{n}{2} = 4l + 2$ There are two isomorphism classes of commuting pairs if -1 is not a square modulo p , their representatives are: $(\theta, \mathrm{Inn}_{L_{n,-1}})$ and $\left(\mathrm{Inn}_{\begin{pmatrix} I_{n-2} & \\ & pI_2 \end{pmatrix}} \theta, \mathrm{Inn}_{L_{n,-1}} \right)$, otherwise there is one isomorphism class given by $(\theta, \mathrm{Inn}_{L_{n,-1}})$.
 - $\frac{n}{2} = 4l$ There are two isomorphism classes of commuting pairs if -1 is not a square modulo p , their representatives are the following: $(\theta, \mathrm{Inn}_{L_{n,-1}})$

and $\left(\text{Inn}_{\begin{pmatrix} I_{n-2} & \\ & pI_2 \end{pmatrix}} \theta, \text{Inn}_{L_{n,-1}} \right)$, otherwise there is one isomorphism class given by $(\theta, \text{Inn}_{L_{n,-1}})$.

• $k = \mathbb{Q}_2$

– $\frac{n}{2} = 4l + 1$ There is one isomorphism class with commuting pair representative: $(\theta, \text{Inn}_{L_{n,-1}})$.

– $\frac{n}{2} = 4l + 3$ There is one isomorphism class with commuting pair representative: $(\theta, \text{Inn}_{L_{n,-1}})$.

– $\frac{n}{2} = 4l + 2$ There are two isomorphism classes of pairs, their representatives are: $(\theta, \text{Inn}_{L_{n,-1}})$ and $\left(\text{Inn}_{\begin{pmatrix} I_{n-2} & \\ & -I_2 \end{pmatrix}} \theta, \text{Inn}_{L_{n,-1}} \right)$.

– $\frac{n}{2} = 4l$ There are two isomorphism classes of pairs, their representatives are: $(\theta, \text{Inn}_{L_{n,-1}})$ and $\left(\text{Inn}_{\begin{pmatrix} I_{n-2} & \\ & -I_2 \end{pmatrix}} \theta, \text{Inn}_{L_{n,-1}} \right)$.

Proof. We start by proving the case for $k = \mathbb{R}$. For $M = M_i = I_{\frac{n}{2}-i,i}$, $i \in \{1, 2, \dots, \lfloor \frac{n}{4} \rfloor\}$. We have,

$$\left(\text{Inn}_{\begin{pmatrix} I_{\frac{n}{2}-i,i} & 0 \\ 0 & I_{\frac{n}{2}-i,i} \end{pmatrix}} \theta, \text{Inn}_{J_n} \right) \approx \left(\text{Inn}_{\begin{pmatrix} I_{\frac{n}{2}-2i,2i} & 0 \\ 0 & -I_{2i} \end{pmatrix}} \theta, \text{Inn}_{L_{n,-1}} \right),$$

for $2i \in \{1, 2, \dots, \frac{n}{2}\}$, but for such i we get distinct semi-congruence class elements $\left(\text{Inn}_{\begin{pmatrix} I_{\frac{n}{2}-2i,2i} & 0 \\ 0 & -I_{2i} \end{pmatrix}} \theta, \text{Inn}_{L_{n,-1}} \right)$, thus the above pairs are in distinct isomorphism classes for $i \in \{1, 2, \dots, \lfloor \frac{n}{4} \rfloor\}$.

We now consider the case for $k = \mathbb{F}_p$. Since there are at most two semi-congruence classes of bilinear forms of \mathbb{F}_p^n it is sufficient to show that the pair $(\theta, \text{Inn}_{J_n})$ is

isomorphic to the pair $\left(\text{Inn} \begin{pmatrix} M_{1,1,s_p} & 0 \\ 0 & M_{1,1,s_p} \end{pmatrix} \theta, \text{Inn}_{J_n} \right)$. The latter is isomorphic to

$$\begin{aligned}
\left(\text{Inn} \begin{pmatrix} I_{n-2} & 0 \\ 0 & s_p I_2 \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right) &\approx \left(\text{Inn} \begin{pmatrix} I_{n-2} & 0 \\ 0 & s_p \begin{pmatrix} a & b \\ -b & a \end{pmatrix}^T \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right) \\
&= \left(\text{Inn} \begin{pmatrix} I_{n-2} & 0 \\ 0 & \begin{pmatrix} s_p^2 & 0 \\ 0 & s_p^2 \end{pmatrix} \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right) \\
&\approx \left(\text{Inn} \begin{pmatrix} I_{n-2} & 0 \\ 0 & I_2 \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right) \\
&\approx \left(\text{Inn} \begin{pmatrix} I_{n-2} & 0 \\ 0 & I_2 \end{pmatrix} \theta, \text{Inn}_{J_n} \right) \\
&= (\theta, \text{Inn}_{J_n}),
\end{aligned}$$

for $a, b \in \mathbb{F}_p$ such that $a^2 + b^2 = s_p$ (which we know exist by lemma 5.3.6). Also, the last isomorphism follows from using Inn_{P_n} , an inner automorphism defined by a permutation matrix, so that $P_n^T P_n = I_n$, and $P_n L_{n,-1} P_n^{-1} = J_n$, as shown in the proof of lemma 4.1.7.

Next we consider the p -adics for $p \neq 2$, considering case by case the size of n . First, suppose $\frac{n}{2} = 4l + 1$. If $-1 \notin \mathbb{Q}_p^{*2}$ then there is only one semi-congruence class and thus only one isomorphism class of commuting involutions of the above form. If $-1 \in \mathbb{Q}_p^{*2}$ then there are two semi-congruence classes given by I_n and M_{p,s_p,ps_p} . In this case, $(\theta, \text{Inn}_{J_n})$, is isomorphic to $\left(\text{Inn} \begin{pmatrix} M_{p,s_p,ps_p} & 0 \\ 0 & M_{p,s_p,ps_p} \end{pmatrix} \theta, \text{Inn}_{J_n} \right)$. This is true for the following reasons: first, the latter of these pairs is isomorphic to

$$\left(\text{Inn} \begin{pmatrix} I_{n-4} & 0 \\ 0 & p I_4 \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right)$$

because, using the inner automorphism defined by the matrix P_n (in lemma 4.1.7),

$$\begin{aligned}
& \left(\text{Inn} \begin{pmatrix} M_{p,s_p,ps_p} & 0 \\ 0 & M_{p,s_p,ps_p} \end{pmatrix} \theta, \text{Inn}_{J_n} \right) \\
& \approx \left(\text{Inn} \begin{pmatrix} I_{n-6} & & 0 \\ & pI_2 & \\ 0 & & s_p I_2 \\ & & & ps_p I_2 \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right) \\
& \approx \left(\text{Inn} \begin{pmatrix} I_{n-6} & & & 0 \\ & pI_2 & & \\ & & s_p \begin{pmatrix} a & b \\ -b & a \end{pmatrix}^T \begin{pmatrix} a & b \\ -b & a \end{pmatrix} & \\ 0 & & & ps_p \begin{pmatrix} a & b \\ -b & a \end{pmatrix}^T \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right) \\
& = \left(\text{Inn} \begin{pmatrix} I_{n-6} & & & 0 \\ & pI_2 & & \\ & & s_p(a^2+b^2)I_2 & \\ 0 & & & ps_p(a^2+b^2)I_2 \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right) \\
& = \left(\text{Inn} \begin{pmatrix} I_{n-6} & & 0 & \\ & \begin{pmatrix} pI_2 & 0 \\ 0 & I_2 \end{pmatrix} & & \\ 0 & & pI_2 & \\ & & & pI_2 \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right) \\
& \approx \left(\text{Inn} \begin{pmatrix} I_{n-6} & & 0 & \\ & \begin{pmatrix} I_2 & 0 \\ 0 & pI_2 \end{pmatrix} & & \\ 0 & & pI_2 & \\ & & & pI_2 \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right)
\end{aligned}$$

since $(s_p, s_p)_p = 1$ and thus, by lemma 5.3.3, there is a solution to $x^2 + y^2 = \frac{1}{s_p}$ over \mathbb{Q}_p , i.e. there elements $a, b \in \mathbb{Q}_p$ such that the 2×2 block $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, which commutes with $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, is such that $A^T A = \frac{1}{s_p} I_2$. Note that the last isomorphism of pairs follows from using the inner involution defined by the permutation matrix

$$P = \sum_{k=1}^{n-6} E_{k,k} + \sum_{k=n-5}^{n-4} (E_{k,k+2} + E_{k+2,k}) + E_{n-1,n-1} + E_{n-2,n-2}.$$

Thus, it remains to show the following isomorphism because it has already been shown that the pairs $(\theta, \text{Inn}_{L_{n,-1}})$ and $(\theta, \text{Inn}_{J_n})$ are isomorphic (via the inner automorphism defined by the permutation matrix P_n as in lemma 4.1.7):

Claim 5.3.12. *The following are isomorphic for all n :*

$$\left(\text{Inn} \begin{pmatrix} I_{n-4} & 0 \\ 0 & pI_4 \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right), \text{ and } \left(\theta, \text{Inn}_{L_{n,-1}} \right)$$

Proof. Since the Hilbert symbol $\left(\frac{1}{s_p}, \frac{1}{s_p} \right)_p = (s_p, s_p)_p = 1$, by lemma 5.3.3 there exists an $a, b \in \mathbb{Q}_p$ such that $a^2 + b^2 = s_p$ so that $(\theta, \text{Inn}_{L_{n,-1}})$ is isomorphic to

$(\text{Inn}_{M_1, s_p, s_p} \theta, \text{Inn}_{L_{n, -1}})$ by using the block $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ in the appropriate place:

$$\begin{aligned}
\theta &= \text{Inn}_{I_n} \theta \\
&\approx \text{Inn} \begin{pmatrix} I_{n-2} & 0 \\ 0 & \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \end{pmatrix} \text{Inn}_{I_n} \theta \text{Inn}^{-1} \begin{pmatrix} I_{n-2} & 0 \\ 0 & \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \end{pmatrix} \\
&= \text{Inn} \begin{pmatrix} I_{n-2} & 0 \\ 0 & \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \end{pmatrix} \text{Inn}_{I_n} \text{Inn} \begin{pmatrix} I_{n-2} & 0 \\ 0 & \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \end{pmatrix}^T \theta \\
&= \text{Inn} \begin{pmatrix} I_{n-2} & 0 \\ 0 & \begin{pmatrix} a & b \\ -b & a \end{pmatrix}^T \end{pmatrix} \begin{pmatrix} I_{n-2} & 0 \\ 0 & \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \end{pmatrix} \theta \\
&= \text{Inn} \begin{pmatrix} I_{n-2} & 0 \\ 0 & \begin{pmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{pmatrix} \end{pmatrix} \theta = \text{Inn}_{M_1, s_p, s_p} \theta
\end{aligned}$$

and noting that the block $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ commutes with the block $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. After using the inner automorphism defined by the permutation matrix:

$$P = \begin{pmatrix} I_{n-4} & 0 \\ 0 & P_4 \end{pmatrix}, P_4 = E_{1,1} + E_{2,3} + E_{3,2} + E_{4,4},$$

the commuting pair, $(\text{Inn}_{M_1, s_p, s_p} \theta, \text{Inn}_{L_{n, -1}})$, is isomorphic to

$$\begin{aligned}
&\left(\text{Inn} \begin{pmatrix} I_{n-4} & 0 \\ 0 & \begin{pmatrix} 1 & & & \\ & s_p & & \\ & & 1 & \\ & & & s_p \end{pmatrix} \end{pmatrix} \theta, \text{Inn} \begin{pmatrix} L_{n-4, -1} & 0 \\ 0 & J_4 \end{pmatrix} \right) \\
&\approx \left(\text{Inn} \begin{pmatrix} I_{n-4} & 0 \\ 0 & Q'^T \begin{pmatrix} 1 & & & \\ & s_p & & \\ & & 1 & \\ & & & s_p \end{pmatrix} Q' \end{pmatrix} \theta, \text{Inn} \begin{pmatrix} L_{n-4, -1} & 0 \\ 0 & Q'^{-1} J_4 Q' \end{pmatrix} \right) \\
&= \left(\text{Inn} \begin{pmatrix} I_{n-4} & 0 \\ 0 & p \begin{pmatrix} 1 & & & \\ & s_p & & \\ & & 1 & \\ & & & s_p \end{pmatrix} \end{pmatrix} \theta, \text{Inn} \begin{pmatrix} L_{n-4, -1} & 0 \\ 0 & J_4 \end{pmatrix} \right),
\end{aligned}$$

for $x^2 + \frac{1}{s_p} y^2 = p$, which exists by lemma 5.3.3 since $\left(\frac{1}{p}, \frac{1}{s_p} \frac{1}{p} \right)_p = 1$ for -1 not a square

modulo p , and

$$Q' = \begin{pmatrix} x & y & 0 & 0 \\ -\frac{1}{s_p}y & x & 0 & 0 \\ 0 & 0 & x & y \\ 0 & 0 & -\frac{1}{s_p}y & x \end{pmatrix}$$

The isomorphism then follows because $(s_p, s_p)_p = 1$, and hence by lemma 5.3.3, there exists a solution over \mathbb{Q}_p to $s_p x^2 + s_p y^2 = 1$ and hence the above pair, which is isomorphic to the pair $\left(\text{Inn} \begin{pmatrix} I_{n-4} & 0 \\ 0 & p \begin{pmatrix} I_2 & 0 \\ 0 & s_p I_2 \end{pmatrix} \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right)$ via the inner automorphism defined by P (given above), is isomorphic to $\left(\text{Inn} \begin{pmatrix} I_{n-4} & 0 \\ 0 & p I_4 \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right)$.

If -1 is a square modulo p then $(p, p)_p = 1$ which implies by lemma 5.3.3 that there is a solution in \mathbb{Q}_p to $a'^2 + b'^2 = p$ so that

$$\begin{aligned} \left(\text{Inn} \begin{pmatrix} I_{n-4} & 0 \\ 0 & p I_4 \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right) &\approx \left(\text{Inn} \begin{pmatrix} I_{n-4} & 0 \\ 0 & p Q''^T I_4 Q'' \end{pmatrix} \theta, \text{Inn}_{Q''^{-1} L_{n,-1} Q''} \right) \\ &= \left(\text{Inn} \begin{pmatrix} I_{n-4} & 0 \\ 0 & I_4 \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right) \\ &= \left(\theta, \text{Inn}_{L_{n,-1}} \right) \text{ for } Q'' = \begin{pmatrix} \begin{pmatrix} a' & b' \\ -b' & a' \end{pmatrix} & \\ & \begin{pmatrix} a' & b' \\ -b' & a' \end{pmatrix} \end{pmatrix}. \end{aligned}$$

□

Therefore, by this claim, there is only one isomorphism class for commuting pairs of involutions of $\text{SL}(n, \mathbb{Q}_p)$ when $\frac{n}{2} = 4l + 1$. Next, suppose $\frac{n}{2} = 4l + 3$. There are two semi-congruence classes given by $I_{\frac{n}{2}}$ and $M_{p, s_p, p s_p}$, but by claim 5.3.12 and the arguments preceding it, there is only one isomorphism class of commuting involutions of $\text{SL}(n, \mathbb{Q}_p)$ for $\frac{n}{2} = 4l + 3$ of the form above. If $\frac{n}{2} = 4l + 2$, we consider first the case that $-1 \notin \mathbb{Q}_p^{*2}$. It is known from [HWD06] that there are five semi-congruence classes given by: $I_{\frac{n}{2}}, M_{1, p, p s_p}, M_{1, 1, p}, M_{1, 1, s_p}, M_{1, 1, p s_p}$. The corresponding pairs are:

$$\left(\text{Inn} \begin{pmatrix} I_{n-4} & 0 \\ 0 & \begin{pmatrix} r I_2 & 0 \\ 0 & s I_2 \end{pmatrix} \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right),$$

for $(r, s) = (1, 1), (p, p s_p), (1, p), (1, s_p), (1, p s_p)$, respectively. Since there is a solution to $a^2 + b^2 = \frac{1}{s_p}$ over the p -adics, as above, these pairs are isomorphic, by lemma 5.3.3

and the techniques above, to pairs of the form

$$\left(\text{Inn} \begin{pmatrix} I_{n-4} & 0 \\ 0 & \begin{pmatrix} uI_2 & 0 \\ 0 & vI_2 \end{pmatrix} \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right),$$

for $(u, v) = (1, 1), (p, p), (1, p), (1, 1), (1, p)$, respectively. But, by claim 5.3.12 this leaves us with two pairs: $(\theta, \text{Inn}_{L_{n,-1}})$ and $\left(\text{Inn} \begin{pmatrix} I_{n-4} & 0 \\ 0 & \begin{pmatrix} I_2 & 0 \\ 0 & pI_2 \end{pmatrix} \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right)$.

Claim 5.3.13. *For $\frac{n}{2}$ even, the following are not isomorphic for -1 not a square modulo p : $(\theta, \text{Inn}_{L_{n,-1}})$ and $\left(\text{Inn} \begin{pmatrix} I_{n-4} & 0 \\ 0 & \begin{pmatrix} I_2 & 0 \\ 0 & pI_2 \end{pmatrix} \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right)$.*

Proof. It is sufficient to show that for $\frac{n}{2}$ even, $\left(\begin{pmatrix} I_{n-4} & 0 \\ 0 & \begin{pmatrix} I_2 & 0 \\ 0 & pI_2 \end{pmatrix} \end{pmatrix}$ and I_n are not semi-congruent, for -1 not a square modulo p , and hence the corresponding outer involutions cannot be isomorphic and therefore, neither can the pairs. Since their determinants differ by a square we need to show, by lemma 5.3.6, that for any $\omega \in \mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$, $c_p(\omega I_n) \neq c_p \begin{pmatrix} I_{n-2} & 0 \\ 0 & pI_2 \end{pmatrix}$. But for $\frac{n}{2}$ even,

$$c_p(\omega I_n) = (\omega, -1^{\frac{n}{2}})_p c_p(I_n) = (\omega, 1)_p c_p(I_n) = c_p(I_n) = (-1, -1)_p.$$

However,

$$c_p \begin{pmatrix} I_{n-2} & 0 \\ 0 & pI_2 \end{pmatrix} = (-1, -1)_p (p, p)_p = -(-1, -1)_p$$

for -1 not a square modulo p . Therefore, since $\begin{pmatrix} I_{n-2} & 0 \\ 0 & pI_2 \end{pmatrix}$ is not congruent to any multiple of I_n for -1 not a square modulo p , the two are not semi-congruent, and the claim follows. \square

Now, if -1 is a square modulo p then, for $\frac{n}{2}$ even, the pairs $(\theta, \text{Inn}_{L_{n,-1}})$, and $\left(\text{Inn} \begin{pmatrix} I_{n-2} & 0 \\ 0 & pI_2 \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right)$ are isomorphic. This follows from the fact that if -1 is a square modulo p then $(p, p)_p = 1$ and thus there is a solution to $a^2 + b^2 = p$ over the p -adics, by lemma 5.3.3, and thus

$$\begin{aligned} \left(\text{Inn} \begin{pmatrix} I_{n-2} & 0 \\ 0 & pI_2 \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right) &\approx \left(\text{Inn} \begin{pmatrix} Q^T I_{n-2} Q & 0 \\ 0 & pI_2 \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right) \\ &= \left(\text{Inn} \begin{pmatrix} pI_{n-2} & 0 \\ 0 & pI_2 \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right) \\ &= (\theta, \text{Inn}_{L_{n,-1}}) \end{aligned}$$

for

$$Q = \begin{pmatrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \end{pmatrix}.$$

Therefore, for $\frac{n}{2} = 4l + 2$ there are two isomorphism classes if -1 is not a square modulo p and one isomorphism class if -1 is a square modulo p . Lastly, we consider when $\frac{n}{2} = 4l$. From [HWD06], it is known that there are five semi-congruence classes (of symmetric bilinear forms) given by: $I_{\frac{n}{2}}, M_{p, s_p, ps_p}, M_{1,1,p}, M_{1,1,s_p}, M_{1,1,ps_p}$. And by the arguments above (including lemma 5.3.12) this leads to two potential isomorphism classes: $(\theta, \text{Inn}_{L_{n,-1}})$, and $(\text{Inn} \begin{pmatrix} I_{n-4} & 0 \\ 0 & \begin{pmatrix} I_2 & 0 \\ 0 & pI_2 \end{pmatrix} \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}})$. As argued above, these two are in distinct isomorphism classes for -1 not a square modulo p and are isomorphic for -1 a square modulo p . Finally, we consider the 2-adics considering case-by-case the size of $\frac{n}{2}$.

Finally, we consider the 2-adics, case by case on the size of $\frac{n}{2}$. First we need a lemma:

Lemma 5.3.14. *The following commuting pairs are isomorphic: $(\theta, \text{Inn}_{L_{n,-1}})$, and $(\text{Inn} \begin{pmatrix} M_{2,3,6} & 0 \\ 0 & M_{2,3,6} \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}})$.*

Proof.

$$\begin{aligned} (\theta, \text{Inn}_{L_{n,-1}}) &= \left(\text{Inn} \begin{pmatrix} I_{n-4} & 0 \\ 0 & I_4 \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right) \\ &\approx \left(\text{Inn} \begin{pmatrix} I_{n-4} & 0 \\ 0 & \begin{pmatrix} I_2 & 0 \\ 0 & -6I_2 \end{pmatrix} \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right), \end{aligned}$$

via $\text{Inn} \begin{pmatrix} I_{n-2} & 0 \\ 0 & \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \end{pmatrix}$ for $a^2 + b^2 = -6$, which we know has a solution over \mathbb{Q}_2 by lemma 5.3.3 since $(-6, -6)_2 = 1$. The above pair is isomorphic to:

$$\begin{aligned} &\left(\text{Inn} \begin{pmatrix} I_{n-4} & & 0 \\ & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -6 \end{pmatrix} & \\ 0 & & & \end{pmatrix} \theta, \text{Inn} \begin{pmatrix} L_{n-4,-1} & 0 \\ 0 & \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \end{pmatrix} \right) \\ &\approx \left(\text{Inn} \begin{pmatrix} I_{n-4} & & 0 \\ & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -6 \end{pmatrix} & \\ 0 & -2 & & \end{pmatrix} \theta, \text{Inn} \begin{pmatrix} L_{n-4,-1} & 0 \\ 0 & \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \end{pmatrix} \right) \end{aligned}$$

via

$$\text{Inn} \begin{pmatrix} I_{n-4} & 0 \\ 0 & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{pmatrix} \quad \text{and} \quad \text{Inn} \begin{pmatrix} I_{n-4} & 0 \\ 0 & \begin{pmatrix} a' & b' & 0 & 0 \\ \frac{1}{6}b' & a' & 0 & 0 \\ 0 & 0 & a' & b' \\ 0 & 0 & \frac{1}{6}b' & a' \end{pmatrix} \end{pmatrix},$$

respectively, for $a'^2 - \frac{1}{6}b'^2 = -2$, which we know exists, by lemma 5.3.3 because $(-\frac{1}{2}, \frac{1}{2}\frac{1}{6})_2 = 1$. The above gives:

$$\begin{aligned} (\theta, \text{Inn}_{L_{n,-1}}) &\approx \left(\text{Inn} \begin{pmatrix} I_{n-4} & 0 \\ 0 & -2 \begin{pmatrix} I_2 & 0 \\ 0 & -6I_2 \end{pmatrix} \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right) \text{ via } \text{Inn} \begin{pmatrix} I_{n-4} & 0 \\ 0 & P_4 \end{pmatrix} \\ &\approx \left(\text{Inn} \begin{pmatrix} I_{n-4} & 0 \\ 0 & -2I_4 \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right) \text{ via } \text{Inn} \begin{pmatrix} I_{n-2} & 0 \\ 0 & \begin{pmatrix} a'' & b'' \\ -b'' & a'' \end{pmatrix} \end{pmatrix} \\ &\approx \left(\text{Inn} \begin{pmatrix} I_{n-4} & 0 \\ 0 & -I_4 \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right) \text{ via } \text{Inn} \begin{pmatrix} I_{n-2} & 0 \\ 0 & \begin{pmatrix} a''' & b''' \\ -b''' & a''' \end{pmatrix} \end{pmatrix} \\ &= \left(\text{Inn} \begin{pmatrix} I_{n-6} & 0 \\ 0 & \begin{pmatrix} I_2 & 0 & 0 \\ 0 & -I_2 & 0 \\ 0 & 0 & -I_2 \end{pmatrix} \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right) \\ &\approx \left(\text{Inn} \begin{pmatrix} I_{n-6} & 0 \\ 0 & \begin{pmatrix} 2I_2 & 0 & 0 \\ 0 & 3I_2 & 0 \\ 0 & 0 & 6I_2 \end{pmatrix} \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right) \text{ via } \text{Inn} \begin{pmatrix} I_{n-6} & 0 \\ 0 & A \end{pmatrix} \\ &\quad \text{for } A = \begin{pmatrix} \begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{pmatrix} & & \\ & \begin{pmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{pmatrix} & \\ & & \begin{pmatrix} x_3 & y_3 \\ -y_3 & x_3 \end{pmatrix} \end{pmatrix} \\ &\approx \left(\text{Inn} \begin{pmatrix} M_{2,3,6} & 0 \\ 0 & M_{2,3,6} \end{pmatrix} \theta, \text{Inn}_{J_n} \right) \text{ via } \text{Inn}_{P_n}. \end{aligned}$$

These isomorphisms follow because $(-6, -6)_2 = (-\frac{1}{6}, -\frac{1}{6})_2 = 1$, $(2, 2)_2 = (\frac{1}{2}, \frac{1}{2})_2 = 1$, and $(-3, -3)_2 = (-\frac{1}{3}, -\frac{1}{3})_2 = 1$, thus by lemma 5.3.3, there exist the following

solutions, over \mathbb{Q}_2 :

$$\begin{aligned} -6a''^2 - 6b''^2 &= 1 \\ 2a'''^2 + 2b'''^2 &= 1 \\ x_1^2 + y_1^2 &= 2 \\ -x_2^2 - y_2^2 &= 3 \\ -x_3^2 - y_3^2 &= 6 \end{aligned}$$

□

This immediately implies that there is only one isomorphism class of commuting involutions (of the above form) of $\mathrm{SL}(n, \mathbb{Q}_2)$ when $\frac{n}{2}$ is odd. Next, we consider (for the 2-adics) when $\frac{n}{2} = 4l$. By lemma 5.3.14 we need to consider only whether the pairs $(\mathrm{Inn}_{M_{1,q,q}} \theta, \mathrm{Inn}_{L_{n,-1}})$, for $q \in \{1, -1, 2, -2, 3, -3, 6, -6\}$ are distinct, since

$$\left(\mathrm{Inn} \begin{pmatrix} M_{1,1,q} & 0 \\ 0 & M_{1,1,q} \end{pmatrix} \theta, \mathrm{Inn}_{J_n} \right) \approx \left(\mathrm{Inn}_{M_{1,q,q}} \theta, \mathrm{Inn}_{L_{n,-1}} \right).$$

Computing the Hilbert symbol for each of the $q \in \{1, -1, 2, -2, 3, -3, 6, -6\}$, gives that $(q, q)_2 = 1$ or $(-q, -q) = 1$. Therefore, by lemma 5.3.3 there exists a solution over \mathbb{Q}_p to $qa^2 + qb^2 = 1$ or $-qa^2 - qb^2 = 1$, hence using the inner automorphism $\mathrm{Inn} \begin{pmatrix} I_{n-2} & 0 \\ 0 & \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \end{pmatrix}$ on the pair $(\mathrm{Inn}_{M_{1,q,q}} \theta, \mathrm{Inn}_{L_{n,-1}})$ we need to determine

only whether or not $(\theta, \mathrm{Inn}_{L_{n,-1}})$ is isomorphic to $(\mathrm{Inn}_{M_{1,-1,-1}} \theta, \mathrm{Inn}_{L_{n,-1}})$.

Claim 5.3.15. $M_{1,-1,-1}$ and I_n , as matrix representatives of symmetric bilinear forms, are not semi-congruent for $\frac{n}{2}$ even.

Proof. $c_p(\alpha I_n) = (\alpha, -1^{\frac{n}{2}})_2 c_p(I_n) = (-1, -1)_2 = -1$ but $c_p(M_{1,-1,-1}) = (-1, -1)_2 \cdot 1 \cdots 1 \cdot (-1, -1)_2 = (-1, -1)_2^2 = 1$, therefore by lemma 5.3.6, the two matrices cannot be semi-congruent, because $M_{1,-1,-1}$ is not congruent to any multiple of I_n . □

This immediately implies that the commuting pair $(\theta, \mathrm{Inn}_{L_{n,-1}})$ is not isomorphic to the commuting pair $(\mathrm{Inn}_{M_{1,-1,-1}} \theta, \mathrm{Inn}_{L_{n,-1}})$ because by [HWD06] $M_{1,-1,-1} \not\cong_s I_n$

implies that the outer involutions $\text{Inn}_{M_{1,-1,-1}} \theta$ and θ are not isomorphic; hence, there are two isomorphism classes of commuting pairs of involutions of $\text{SL}(n, \mathbb{Q}_2)$ for $\frac{n}{2} = 4l$. Also, there are two isomorphism classes of commuting pairs of involutions of $\text{SL}(n, \mathbb{Q}_2)$ for $\frac{n}{2} = 4l + 2$ by the above lemma and the fact that

$$\begin{aligned} \left(\text{Inn} \begin{pmatrix} M_{2,3,-6} & 0 \\ 0 & M_{2,3,-6} \end{pmatrix} \theta, \text{Inn}_{J_n} \right) &\approx \left(\text{Inn} \begin{pmatrix} I_{n-6} & 0 \\ 0 & \begin{pmatrix} 2I_2 & 0 & 0 \\ 0 & 3I_2 & 0 \\ 0 & 0 & -6I_2 \end{pmatrix} \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right) \\ &\approx \left(\text{Inn} \begin{pmatrix} I_{n-6} & 0 \\ 0 & \begin{pmatrix} I_2 & 0 & 0 \\ 0 & -I_2 & 0 \\ 0 & 0 & I_2 \end{pmatrix} \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right) \\ &\approx \left(\text{Inn} \begin{pmatrix} I_{n-4} & 0 \\ 0 & -I_2 \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}} \right), \end{aligned}$$

the first isomorphism follows from using the inner automorphism Inn_{P_n} , the second from the proof of lemma 5.3.14, lemma 5.3.3 and $(2, 2)_2, (-6, -6)_2$ and $(-3, -3)_2 = 1$. The third isomorphism follows from the inner automorphism Inn_P , for P the permutation matrix: $\sum_{i=1}^{n-4} E_{i,i} + \sum_{i=n-3}^n E_{i,i+2} + E_{i+2,i}$. \square

To complete the classification of commuting pairs (one outer, one inner) of involutions of $\text{SL}(n, k)$, we need to consider those of type (4) from corollary 5.3.1, namely, those of the form:

$$\left(\text{Inn} \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \theta, \text{Inn}_{J_n} \right),$$

for D a diagonal matrix. Since $\text{Inn} \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ preserves Inn_{J_n} , we can take D to be a matrix representative of a semi-congruence class of symmetric bilinear form of $k^{\frac{n}{2}}$, i.e. take $A = Q$ and $B = 0$ for Q such that $Q^T D Q = \alpha M$ for M a matrix representative of a semi-congruence class of bilinear forms. However, this is equivalent to classifying outer/inner commuting pairs of type (3) in corollary 5.3.1:

Lemma 5.3.16. *The classification of each of the commuting pairs of involutions of the forms below is equivalent:*

$$\left(\text{Inn} \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} \theta, \text{Inn}_{J_n} \right), \left(\text{Inn} \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \theta, \text{Inn}_{J_n} \right).$$

Proof. First, suppose $\left(\text{Inn}_{\begin{pmatrix} D_1 & 0 \\ 0 & D_1 \end{pmatrix}} \theta, \text{Inn}_{J_n}\right) \approx \left(\text{Inn}_{\begin{pmatrix} D_2 & 0 \\ 0 & D_2 \end{pmatrix}} \theta, \text{Inn}_{J_n}\right)$. This implies there exists a nonsingular $X \in \text{GL}(n, k)$ such that $X^{-1}J_nX = \alpha J_n$ and $X^T \begin{pmatrix} D_1 & 0 \\ 0 & D_1 \end{pmatrix} X = \beta \begin{pmatrix} D_2 & 0 \\ 0 & D_2 \end{pmatrix}$ for $\alpha, \beta \in k^*$. This, in turn, implies that

$$\begin{aligned} \left(\text{Inn}_{\begin{pmatrix} 0 & D_1 \\ -D_1 & 0 \end{pmatrix}} \theta, \text{Inn}_{J_n}\right) &= \left(\text{Inn}_{J_n} \begin{pmatrix} D_1 & 0 \\ 0 & D_1 \end{pmatrix} \theta, \text{Inn}_{J_n}\right) \\ &\approx \left(\text{Inn}_{X^T J_n \begin{pmatrix} D_1 & 0 \\ 0 & D_1 \end{pmatrix} X} \theta, \text{Inn}_{X^{-1} J_n X}\right) \\ &= \left(\text{Inn}_{\alpha J_n X^T \begin{pmatrix} D_1 & 0 \\ 0 & D_1 \end{pmatrix} X} \theta, \text{Inn}_{J_n}\right) \\ &= \left(\text{Inn}_{\beta J_n \begin{pmatrix} D_2 & 0 \\ 0 & D_2 \end{pmatrix}} \theta, \text{Inn}_{J_n}\right) \\ &= \left(\text{Inn}_{\begin{pmatrix} 0 & D_2 \\ -D_2 & 0 \end{pmatrix}} \theta, \text{Inn}_{J_n}\right). \end{aligned}$$

Equivalence follows from using the above argument in reverse: Suppose that

$$\left(\text{Inn}_{\begin{pmatrix} 0 & D_1 \\ -D_1 & 0 \end{pmatrix}} \theta, \text{Inn}_{J_n}\right) \approx \left(\text{Inn}_{\begin{pmatrix} 0 & D_2 \\ -D_2 & 0 \end{pmatrix}} \theta, \text{Inn}_{J_n}\right).$$

This implies there exists a nonsingular $X \in \text{GL}(n, k)$ such that $X^{-1}J_nX = \alpha J_n$ and $X^T \begin{pmatrix} 0 & D_1 \\ -D_1 & 0 \end{pmatrix} X = \beta \begin{pmatrix} 0 & D_2 \\ -D_2 & 0 \end{pmatrix}$ for $\alpha, \beta \in k^*$. This, in turn, implies that

$$\begin{aligned} \left(\text{Inn}_{\begin{pmatrix} D_1 & 0 \\ 0 & D_1 \end{pmatrix}} \theta, \text{Inn}_{J_n}\right) &= \left(\text{Inn}_{J_n} \begin{pmatrix} 0 & D_1 \\ -D_1 & 0 \end{pmatrix} \theta, \text{Inn}_{J_n}\right) \\ &\approx \left(\text{Inn}_{X^T J_n \begin{pmatrix} 0 & D_1 \\ -D_1 & 0 \end{pmatrix} X} \theta, \text{Inn}_{X^{-1} J_n X}\right) \\ &= \left(\text{Inn}_{\alpha J_n X^T \begin{pmatrix} 0 & D_1 \\ -D_1 & 0 \end{pmatrix} X} \theta, \text{Inn}_{J_n}\right) \\ &= \left(\text{Inn}_{\beta J_n \begin{pmatrix} 0 & D_2 \\ -D_2 & 0 \end{pmatrix}} \theta, \text{Inn}_{J_n}\right) \\ &= \left(\text{Inn}_{\begin{pmatrix} D_2 & 0 \\ 0 & D_2 \end{pmatrix}} \theta, \text{Inn}_{J_n}\right). \end{aligned}$$

□

Chapter 6

Classification of Commuting Pairs of Outer Involutions

In this chapter we consider isomorphism classes of commuting involutions of the group $\mathrm{SL}(n, k)$ such that both involutions in the pair are outer involutions. We will see that the classification of such pairs is equivalent to classifying inner/outer pairs.

Lemma 6.0.17. *The pair: $(\mathrm{Inn}_M \theta, \mathrm{Inn}_{M'} \theta)$ is a commuting pair of involutions of $\mathrm{SL}(n, k)$ if and only if $(\mathrm{Inn}_M \theta, \mathrm{Inn}_{M^{-1}M'})$ is a commuting pair of involutions of $\mathrm{SL}(n, k)$.*

Proof. First, suppose that $(\mathrm{Inn}_M \theta, \mathrm{Inn}_{M'} \theta)$ is a commuting pair of involutions of $\mathrm{SL}(n, k)$. Then, M is a matrix representative of a nondegenerate symmetric or skew-symmetric bilinear form on k^n , and M' is a matrix representative of a nondegenerate symmetric or skew-symmetric bilinear form on k^n . Also, because the above is a

commuting pair we have:

$$\begin{aligned}
& \text{Inn}_{M'} \theta \text{Inn}_M \theta = \text{Inn}_M \theta \text{Inn}_{M'} \theta \\
& \rightarrow \theta \text{Inn}_{M'^{T-1}} \text{Inn}_M \theta = \theta \text{Inn}_{M^{T-1}} \text{Inn}_{M'} \theta \\
& \rightarrow \text{Inn}_{M'^{T-1}} \text{Inn}_M = \text{Inn}_{M^{T-1}} \text{Inn}_{M'} \\
& \rightarrow \text{Inn}_{MM'^{T-1}} = \text{Inn}_{M'M^{T-1}} \\
& \rightarrow MM'^{T-1} = \alpha M' M^{T-1} \\
& \rightarrow MM'^{-1} = \pm \alpha M' M^{-1} \\
& \rightarrow MM'^{-1} M = \pm \alpha M' \\
& \rightarrow \text{Inn}_{MM'^{-1}M} = \text{Inn}_{M'},
\end{aligned}$$

for $\alpha \in k^*$. This gives the following:

$$\begin{aligned}
\text{Inn}_{M^{-1}M'} \text{Inn}_M \theta &= \text{Inn}_{MM^{-1}M'} \theta \\
&= \text{Inn}_{M'} \theta \\
&= \text{Inn}_{MM'^{-1}M} \theta \\
&= \text{Inn}_M \text{Inn}_{MM'^{-1}} \theta \\
&= \text{Inn}_M \theta \text{Inn}_{M^{-1}M'},
\end{aligned}$$

which implies that the automorphism $\text{Inn}_{M^{-1}M'}$ commutes with the involution $\text{Inn}_M \theta$.

The commuting pair also gives:

$$\begin{aligned}
& \text{Inn}_{M'} \theta \text{Inn}_M \theta = \text{Inn}_M \theta \text{Inn}_{M'} \theta \\
& \rightarrow \text{Inn}_{M'} \theta \theta \text{Inn}_{M^{T-1}} = \text{Inn}_M \theta \theta \text{Inn}_{M'^{T-1}} \\
& \rightarrow \text{Inn}_{M'} \text{Inn}_{M^{T-1}} = \text{Inn}_M \text{Inn}_{M'^{T-1}} \\
& \rightarrow \text{Inn}_{M^{T-1}M'} = \text{Inn}_{M'^{T-1}M} \\
& \rightarrow M^{T-1} M' = \beta M'^{T-1} M \\
& \rightarrow M^{-1} M' = \pm \beta M'^{-1} M \\
& \rightarrow (M^{-1} M')^2 = \pm \beta I_n
\end{aligned}$$

and this implies that $\text{Inn}_{M^{-1}M'}$ is an involution of $\text{SL}(n, k)$. Therefore, if the pair

$(\text{Inn}_M \theta, \text{Inn}_{M'} \theta)$ is a commuting pair of involutions of $\text{SL}(n, k)$ then the above shows that $(\text{Inn}_M \theta, \text{Inn}_{M^{-1}M'})$ is a commuting pair of involutions of $\text{SL}(n, k)$.

Next, suppose $(\text{Inn}_M \theta, \text{Inn}_{M^{-1}M'})$ is a commuting pair of involutions of $\text{SL}(n, k)$. Then M is a matrix representative of a nondegenerate symmetric or skew-symmetric bilinear form on k^n , and $\text{Inn}_{M^{-1}M'}$ is an involution, thus:

$$\begin{aligned} \text{Inn}_{M^{-1}M'}^2 &= Id \\ \rightarrow M^{-1}M'M^{-1}M' &= \alpha I_n \\ \rightarrow M^{-1}M'M^{-1} &= \alpha M'^{-1} \\ \rightarrow \alpha MM'^{-1}M &= M' \end{aligned}$$

for $\alpha \in k^*$. Also, because the above pair is a commuting pair, this gives:

$$\begin{aligned} \text{Inn}_{M^{-1}M'} \text{Inn}_M \theta &= \text{Inn}_M \theta \text{Inn}_{M^{-1}M'} \\ \rightarrow \text{Inn}_{M'} \theta &= \text{Inn}_{MM^{-1}M'} \theta = \text{Inn}_M \text{Inn}_{(M^{-1}M')^{T-1}} \theta \\ \rightarrow \text{Inn}_{M'} &= \text{Inn}_{(M^{-1}M')^{T-1}M} = \text{Inn}_{M^T M'^{T-1}M} = \text{Inn}_{MM'^{T-1}M} \\ \rightarrow M' &= \beta MM'^{T-1}M \end{aligned}$$

Therefore, this shows that $\alpha MM'^{-1}M = M' = \beta MM'^{T-1}M$, which implies that $M'^T = \alpha M^T M'^{T-1}M^T = \alpha MM'^{T-1}M = \alpha \frac{1}{\beta} M'$ so that $M' = \frac{\alpha}{\beta} M'^T$. Transposing both sides of the preceding equation gives $M'^T = \frac{\alpha}{\beta} M'$, so that $M' = \frac{\alpha}{\beta} M'^T = \frac{\alpha}{\beta} \frac{\alpha}{\beta} M' = \left(\frac{\alpha}{\beta}\right)^2 M'$. This implies that $\left(\frac{\alpha}{\beta}\right)^2 = 1$, which gives $M' = \pm M'^T$. This implies that M' is symmetric or skew-symmetric and, hence, $\text{Inn}_{M'} \theta$ is an involution. Furthermore,

$$\begin{aligned} \text{Inn}_{M'} \theta \text{Inn}_M \theta &= \theta \text{Inn}_{M'^{T-1}} \text{Inn}_M \theta \\ &= \theta \text{Inn}_{M'^{-1}} \text{Inn}_M \theta \\ &= \theta \text{Inn}_{MM'^{-1}} \theta \\ &= \theta \text{Inn}_{M'M^{-1}} \theta \\ &= \theta \text{Inn}_{M^{-1}} \text{Inn}_{M'} \theta \\ &= \text{Inn}_M \theta \text{Inn}_{M'} \theta, \end{aligned}$$

because, as shown above, M' is either symmetric or skew-symmetric and by supposition, $\text{Inn}_{M^{-1}M'}$ is an involution, so that $M'M^{-1} = \alpha MM'^{-1}$. Therefore, the involutions $\text{Inn}_M \theta$ and $\text{Inn}_{M'} \theta$ commute and thus $(\text{Inn}_M \theta, \text{Inn}_{M'} \theta)$ is a commuting pair of involutions of $\text{SL}(n, k)$, if $(\text{Inn}_M \theta, \text{Inn}_{M^{-1}M'})$ is a commuting pair of involutions of $\text{SL}(n, k)$. \square

To finish showing the equivalence of classification of inner/outer pairs with the classification of outer/outer pairs, the following lemma is needed:

Lemma 6.0.18. *The commuting pairs of outer involutions of $\text{SL}(n, k)$,*

$$(\text{Inn}_M \theta, \text{Inn}_{M_1} \theta) \text{ and } (\text{Inn}_M \theta, \text{Inn}_{M_2} \theta)$$

are isomorphic if and only if their corresponding outer/inner pairs are isomorphic, i.e., if and only if

$$(\text{Inn}_M \theta, \text{Inn}_{M^{-1}M_1}) \text{ and } (\text{Inn}_M \theta, \text{Inn}_{M^{-1}M_2})$$

are isomorphic.

Proof. First, suppose that $(\text{Inn}_M \theta, \text{Inn}_{M_1} \theta) \approx (\text{Inn}_M \theta, \text{Inn}_{M_2} \theta)$. Then there exists a $C \in \text{GL}(n, k)$ such that $C^T M C = \alpha M$ and $C^T M_1 C = \beta M_2$ which implies that $C^{-1} M^{-1} = \frac{1}{\alpha} M^{-1} C^T$ so that

$$\begin{aligned} (\text{Inn}_M \theta, \text{Inn}_{M^{-1}M_1}) &\approx (\text{Inn}_C \text{Inn}_M \theta \text{Inn}_{C^{-1}}, \text{Inn}_C \text{Inn}_{M^{-1}M_1} \text{Inn}_{C^{-1}}) \\ &= (\text{Inn}_{C^T M C} \theta, \text{Inn}_{C^{-1} M^{-1} M_1 C}) \\ &= (\text{Inn}_M \theta, \text{Inn}_{M^{-1} C^T M_1 C}) \\ &= (\text{Inn}_M \theta, \text{Inn}_{M^{-1} M_2}) \end{aligned}$$

Next, suppose that $(\text{Inn}_M \theta, \text{Inn}_{M^{-1}M_1}) \approx (\text{Inn}_M \theta, \text{Inn}_{M^{-1}M_2})$. Then there exists a $C' \in \text{GL}(n, k)$ such that $C'^T M C' = \alpha M$ and

$$\begin{aligned} C'^{-1} M^{-1} M_1 C' &= \beta M^{-1} M_2 \\ \rightarrow C'^{-1} M^{-1} &= \frac{1}{\alpha} M^{-1} C'^T \\ \rightarrow \frac{1}{\alpha} M^{-1} C'^T M_1 C' &= \beta M^{-1} M_2 \\ \rightarrow C'^T M_1 C' &= \alpha \beta M_2 \end{aligned}$$

so that

$$\begin{aligned}
(\text{Inn}_M \theta, \text{Inn}_{M_1} \theta) &\approx (\text{Inn}'_C \text{Inn}_M \theta \text{Inn}_{C'^{-1}}, \text{Inn}'_C \text{Inn}_{M_1} \theta \text{Inn}_{C'^{-1}}) \\
&= (\text{Inn}_{C'^T M C'} \theta, \text{Inn}_{C^T M_1 C} \theta) \\
&= (\text{Inn}_M \theta, \text{Inn}_{M_2} \theta)
\end{aligned}$$

Therefore, this gives that the commuting pairs of involutions $(\text{Inn}_M \theta, \text{Inn}_{M^{-1}M_1})$ and $(\text{Inn}_M \theta, \text{Inn}_{M^{-1}M_2})$ are isomorphic if and only if the commuting pairs of involutions $(\text{Inn}_M \theta, \text{Inn}_{M_1} \theta)$ and $(\text{Inn}_M \theta, \text{Inn}_{M_2} \theta)$ are isomorphic. \square

Corollary 6.0.19. *Classifying commuting pairs of outer involutions of $\text{SL}(n, k)$ is equivalent to classifying outer/inner pairs of commuting involutions of $\text{SL}(n, k)$.*

Proof. This is an immediate consequence of the above two lemmas. \square

The resulting isomorphism classes of commuting outer involutions, being derived from the isomorphism classes of inner/outer commuting pairs, will be summarized in the next section.

Chapter 7

Summary of Results

In this section we summarize the isomorphism classes of commuting pairs of involutions of $\mathrm{SL}(n, k)$, for specific types of fields, namely, for algebraically closed fields, \mathbb{R} , \mathbb{F}_p (with characteristic not two), and \mathbb{Q}_p .

7.1 $k = \bar{k}$: Algebraically Closed

1. If n is even, divisible by 4, we have the inner pairs:

$$\left(\mathrm{Inn}_{I_{n-i,i}}, \mathrm{Inn} \begin{pmatrix} I_{n-i-j,j} & 0 \\ 0 & I_{i-k,k} \end{pmatrix} \right).$$

Here, $i \in \{1, 2, \dots, \frac{n}{2}\}$, and these are distinct for $j \in 0, 1, \dots, \lfloor \frac{n-i}{2} \rfloor$, $k \in \{0, 1, \dots, i\}$, j, k not both zero, unless $n - i - j = j$, in which case $k \in \{0, 1, \dots, \lfloor \frac{i}{2} \rfloor\}$. We also have the inner/outer commuting pairs $(\theta, \mathrm{Inn}_{I_{n-i,i}})$, for $i \in \{1, 2, \dots, \frac{n}{2}\}$, and

$$\left(\mathrm{Inn} \begin{pmatrix} J_{n-i} & 0 \\ 0 & J_i \end{pmatrix} \theta, \mathrm{Inn}_{n-i,i} \right),$$

for $i \in \{2, 4, \dots, \frac{n}{2}\}$, $n - i$ and i even, and hence, the outer pairs $(\theta, \mathrm{Inn}_{I_{n-i,i}} \theta)$, for $i \in \{1, 2, \dots, \frac{n}{2}\}$, and $(\mathrm{Inn}_{J_n} \theta, \mathrm{Inn}_{J_n C I_{n-i,i} C^{-1}} \theta)$, for $i \in \{2, 4, \dots, \frac{n}{2}\}$, $n - i$

and i even, for

$$C = \begin{pmatrix} 0_{\frac{n}{2} \times \frac{n-i}{2}} & I_{\frac{n}{2}} & 0_{\frac{n}{2} \times i} \\ -I_{\frac{n-i}{2}} & 0_{\frac{n-i}{2} \times \frac{n}{2}} & 0_{\frac{n-i}{2} \times \frac{i}{2}} \\ 0_{\frac{i}{2} \times \frac{n-i}{2}} & 0_{\frac{i}{2} \times \frac{n}{2}} & I_{\frac{i}{2}} \end{pmatrix}$$

2. If n is even, not divisible by 4, we get the same form of pairs as above (for n even, divisible by 4).
3. If n is odd we get the inner pairs: $\left(\text{Inn}_{I_{n-i,i}}, \text{Inn} \begin{pmatrix} I_{n-i-j,j} & 0 \\ 0 & I_{i-k,k} \end{pmatrix} \right)$. Here, $i \in \{1, 2, \dots, \frac{n}{2}\}$, and these are distinct for $j \in 0, 1, \dots, \lfloor \frac{n-i}{2} \rfloor$, $k \in \{0, 1, \dots, i\}$, j, k not both zero, unless $n - i - j = j$, in which case $k \in \{0, 1, \dots, \lfloor \frac{i}{2} \rfloor\}$. We also have the inner/outer pairs: $(\theta, \text{Inn}_{I_{n-i,i}})$, for $i \in \{1, 2, \dots, \frac{n}{2}\}$, and hence the outer pairs: $(\theta, \text{Inn}_{I_{n-i,i}} \theta)$, for $i \in \{1, 2, \dots, \frac{n}{2}\}$.

7.2 $k = \mathbb{R}$: The Real Numbers

1. If n is even, divisible by 4, we have the inner pairs: $\left(\text{Inn}_{L_{n,-1}}, \text{Inn}_A \right)$, for $A = P^T L_{n,-1} P$, $P^T I_{\frac{n}{2}, \frac{n}{2}} P$, $P^T I_{\frac{n}{2}, \frac{n}{2}} L_{n,-1} P$, $P^T \begin{pmatrix} I_{\frac{n}{2}-i,i} & 0 \\ 0 & I_{\frac{n}{2}-i,i} \end{pmatrix} P$, or $I_{n-2i, 2i} L_{n,-1}$, $i \in \{1, 2, \dots, \frac{n}{4}\}$, for $P = \sum_{i=1}^{\frac{n}{2}} (E_{i, 2i-1} + E_{\frac{n}{2}+i, 2i})$. Also, we have the inner pairs: $\left(\text{Inn}_{I_{n-i,i}}, \text{Inn}_B \right)$, for $B = \begin{pmatrix} I_{n-i-j,j} & 0 \\ 0 & I_{i-k,k} \end{pmatrix}$, $i \in \{1, 2, \dots, \frac{n}{2}\}$, and these are distinct for $j \in 0, 1, \dots, \lfloor \frac{n-i}{2} \rfloor$, $k \in \{0, 1, \dots, i\}$, j, k not both zero, unless $n - i - j = j$, in which case $k \in \{0, 1, \dots, \lfloor \frac{i}{2} \rfloor\}$, or $B = L_{n,-1}$, if $n - i$ and i are even, or if $n - i = i$, $B = P L_{n,-1} P^T$. We also have the inner/outer pairs: $(\text{Inn}_M \theta, \text{Inn}_{I_{n-i,i}})$, for $i \in \{1, 2, \dots, \frac{n}{2}\}$, and $M = \begin{pmatrix} I_{n-i-j,j} & 0 \\ 0 & I_{i-k,k} \end{pmatrix}$, for $j \in \{0, 1, \dots, \lfloor \frac{n-i}{2} \rfloor\}$, and $k \in \{0, 1, \dots, i\}$ unless $n - i - j = j$, in which case $k \in \{0, 1, \dots, \lfloor \frac{i}{2} \rfloor\}$. For $n - i \neq i$ these are distinct, and for $n - i = i$ these are distinct for distinct $\{j, k\}$, or $M = \begin{pmatrix} J_{n-i} & 0 \\ 0 & J_i \end{pmatrix}$, for $i \in \{2, 4, \dots, \frac{n}{2}\}$, $n - i$ and i even. We have the inner/outer pairs: $(\text{Inn}_N \theta, \text{Inn}_{J_n})$, for $N = \begin{pmatrix} I_{\frac{n}{2}-i,i} & 0 \\ 0 & I_{\frac{n}{2}-i,i} \end{pmatrix}$, $\begin{pmatrix} 0 & I_{\frac{n}{2}-i,i} \\ I_{\frac{n}{2}-i,i} & 0 \end{pmatrix}$, for $i \in \{1, 2, \dots, \frac{n}{4}\}$. These, in turn, give us the commuting outer pairs: $\left(\text{Inn}_{I_{n-(j+k), j+k}} \theta, \text{Inn}_{I_{n-(j+k), j+k}} C I_{n-i,i} C^T \theta \right)$, for $i \in \{1, 2, \dots, \frac{n}{2}\}$, $j \in \{0, 1, \dots, \lfloor \frac{n-i}{2} \rfloor\}$, and $k \in \{0, 1, \dots, \lfloor \frac{i}{2} \rfloor\}$, if $n - i - j = j$, otherwise $k \in$

$\{0, 1, \dots, i\}$, and j, k not both zero.

$$C = \begin{pmatrix} I_{n-i-j \times n-i-j} & 0_{n-i-j \times j} & 0_{n-i-j \times i-k} & 0_{n-i-j \times k} \\ 0_{i-k \times n-i-j} & 0_{i-k \times j} & I_{i-k \times i-k} & 0_{i-k \times k} \\ 0_{j \times n-i-j} & I_{j \times j} & 0_{j \times i-k} & 0_{j \times k} \\ 0_{k \times n-i-j} & 0_{k \times j} & 0_{k \times i-k} & I_{k \times k} \end{pmatrix},$$

and $(\text{Inn}_{J_n} \theta, \text{Inn}_{J_n C' I_{n-i, i} C'^{-1}} \theta)$, for $i \in \{2, 4, \dots, \frac{n}{2}\}$, $n-i$ and i even,

$$C' = \begin{pmatrix} 0_{\frac{n}{2} \times \frac{n-i}{2}} & I_{\frac{n}{2}} & 0_{\frac{n}{2} \times i} \\ -I_{\frac{n-i}{2}} & 0_{\frac{n-i}{2} \times \frac{n}{2}} & 0_{\frac{n-i}{2} \times \frac{i}{2}} \\ 0_{\frac{i}{2} \times \frac{n-i}{2}} & 0_{\frac{i}{2} \times \frac{n}{2}} & I_{\frac{i}{2}} \end{pmatrix}.$$

We also have the commuting outer pairs: $(\text{Inn}_{I_{n-2i, 2i}} \theta, \text{Inn}_{I_{n-2i, 2i} C_1 J_n C_1^T} \theta)$, and $(\text{Inn}_{J_n} \theta, \text{Inn}_{J_n I_{n-i, i} J_n I_{n-i, i}} \theta)$, for $i \in \{0, 1, \dots, \frac{n}{4}\}$, with

$$C_1 = \begin{pmatrix} I_{\frac{n}{2}-i \times \frac{n}{2}-i} & 0_{\frac{n}{2}-i \times i} & 0_{\frac{n}{2}-i \times \frac{n}{2}-i} & 0_{\frac{n}{2}-i \times i} \\ 0_{\frac{n}{2}-i \times \frac{n}{2}-i} & 0_{\frac{n}{2}-i \times i} & I_{\frac{n}{2}-i \times \frac{n}{2}-i} & 0_{\frac{n}{2}-i \times i} \\ 0_{i \times \frac{n}{2}-i} & I_{i \times i} & 0_{i \times \frac{n}{2}-i} & 0_{i \times i} \\ 0_{i \times \frac{n}{2}-i} & 0_{i \times i} & 0_{i \times \frac{n}{2}-i} & I_{i \times i} \end{pmatrix},$$

2. If n is even, not divisible by 4, we get the inner pairs: $(\text{Inn}_{L_{n,-1}}, \text{Inn}_A)$ for

$$A = \begin{pmatrix} I_2 & 0 \\ 0 & P_{n-2} \end{pmatrix}^T \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 \\ 0 & I_{\frac{n-2}{2}, \frac{n-2}{2}} \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ 0 & P_{n-2} \end{pmatrix},$$

$P_n^T \begin{pmatrix} I_{\frac{n}{2}-i} & 0 \\ 0 & I_{\frac{n}{2}-i} \end{pmatrix} P_n$, or $I_{n-2i, 2i} L_{n,-1}$, for $i \in \{1, 2, \dots, \lfloor \frac{n}{4} \rfloor\}$, and for P_k as defined above. We also get the inner pairs: $(\text{Inn}_{I_{n-i, i}}, \text{Inn}_B)$, for $i \in \{1, 2, \dots, \frac{n}{2}\}$, with $B = \begin{pmatrix} L_{n-i, -1} & 0 \\ 0 & L_{i, -1} \end{pmatrix}$, for $n-i$ and i both even, or $B = \begin{pmatrix} I_{n-i-j, j} & 0 \\ 0 & I_{i-k, k} \end{pmatrix}$, for $j \in \{1, 2, \dots, \lfloor \frac{n-i}{2} \rfloor\}$ and $k \in \{1, 2, \dots, i\}$ if $n-i-j \neq j$, otherwise $k \in \{1, 2, \dots, \lfloor \frac{i}{2} \rfloor\}$, or $B = P_n^T L_{n,-1} P$ if $n-i = i$, i.e. if $i = \frac{n}{2}$. The inner/outer pairs and outer pairs are the same type as for n divisible by 4.

3. If n is odd we get the inner pairs: $(\text{Inn}_{I_{n-i, i}}, \text{Inn} \begin{pmatrix} I_{n-i-j, j} & 0 \\ 0 & I_{i-k, k} \end{pmatrix})$. Here, $i \in \{1, 2, \dots, \frac{n}{2}\}$, and these are distinct for $j \in 0, 1, \dots, \lfloor \frac{n-i}{2} \rfloor$, $k \in \{0, 1, \dots, i\}$, j, k

not both zero, unless $n - i - j = j$, in which case $k \in \{0, 1, \dots, \lfloor \frac{i}{2} \rfloor\}$. Also, for n odd, the inner/outer pairs and outer pairs are the same form above, but note that not both $n - i$ and i can be even, so there are no pairs with an involution of the form $\begin{pmatrix} J_{n-i} & 0 \\ 0 & J_i \end{pmatrix} \theta$.

7.3 $k = \mathbb{F}_p$: Finite Fields (of characteristic not two)

1. If n is even, divisible by 4, we have the inner pairs $(\text{Inn}_{L_{n,q}}, \text{Inn}_A)$, for $A = P^T L_{n,q'} P$, $P^T I_{\frac{n}{2}, \frac{n}{2}} P$, $P^T I_{\frac{n}{2}, \frac{n}{2}} L_{n,q'} P$, or $P^T \begin{pmatrix} I_{\frac{n}{2}-i, i} & 0 \\ 0 & I_{\frac{n}{2}-i, i} \end{pmatrix} P$, $I_{n-2i, 2i} L_{n,q}$, for $i \in \{1, 2, \dots, \frac{n}{4}\}$, $q, q' \in \mathbb{F}_p^*/\mathbb{F}_p^{*2}$, $q, q' \not\equiv 1 \pmod{\mathbb{F}_p^{*2}}$ and for $P = \sum_{i=1}^{\frac{n}{2}} (E_{i, 2i-1} + E_{\frac{n}{2}+i, 2i})$. Also, we have the inner pairs $(\text{Inn}_{I_{n-i, i}}, \text{Inn}_B)$, for $B = \begin{pmatrix} I_{n-i-j, j} & 0 \\ 0 & I_{i-k, k} \end{pmatrix}$ and for $i \in \{1, 2, \dots, \frac{n}{2}\}$, and these are distinct for $j \in 0, 1, \dots, \lfloor \frac{n-i}{2} \rfloor$, $k \in \{0, 1, \dots, i\}$, j, k not both zero, unless $n - i - j = j$, in which case $k \in \{0, 1, \dots, \lfloor \frac{i}{2} \rfloor\}$, or $B = L_{n,q'}$, if $n - i$ and i are even or if $n - i = i P L_{n,q'} P^T$. We also have the inner/outer pairs: $(\text{Inn}_M \theta, \text{Inn}_{I_{n-i, i}})$, for $i \in \{1, 2, \dots, \frac{n}{2}\}$, and $M = \begin{pmatrix} M_1 & 0 \\ 0 & \alpha_{N_1} \end{pmatrix}$, for M_1 a representative of a semi-congruence class of symmetric bilinear form of \mathbb{F}_p^{n-i} , N_1 a semi-congruence class of symmetric bilinear form of \mathbb{F}_p^i and $\alpha \in \mathbb{F}_p^*/\mathbb{F}_p^{*2}$, where distinct pairs are given by α listed in table 1, or $M = \begin{pmatrix} J_{n-i} & 0 \\ 0 & J_i \end{pmatrix}$, for $i \in \{2, 4, \dots, \frac{n}{2}\}$, $n - i$ and i even. If -1 is not a square in \mathbb{F}_p then, we also have the inner/outer pairs $(\theta, \text{Inn}_{L_{n,-1}})$, and $(\text{Inn}_{J_n} \theta, \text{Inn}_{J_n})$. These give us the outer pairs: $(\text{Inn}_{J_n} \theta, \text{Inn}_{J_n C' I_{n-i, i} C'^{-1}} \theta)$, for $i \in \{2, 4, \dots, \frac{n}{2}\}$, $n - i$ and i even, for

$$C' = \begin{pmatrix} 0_{\frac{n}{2} \times \frac{n-i}{2}} & I_{\frac{n}{2}} & 0_{\frac{n}{2} \times i} \\ -I_{\frac{n-i}{2}} & 0_{\frac{n-i}{2} \times \frac{n}{2}} & 0_{\frac{n-i}{2} \times \frac{i}{2}} \\ 0_{\frac{i}{2} \times \frac{n-i}{2}} & 0_{\frac{i}{2} \times \frac{n}{2}} & I_{\frac{i}{2}} \end{pmatrix}.$$

Also we have, for $i \in \{1, 2, \dots, \frac{n}{2}\}$, the outer pairs:

$$\left(\text{Inn}_{M_{n,1,1,sp}} \theta, \text{Inn}_{M_{n,1,1,sp} A I_{n-i, i} A^{-1}} \theta \right)$$

for $A = I_n$, or $(I_n - E_{n-i, n-i} - E_{i, i} + E_{n-i, i} + E_{i, n-i})$, and the outer pairs: $(\theta, \text{Inn}_{B I_{n-i, i} B^{-1}} \theta)$ for $B = I_n$, or $\begin{pmatrix} I_{n-2} & 0 \\ 0 & \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \end{pmatrix} (I_n - E_{n-i, n-i} - E_{i-1, i-1} +$

$E_{n-i,i-1} + E_{i-1,n-i}$), such that $a^2 + b^2 = \frac{1}{s_p}$, which we know exists from [Sch85]. Also, if -1 is not a square in \mathbb{F}_p , then we also have the outer pairs: $(\theta, \text{Inn}_{L_{n,-1}} \theta)$ and $(\text{Inn}_{J_n} \theta, \theta)$.

2. If n is even, not divisible by 4, we get the inner pairs: $(\text{Inn}_{L_{n,q}}, \text{Inn}_A)$ for

$$A = \begin{pmatrix} I_2 & 0 \\ 0 & P_{n-2} \end{pmatrix}^T \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 \\ 0 & I_{\frac{n-2}{2}, \frac{n-2}{2}} \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ 0 & P_{n-2} \end{pmatrix}, \text{ or}$$

$$A = \begin{pmatrix} I_2 & 0 \\ 0 & P_{n-2} \end{pmatrix}^T \begin{pmatrix} \begin{pmatrix} a & b \\ -qb & -a \end{pmatrix} & 0 \\ 0 & I_{\frac{n-2}{2}, \frac{n-2}{2}} L_{n-2,q'} \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ 0 & P_{n-2} \end{pmatrix}$$

such that $a, b \in \mathbb{F}_p$, $a^2 - qb^2 = q'$, $q, q' \not\equiv 1 \pmod{\mathbb{F}_p^{*2}}$ for some fixed a, b , also A can be

$P_n^T \begin{pmatrix} I_{\frac{n-i}{2}} & 0 \\ 0 & I_{\frac{n-i}{2}} \end{pmatrix} P_n$, or $I_{n-2i,2i} L_{n,q}$, for $i \in \{1, 2, \dots, \lfloor \frac{n}{4} \rfloor\}$, for P_k as defined above, i.e. $\sum_{i=1}^{\frac{k}{2}} (E_{i,2i-1} + E_{\frac{k}{2}+i,2i})$, and for $q' \in \mathbb{F}_p^*/\mathbb{F}_p^{*2}$. We also get the inner pairs: $(\text{Inn}_{I_{n-i,i}}, \text{Inn}_B)$, for $i \in \{1, 2, \dots, \frac{n}{2}\}$ and for $B = \begin{pmatrix} L_{n-i,q'} & 0 \\ 0 & L_{i,q'} \end{pmatrix}$, for $n-i$ and i even, or $B = \begin{pmatrix} I_{n-i-j,j} & 0 \\ 0 & I_{i-k,k} \end{pmatrix}$, for $j \in \{1, 2, \dots, \lfloor \frac{n-i}{2} \rfloor\}$ and $k \in \{1, 2, \dots, i\}$ if $n-i-j \neq j$, otherwise $k \in \{1, 2, \dots, \lfloor \frac{i}{2} \rfloor\}$, or $P_n^T L_{n,q'} P$ if $n-i = i$, i.e. if $i = \frac{n}{2}$. The inner/outer pairs and outer pairs are the same type as for n divisible by 4.

3. If n is odd we get the inner pairs: $(\text{Inn}_{I_{n-i,i}}, \text{Inn} \begin{pmatrix} I_{n-i-j,j} & 0 \\ 0 & I_{i-k,k} \end{pmatrix})$. Here, $i \in \{1, 2, \dots, \frac{n}{2}\}$, and these are distinct for $j \in 0, 1, \dots, \lfloor \frac{n-i}{2} \rfloor$, $k \in \{0, 1, \dots, i\}$, j, k not both zero, unless $n-i-j = j$, in which case $k \in \{0, 1, \dots, \lfloor \frac{i}{2} \rfloor\}$. Also, for n odd, the inner/outer pairs and outer pairs are the same form above, but note that not both $n-i$ and i can be even, so there are no pairs with an involution of the form $\begin{pmatrix} J_{n-i} & 0 \\ 0 & J_i \end{pmatrix} \theta$.

7.4 $k = \mathbb{Q}_p$: P-adic Numbers ($p \neq 2$)

1. If n is even, divisible by 4, we have the inner pairs: $(\text{Inn}_{L_{n,q}}, \text{Inn}_A)$, for $A = P_n^T L_{n,q'} P$, $P_n^T I_{\frac{n}{2}, \frac{n}{2}} P$, $P_n^T I_{\frac{n}{2}, \frac{n}{2}} L_{n,q'} P$, or $P_n^T \begin{pmatrix} I_{\frac{n-i}{2}} & 0 \\ 0 & I_{\frac{n-i}{2}} \end{pmatrix} P$, $I_{n-2i,2i} L_{n,q}$, for $i \in \{1, 2, \dots, \frac{n}{4}\}$, $q, q' \in \mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$, $q, q' \not\equiv 1 \pmod{\mathbb{Q}_p^{*2}}$ and for $P = \sum_{i=1}^{\frac{n}{2}} (E_{i,2i-1} +$

$E_{\frac{n}{2}+i,2i}$). Also, we have the inner pairs: $(\text{Inn}_{I_{n-i,i}}, \text{Inn}_B)$ for $B = \begin{pmatrix} I_{n-i-j,j} & 0 \\ 0 & I_{i-k,k} \end{pmatrix}$ and for $i \in \{1, 2, \dots, \frac{n}{2}\}$, and these are distinct for $j \in 0, 1, \dots, \lfloor \frac{n-i}{2} \rfloor$, and $k \in \{0, 1, \dots, i\}$, j, k not both zero, unless $n - i - j = j$, in which case $k \in \{0, 1, \dots, \lfloor \frac{i}{2} \rfloor\}$, or $B = L_{n,q'}$, if $n - i$ and i are even or if $n - i = i$ $PL_{n,q'}P^T$. We also have the inner/outer pairs: $(\text{Inn}_M \theta, \text{Inn}_{I_{n-i,i}})$, for $i \in \{1, 2, \dots, \frac{n}{2}\}$, and $M = \begin{pmatrix} M_1 & 0 \\ 0 & \alpha N_1 \end{pmatrix}$, for M_1 a representative of a semi-congruence class of symmetric bilinear form of \mathbb{Q}_p^{n-i} , N_1 a semi-congruence class of symmetric bilinear form of \mathbb{Q}_p^i and $\alpha \in \mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$, where distinct pairs are given by α listed in tables 2 through 4. Also, M can be $\begin{pmatrix} J_{n-i} & 0 \\ 0 & J_i \end{pmatrix}$, for $i \in \{2, 4, \dots, \frac{n}{2}\}$, $n - i$ and i even. If -1 is not a square in \mathbb{Q}_p then, we also have the inner/outer pairs: $(\theta, \text{Inn}_{L_{n,-1}})$, and $(\text{Inn} \begin{pmatrix} I_{n-2} & 0 \\ 0 & pI_2 \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}})$, and these two are distinct if $\frac{n}{2}$ is even. Also, there are the inner/outer pairs: $(\text{Inn}_{L_{n,-1}} \theta, \text{Inn}_{L_{n,-1}})$, and $(\text{Inn} \begin{pmatrix} L_{n-2,-1} & 0 \\ 0 & pL_{2,-1} \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}})$, and these two are distinct if $\frac{n}{2}$ is even, as well. These give us the outer pairs: $(\text{Inn}_{J_n} \theta, \text{Inn}_{J_n C' I_{n-i,i} C'^{-1}} \theta)$, for $i \in \{2, 4, \dots, \frac{n}{2}\}$, $n - i$ and i even, for

$$C' = \begin{pmatrix} 0_{\frac{n}{2} \times \frac{n-i}{2}} & I_{\frac{n}{2}} & 0_{\frac{n}{2} \times i} \\ -I_{\frac{n-i}{2}} & 0_{\frac{n-i}{2} \times \frac{n}{2}} & 0_{\frac{n-i}{2} \times \frac{i}{2}} \\ 0_{\frac{i}{2} \times \frac{n-i}{2}} & 0_{\frac{i}{2} \times \frac{n}{2}} & I_{\frac{i}{2}} \end{pmatrix}.$$

Also, we have the outer pairs stemming from the outer/inner pairs from tables 2 through 4 and, $(\text{Inn}_{L_{n,-1}} \theta, \theta)$, $(\theta, \text{Inn}_{L_{n,-1}} \theta)$, as well as (if $\frac{n}{2}$ is even) the commuting pair: $(\text{Inn} \begin{pmatrix} I_{n-2} & 0 \\ 0 & pI_2 \end{pmatrix} \theta, \text{Inn} \begin{pmatrix} I_{n-2} & 0 \\ 0 & pI_2 \end{pmatrix}_{L_{n,-1}} \theta)$ and the commuting pair $(\text{Inn} \begin{pmatrix} L_{n-2,-1} & 0 \\ 0 & pL_{2,-1} \end{pmatrix} \theta, \text{Inn} \begin{pmatrix} L_{n-2,-1} & 0 \\ 0 & pL_{2,-1} \end{pmatrix}_{L_{n,-1}} \theta)$.

2. If n is even, not divisible by 4, we get the inner pairs: $(\text{Inn}_{L_{n,q}}, \text{Inn}_A)$ for

$$A = \begin{pmatrix} I_2 & 0 \\ 0 & P_{n-2} \end{pmatrix}^T \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 \\ & I_{\frac{n-2}{2}, \frac{n-2}{2}} \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ 0 & P_{n-2} \end{pmatrix},$$

$$\begin{pmatrix} I_2 & 0 \\ 0 & P_{n-2} \end{pmatrix}^T \begin{pmatrix} \begin{pmatrix} a & b \\ -qb & -a \end{pmatrix} & 0 \\ & I_{\frac{n-2}{2}, \frac{n-2}{2}} L_{n-2,q'} \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ 0 & P_{n-2} \end{pmatrix}$$

such that $a, b \in \mathbb{Q}_p$, $a^2 - qb^2 = q'$, $q, q' \not\equiv 1 \pmod{\mathbb{Q}_p^{*2}}$ for some fixed a, b ,

$P_n^T \begin{pmatrix} I_{\frac{n}{2}-i} & 0 \\ 0 & I_{\frac{n}{2}-i} \end{pmatrix} P_n$, or $I_{n-2i,2i} L_{n,q}$, for $i \in \{1, 2, \dots, \lfloor \frac{n}{4} \rfloor\}$, for P_k as defined above, i.e. $\sum_{i=1}^{\frac{k}{2}} (E_{i,2i-1} + E_{\frac{k}{2}+i,2i})$, and for $q' \in \mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$. We also get the inner pairs: $(\text{Inn}_{I_{n-i,i}}, \text{Inn}_B)$, for $i \in \{1, 2, \dots, \frac{n}{2}\}$ and for $B = \begin{pmatrix} L_{n-i,q'} & 0 \\ 0 & L_{i,q'} \end{pmatrix}$, for $n-i$ and i even, or $\begin{pmatrix} I_{n-i-j,j} & 0 \\ 0 & I_{i-k,k} \end{pmatrix}$, for $j \in \{1, 2, \dots, \lfloor \frac{n-i}{2} \rfloor\}$ and $k \in \{1, 2, \dots, i\}$ if $n-i-j \neq j$, otherwise $k \in \{1, 2, \dots, \lfloor \frac{i}{2} \rfloor\}$, or $P_n^T L_{n,q'} P$ if $n-i = i$, i.e. if $i = \frac{n}{2}$. The inner/outer pairs and outer pairs are the same type as for n divisible by 4.

3. If n is odd we get the inner pairs: $(\text{Inn}_{I_{n-i,i}}, \text{Inn}(\begin{pmatrix} I_{n-i-j,j} & 0 \\ 0 & I_{i-k,k} \end{pmatrix}))$. Here, $i \in \{1, 2, \dots, \frac{n}{2}\}$, and these are distinct for $j \in \{0, 1, \dots, \lfloor \frac{n-i}{2} \rfloor\}$, $k \in \{0, 1, \dots, i\}$, j, k not both zero, unless $n-i-j = j$, in which case $k \in \{0, 1, \dots, \lfloor \frac{i}{2} \rfloor\}$. Also, for n odd, the inner/outer pairs and outer pairs are the same form above, but note that not both $n-i$ and i can be even, so there are no pairs with an involution of the form $\begin{pmatrix} J_{n-i} & 0 \\ 0 & J_i \end{pmatrix} \theta$.

7.5 $k = \mathbb{Q}_2$: 2-adic Numbers

1. If n is even, divisible by 4, we have the inner pairs: $(\text{Inn}_{L_{n,q}}, \text{Inn}_A)$, for $A = P^T L_{n,q'} P$, $P^T I_{\frac{n}{2}, \frac{n}{2}} P$, $P^T I_{\frac{n}{2}, \frac{n}{2}} L_{n,q'} P$, $P^T \begin{pmatrix} I_{\frac{n}{2}-i,i} & 0 \\ 0 & I_{\frac{n}{2}-i,i} \end{pmatrix} P$, or $I_{n-2i,2i} L_{n,q}$, for $i \in \{1, 2, \dots, \frac{n}{4}\}$, $q, q' \in \mathbb{Q}_2^*/\mathbb{Q}_2^{*2}$, $q, q' \not\equiv 1 \pmod{\mathbb{Q}_2^{*2}}$ and for $P = \sum_{i=1}^{\frac{n}{2}} (E_{i,2i-1} + E_{\frac{n}{2}+i,2i})$. Also, we have the inner pairs: $(\text{Inn}_{I_{n-i,i}}, \text{Inn}_B)$ for $B = \begin{pmatrix} I_{n-i-j,j} & 0 \\ 0 & I_{i-k,k} \end{pmatrix}$ and for $i \in \{1, 2, \dots, \frac{n}{2}\}$, and these are distinct for $j \in \{0, 1, \dots, \lfloor \frac{n-i}{2} \rfloor\}$, $k \in \{0, 1, \dots, i\}$, j, k not both zero, unless $n-i-j = j$, in which case $k \in \{0, 1, \dots, \lfloor \frac{i}{2} \rfloor\}$, or $B = L_{n,q'}$, if $n-i$ and i are even, or if $n-i = i$, $B = P L_{n,q'} P^T$. We also have the inner/outer pairs: $(\text{Inn}_M \theta, \text{Inn}_{I_{n-i,i}})$, for $i \in \{1, 2, \dots, \frac{n}{2}\}$, and $M = \begin{pmatrix} M_1 & 0 \\ 0 & \alpha N_1 \end{pmatrix}$, for M_1 a representative of a semi-congruence class of symmetric bilinear form of \mathbb{Q}_2^{n-i} , N_1 a semi-congruence class of symmetric bilinear form of \mathbb{Q}_2^i and $\alpha \in \mathbb{Q}_2^*/\mathbb{Q}_2^{*2}$, where distinct pairs are given by α listed in tables 5 through 9. M can also equal: $\begin{pmatrix} J_{n-i} & 0 \\ 0 & J_i \end{pmatrix}$, for $i \in \{2, 4, \dots, \frac{n}{2}\}$, $n-i$ and i even. There are also the pairs: $(\theta, \text{Inn}_{L_{n,-1}})$, and $(\text{Inn}(\begin{pmatrix} I_{n-2} & 0 \\ 0 & -I_2 \end{pmatrix}) \theta, \text{Inn}_{L_{n,-1}})$, and these two

are distinct if $\frac{n}{2}$ is even. Also, there are the pairs: $(\text{Inn}_{L_{n,-1}} \theta, \text{Inn}_{L_{n,-1}})$, and $\left(\text{Inn} \begin{pmatrix} L_{n-2,-1} & 0 \\ 0 & -L_{2,-1} \end{pmatrix} \theta, \text{Inn}_{L_{n,-1}}\right)$, and these two are distinct if $\frac{n}{2}$ is even. These give us the outer pairs: $(\text{Inn}_{J_n} \theta, \text{Inn}_{J_n C' L_{n-i,i} C'^{-1}} \theta)$, for $i \in \{2, 4, \dots, \frac{n}{2}\}$, $n-i$ and i even, for

$$C' = \begin{pmatrix} 0_{\frac{n}{2} \times \frac{n-i}{2}} & I_{\frac{n}{2}} & 0_{\frac{n}{2} \times i} \\ -I_{\frac{n-i}{2}} & 0_{\frac{n-i}{2} \times \frac{n}{2}} & 0_{\frac{n-i}{2} \times \frac{i}{2}} \\ 0_{\frac{i}{2} \times \frac{n-i}{2}} & 0_{\frac{i}{2} \times \frac{n}{2}} & I_{\frac{i}{2}} \end{pmatrix}.$$

Also, we have the outer pairs stemming from the outer/inner pairs from tables 5 through 9 and the pairs: $(\text{Inn}_{L_{n,-1}} \theta, \theta)$, $(\theta, \text{Inn}_{L_{n,-1}} \theta)$, as well as (if $\frac{n}{2}$ is even) the commuting pair: $\left(\text{Inn} \begin{pmatrix} I_{n-2} & 0 \\ 0 & pI_2 \end{pmatrix} \theta, \text{Inn} \begin{pmatrix} I_{n-2} & 0 \\ 0 & pI_2 \end{pmatrix}_{L_{n,-1}} \theta\right)$ and the commuting pair $\left(\text{Inn} \begin{pmatrix} L_{n-2,-1} & 0 \\ 0 & pL_{2,-1} \end{pmatrix} \theta, \text{Inn} \begin{pmatrix} L_{n-2,-1} & 0 \\ 0 & pL_{2,-1} \end{pmatrix}_{L_{n,-1}} \theta\right)$.

2. If n is even, not divisible by 4, we get the inner pairs: $(\text{Inn}_{L_{n,q}}, \text{Inn}_A)$ for

$$A = \begin{pmatrix} I_2 & 0 \\ 0 & P_{n-2} \end{pmatrix}^T \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 \\ 0 & I_{\frac{n-2}{2}, \frac{n-2}{2}} \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ 0 & P_{n-2} \end{pmatrix}, \\ \begin{pmatrix} I_2 & 0 \\ 0 & P_{n-2} \end{pmatrix}^T \begin{pmatrix} \begin{pmatrix} a & b \\ -qb & -a \end{pmatrix} & 0 \\ 0 & I_{\frac{n-2}{2}, \frac{n-2}{2}} L_{n-2,q'} \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ 0 & P_{n-2} \end{pmatrix}$$

for $a, b \in \mathbb{Q}_2$, $a^2 - qb^2 = q'$, $q, q' \not\equiv 1 \pmod{\mathbb{Q}_2^{*2}}$ for some fixed a, b , as well as $P_n^T \begin{pmatrix} I_{\frac{n-i}{2}} & 0 \\ 0 & I_{\frac{n-i}{2}} \end{pmatrix} P_n$, or $I_{n-2i, 2i} L_{n,q}$, for $i \in \{1, 2, \dots, \lfloor \frac{n}{4} \rfloor\}$, for P_k as defined above, i.e. $\sum_{i=1}^{\frac{k}{2}} (E_{i, 2i-1} + E_{\frac{k}{2}+i, 2i})$, and for $q' \in \mathbb{Q}_2^*/\mathbb{Q}_2^{*2}$. We also get the inner pairs: $(\text{Inn}_{I_{n-i,i}}, \text{Inn}_B)$, for $i \in \{1, 2, \dots, \frac{n}{2}\}$ and for $B = \begin{pmatrix} L_{n-i,q'} & 0 \\ 0 & L_{i,q'} \end{pmatrix}$, for $n-i$ and i even, or $\begin{pmatrix} I_{n-i-j,j} & 0 \\ 0 & I_{i-k,k} \end{pmatrix}$, for $j \in \{1, 2, \dots, \lfloor \frac{n-i}{2} \rfloor\}$ and $k \in \{1, 2, \dots, i\}$ if $n-i-j \neq j$, otherwise $k \in \{1, 2, \dots, \lfloor \frac{i}{2} \rfloor\}$, or $B = P_n^T L_{n,q'} P$ if $n-i = i$, i.e. if $i = \frac{n}{2}$. The inner/outer pairs and outer pairs are the same type as for n divisible by 4.

3. If n is odd we get the inner pairs: $\left(\text{Inn}_{I_{n-i,i}}, \text{Inn} \begin{pmatrix} I_{n-i-j,j} & 0 \\ 0 & I_{i-k,k} \end{pmatrix}\right)$. Here, $i \in \{1, 2, \dots, \frac{n}{2}\}$, and these are distinct for $j \in \{0, 1, \dots, \lfloor \frac{n-i}{2} \rfloor\}$, $k \in \{0, 1, \dots, i\}$, j, k not both zero, unless $n-i-j = j$, in which case $k \in \{0, 1, \dots, \lfloor \frac{i}{2} \rfloor\}$. Also, for n odd, the inner/outer pairs and outer pairs are the same form above, but note

that not both $n - i$ and i can be even, so there are no pairs with an involution of the form $\begin{pmatrix} J_{n-i} & 0 \\ 0 & J_i \end{pmatrix} \theta$.

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