UNIVARIATE, MULTIVARIATE AND INCOMPLETE MULTIVARIATE NON-PARAMETRIC MIXED MODELS IN THE ANALYSIS OF INCOMPLETE AND/OR ORTHOGONAL TWO-WAY LAYOUT EXPERIMENTS

by

Elsa C. Servy

Institute of Statistics
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1. REVIEW OF LITERATURE: RANK TESTS AND MIXED MODELS FOR THE ANALYSIS OF VARIANCE PROBLEM

1.1 Rank Tests

1.1.1 Introduction

The theory of rank tests has developed in a compact system that forms, at present, a specialized part of the theory of testing statistical hypothesis.

Rank tests were regarded rather skeptically at first because their use was justified only on intuitive grounds by those who introduced them. Later on, the works by Terry (1952), Lehmann (1953) and others showed that the asymptotic power of these tests is high when compared with the power of the standard tests used for solving analogous parametric problems.

From then on, many works have been devoted to the search for new rank tests and to the study of their properties. The bibliography compiled by Savage (1962) includes valuable references on rank tests.

Some chapters on rank tests are included in general books of statistics (Wilks (1962) and Lehmann (1959)), and other books that treat the subject in detail have been written by Kendall (1948), Siegel (1956), Fraser (1957), Hájek and Sidák (1967), Hájek (1969) and Puri and Sen (1970).

As recently as June, 1969, the First International Symposium on Non-parametric Techniques in Statistical Inference was held at

Indiana University, Bloomington, Indiana. Interesting articles on rank tests can be found in the proceedings of the symposium edited by M. L. Puri (1970).

Some of the advantages offered by rank tests are:

i) They can be applied even in those cases in which only the order of the observations are known.

ii) They are suitable for testing hypothesis that involve quite general assumptions.

iii) For many statistical problems, the asymptotic power of the tests is high as compared with that of the tests developed for parametric models.

Rank tests, like other non-parametric methods, are now widely applied to many fields of biometrics, agriculture, economics, industry, etc. But to avoid over-estimation of their value, it has to be said that:

i) The asymptotic results are only applicable if the sample is large enough.

ii) The tests available at the present assume the existence of simple experimental designs.

Looking for ways to overcome this second drawback, this thesis is oriented toward the search for methods that can make possible the application of rank tests to experiments that exhibit awkward patterns.

To this end special attention has been paid to tests known as randomization-rank tests whose underlying theory is described in the works of Chatterjee, Puri, Sen, Koch, Gerig and others.

These tests, based on the order properties of the observations, share common characteristics with the conditional tests, permutation tests, and invariant tests that are considered in the next sections.

1.1.2 Conditional Tests

Some hypothesis testing problems are difficult to handle on the whole sample space, but become simple in a subspace of it. The name conditional tests is given to those tests performed in one of the cells of a partition of the sample space, the cell being determined by the actual sample obtained and the inference being made by the use of a distribution constructed from the results of the experiment.

The size of the test inside the cell is chosen by trying to keep the total size of the test over the whole sample space equal to some pre-assigned coefficient α.

If the test-function obtained conditionally does not depend on the cell of the partition, the test is said to be "unconditional" in relation to the whole sample space. But, even if the test-function depends on the partition, the results can be re-interpreted, considering again the total space and in many situations the optimal properties of the test inside the cell imply optimal properties of the test in the general sample space.

1.1.3 Permutation and Randomization Tests

Permutation tests can be used for testing a hypothesis (H) that is invariant under a finite group G of (M) permutations. Invariance of H under G means that if y has a distribution that belongs to H, then y and $g(y)(g \in G)$ are identically distributed random variables.

Permutation tests are conditional tests in which the cell where the test is carried out is the set of (M) points $g(y_o)$ obtained applying to the actual result of the experiment the M transformations of G. The cell can be identified as $S(y_o)$. Inside that cell, the conditional probability, given $S(y_o)$, (under the hypothesis is $\frac{1}{M}$, whatever measure in H is taken into account. The α-level test of the hypothesis H can be defined as a rule that determines, for every possible cell $S(y_o)$, a subset $S'(y_o)$ of s_α points $(s_\alpha \leq \alpha M)$, in such a way that the hypothesis is rejected if, and only if, the observed sample point lies in $S'(y_o)$.

If the group G of transformations is the group of n! permutations of the observations of the sample $\{Y_1, Y_2, \ldots, Y_n\}$, the symmetry of the measures of the hypothesis under G can be obtained by:

i) the assumption that, under H, the observations at hand are a random sample from a certain population. In this case, H is said to be a hypothesis of randomness;

ii) by an actual randomization introduced in the experiment. In this case, it is not necessary to assume that the observations are independent and identically distributed and H is called a hypothesis of interchangeability.

Some authors differentiate the tests connected with case i) from those connected with case ii) by using the names permutation tests and randomization tests respectively, while for others both types are called indiscriminantly by one or the other denomination.

An example of the application of the permutation (or randomization) principle is offered by the two-sample test of Fisher and Pitman, (Fisher (1925) and Pitman (1937)).

Fisher was the first to emphasize the importance of randomization and to conceive of the permutation tests. The randomization model for complete block design was formulated by Neyman (1935) and randomization models have been formulated for other designs by Kempthorne (1952 and 1955), Wilk (1955) and Wilk and Kempthorne (1955).

The basic ideas involved in randomization tests have been formalized by Lehmann and Stein (1949). The hypothesis under test is defined as a hypothesis of invariance under a certain partition Π of the sample space. Permutation tests are defined as similar α tests having the $S(\alpha)$ structure.

This last condition means that their test-functions, call them Φ, are such that

$$\sum_{y' \in S(y)} \Phi(y') = M\alpha \quad \forall \quad y \in \mathcal{Y} .$$

where α is the size of the test. $S(y)$ is the set of points that lie in the same cell of the partition to which y belongs.

1.1.4 Invariant Tests

There are many examples of statistical problems whose essence does not change if some group of transformations are applied to the variables involved. For instance, in testing the hypothesis about the difference in magnitude of two means, the unit of measurement used is alien to the substance of the problem. So, it is logical that in order to solve it, the search for the correct procedure has to be restricted to those tests that are independent or invariant under the changes of the unit of measurement.

In general, the principle of invariance confines attention to test functions that are invariant with respect to the group of transformations of interest.

Formally, a test function $\Phi(x)$ is invariant with respect to a group G of transformations if $\Phi(g(y)) = \Phi(y)$ for all $g \in G$.

Every invariant (re G) statistic can be expressed as a function of another statistic of the same family (i.e., invariant under G, say $m(y)$) that is called maximal invariant (re G). So, in looking for an invariant test only functions of $m(y)$ come into consideration, and this amounts to restating the original hypothesis problem in terms of m. Making this transformation has the advantage of allowing one to deal directly with invariant functions, and besides this, it produces in many cases mathematical simplications in the setting up of the problem.

If this is the situation, invariance is a logical property to require of a test statistic as well as a means of making the problem at hand more manageable. In certain problems, invariance reduces the data so completely that the actual values of the observations are discarded and only order relations between group of variables are retained. The related tests are based on the ranks of the observations.

1.1.5 Rank Tests and Rank-Permutation-Type Tests

In connection with the invariance principle, rank tests can be defined as tests that are invariant under the group of transformation G where $g(\epsilon G)$ is a continuous and strictly increasing function. In particular, they are invariant under changes of location and scale parameters.

If the hypothesis under consideration is the hypothesis of symmetry under the group G of n! permutations of the sample values $\{Y_1, Y_2, \ldots, Y_n\}$, rank tests can be defined as permutation tests that are independent of the order statistic of the sample (depending, as a consequence, only on the ranks of the observations).

For other hypotheses, other definitions of rank tests have to be given. Hájek (1967) gives careful definitions of rank tests for the hypothesis of interchangeability, randomness, symmetry, independence and random blocks.

In order to avoid technicalities, the term rank tests will be used when the test-functions are based primarily on order properties of the observations, depending implicitly or explicitly on some ranking defined over them. This work will concentrate on rank tests that are generalizations of the ones created by Friedman, Wilcoxon and Kruskal-Wallis and whose characteristics can be outlined as follows:

i) A certain null hypothesis is specified on the distribution of a scalar or vector variable Y.

ii) Certain functions, say $Z_i = f_i(Y)$ of the observations (the observations themselves, their differences, etc.) are considered. They are chosen in such a way that their distributions depend on the pertinent hypothesis.

iii) Some rules are given for ranking the values of Z and accordingly, the original sample space is transformed into a finite set R whose elements are functions of those ranks.

iv) The testing hypothesis problem is restated in terms of R and the permutation principle.

v) When the sample is large, an approximation to the distribution of the test statistic is found restricted to the cell of R in which the test has been constructed, and a conditional large sample test is derived. If the asymptotic test does not depend on the partition of R, it is an approximate test unconditionally.

These tests will be identified as rank randomization type tests. They are very useful for solving analysis of variance problems about which we are concerned. In the next section a review of the procedures based on ranks that are suitable for testing analysis of variance hypotheses will be presented and emphasis will be put on those that belong to this rank randomization class.

1.1.6 Rank Tests for the Analysis of Variance Problem

1.1.6.1 Introduction. The most famous tests in the literature are known by the following names:

1) Sign test
2) Wilcoxon's signed rank test
3) Wilcoxon's sum-rank test
4) Mann-Whitney test
5) Kruskal-Wallis test
6) Friedman test.

They were introduced by different statisticians around the decade of the forties. As rank tests gained in reputation, a lot of work was done in order to extend them to more complex or more general situations.

The extentions have been accomplished in different directions, in order

i) to be able to deal with tied observations;
ii) to consider more complex experimental situations: for instance, incomplete block and factorial designs;
iii) to consider test-functions that depend on the ranks through some functions called "scores" and that can

replace the ranks in the algebraic expressions of the classical test statistics;

iv) to include multivariate responses.

The review of rank tests for testing analysis of variance type of hypothesis to be presented in the following section is not intended to be complete or exhaustive. No mention is made to the analysis of covariance procedures or to the problems of multi-comparisons of treatments. The tests are classified according to the type of experiment for which they are suited, i.e.:

i) Rank tests for one-way layout experiments;

ii) Rank tests for two-way layout experiments;

iii) Rank tests for complex experiments.

1.1.6.2 Rank Tests for One-way Layout Experiments. Let Y_{ij} $(j = 1, \dots, N_i)$ represent N_i independent identically distributed random variables from a continuous cumulative distribution function $F_i = G(y - \alpha_i)$. Let i vary from 1 to v. The hypothesis to be tested is

$$H_o: \ \alpha_i = \alpha \quad \text{for } i = 1, 2, \dots, v$$

and the alternatives are

$$H_1: \text{ at least one } \alpha_i \neq \alpha .$$

Ranks are assigned to the observations, i.e.,

$$R_{ij} = \text{Rank}\ \{Y_{ij}: Y_{i'j'}, \quad i' = 1, 2, \dots, v; \quad j' = 1, \dots, N_i\} .$$

Ties are handled by the mid-ranks method. This method can be described as a rule, that, assigns to each observation in a tie, the average of ranks that the tied observations would have had if they would have been distinguishable.

Under H_o, the distribution of the vector of ranks is invariant under the $N!$ $(N = \Sigma N_i)$ possible allocations of the ranks of the members of the v populations. Conditional on the magnitude of the Y's, the probability attached to any permutation of the ranks is $\frac{1}{N!}$ no matter what measure in H_o is taken into consideration. A permutation test is performed using this invariance property; its test function is given by the statistic:

$$L = \frac{N-1}{N\sigma^2_{NR}} \sum_{i=1}^{v} N_i(\bar{R}_i - \frac{N+1}{2})^2$$

with

$$\bar{R}_i = \frac{1}{N_i} \sum_{j=1}^{N_i} R_{ij}$$

and

$$\sigma^2_{NR} = \frac{1}{N} \sum_{i=1}^{v} \sum_{j=1}^{N_i} (R_{ij} - \frac{N+1}{2})^2 .$$

If N and the N_i's are simultaneously made large, L is, under H_o, asymptotically distributed as χ^2_{v-1}, conditionally on the permutation structure. Since the χ^2 distribution does not depend on it, the approximation is also satisfactory unconditionally.

Specialization of the Test

If there are no ties, the test becomes the one proposed by Kruskal-Wallis (1952). In this case,

$$\sigma^2_{NR} = \frac{N^2-1}{12}$$

and

$$L = \frac{12}{N(N+1)} \sum_{i=1}^{v} N_i(\bar{R}_i - \frac{N+1}{2})^2 .$$

Also, Wilcoxon's rank sum and the Mann-Whitney test can be seen, in turn, as special cases of the Kruskal-Wallis test. The Wilcoxon's sum rank test (Wilcoxon, (1945)) was created for the case in which the subsamples are of equal size. Mann and Whitney (1947) extended it to unequal subsample sizes. The test function used by Wilcoxon is the minimum sum of the ranks of both samples, and the one for the Mann-Whitney test is

$$U = \sum_{h=1}^{n_1} \sum_{h'=1}^{n_2} Z_{hh'}$$

where

$$Z_{hh'} = \begin{cases} 1, & \text{if } Y_h < Y_{h'} \\ 0, & \text{if } Y_h \geq Y_{h'} \end{cases} .$$

When two-sided alternatives are considered, these statistics give the same critical regions that occur when L is taken for v=2.

Other Related Tests

Chatterjee and Sen (1964) considered two procedures for testing non-parametric hypotheses that are counterparts of the Mahalabis' D^2 and Hotelling's T^2 for testing the equality of means of two bivariate normal distributions. Later on, the same authors (Chatterjee and Sen, (1965)) extended the Kruskal-Wallis test to multivariate responses. In 1966, Puri and Sen developed a rank-order test with analogous purposes, by using a statistic of the Chernoff-Savage type.

1.1.6.3 Rank Tests for Two-way Layout Experiments. The case in which the number of treatments to be compared is two and the case in which the number is greater than two will be considered separately

i) two treatments (paired samples);

Sign Test (Dixon and Mood, 1946)

Let (Y_{i1}, Y_{i2}); $(i = 1, 2, \ldots, b)$ be stochastically independent pairs. Also, let Y_{i1} and Y_{i2} be independent with cumulative distribution functions F_{i1}, F_{i2} such that

$$F_{i1}(y_{i1}) = G_i(y_{i1} - \alpha_1) \qquad F_{i2}(y_{i2}) = G_i(y_{i2} - \alpha_2)$$

with

$$\alpha_2 = \mu + \Delta \quad \text{and} \quad \alpha_2 = \mu .$$

The hypothesis to be tested is $\Delta = 0$, so that the test is directed to location alternatives, i.e., to $\Delta \neq 0$ (or $\Delta > 0$, or $\Delta < 0$).

Two-sided as well as one-sided tests can be obtained by transforming the original variables to

$$\{Z_i = (Y_{i1} - Y_{i2}); \quad i = 1, 2, \ldots, b\}.$$

Restated in terms of the Z's, the hypothesis is a hypothesis of symmetry of the distributions of the Z_i's. The test statistic is:

$$S = \sum_{i=1}^{b} \mu(Z_i) .$$

where

$$\mu(X_i) = \begin{cases} 1, & \text{if } X_i > 0 \\ 0, & \text{if } X_i = 0 \end{cases} .$$

Under the hypothesis, S follows the binomial law of probability with $p = \frac{1}{2}$.

Wilcoxon's Signed Rank Tests (Wilcoxon, (1945))

Let the vectors $\{(Y_{i1}, Y_{i2})\}$ be stochastically independent for $i = 1, 2, \ldots, b$. Let the members of each pair be independent with marginals F_{i1}, F_{i2} respectively, and such that

$$F_{i1} = G(y - \alpha_1)$$

and

$$F_{i2} = G(y - \alpha_2) \ ,$$

with

$$\alpha_1 = \mu + \Delta \quad \text{and} \quad \alpha_2 = \mu \ .$$

The hypothesis to be tested is $\Delta = 0$, and the test is directed to location alternatives. ($\Delta \neq 0$, $\Delta > 0$ or $\Delta < 0$).

Let $Z_i = Y_{i1} - Y_{i2}$ $(i = 1, \ldots, b)$. In terms of the vector $(Z_1, Z_2, \ldots, Z_b)$ the hypothesis becomes a hypothesis of randomness and symmetry.

Let us define

$$S_i = \text{Sign } Z_i \cdot \{\text{Rank } |Z_i| : |Z_1|, \ldots, |Z_b|\} \ ,$$

where ties are handled by the mid-ranks method and zero is assigned to zero values.

Conditional on the magnitude of the ranks assigned to the $|Z|$'s, the sign of each S_i has equal probability of being plus or minus under the hypothesis. This randomization model has 2^b possible (equally probable) realizations for sign invariance.

The test statistic used for the test is

$$S = \sum_i^b S_i \ .$$

Under the hypothesis, the statistic S is approximately normally distributed for large b. One-sided as well as two-sided tests can be obtained by using this normal approximation.

ii) More than two treatments (random blocks)

Let

$$\{Y_{ij}; \quad i = 1, 2, \ldots, b; \quad j = 1, 2, \ldots, p\}$$

be independent random variables from some continuous cumulative distribution function $F_{ij} = G_i(y - \alpha_j)$. The hypothesis to be tested is

$$H_o: \alpha_j = \alpha \quad \text{for } j = 1, 2, \ldots, p$$

and the test is directed to location alternatives, provided there is no treatment x block interaction with respect to location. (i stands for blocks, j stands for treatments).

Let

$$R_{ij} = \text{Rank}\ \{Y_{ij}: Y_{i1}, \ldots, Y_{ip}\}$$

with ties handled by the mid-ranks method.

Conditional on the magnitudes of the Y's, a randomization-type model can be formulated considering the $(p!)^b$ equally probable matrices of ranks, $((R_{ij}))$, that can be obtained under H_o when all the possible permutations of the ranks in each block are taken into account. A permutation test has been developed using the conditional uniform distribution (under H_o) $\frac{1}{(p!)^b}$, and its test statistic is the quadratic form

$$Q = \frac{b(p-1)}{p\,\sigma^2_{P,R}} \sum_{j=1}^{p} (\bar{R}_{ij} - \frac{p+1}{2})^2$$

with

$$\bar{R}_{\cdot j} = \frac{\sum_{i=1}^{b} R_{ij}}{b}$$

and

$$\sigma^2_{P,R} = \frac{1}{bp} \sum_{i=1}^{b} (R_{ij} - \frac{p+1}{2})^2.$$

Under H_o, and conditionally on the actual matrix of ranks $((R_{ij}))$, Q is asymptotically distributed as a χ^2 with (p-1) d.f. This approximation applies unconditionally as well.

This test was the result of work by Kendall (1948).

Specializations of the Tests

If there are no ties, Q takes the form

$$\frac{12b}{p(p+1)} \sum_{j=1}^{p} (\bar{R}_{ij} - \frac{p+1}{2})^2$$

that is the statistics of Friedman (1937) for the problem of b rankings.

If p=2, the problem of random blocks is reduced to the one related to the sign test, and the critical region given by Q coincides with the one afforded by S for two-sided alternatives.

<u>Other Tests Related to the Rank Tests for Two-way Layout Experiments</u>

As a generalization of Friedman's procedure, Durbin (1951) presented a test suitable for the problem of random blocks when the experimental data form a balanced incomplete block design and, two years later, Benard and van Elteren (1953) extended Kendall's statistics to a two-way experimental array of data with arbitrary frequencies in the cells.

With respect to incomplete two-way layouts, the methods of paired comparisons should be mentioned. These methods are used for the comparison of p treatments presented in pairs to obtain N_{ij} responses in relation with treatments i,j; (i,j = 1, 2, ..., p).

A monograph on the subject has been written by David (1963). It presents several (univariate) models and suitable procedures for the problems of hypothesis testing and estimation. The monograph describes the works of Bradley and Terry (1952), Scheffé (1952) and others. These methods are based on intro-pair rankings and, for this reason, they can be seen as extensions of the classical sign-test. On the other hand, the test developed by Puri and Sen for the same problem (1969) uses intra and interblock information; when p=2, it coincides with the Wilcoxon's sign test.

There are several rank tests available for multivariate cross-classified data.

The Friedman test has been extended by Gerig (1969) in the case of multivariate responses.

The same has been done for the sign test: several tests go by the name of bivariate sign tests, notably those by Hodges (1955), Blumen (1968) and Chatterjee (1966). A multivariate sign test is due to Bennet (1962). Bickel (1965) has found two statistics for testing the nullity of a location parameter of a multivariate population; one is asymptotically equivalent to the sign statistics, and the other asymptotically equivalent to the Wilcoxon statistic; Sen and Puri (1967) have constructed a test for the same problem that can be seen as a multivariate version of the Wilcoxon sign test.

The development of multivariate tests for paired comparisons is just starting. The works by Sen and David (1968) and Davidson and Bradley (1969) solve the problem by using intra-block ranking, while two recent papers by Shane and Puri (1969) and Puri and Shane (1970) present tests that also use the inter-block information contained in the samples.

In this review of rank tests for two-way classified data, the articles by Sen (1968 and 1969) and Koch and Sen (1968) must be included. Nevertheless, the discussion of these papers will be postponed until the next sections since they need special treatment in view of their connection to the main subject of this research.

1.1.6.4 Rank Tests for Complex Experiments. We include both factorial designs and split-plot designs under the name of complex experiments because their location parameters under the alternatives can be expressed as rather complex linear combinations of effects.

Separate tests on the individual effects or group of effects are of interest in these experiments, the hypotheses being of the same kind as the ones of interest in the analogous parametric problems.

Mehra and Sen (1969) presented a test for interaction in a factorial experiment replicated in blocks. A permutation model is applied to the (aligned) observations, and the resulting test statistic has the same structure as the one used for testing the similar parametric hypothesis. Later on the test was extended (Puri and Sen, (1970)) for analyzing multi-response factorial experiments.

The discussion of split plot type designs from the point of view of non-parametric theory has been done by Koch in two recent papers (Koch, (1969) and (1970)). A certain number of hypotheses are considered. They are analyzed under several different assumptions concerning the joint distribution of the components of the observation vector, and different rank tests are proposed.

1.2 Mixed Models

1.2.1 Introduction

In the preceding sections most of the tests have been classified primarily according to the design of the experiment and the kind of responses (univariate of multivariate) involved, and no emphasis has been put on the model in which they are based. Oppositely, in this section special attention will be paid to the discussion of the models.

This study will be concerned with two-way layout experiment models whose underlying assumptions are non-parametric counterparts of those involved in the traditional mixed linear models. Non-parametric mixed models are the subject matter of this work, and in order that their relation with the classic analysis of variance type models be understood, a brief review of the parametric mixed models will be done in the next section.

1.2.2 Mixed Linear Models

According to the traditional point of view, to describe the data obtained from an experiment by an analysis of variance model is to assume that the observations consist of linear combinations of effects. Under the term effect what are usually called "general mean", "main effects", "interactions" and "errors" have to be understood. The effects (not directly observable quantities) are idealized formulations of some properties of interest to the investigator in the phenomena underlying the observations and in relation to these properties each effect is an unknown constant or a random variable.

The model is called a fixed effect model if the only random effect in the model equations is the error term. It is called a random effect model if all effects, except the additive constant (if there is one), are random effects. A case which falls into neither of these categories is called a mixed model.

The main effect of a level of a factor is considered a random variable if the set of levels actually tested can be thought of as a sample from a larger set of similar levels and the conclusions are to be extended to the whole class of levels and not only to the ones tested. Otherwise, if the levels under study are the only ones of interest, their effects are considered parameters of the model. It is customary to consider the interactions among random effects or among fixed and random effects as random.

When an experimental situation can be modeled by a mixed linear model, the researcher can have in mind two different kinds of inferences.

i) Estimation and tests of hypothesis on the population variances of the random effects of the model;

ii) estimation and tests of hypothesis on the fixed effects.

If the interest is in the type i) inference, the problem is identified as a variance component problem. From this point of view, the fixed effects of the model are nuisance parameters whose influence on the conclusions should be eliminated.

On the other hand, for problem ii), the variances of the random effects are taken into account only to the extent that the errors associated with the estimates of the fixed effect treatments are functions of them.

The methods of analysis for handling both problems depend on the model equations and on the assumptions about the distributional properties of the random effects.

Among the many models that can be generated by these means, only two are going to be considered here: the one presented by Eisenhart and the one given by Scheffé. Solely the related test of hypothesis will be mentioned, since this thesis is addressed only to hypothesis testing problems.

Both models will be discussed in relation to a two-way layout experiment. In the sequel Y_{ij} will represent the yield or the response obtained when the $j^{\underline{th}}$ treatment is applied to the $i^{\underline{th}}$ block ($j = 1, 2, \ldots, p$; $i = 1, 2, \ldots, b$).

1.2.2.1 Eisenhart Models. Eisenhart (1947) defined a mixed model or model III as a mixture of his models I and II described below:

Model I

The numbers Y_{ij} are (observed values of) random variables distributed about the true mean m_{ij} that is a fixed constant and it is assumed that

$$Y_{ij} = m_{ij} + e_{ij} = \alpha_i^B + \alpha_j^A + e_{ij} \qquad (i=1, \ldots b;\ j = 1, \ldots p) \tag{1.2.2.1.1}$$

where α_i^B is the effect of the block and α_j^A is the treatment effect (both considered fixed) and the error terms e's are independent, homoscedastic and normally distributed.

Model II

The random variables Y_{ij} are assumed to be the sum of a fixed constant μ and random variables a_i^B, a_j^A and e_{ij} that reflect the random effects of the blocks, treatments and the experimental error respectively, that is

$$Y_{ij} = \mu + a_i^B + a_j^A + e_{ij} \tag{1.2.2.1.2}$$

$$(i = 1, \ldots, b;\ j = 1, \ldots, p).$$

It is assumed that the a's, b's and e's are jointly independent and normally distributed random variables with zero means and variances σ_B^2, σ_A^2 and σ^2, respectively.

The assumptions of Eisenhart govern the methods of analysis described in most of the current textbooks and, only when defining the interactions, do some divergences occur. We will say that a model belongs to the Eisenhart class if all the random components (main effects, interactions, errors) are jointly normally distributed with zero mean and diagonal covariance matrix. It should be noted that the observable variables Y_{ij} described by a mixed model are correlated even if the errors are independent. Let us consider the covariance matrix of the Y's when the model equation is

$$Y_{ij\ell} = \mu + \alpha_j^A + a_i^B + a_{ij}^{AB} + e_{ij\ell} \quad \begin{pmatrix} i = 1, 2, \ldots, b \\ j = 1, 2, \ldots, p \\ \ell = 1, 2, \ldots, c \end{pmatrix} \tag{1.2.2.1.3}$$

where a_{ij}^{AB} represents the (random) effect of the interaction between the fixed factor (A) and the random one (B). The usual side conditions imposed on the effects are

$$\Sigma \alpha_j^A = 0$$

and

$$\sum_{j=1}^{p} a_{ij} = 0 \quad \forall \quad i = 1, 2, \ldots, b .$$

Let the Y's to be ordered as in $\underset{\sim}{Y}'$

$$\underset{\sim}{Y}' = (Y_{111}, \ldots, Y_{11c}, \ldots, Y_{1p1}, \ldots, Y_{1pc}, \ldots, \qquad (1.2.2.1.3)$$

$$Y_{b11}, \ldots, Y_{b1c}, \ldots, Y_{bp1}, \ldots, Y_{bpc})$$

and let $\underset{\sim}{\Sigma}$ represent its covariance matrix. It can be expressed as follows:

$$\underset{\sim}{\Sigma} = \text{diag}(\underset{\sim}{D}, \underset{(b)}{\ldots}, \underset{\sim}{D}) \qquad (1.2.2.1.4)$$

with

$$\underset{\sim}{D} = \begin{pmatrix} \underset{\sim}{C} & \underset{\sim}{B} \cdots \underset{\sim}{B} \\ \underset{\sim}{B} & \underset{\sim}{C} \cdots \underset{\sim}{B} \\ \cdots\cdots & \cdots\cdots \\ \underset{\sim}{B} & \underset{\sim}{B} \cdots \underset{\sim}{C} \end{pmatrix} , \qquad (1.2.2.1.5)$$

where $\underset{\sim}{D}$ has dimension (pc x pc) and the $\underset{\sim}{B}$'s and $\underset{\sim}{C}$'s are (c x c) matrices as defined below:

$$\underset{\sim}{B} = (\rho_o \sigma_o^2) \underset{c\sim c}{J} \qquad (1.2.2.1.6)$$

with

$$\rho_o = \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2}$$

and

$$\sigma_o^2 = \sigma_A^2 + \sigma_B^2$$

$$\underset{\sim}{C} = (1 - \rho_1)\,\sigma_1^2\,\underset{\sim}{I}_c + \rho_1\sigma_1^2\,\underset{\sim}{J}_{cc}\ , \qquad (1.2.2.1.7)$$

where

$$\rho_1 = \frac{\sigma_B^2 + \sigma_{AB}^2}{\sigma_B^2 + \sigma_{AB}^2 + \sigma^2}$$

and

$$\sigma_1^2 = \sigma_B^2 + \sigma_{AB}^2 + \sigma^2 .$$

The symbols $\underset{\sim}{J}_{cc}$ and $\underset{\sim}{I}_c$ stand for a c x c matrix with all its entries equal to 1 and for the c x c unity matrix, respectively.

When the hypothesis

$$H_o\colon\ \alpha_j^A = 0 \quad \forall \quad j = 1 \ldots, p$$

is considered, the classical test is based on the vector of cell means

$$\underset{\sim}{\overline{Y}}' = (\overline{Y}_{11.}, \ldots, \overline{Y}_{1p.}, \overline{Y}_{21.}, \ldots, \overline{Y}_{2p.}, \ldots, \overline{Y}_{b1}, \ldots, \overline{Y}_{bp}) \qquad (1.2.2.1.8)$$

whose covariance matrix, call it $\bar{\underset{\sim}{\Sigma}}$, exhibit the following pattern

$$\bar{\underset{\sim}{\Sigma}} = \text{diag}(\bar{\underset{\sim}{D}}, \underset{(b)}{\ldots}, \bar{\underset{\sim}{D}}) \qquad (1.2.2.1.9)$$

$$\bar{\underset{\sim}{D}} = (1 - \rho_2)\, \sigma_2^2\, \underset{\sim}{I}_p + \rho_2 \sigma_2^2\, {}_p\underset{\sim}{J}_p \qquad (1.2.2.1.10)$$

with

$$\rho_2 = \frac{\sigma_B^2}{\sigma_B^2 + \sigma_{AB}^2 + \sigma^2/c}$$

and

$$\sigma_2^2 = \sigma_B^2 + \sigma_{AB}^2 + \sigma^2/c \quad .$$

Matrices like $\underset{\sim}{C}$ (1.2.2.1.7) and $\bar{\underset{\sim}{D}}$ (1.2.2.1.10) that can be expressed as the linear combination $(a-b)\,\underset{\sim}{I} + b\,\underset{\sim}{J}$ of the matrices $\underset{\sim}{I}$ and $\underset{\sim}{J}$ are called uniform matrices.

The test of the hypothesis H_o is made by using the test statistic

$$\frac{(p-1)(b-1)}{(p-1)} \; \frac{\sum_{j=1}^{p} (\bar{Y}_{ij.} - \bar{Y}_{...})^2}{\sum_{i=1}^{b} \sum_{j=1}^{p} (\bar{Y}_{ij.} - \bar{Y}_{.j.} - \bar{Y}_{i..} + \bar{Y}_{...})^2} \qquad (1.2.2.1.11)$$

where the $\bar{Y}$'s represent the means of the Y's with respect to the subscripts for which dots have been substituted.

If the assumptions of the Eisenhart model hold, the test statistic is distributed as an F-variable with (p-1) and (p-1)(b-1) d.f. and it provides an exact test for the hypothesis H_o.

1.2.2.2 Scheffé's Model. The results obtained under Eisenhart's assumptions have led to the use of these models even in cases in which the validity of the assumptions can hardly be defended.

Scheffé points out that the assumptions of Eisenhart's models are unsatisfactory and *ad hoc*, and he presents a new mixed model for two-way classified data that is less restrictive and more complete than Eisenhart's mixed model. The model equations are

$$Y_{ijk} = m_{ij} + e_{ijk} \quad \begin{pmatrix} i = 1, \ldots, b \\ j = 1, \ldots, p \\ k = 1, \ldots, c \end{pmatrix} \qquad (2.2.2.1)$$

where m_{ij} is the true cell mean and e_{ijk} is a random variable that represents an error. m_{ij} is a function of the fixed levels of a factor A and the random levels of a factor B. The b x p matrix

$$((m_{ij})) \quad \begin{pmatrix} i = 1, \ldots, b \\ \\ j = 1, \ldots, p \end{pmatrix}$$

is a random matrix and each row-vector is considered a random sample from a normal distribution with the covariance matrix $((\sigma_{jj'}))$.

The errors

$$\{e_{ijk} : i = 1, \ldots, b; \; j = 1, \ldots, p; \; k = 1, \ldots, c\}$$

constitute a random sample from a normal distribution $N(0,\sigma^2)$ and are independent from the means

$$\{m_{ij}\colon\ k = 1, \ldots, b;\ \ j = 1, \ldots, p\}\ .$$

The effects of the levels of the factors A,B and their interactions are defined in terms of the basic m's as follows:

$$\begin{aligned}
&\text{General mean:}\ \ \mu = m_{\cdot\cdot} \\
&\text{Fixed effect (or A):}\ \ \alpha_j^A = m_{\cdot j} - m_{\cdot\cdot} \\
&\text{Random effect (or B):}\ \ a_i^B = m_{i\cdot} - m_{\cdot\cdot} \\
&\text{Interaction:}\ \ a_{ij}^{AB} = m_{ij} = m_{i\cdot} - m_{\cdot j} + m_{\cdot\cdot}
\end{aligned} \qquad (1.2.2.2)$$

A dot replacing a j-subscript indicates summation on the levels of A and a dot substituting an i-subscript means expectation with respect to the levels of B. The model can be represented by the equation

$$Y_{ijk} = \mu + \alpha_j^A + a^B + a_{ij}^{AB} + e_{ijk} \qquad (1.2.2.2.3)$$

The distributional properties of the effects and their side conditions are a consequence of their functional relations with the variables $\{m_{11}, \ldots, m_{bp}\}$.

a) $\sum\limits_j \alpha_j^A = 0 \qquad \sum\limits_j a_{ij}^{AB} = 0$

b) the a_i^B and a_{ij}^{AB} are jointly normally distributed with a zero mean and with the following covariances:

$$\mathrm{Cov}(a_i^B , a_{i'}^B) = \partial_{ii'} \sigma_{..}$$

$$\mathrm{Cov}(a_{ij}^{AB} , a_{i'j'}^{AB}) = \partial_{ii'}(\sigma_{jj'} - \sigma_{.j} - \sigma_{.j'} + \sigma_{..}) \quad (1.2.2.2.3)$$

$$\mathrm{Cov}(a_i^B , a_{i'j}^{AB}) = \partial_{ii'} (\sigma_{.j} - \sigma_{..}) ,$$

where $\partial_{ii'}$ is the Kronecker index and the dots indicate either summation on the rows or columns of the matrix $((\sigma_{jj'}))$.

It can be seen that in this model the interactions and main effects of the random factor, as well as the interactions associated with the same random level of B, are or can be correlated. This is the basic difference between the earlier models and the one proposed by Scheffé.

These assumptions generate a covariance matrix for the observations that differs from the one obtained with Eisenhart's model. Let the vector of observations $\underset{\sim}{Y}'$ defined as in (1.2.2.1.3); then its covariance matrix under the Scheffé's model is:

$$\underset{\sim}{\Sigma} = \mathrm{diag}(\underset{\sim}{D}_{(b)}, \ldots, \underset{\sim}{D})$$

$$\underset{\sim}{D} = \begin{pmatrix} \underset{\sim}{D}_{11} & \underset{\sim}{D}_{12} \cdots & \underset{\sim}{D}_{1p} \\ \underset{\sim}{D}_{21} & \underset{\sim}{D}_{22} \cdots & \underset{\sim}{D}_{2p} \\ \cdots & \cdots\cdots & \cdots \\ \underset{\sim}{D}_{p1} & \underset{\sim}{D}_{p2} \cdots & \underset{\sim}{D}_{pp} \end{pmatrix} = ((D_{jj'})) . \quad (1.2.2.2.4)$$

where

$$\underset{\sim}{D}_{jj'} \begin{cases} \underset{\sim c}{J}_c \sigma_{jj'} + \underset{\sim}{I}_c \sigma^2 & \text{if } j = j' \\ \underset{\sim c}{J}_c \sigma_{jj'} & \text{if } j \neq j' \end{cases} .$$

Scheffé constructs a test of the hypothesis H_o, based on the vector of means

$$(\overline{Y}_{11.}, \ldots, \overline{Y}_{1p.}, \overline{Y}_{21.}, \ldots, \overline{Y}_{2p.}, \ldots, \overline{Y}_{b1.}, \ldots, \overline{Y}_{bp}) \quad (1.2.2.2.5)$$

whose covariance matrix is

$$\underset{\sim}{\Sigma} = \operatorname{diag}(\underbrace{\underset{\sim}{D}, \underset{\sim}{D}, \ldots, \underset{\sim}{D}}_{(b)})$$

with

$$\underset{\sim}{D} = ((\sigma_{jj'})) + {}_p\underset{\sim}{I}_p \sigma^2/c \quad . \qquad (1.2.2.2.6)$$

The corresponding test statistic is

$$T^2 = b\, \underset{\sim}{\overline{Y}}' \underset{\sim}{C}' (\underset{\sim}{C}\, \underset{\sim}{V}\, \underset{\sim}{C}')^{-1} \underset{\sim}{C}\, \underset{\sim}{\overline{Y}} \quad . \qquad (1.2.2.2.7)$$

where

$$\overline{Y}' = (\overline{Y}_{.1.}, \overline{Y}_{.2.}, \ldots, \overline{Y}_{.p.}) \quad ,$$

$V = ((v_{jj'}))$ is a p x p matrix with entries ,

$$v_{jj'} = \frac{1}{(b-1)} \sum_{i=1}^{b} (\overline{Y}_{ij.} - \overline{Y}_{.j.})(\overline{Y}_{ij'.} - \overline{Y}_{.j'.}) \quad ,$$

and $\underset{\sim}{C}$ is a (p-1) x p matrix of linearly independent contrasts.

If the null hypothesis is true $\frac{b-p+1}{(b-1)(p-1)}$, T^2 is distributed as the Snedecor F variable with (b-p+1) and (b-1)(p-1) d.f. (It is assumed that $b > p$).

Note that if

$$\sigma_{jj'} = \begin{cases} \rho\,\sigma^2 & j \neq j' \\ \sigma^2 & j = j' \end{cases}$$

then the matrices (1.2.2.1.10) and (1.2.2.2.6) coincide, with

$$\sigma^2 = \sigma_{AB}^2 + \sigma_B^2$$

$$\rho = \frac{\sigma_B^2}{\sigma_{AB}^2 + \sigma_B^2}$$

and, in this situation, the test given by (1.2.2.1.11) is applicable.

Before entering the discussion on non-parametric mixed models, let us say that mixed linear models can be also defined for multi-dimensional data. Roy and Gnanadesikan (1959) have defined and studied multivariate linear models, fixed and random, that are extensions of the Eisenhart's models. The multivariate mixed linear model can be studied by using methods which are, essentially, a combination of the methods given for the separate models, I and II.

Another paper by Roy and Cobb (1960) deals with mixed models with non-normal random effects. But their basic interest lies on the variance component problem an do not consider the test for the fixed effects.

1.2.3 Non-parametric Mixed Models

Mixed models were discussed in connection with linear models, but their essential characteristics make it possible to extend the concept beyond this original context.

For the two-way layout experiments, these basic characteristics can be summarized as follows:

i) The observations in the same block are correlated, so that in spite of their univariate appearance the blocks have a multivariate nature;

ii) the location parameters (expectations in the normal case) of the components of each block vary according to the levels of a certain fixed factor.

The following definition of a non-parametric mixed model is free of many of the assumptions understood in the parametric model, but still keeps the basic characteristics cited in the previous paragraphs.

Given the matrix of observable random variables

$$\begin{pmatrix} Y_{11} & Y_{12} & \cdots & Y_{1p} \\ Y_{21} & Y_{22} & \cdots & Y_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ Y_{b1} & Y_{b2} & \cdots & Y_{bp} \end{pmatrix}, \qquad (1.2.3.1)$$

it will be said that it conforms to a mixed model if:

The row-vectors

$$\{(Y_{i1}, Y_{i2}, \ldots, Y_{ip}): \quad i = 1, 2, \ldots, b\}$$

are mutually independent random variables from some continuous joint cumulative distribution function $F_i(\underset{\sim}{Y})$

$$F_i(\underset{\sim}{Y}) = G_i(\underset{\sim}{Y} - \underset{\sim}{m}_i) = G_i(\underset{\sim}{e}) \qquad (1.2.3.2)$$

with location parameter $\underset{\sim}{m}_i = (m_{i1}, m_{i2}, \ldots, m_{ip})$ such that

$$m_{ij} = b_i + \tau_j \quad (i = 1, \ldots, b; \; j = 1, \ldots, p). \qquad (1.2.3.3)$$

Koch and Sen (1968) describe four types of mixed models, that arise in relation with the assumptions that can be (or cannot be) made about the nature of the distribution functions F_i. The basic assumption that holds in all the cases is:

A_1: The joint distribution of any linearly independent set of contrasts among the observations on any particular vector $\underset{\sim}{Y}_i$ is diagonally symmetric.

Two additional assumptions which may or may not be imposed are:

A_2: The "additivity" of block effects. That is to say,

$$G_1 = G_2 = \cdots = G_b = G$$

A_3: The "compound symmetry" of the error vectors. That means that $G_i(x_1, x_2, \ldots, x_p)$ is symmetric in $(x_1, \ldots, x_p)$ $\quad i = 1, \ldots, b$.

The presence or absence of A_2 and A_3 among the assumptions determines four cases of interest, that are described in the following table.

	Not A_2	A_2
Not A_3	Case I	Case III
A_3	Case II	Case IV

Assumption A_1 means that if

$$\underset{\sim}{U} = \underset{\sim}{C}_1 \underset{\sim}{Y}$$

is a set of (p-1) independent contrasts with location parameter θ, then $(\underset{\sim}{U}-\underset{\sim}{\theta})$ and $(\underset{\sim}{\theta}-\underset{\sim}{U})$ have the same distribution. This assumption is less restrictive than the multinormality of the $\underset{\sim}{Y}$'s. Because if $\underset{\sim}{Y}$ is normally distributed, $\underset{\sim}{U} = \underset{\sim}{C}_1 \underset{\sim}{Y}$ has a (p-1) dimensional normal distribution and, then, not only $(\underset{\sim}{U}-\underset{\sim}{\theta})$ and $(\underset{\sim}{\theta}-\underset{\sim}{U})$ have the same distribution but this is the case with all the points on the ellipse (of center $\underset{\sim}{\theta}$)

$$Q(\underset{\sim}{U}-\underset{\sim}{\theta}) = K$$

where K represents some constant and Q stands for the exponent of the multinormal density function.

When the assumption of normality is made, it is also assumed that the error vectors $\underset{\sim}{e}_i$ have the same distribution for any i. This condition is not required in cases I and II. But in cases III and IV, G_i has to be equal to G for any i. This property, known as the additivity property, does not hold for heteroscedastic experiments.

The concept of compound symmetry is connected with the concept of interaction. Compound symmetry means that the errors are independent of the columns (treatments) with which they are associated. If compound symmetry exists, all the marginal distribution functions that can be obtained from $G_i(\underset{\sim}{e})$ are equal, provided they have the same number of coordinates. This implies that the covariance matrix of $\underset{\sim}{e}_i$ is uniform, i.e., it can be expressed as

$$a\underset{\sim}{I} + (b-a)\,\underset{\sim}{J} \; .$$

In accordance with the four types of mixed models, Koch and Sen present four different tests for the hypothesis H_o of no treatment effects. In symbols,

$$H_o: \;\; \tau_1 = \tau_2 = \cdots = \tau_p = 0 \; .$$

All the tests are addressed to translation-type of alternatives and belong to the class of rank-permutation tests described in the first section of this chapter.

In the following chapters some extensions of the cases I, II and IV will be considered. But before discussing these extensions in detail, some remarks will be made in the next section that apply to all the chapters of this thesis.

1.2.4 General Remarks and Notation used in Relation to the Extensions of Non-parametric Mixed Models

The extensions of mixed models to be discussed in the next section go in two directions:

i) The number of results per treatment in each block is allowed to be different from 1;

ii) The responses of the treatments can be multivariate.

Several extensions are presented; their descriptions require the use of sets of parallel symbols in relation with parallel concepts involved in the various models. But in order to simplify the writing, the same symbols are used for similar ideas in different situations. So, they have to be interpreted at the light of the context in which they appear.

The notation, as well as the remarks that follow, are valid for all the cases for which they are applicable.

i) Notation

Either the terms "block" or "subject" is used indistinctly to designate the set of observations that constitute an independent replication of the experiment. b and p stand for the number of blocks and treatments in the experiment. c indicates the dimension of each response. N_{ij} indicates the number of results obtained under treatment j for subject i. $(N_{ij} \geq 0)$.

$$\sum_{j=1}^{p} N_{ij} = N_i ,$$

the number of results associated with subject "i"

$$\sum_{i=1}^{b} N_i = N ,$$

number of results in the experiment; b_j indicates the number of blocks in which treatment j appears $(0 < b_j \leq b)$, and S_{bj} stands for the set of blocks in which treatment j is present.

ii) Rank totals, their covariance and correlation matrix

With only one exception, all the test statistics to be considered in the sequel are based on totals of ranks obtained for each treatment and variable separately. With p treatments and c-variate responses, the vector of rank totals has pc coordinates, and they are assumed to be ordered in the following way

$$\underset{\sim}{T}_b' = (T_{b1}^{(1)}, \ldots, T_{bp}^{(1)}, T_{b1}^{(2)}, \ldots, T_{bp}^{(2)}, \ldots, T_{b1}^{(i)}, \ldots, T_{bp}^{(i)}) \qquad (1.2.4.1)$$

where

$$T_{bj}^{(k)} \quad (k = 1, \ldots, c; \ j = 1, \ldots, p)$$

stands for the total that correspond to the $k^{\underline{th}}$

variable and the $j^{\underline{th}}$ treatment. The subscript b means that the total is computed over a number of blocks that depends on b. The covariance matrix of $\underset{\sim}{T}'_b$ is identified as

$$\underset{\sim}{V}_b = ((\mathrm{Cov}(T^{(k)}_{bj},\ T^{(k')}_{bj'} | \rho_b))) .$$

$\underset{\sim}{V}_b$ can be partitioned as follows:

$$\underset{\sim}{V}_b = \begin{pmatrix} \underset{\sim}{V}_b^{(11)} & \underset{\sim}{V}_b^{(12)} & \cdots & \underset{\sim}{V}_b^{(1c)} \\ \underset{\sim}{V}_b^{(21)} & \underset{\sim}{V}_b^{(22)} & \cdots & \underset{\sim}{V}_b^{(2c)} \\ \cdot & \cdot & \cdots & \cdot \\ \underset{\sim}{V}_b^{(c1)} & \underset{\sim}{V}_b^{(c2)} & \cdots & \underset{\sim}{V}_b^{(cc)} \end{pmatrix} = ((\underset{\sim}{V}_b^{(kk')})) \quad (1.2.4.2)$$

where

$$\underset{\sim}{V}_b^{(kk')} \qquad (k,k' = 1, \ldots, c)$$

represents the matrix of covariance of those pairs of totals in which one total refers to the variable "k" and the other to the variable "k′" .

The correlation coefficient of

$$(T^{(k)}_{bj} \ ,\ \ T^{(k')}_{bj'})$$

is represented by

$$\rho_{bjj'} = \frac{\mathrm{Cov}(T^{(k)}_{bj} \ ,\ T^{(k')}_{bj'} | \rho_b)}{[\mathrm{Var}(T^{(k)}_{bj} | \rho_b)\ \mathrm{Var}(T^{(k')}_{bj'} | \rho_b)]^{1/2}} \quad . \qquad (1.2.4.3)$$

The symbol

$$\sigma_{bj}^{(k)\ 2}$$

is also used for

$$\mathrm{Var}(T_{bj}^{(k)} | \rho_b) \ .$$

$\underset{\sim}{T}_{b*}$ designates the vector obtained from $\underset{\sim}{T}$ by omitting the c coordinates that correspond to the $p^{\underline{th}}$ treatment. $\underset{\sim}{V}_{b*}$ denotes the covariance matrix of $\underset{\sim}{T}_{b*}$.

iii) Non-compared set of treatments

If there exist two or more complementary sets of treatments such that any two treatments belonging to different sets do not appear together in any block, that sets of treatments are called "non-compared sets". If there is more than one non-compared set of treatments the design is disconnected.

For any two treatments j, j′ belonging to different non-compared sets

$$\sum_{i=1}^{b} N_{ij} N_{ij'} = 0 \ .$$

The number of non-compared sets is called S and the number of treatments in the $s^{\underline{th}}$ set is called

$$p_s \quad (s = 1, 2, \ldots, S) \ .$$

If there are S non-compared sets of treatments $\underset{\sim}{V}_b$ can be written by permuting rows and columns as

$$\underset{\sim}{V}_b = \begin{pmatrix} \underset{\sim}{V}_{b_1} & \underset{\sim}{0} & \cdots & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{V}_{b_2} & \cdots & \underset{\sim}{0} \\ \cdot & \cdot & \cdots & \cdot \\ \underset{\sim}{0} & \underset{\sim}{0} & \cdots & \underset{\sim}{V}_{b_s} \end{pmatrix} \qquad (1.2.4.4)$$

where $\underset{\sim}{V}_{b_s}$ stands for the covariance matrix of the cp_s-vector of treatment totals in the set "s". ($c \geq 1$).

iv) Ties

All the models discussed in this work assume that the data of the experiment are observed values of random variables that have a continuous cumulative distribution function. For that reason ties are assumed to happen with probability zero, and they are not considered in the theoretical discussion.

If ties occur in practice, they can be managed by the mid-ranks method.

v) Superfluous Blocks

When the ranking is made individually for each block those blocks such that $N_{ij} = 0$ for all j except at most for only one treatment are omitted from the experimental data to be analyzed. This implies that for any block N_i must be at least equal to 2.

In practice, the vectors with all tied observations are omitted as well.

vi) Distribution of the Test Statistic

When the frequencies of the treatments are unequal, the computation of the permutation distribution of the test statistic becomes very difficult because of the large number of parameters involved. For this reason, in all the types of mixed models that are considered in this work only the asymptotic (permutational) distribution of the test criterion is considered.

In the discussion of the limit distribution, the rank of

$$\lim_{b \to \infty} \left(\left(\rho_{bjj'}^{(kk')} \right)\right)$$

comes into consideration. The question of under which conditions the rank of this limit has a given value is not discussed here. For practical purposes, it is assumed that the sample is large enough so that the actual rank of

$$\left(\left(\rho_{bjj'}^{(kk')} \right)\right)$$

coincides with the rank of its limit.

2. MIXED MODELS FOR THE CASE OF SEVERAL (EVENTUALLY ONE OR ZERO) RESULTS PER TREATMENT. UNIVARIATE RESPONSES

2.1 Introduction

Linear mixed models for two-way classified data such that the number of results per treatment in the same block is (or can be) different from one arise in relation to randomized block experiments of several kinds.

If the number of results per treatment takes only the values zero and one, the design of the experiment is named incomplete randomized block design.

If the number of results per treatment is different from one and bigger than zero for at least one treatment, the design can be of the type either of a generalized randomized block design or of a randomized block design with several observations per cell. In the first kind of experiment, the various results associated with a treatment in the same block come from different experimental units, for instance different plots of the same block of land or different trials performed on the same patient. On the other hand, in the second type of experiment, the (several) results that correspond to the same treatment are obtained from the same experimental unit, say, by selecting several samples from the plot of land or by performing several determinations after the patient has been submitted to a unique trial of the treatment.

The theory for the analysis of parametric mixed linear models of the type described above has been fully studied for balanced designs or designs with a high degree of symmetry. (The examples that illustrate the models worked by Eisenhart and Scheffé are applicable to randomized block designs with several, equal in number, observations per cell.) But, although a lot of work has been done for the case of unbalanced designs, a complete and satisfactory answer is still lacking.

In extending the non-parametric mixed models to the case of several results per treatment, it is convenient to keep in mind the type of experiment that lies in the background of each model. There is not a one to one correspondence between the models that will be presented in the next sections and the designs mentioned in the previous paragraphs. It can be anticipated that the extension of case I to be given in Section 2.2.2 can be applied, according to the circumstances, to any type of experiment. On the contrary, the extension of model IV (to be developed in Section 2.2.3) is only suitable for either experiments of the generalized block design type, or incomplete block design type.

In Section 2.3, two rank tests will be presented that are related to a model, that is the non-parametric counterpart of the mixed linear model underlying a complete randomized block design with several observations per cell.

2.2 Mixed Models for the Case of Several (eventually one or zero) Results per Treatment. Extensions of Models I and IV

2.2.1 Introduction

Let

$$\underset{\sim}{Y}_i' = (Y_{i11}, \ldots, Y_{i1N_{i1}}, Y_{i21}, \ldots,$$

$$Y_{i2N_{i2}}, \ldots, Y_{ip1}, \ldots, Y_{ipN_{ip}}) \qquad (2.2.1.1)$$

$$(i = 1, \ldots, b)$$

be b stochastically independent vectors where

$$Y_{ij\ell} \quad (j = 1, \ldots, p; \ \ell = 1, \ldots, N_{ij})$$

represents the response of the ℓ^{th} trial of the j^{th} treatment in the i^{th} subject or block.

It is assumed that $\underset{\sim}{Y}_i$ has a continuous N_i-variate cumulative distribution function $F_i(\underset{\sim}{y})$ such that

$$F_i(\underset{\sim}{y}) = G_i(\underset{\sim}{y} - \underset{\sim}{m}_i), \quad \underset{\sim}{m}_i' = (m_{i11}, \ldots, m_{iN_{i1}}, \ldots, m_{ipN_{ip}}) \qquad (2.2.1.2)$$

with

$$m_{ij\ell} = b_i + \tau_j, \qquad (2.2.1.3)$$

$$(i = 1, \ldots, b; \ j = 1, \ldots, p; \ \ell = 1, \ldots, N_{ij})$$

The hypothesis to be tested is the hypothesis of homogeneity of treatment effects

$$H_o: \tau_1 = \tau_2 = \cdots = \tau_p = 0$$

against

H_1: at least one τ is different from zero.

Together with assumptions (2.2.1.2) and (2.2.1.3) some additional assumptions may or may not be added.

These optional assumptions are

A_1: Diagonal symmetry

A_2: Additivity of subject effects

A_3: Compound symmetry of the error vectors of the same block.

The meaning of these assumptions are obvious extensions of the ones considered in the case $N_{ij} = 1$. Only that now the number of coordinates is N_i instead of p.

With respect to A_2, it has to be added that in this case the equality of the cumulative distribution functions G_i has to be understood as equality of the marginal of the same group of variables.

2.2.2 Case UI

Together with (2.2.1.2) and (2.2.1.3), it is considered that assumption A_1 holds.

In order to test H_o, let us define

$$Z_{ij\ell} = Y_{ij\ell} - \bar{Y}_{i\cdot\cdot} \tag{2.2.2.1}$$

$$(i = 1, \ldots, p;\ j = 1, \ldots, p;\ \ell = 1, \ldots, N_{ij})$$

and

$$R_{ij\ell} = \text{rank}\{Z_{ij\ell}:\ Z_{i11}, \ldots, Z_{ipN_{ip}}\} \tag{2.2.2.2}$$

$$(i = 1, \ldots, b;\ j = 1, \ldots, p;\ \ell = 1, \ldots, N_{ij}).$$

The Z's are contrasts; hence, under H_o and assumption A_1,

$$\underset{\sim}{Z}_i' \quad \text{(and} \quad -\underset{\sim}{Z}_i' \quad (i = 1, \ldots, b)\ ,$$

where

$$\underset{\sim}{Z}_i' = (Z_{i11}, \ldots, Z_{i1N_{i1}}, Z_{i21}, \ldots, Z_{i2N_{i2}}, \ldots, Z_{ip1}, \ldots, Z_{ipN_{ip}})\ , \tag{2.2.2.3}$$

have the same distribution. As a consequence, the vectors of ranks

$$\underset{\sim}{R}_i' = (R_{i11}, \ldots, R_{i1N_{i1}}, R_{i21}, \ldots, R_{i2N_{i2}}, \ldots, R_{ip1}, \ldots, R_{ipN_{ip}})\ , \tag{2.2.2.4}$$

are (conditionally) equally probable, each one occurring with conditional probability 1/2 for any value of i. Then, given the actual vectors $\{\underset{\sim}{R}_i:\ i = 1, \ldots, b\}$, this sign invariance generates a set of 2^b equally likely realizations. Let us call $\mathcal{P}_b$ the related conditional probability law.

Under $\mathcal{P}_b$,

$$E(R_{ij\ell}|\mathcal{P}_b) = \frac{R_{ij\ell} + (N_i + 1 - R_{ij\ell})}{2} = \frac{N_i + 1}{2} \qquad (2.2.2.5)$$

$$E(R_{ij\ell}R_{i'j'\ell'}|\mathcal{P}_b) = \begin{cases} R_{ij\ell}R_{ij'\ell'} + (N_i + 1 - R_{ij\ell})(N_i + 1 - R_{ij'\ell'}) & \text{if } i = i' \\ \left(\frac{N_i + 1}{2}\right)\left(\frac{N_{i'} + 1}{2}\right) & \text{if } i \neq i' \end{cases} \qquad (2.2.2.6)$$

and

$$\mathrm{Cov}(R_{ij\ell}, R_{i'j'\ell'}|\mathcal{P}_b) = \begin{cases} \left(R_{ij\ell} - \frac{N_i + 1}{2}\right)\left(R_{ij'\ell'} - \frac{N_{i+1}}{2}\right) & \text{if } i = i' \\ 0 & \text{if } i \neq i' \end{cases} \qquad (2.2.2.7)$$

for all

$$i, i' = 1, \ldots, b; \quad j, j' = 1, \ldots, p;$$

and

$$\ell = 1, N_{ij}, \ \ell' = 1, \ldots, N_{i'j'} \ .$$

The test statistic to be used to test H_o is based on the totals

$$T_{bj} = \sum_{i=1}^{b} \sum_{\ell=1}^{N_{ij}} R_{ij\ell} \quad (j = 1, \ldots, p) \qquad (2.2.2.8)$$

that have the following first and second moments

$$E(T_{b_j} | \rho_b) = \sum_{i=1}^{b} N_{ij}\left(\frac{N_i + 1}{2}\right) \tag{2.2.2.9}$$

$$\mathrm{Cov}(T_{b_j}, T_{b_{j'}} | \rho_b) = \sum_{i=1}^{b} \left[\sum_{\ell=1}^{N_{ij}} \left(R_{ij\ell} - \frac{N_i+1}{2}\right)\right]\left[\sum_{\ell=1}^{N_{ij'}} \left(R_{ij'\ell} - \frac{N_i+1}{2}\right)\right], \tag{2.2.2.10}$$

the summation (over i) being extended only on those blocks in which both treatments j and j′ are present.

The T's are subject to a linear constrain

$$\sum_{j=1}^{p} T_j = \sum_{i=1}^{b} \frac{N_i(N_i + 1)}{2} \tag{2.2.2.11}$$

and also can be seen that

$$\sum_{j'=1}^{p} \mathrm{Cov}(T_{b_{j'}}, T_{b_{j'}} | \rho_b) = 0 \qquad (j = 1, \ldots, p) \tag{2.2.2.12}$$

So that the rank of $\underset{\sim}{V}_b$ is at most (p-1). If there are S non-compared sets of treatments the rank is, at most (p-S).

If the rank of $\underset{\sim}{V}_b$ is (p-1) the test statistic is taken to be

$$W_b = [\underset{\sim}{T}_b - E(\underset{\sim}{T}_b|\rho_b)]'\underset{\sim}{C}'[\underset{\sim}{C}\underset{\sim}{V}_b\underset{\sim}{C}']^{-1}\underset{\sim}{C}[\underset{\sim}{T}_b - E(\underset{\sim}{T}_b|\rho_b)] \tag{2.2.2.13}$$

where $\underset{\sim}{T}_b$ is defined as in (1.2.4.1) and $\underset{\sim}{C}$ is any (p-1) x p matrix of linearly independent contrasts.

In particular, $\underset{\sim}{C}$ can have the form

$$\underset{\sim}{C} = [\underset{\sim}{J}_{p-1} \vdots {}_{p-1}\underset{\sim}{0}_1] - \frac{1}{p}\, {}_{p-1}\underset{\sim}{J}_p \tag{2.2.2.14}$$

and in this case,

$$W_b = [\underset{\sim}{T}_{b*} - E(\underset{\sim}{T}_{b*}|\rho_b)]'\underset{\sim}{V}_{b*}^{-1}[\underset{\sim}{T}_{b*} - E(\underset{\sim}{T}_{b*}|\rho_b)] \tag{2.2.2.15}$$

where $\underset{\sim}{T}_{b*}$ is the vector formed by the first (p-1) coordinates of $\underset{\sim}{T}_b$, and $\underset{\sim}{V}_{b*}$ is its covariance matrix.

If there are $S(\neq 1)$ non-compared sets and the covariance matrix of the $S^{\underline{th}}$ set has rank (p_s-1), W_b still has the form (2.2.2.13) with

$$\underset{\sim}{C} = \text{diag}\ (\underset{\sim}{C}_1, \underset{\sim}{C}_2, \cdots, \underset{\sim}{C}_S) \tag{2.2.2.16}$$

with

$$\underset{\sim}{C}_s \ (s = 1, \ldots, S)$$

being a matrix of (p_s-1) linear independent contrasts among the p_s treatments in the same set.

Also, in this case W_b can be written as

$$W_b = \sum_{s=1}^{S} [\underset{\sim}{T}_{b(s)} - E(\underset{\sim}{T}_{b(s)}|\rho_b)]'\underset{\sim}{C}'_{(s)}[\underset{\sim}{C}_{(s)}\underset{\sim}{V}_{b(s)}\underset{\sim}{C}'_{(s)}]^{-1}$$

$$\underset{\sim}{C}_{(s)}[\underset{\sim}{T}_{b(s)} - E(\underset{\sim}{T}_{b(s)}|\rho_b)] \tag{2.2.2.17}$$

where $\underset{\sim}{T}_{b(s)}$ (s = 1, ..., S) is the vector of the totals that correspond to the $s^{\underline{th}}$ set of non-compared sets.

In this situation, W_b tests the hypothesis of homogeneity of the treatments inside each set and it is insensitive to the differences of treatments in different sets.

The distribution of W_b (2.2.2.13) is given by the following theorem.

Under the hypothesis H_o and assumptions (2.2.1.2), (2.2.1.3) and A_1, if

i) $\lim_{b \to \infty} ((\rho_{bjj'})) = ((\rho_{jj'}))$ has rank (p-1)

ii) The N_i's are uniformly bounded

iii) For each j,

$$\mathrm{Var}(T_{bj} | P_b) > \epsilon > 0$$

and

$$\lim_{b \to \infty} \mathrm{Var}(T_{bj} | P_b) = \infty$$

the permutation distribution of W_b, as defined in (2.2.2.13) is asymptotically the distribution of a χ^2 with (p-1) d.f. [The proof follows from theorems 7.1.1 and 7.1.2 of Chapter 7, taking c=1.]

Also, it can be easily seen that the one of (2.2.2.17) is the distribution of a χ^2 with

$$\sum_{s=1}^{S} (p_s - 1) \text{ d.f.}$$

It follows also from the remarks at the end of theorem 7.1.12 of Chapter 7 that if conditions i), ii), and iii) are fulfilled but the rank of $\underset{\sim}{V}_b$ (that is assumed to coincide with the rank of $((\rho_{jj'}))$) is $r < (p-S)$, the distribution of

$$W_b = [\underset{\sim}{T}_b - E(\underset{\sim}{T}_b|\rho_b)]'C'\underset{\sim}{V}_b C'[C\ \underset{\sim}{V}_b C']^{-1}C[T_b - E(\underset{\sim}{T}_b|\rho_b)]$$

with C $(r \times p)$ such that $\underset{\sim}{C}\ \underset{\sim}{V}_b \underset{\sim}{C}'$ is non-singular, is asymptotically that of a χ^2-variable with r degrees of freedom. Nevertheless, in this case the invariance property of W_b under different choices of the matrix of contrasts $\underset{\sim}{C}$ is lost and the value of W_b will depend on the particular selected matrix.

Condition iii) is difficult to discuss in a general set-up, since it cannot be asserted without further research that $\mathrm{Var}(T_{bj}|\rho_b)$ will be bigger than zero for any treatment.

The situation is more clear when N_{ij} is chosen to be 0 or 1; if so, $\mathrm{Var}\ (T_{bj}|\rho_b) > 0$ unless

$$R_{ij} = \frac{N_i + 1}{2}$$

for all the blocks in the experiment and this would imply that some functional relationship among the set of variables $\{Y_{ij}: \ j = 1, 2, \ldots, p\}$ hold with probability one, a case that is not of interest in relation to this work.

In practice we expect that $\mathrm{Var}(T_{bj}|\rho_b)$ will be bigger than zero; if it is not that way, the treatment can be eliminated from the data and the analysis performed for the rest of treatments.

The asymptotic permutation test for H_o is given by the rule

$$\text{Reject } H_o \quad \text{if } W_b \geq \chi^2_{r,\alpha}$$

$$\text{Accept } H_o \quad \text{if } W_b < \chi^2_{r,\alpha} \quad ,$$

r is the rank of $\underset{\sim}{V}_b$ and $\chi^2_{r,\alpha}$ is such that

$$P\{\chi^2_r \leq \chi^2_{r,\alpha}\} = 1-\alpha \qquad (0 < \alpha < 1)$$

for a desired level of significance.

2.2.3 Case U IV

The assumptions that characterize this case are (2.2.1.2), (2.2.1.3), A_2 and A_3.

The conditions A_2 and A_3 together imply that all the marginals that can be obtained from G_i are equal if they have the same number of coordinates.

Let us define the aligned yields and errors by

$$Z_{ij\ell} = Y_{ij\ell} - \overline{Y}_i$$

with

$$\overline{Y}_i = \frac{1}{N_i} \sum_{j=1}^{p} \sum_{\ell=1}^{N_{ij}} Y_{ij\ell} \tag{2.2.3.1}$$

$$\begin{pmatrix} i = 1, \ldots, b \\ j = 1, \ldots, p \\ \ell = 1, \ldots, N_{ij} \end{pmatrix}$$

and

$$\epsilon_{ij\ell} = e_{ij\ell} - \bar{e}_i$$

with

$$\bar{e}_i = \frac{1}{N_i} \sum_{j=1}^{p} \sum_{\ell=1}^{N_{ij}} e_{ij\ell} \qquad (2.2.3.2)$$

$$\begin{pmatrix} i = 1, \ldots, b \\ j = 1, \ldots, p \\ \ell = 1, \ldots, N_{ij} \end{pmatrix}$$

From the properties of the e's, it follows that the vectors

$$\{\underset{\sim}{\epsilon}_i = (\epsilon_{i11}, \ldots, \epsilon_{ipN_{ip}}): \; i = 1, \ldots, b\}$$

are independent, identically distributed with a distribution function that is symmetric in its N_i arguments.

Under H_o it can be written that

$$Z_{ij\ell} = \epsilon_{ij\ell} \quad \begin{pmatrix} i = 1, \ldots, b \\ j = 1, \ldots, p \\ \ell = 1, \ldots, N_{ij} \end{pmatrix} \qquad (2.2.3.3)$$

So, the test of the hypothesis H_o reduces to testing the interchangeability of

$$(Z_{i11}, \ldots, Z_{ipN_{ip}}) .$$

The test is based on inter-block ranking. Ranking is carried out over the

$$N = \sum_{i=1}^{b} N_i$$

observations.

Let

$$r_{ij\ell} = \text{rank}\{Z_{ij\ell}: Z_{111}, \ldots, Z_{ipN_{ip}}, \ldots, Z_{bp1}, \ldots, Z_{bpN_{bp}}\}. \qquad (2.2.3.4)$$

Given the values of

$$\{r_{ij\ell}: i = 1, \ldots, b; \; j = 1, \ldots, p; \; \ell = 1, \ldots, N_{ij}\}$$

it is assumed that under H_o, any j^{th} treatment in the i^{th} block, could be related to any combination of N_{ij} ranks present in the i^{th} block.

The test is based on intra-block permutation of the ranks of each block. The number of configurations derived from a given set of data is

$$\prod_{i=1}^{b} (N_i)!$$

each configuration having (conditionally) the same probability under H_o.

Let us denote with $\mathcal{P}_b$ this uniform (conditional) probability law.

The test statistic is a function of the totals

$$T_{bj} = \sum_{i=1}^{b} \sum_{\ell=1}^{N_{ij}} r_{ij\ell} \qquad (j = 1, \ldots, p) \tag{2.2.3.5}$$

$$E(r_{ij\ell}|\mathcal{P}_b) = \frac{1}{N_i} \sum_{t=1}^{p} \sum_{h=1}^{N_{it}} r_{ith} = \bar{r}_i \tag{2.2.3.6}$$

$$E(r_{ij\ell}r_{i'j'\ell'}|\mathcal{P}_b) = \begin{cases} \dfrac{1}{N_i} \displaystyle\sum_{t=1}^{p} \sum_{h=1}^{N_{it}} r_{ith}^2 & \text{if } i = i', \ j = j', \ \ell = \ell' \\[2ex] \dfrac{-1}{N_i} \left[\displaystyle\sum_{t=1}^{p} \sum_{h=1}^{N_{it}} r_{ith}^2 - \left(\sum_{t=1}^{p} \sum_{h=1}^{N_{it}} r_{ith} \right)^2 \right] & \\[2ex] & \text{if } i = i' \\ & \text{and } \begin{cases} j = j' \text{ and } \ell \neq \ell' \\ \text{or } j \neq j' \end{cases} \\[2ex] \bar{r}_i \bar{r}_{i'} & \text{if } i \neq i' \end{cases} \tag{2.2.3.7}$$

for any $i, i' = 1, 2, \ldots, b$; $j, j' = 1, \ldots, p$; $\ell = 1, \ldots, N_{ij}$; $\ell' = 1, \ldots, N_{i'j'}$.

Proof

Under $\mathcal{P}_b$ the N_i orderings

$$\{r_{i11}, \ldots, r_{ipN_{ip}}\}$$

are equally likely, and it implies that each $r_{ij\ell}$ has a probability of occurrence equal to $\frac{1}{N_i}$. Hence,

$$E(r_{ij\ell}|\rho_b) = \frac{1}{N_i} \sum_{t=1}^{p} \sum_{h=1}^{N_{it}} r_{ith} = \bar{r}_i$$

and

$$E(r^2_{ij\ell}|\rho_b) = \frac{1}{N_i} \sum_{t=1}^{p} \sum_{h=1}^{N_{it}} r^2_{ith} \ .$$

If $i \neq i'$ the random vectors

$$(r_{i11}, \ldots, r_{ipN_{ip}})$$

and

$$(r_{i'11}, \ldots, r_{i'pN_{i'p}})$$

are independent, then

$$E(r_{ij\ell}r_{i'j'\ell'}|\rho_b) = E(r_{ij\ell}|\rho_b)E(r_{i'j'\ell'}|\rho_b) = \bar{r}_i \ \bar{r}_{i'} \ .$$

If $r_{ij\ell}$ and $r_{ij'\ell'}$ are different coordinates of

$$(r_{i11}, \ldots, r_{ipN_{ip}})$$

(i.e., $j \neq j'$ or $j = j'$ and $\ell \neq \ell'$),

$$E(r_{ij\ell} r_{ij'\ell'} | \rho_b) = \sum_{t=1}^{p} \sum_{h=1}^{N_{it}} E(r_{ij\ell} | r_{ith}) \frac{r_{ith}}{N_i}$$

$$= \sum_{t=1}^{p} \sum_{h=1}^{N_{it}} \left[\left(\frac{\sum_{t'=1}^{p} \sum_{h'=1}^{N_{it'}} r_{it'h'} - r_{ith}}{N_i - 1} \right) \frac{r_{ith}}{N_i} \right]$$

$$= \frac{1}{N_i(N_i - 1)} \left[\left(\sum_{t=1}^{p} \sum_{h=1}^{N_{it}} r_{ith} \right)^2 - \sum_{t=1}^{p} \sum_{h=1}^{N_{it}} r_{ith}^2 \right].$$

Under the conditional law ρ_b,

$$E(T_{bj} | \rho_b) = \sum_{i=1}^{b} N_{ij} \bar{r}_i , \tag{2.2.3.8}$$

$$\mathrm{Cov}(T_{bj}, T_{bj'} | \rho_b) = \sum_{i=1}^{b} N_{ij'} (\partial_{jj'} N_i - N_{ij}) \mathcal{R}_i \tag{2.2.3.9}$$

where $\partial_{jj'}$ is the Kronecker index and

$$\mathcal{R}_i = \frac{\sum_{t=1}^{p} \sum_{h=1}^{N_{it}} (r_{ith} - \bar{r}_i)^2}{N_i(N_i - 1)} , \tag{2.2.3.10}$$

The T_{bj}'s satisfy the relation

$$\sum_{j=1}^{p} T_{bj} = \sum_{i=1}^{b} N_i \bar{r}_i$$

that is a fixed number. Also, it can be seen that

$$\sum_{j'=1}^{p} \mathrm{Cov}(T_{bj}, T_{bj'} | \rho_b) = 0 \quad \forall \quad j \ ,$$

so the rank of $\underset{\sim}{V}_b$ will be at most (p-1).

If superfluos rows are omitted, it can be seen that a necessary and sufficient condition for $\mathrm{Cov}(T_{bj}, T_{bj'})$ to be different from zero is that

$$\sum_{i}^{b} N_{ij}N_{ij'} > 0 \ ,$$

that implies that at least in one block treatments j and j′ appear together.

Also, it can be shown as in Benard and van Elteren (1953) that:

> "If and only if there are S non-compared sets of treatments the rank of the matrix $\underset{\sim}{V}_b$, is, for any b, p-S."

The proof is based on the Tausski's Theorem (Tausski (1949)). If the design is connected, the test statistic is taken as

$$W_b = [\underset{\sim}{T}_b - E(\underset{\sim}{T}_b | \rho_b)]' \underset{\sim}{C}' [\underset{\sim}{C} \, \underset{\sim}{V}_b \underset{\sim}{C}']^{-1} \underset{\sim}{C} [\underset{\sim}{T}_b - E(\underset{\sim}{T}_b | \rho_b)] \ , \qquad (2.2.3.11)$$

where $\underset{\sim}{T}_b$ is the vector of treatment totals, $\underset{\sim}{V}_b$ its covariance matrix, and $\underset{\sim}{C}$ is a (p-1) x p matrix of linearly independent contrasts.

If the design consists of S(> 1) non-compared sets, W_b becomes, as in (2.2.2.17)

$$\sum_{s=1}^{S} [\underset{\sim}{T}_{b(s)} - E(\underset{\sim}{T}_{b(s)}|\rho_b)]'\underset{\sim}{C}'_{(s)}[\underset{\sim}{C}_{(s)}\underset{\sim}{V}_{b(s)}\underset{\sim}{C}'_{(s)}]^{-1} \times \tag{2.2.3.12}$$

$$\times \quad \underset{\sim}{C}_{(s)}[\underset{\sim}{T}_{b(s)} - E(\underset{\sim}{T}_{b(s)}|\rho_b)]$$

where $\underset{\sim}{T}_{b(s)}$ and $\underset{\sim}{V}_{b(s)}$ have analogous meaning to those in Case I.

The distribution of W_b is stated in the following Theorem.

Under the hypothesis H_o and assumptions (2.2.1.2), (2.2.1.3), A_2 and A_3, if

i) $\lim_{b \to \infty} ((\rho_{bjj'})) = ((\rho_{jj'}))$ has rank (p-1)

ii) the N_i's are uniformly bounded;

iii) $\lim_{b \to \infty} \frac{b_j}{b} = h \neq 0$,

then the permutation distribution of W_b defined in (2.2.3.11) is, asymptotically, the distribution of a χ^2 with (p-1) d.f. [The proof follows from theorems 7.1.5 and 7.1.6 of Chapter 7 taking c=1.] Also, it can be seen that W_b, as defined in (2.2.3.12), has the distribution of a χ^2 with

$$\sum_{s=1}^{S} (p_s - 1) = p-S \text{ d.f.}$$

The asymptotic permutation test for H_o is given by the rule

$$\text{Reject } H_o \quad \text{if } W_b \geq \chi^2_{r,\alpha}$$

$$\text{Accept } H_o \quad \text{if } W_b < \chi^2_{r,\alpha} \quad ,$$

r is the rank of $\underset{\sim}{V}_b$ and $\chi^2_{r,\alpha}$ is such that

$$P\{\chi^2_r \leq \chi^2_{r,\alpha}\} = k-\alpha \qquad (0 < \alpha < 1)$$

for a desired level of significance.

2.3 Mixed Model for the Case of Several (Different from zero) Results per Cell. Extension of Model II (Case U II$_o$)

Let the data be represented by a set of stochastically independent vectors

$$\underset{\sim}{Y}_i' = (Y_{i11}, \ldots, Y_{i1N_{i1}}, \ldots, Y_{i21}, \ldots, \qquad (2.3.1)$$

$$Y_{i2N_{i2}}, \ldots, Y_{ip1}, \ldots, Y_{ipN_{ip}}) \qquad (i = 1, \ldots, b)$$

Let the set of treatments under test be divided in disjoint subsets

$$S^i_1, S^i_2, \ldots, S^i_{U_i}$$

of $p_1, p_2, \ldots, p_{U_i}$ treatments, respectively

$$(\sum_{u=1}^{U_i} p_u = p)$$

in the i^{th} block. If treatment j belongs to S^i_1, let N_{ij} be equal to M^i_1; if it belongs to S^i_2, let N_{ij} be M^i_2 and so on.

In each block the relation

$$\sum_{u=1}^{U_i} p^i_u M^i_u = N_i$$

is assumed to hold and all the numbers M_u^i in the same block are assumed to be bigger than zero and different from one to each other.

The numbers

$$U_i, p_1, p_2, \ldots, p_{U_i},$$

as well as

$$M_1^i, M_2^i, \ldots, M_{U_i}^i$$

are fixed by design and they may vary from block to block subject to the relations previously stated.

The Y's are considered to arise from a two-stage random process that can be described as follows.

At the first random stage, b partitions

$$\{\Pi_i: S_1^i, S_2^i, \ldots, S_{U_i}^i; \quad i = 1, \ldots, b\}$$

of the treatments are selected at random, individually for each block. In the i^{th} block, to all the

$$\frac{p!}{p_1!, p_2!, \ldots, p_{U_i}!}$$

possible partitions, equal chances of being selected are given. After the selection of Π_i, the decision of taking M_u^i replicated observations in all that treatments that form the set S_u^i is made. So, in this model N_{ij} is a random variable that takes values in the set

$$\{M_1^i, M_2^i, \ldots, M_{U_i}^i\}$$

and whose distribution of probability is determined by the fact that

$$N_{ij} = M_u^i$$

iff treatment j belongs to the random set S_u^i.

The cumulative distribution function of $\underset{\sim}{Y}_i$ is assumed to be a N_i-variate continuous function

$$F_i(\underset{\sim}{y}) = G_i(\underset{\sim}{y} - \underset{\sim}{m}_i) = G_i(\underset{\sim}{e})$$

with

$$\underset{\sim}{m}_i' = (m_{i11}, \ldots, m_{i1N_{i1}}, \ldots, m_{ip1}, \ldots, m_{ipN_{ip}})$$

and

$$m_{ij\ell} = b_i + \tau_j$$

$$(i = 1, \ldots, b; \quad j = 1, \ldots, p; \quad \ell = 1, \ldots, N_{ij}) . \qquad (2.3.3)$$

The second stage of the random process associated with this model refers to the selection of the actual Y's from the distribution function F_i that depends on Π_i through the variables

$$N_{ij} (j = 1, 2, \ldots, p) .$$

A further assumption on the distribution of the errors $\underset{\sim}{e}$'s completes the model: for any given block the errors related to a certain treatment are interchangeable random variables and so are the vectors of errors.

$$(e_{ij1}, e_{ij2}, \cdots, e_{ijN_{ij}})$$

and

$$(e_{ij'1}, e_{ij'2}, \cdots, e_{ij'N_{ij'}})$$

for all j and j′ in the same subset S_u^i of treatments. This assumption will be called "fragmented compound symmetry" and symbolized by A_4.

The hypothesis to be considered is, as usual

$$H_o: \tau_1 = \tau_2 = \cdots = \tau_p = 0 .$$

The procedure for testing H_o is as follows:

Let

$$R_{ij\ell} = \text{rank}\{Y_{ij\ell}: Y_{i11}, \cdots, Y_{i1N_{i1}}, \cdots, Y_{ip1}, \cdots, Y_{ipN_{ip}}\}$$

$$\begin{pmatrix} i = 1, \cdots, b \\ j = 1, \cdots, p \\ \ell = 1, \cdots, N_{ij} \end{pmatrix} \tag{2.3.5}$$

and

$$\underset{\sim}{R}_i' = (R_{i11}, \cdots, R_{i1N_{i1}}, \cdots, R_{ip1}, \cdots, R_{ipN_{ip}})$$

$$(i = 1, \cdots, b) . \tag{2.3.6}$$

Under H_o the vectors of ranks have the property of fragmented compound symmetry. It will be associated to each set

$$\{R_i: \; i = 1, \ldots, b\}$$

the

$$\prod_{i=1}^{b} \prod_{u=1}^{U_i} (M_u^i \,!)^{p_u^i} =$$

$$= \prod_{i=1}^{b} \prod_{u=1}^{U_i} (M_u^i \,!)^{p_u^i} \frac{p!}{\prod_{u=1}^{U_i} p_u \,!} \cdot \prod_{u=1}^{U_i} p_u$$

equally probable (under H_o) results that come into consideration observing that if H_o is true,

i) the actual ranks in $\underset{\sim}{R}_i$ could have been obtained under any of the equally probable partitions Π_i;

ii) for a given partition any permutation of the vectors of ranks corresponding to the treatments in the same set S_u^i are equally probable;

iii) for any individual treatment, all the permutations of its associated ranks are equally probable (and this is true even under hypotheses different from H_o).

Conditionally to this set of

$$\prod_{i=1}^{b} \prod_{u}^{U_i} (M_u^i \,!)^{p_u^i} \, p \,!$$

results, a uniform probability distribution can be defined. Let us call it $\mathcal{P}_b$.

Two statistics for the test of H_o can be suggested. One based on the totals

$$T_{bj} = \sum_{i=1}^{b} \sum_{\ell=1}^{N_{ij}} R_{ij\ell} \tag{2.3.7}$$

and the other based on the unweighted sum of rank averages

$$T_{b\bar{R}j} = \sum_{i=1}^{b} \bar{R}_{ij} , \tag{2.3.8}$$

with

$$\bar{R}_{ij} = \frac{\sum_{\ell=1}^{N_{ij}} R_{ij\ell}}{N_{ij}} . \tag{2.3.9}$$

Under $\mathcal{P}_b$,

$$E(T_{bj}|\mathcal{P}_b) = \frac{N_i}{p} \left(\frac{N_i + 1}{2} \right) , \tag{2.3.10}$$

$$\mathrm{Cov}(T_{bj}, T_{bj'}|\mathcal{P}_b) = \frac{\partial_{jj'}\, p-1}{p(p-1)} \times$$

$$\times \sum_{i=1}^{b} \sum_{t=1}^{p} [R_{it.}^2 - ((\frac{N_i}{p})^2 (\frac{N_i + 1}{2})] , \tag{2.3.110}$$

∂ being the Kronecker index and

$$R_{ij.} = \sum_{\ell=1}^{N_{ij}} R_{ij\ell} . \tag{2.3.12}$$

Also

$$E(T_{b\bar{R}j}|\rho_b) = \sum_{i=1}^{b} \frac{1}{p} \sum_{t=1}^{p} \bar{R}_{it} \qquad (2.3.13)$$

and

$$\mathrm{Cov}(T_{b\bar{R}j}, T_{b\bar{R}j'}|\rho_b) = \frac{\partial_{jj'}\, p-1}{p(p-1)} \times$$

$$\times \sum_{i=1}^{b} \sum_{t=1}^{p} \left[\bar{R}_{it}^{2} - \frac{\sum_{t=1}^{p} \bar{R}_{ij}}{p} \right]^{2} \qquad (2.3.14)$$

In Chapter 7 (Section 7.2) it is shown how these moments have been obtained.

If $\underset{\sim}{V}_b$ and $\underset{\sim}{V}_{b\bar{R}}$ represent the covariance matrices of the T_b 's and $T_{b\bar{R}}$'s respectively, the related statistics for testing H_o are

$$W_b = [\underset{\sim}{T}_b - E(\underset{\sim}{T}_b|\rho_b)]'\underset{\sim}{C}'(\underset{\sim}{C}\,\underset{\sim}{V}_b\underset{\sim}{C}')^{-1}\underset{\sim}{C}[\underset{\sim}{T}_b - E(\underset{\sim}{T}_b|\rho_b)] \qquad (2.3.15)$$

and

$$W_{b\bar{R}} = [\underset{\sim}{T}_{b\bar{R}} - E(\underset{\sim}{T}_{b\bar{R}}|\rho_b)]'\underset{\sim}{C}'[\underset{\sim}{C}\,\underset{\sim}{V}_{b\bar{R}}\underset{\sim}{C}']^{-1}\underset{\sim}{C}[\underset{\sim}{T}_{b\bar{R}} - E(\underset{\sim}{T}_{b\bar{R}}|\rho_b)] \qquad (2.3.16)$$

where $\underset{\sim}{T}_b$ is the vector of the totals defined in (2.3.7) and $\underset{\sim}{T}_{b\bar{R}}$ is the vector of the unweighted sum of rank averages defined in (2.3.8).

$\underset{\sim}{C}$ is any (p-1) x p matrix of linearly independent contrasts.

After some algebra, it can be shown that

$$W_b = \frac{(p-1)\sum_{j=1}^{p}[T_{bj} - E(T_{bj}|\mathcal{P}_b)]^2}{\sum_{i=1}^{b}\sum_{j=1}^{p}[R_{ij.}^2 - (\frac{N_i}{p})^2(\frac{N_i+1}{2})^2]} \qquad (2.3.17)$$

and

$$W_{b\overline{R}} = \frac{(p-1)\sum_{j=1}^{p}[T_{b\overline{R}j} - E(T_{b\overline{R}j}|\mathcal{P}_b)]^2}{\sum_{i=1}^{b}\sum_{j=1}^{p}[\overline{R}_{ij}^2 - (\sum_{j=1}^{p}\frac{R_{ij}}{p})^2]} \; . \qquad (2.3.18)$$

It can be seen from (2.3.11) and (2.3.14) that the columns of either $\underset{\sim}{V}_{bT}$ or $\underset{\sim}{V}_{b\overline{R}}$ add to zero. So, they have, at most, rank (p-1).

The distribution of W_b (2.3.15) is given by the following theorem.

Under the hypothesis H_o and assumptions (2.3.2), (2.3.3) and A_4 if

i) $\lim_{b\to\infty} ((\rho_{bjj'})) = ((\rho_{jj'}))$ has rank (p-1)

ii) the N_i's are uniformly bounded

iii) $\lim_{b\to\infty} \mathrm{Var}(T_{bj}|\mathcal{P}_b) = \infty$, then

the permutation distribution of W_b defined in (2.3.15) is asymptotically the distribution of a χ^2 variable with (p-1) d.f.

A similar theorem holds for $W_{b\bar{R}}$ (2.3.16). The proofs of these theorems are given in Chapter 7 (theorems 7.1.8 and 7.1.9).

The asymptotic permutation for H_o is:

$$\text{Reject } H_o \quad \text{if } W_b \quad (W_{b\bar{R}}) \geq \chi^2_{r\alpha}$$

$$\text{Accept } H_o \quad \text{if } W_b \quad (W_{b\bar{R}}) < \chi^2_{r\alpha} \quad ,$$

r being the rank of $\underset{\sim}{V}_b(\underset{\sim}{V}_{b\bar{R}})$ and $\chi^2_{r\alpha}$ chosen such that

$$P\{\chi^2_r \leq \chi^2_{r\alpha}\} = 1-\alpha \qquad (0 < \alpha < 1)$$

for the level α of significance.

2.4 A Numerical Example

To illustrate some of the techniques of this chapter, a set of data has been analyzed. The data represent the results of replicated blood analysis in a group of six swines submitted to some experimental diet.

The blood samples were taken monthly from October to February. At each time several determinations of the proportion of Calcium in serum (mg per 100 ml serum) were performed. The following table shows the outcome of the experiment.

Blood Analyses

Swine	Month									
	Oct.		Nov.		Dec.		Jan.		Feb.	
1	10.8 11.0 11.2	12.0 11.8	9.8 10.2 10.5	11.1 10.8 9.9	9.5 10.0	10.7	9.5 9.9	10.0	9.2 9.5	10.0 9.9
2	10.5 11.1	11.6 11.2	10.3 10.5	10.7	10.3 10.6 10.7	10.9 10.4	10.4 10.5	10.7 10.6	10.4 10.6	10.7 10.5
3	10.9 12.2	12.5	10.3 10.7 11.0	11.1 10.4	9.9 10.0	10.5 10.3	9.9 10.4 10.2	10.7 10.1 10.6	10.2 10.5	10.4
4	10.3 11.2 11.8	11.9 11.4	11.0 11.7	10.5	10.2 10.6 10.8	10.9 10.4	10.7 11.0 10.9	10.4 10.5	10.8 11.2 11.1	10.5 10.6 10.7
5	10.8 11.8	13.4 12.2	10.6 10.5	10.0 11.0	10.5 11.4 10.8	9.7 10.2	10.4 10.8 10.3	9.8 10.1	9.8 10.1	10.4
6	11.4 12.0 12.3	12.5 11.9	10.9 11.0	10.3 10.8	9.8 10.6	10.9 10.5	10.8 11.0 10.9	10.6 10.7	10.8 10.9	10.6 10.7

Several models were applied to this set of data and their corresponding tests for the hypotheses of no effects of months were constructed. The lists of models and the results of the tests are shown below:

i) Case I of Koch and Sen applied to the average of the cells $W_b = 6.000$

ii) Case U I
(Section 2.2). $W_b = 5.950$

iii) Case U II
(Section 2.4) $W_b = 14.00$

$$W_{b\bar{R}} = 20.15$$

The number of blocks (b=6) is hardly "sufficiently large" to justify using a large sample approximation and any conclusion based on these results has to be taken with reticense.

3. MIXED MODELS FOR THE CASE OF SEVERAL (EVENTUALLY ONE OR ZERO) RESULTS PER TREATMENT AND MULTIVARIATE RESPONSES

3.1 Introduction

Extensions to multivariate responses of Case UI, Case UII (that underlies the Benard and van Elteren test) and Case UIV are presented in this chapter.

Case UII_o has not been extended to multidimensional responses, but the procedure for obtaining such extension can be done following the same steps that are used in the univariate situation.

For the type of problems to be considered in this section, the outcome of the experiment can be represented by the set of stochastically independent matrices (each matrix being associated with a different subject) $\{\underset{\sim}{Y}_1, \underset{\sim}{Y}_2, \ldots, \underset{\sim}{Y}_b\}$, with

$$\underset{\sim}{Y}_i = \begin{bmatrix} Y^{(1)}_{i11}, & \ldots, & Y^{(1)}_{i1N_{i1}}, & Y^{(1)}_{i21}, & \ldots, & Y^{(1)}_{i2N_{i2}}, \ldots, & Y^{(1)}_{ip1}, & \ldots, & Y^{(1)}_{ipN_{ip}} \\ Y^{(2)}_{i11}, & \ldots, & Y^{(2)}_{i1N_{i1}}, & Y^{(2)}_{i21}, & \ldots, & Y^{(2)}_{i2N_{i2}}, \ldots, & Y^{(2)}_{ip1}, & \ldots, & Y^{(2)}_{ipN_{ip}} \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \quad \cdots & \cdot & \cdots & \cdot \\ Y^{(c)}_{i11}, & \ldots, & Y^{(c)}_{i1N_{i1}}, & Y^{(c)}_{i21}, & \ldots, & Y^{(c)}_{i2N_{i2}}, \ldots, & Y^{(c)}_{ip1}, & \ldots, & Y^{(c)}_{ipN_{ip}} \end{bmatrix} \tag{3.1.1}$$

where

$$\underset{\sim}{Y}'_{ij\ell} = (Y^{(1)}_{ij\ell}, Y^{(2)}_{ij\ell}, \ldots, Y^{(c)}_{ij\ell})$$

stands for the ℓ^{th} result of treatment j taken on the i^{th} subject.

It is assumed that each matrix comes from some $(c \times N_i)$-valued continuous cumulative distribution function

$$F_i(\underset{\sim}{y}) = G_i(\underset{\sim}{y}-\underset{\sim}{m}_i) = G_i(\underset{\sim}{e}) \tag{3.1.2}$$

with $\underset{\sim}{m}_i = ((m_{ij}^{(k)}))$ being a $(c \times N_i)$ matrix such that

$$m_{ij\ell}^{(k)} = b_i^{(k)} + \tau_j^{(k)} \quad \begin{pmatrix} i = 1, \ldots, b \\ j = 1, \ldots, p \\ \ell = 1, \ldots, N_{ij} \end{pmatrix}. \tag{3.1.3}$$

The hypothesis to be tested is

$$H_o: \ \tau_1^{(k)} = \tau_2^{(k)} = \ldots = \tau_p^{(k)} = 0 \qquad \text{for } k = 1, 2, \ldots, c.$$

The assumptions A_1, A_2 and A_3 that have been defined for univariate models can be adapted to the multivariate case in the following way:

> A_1 Skew symmetry: The joint distribution of any linearly independent set of contrasts among the vectors $\underset{\sim}{Y}_{ij\ell}$ is (diagonally) symmetrically distributed about its location parameter.

In other words, let $\underset{\sim}{C}$ be a $(N_i-1)xN_i$ matrix of contrasts. Then A_1 states that

$$\underset{\sim}{C}\underset{\sim}{Y}_i' \quad \text{and} \quad [\underset{\sim}{\theta} - \underset{\sim}{C}\underset{\sim}{Y}_i']$$

have the same distribution for any i, where $\underset{\sim}{\theta}$ is the location parameter of $\underset{\sim}{C}\underset{\sim}{Y}_i'$.

A_2 Additivity of subject effects: Any marginal obtained from G_i is the same (for the same group of variables) whatever block is considered.

A_3 Compound symmetry of the error columns: This means that

$$G_i(\underset{\sim}{e}_{i1}, \underset{\sim}{e}_{i2}, \ldots, \underset{\sim}{e}_{iN_i}) = G_i(\underset{\sim}{e}_{ih_1}, \underset{\sim}{e}_{ih_2}, \ldots, \underset{\sim}{e}_{ih_{N_i}})$$

where $h_1, h_2, \ldots, h_{N_i}$ is any permutation of the subscripts $\{1, 2, \ldots, N_i\}$.

The three cases to be presented are identified as cases MUI, MUII and MUIV; for all of them, assumptions (3.1.2) and (3.1.3) are supposed to hold, and for:

Case MUI: A_1 holds (A_2 and A_3 may not)

Case MUII: A_3 holds (A_1 and A_2 may not)

Case MUIV: A_2 and A_3 hold (A_1 may not).

3.2 Case MUI

To test H_o, the following matrices are considered:

$$\underset{\sim}{Z}_i = (\underset{\sim}{Z}_{i11}, \ldots, \underset{\sim}{Z}_{i1N_{i1}}, \ldots, \underset{\sim}{Z}_{ip1}, \ldots, \underset{\sim}{Z}_{ipN_{ip}}), \qquad (3.2.1)$$

where

$$\underset{\sim}{Z}_{ij\ell} = \underset{\sim}{Y}_{ij\ell} - \bar{\underset{\sim}{Y}}_i \quad \begin{pmatrix} i = 1, \ldots, b \\ j = 1, \ldots, p \\ \ell = 1, \ldots, N_{ij} \end{pmatrix} \qquad (3.2.2)$$

with

$$\bar{\underset{\sim}{Y}}_i = \frac{1}{N_i} \sum_{j=1}^{p} \sum_{\ell=1}^{N_{ij}} \underset{\sim}{Y}_{ij\ell}. \tag{3.2.3}$$

If H_o holds, the distribution of $\{\underset{\sim}{Z}_i : i = 1, \ldots, b\}$ remains invariant under the group of transformations g such that $g \in G$ is defined as

$$g(\underset{\sim}{Z}) = g(\underset{\sim}{Z}_1, \underset{\sim}{Z}_2, \ldots, \underset{\sim}{Z}_b) =$$

$$= ((-1)^{\alpha_1}\underset{\sim}{Z}_1, (-1)^{\alpha_2}\underset{\sim}{Z}_2, \ldots, (-1)^{\alpha_b}\underset{\sim}{Z}_b),$$

where α_i $(i = 1, \ldots, b)$ can take the values 0 or 1.

As a consequence, the matrices

$$\underset{\sim}{R}_i \quad \text{and} \quad (N_i + 1)\underset{\sim}{J} - \underset{\sim}{R}_i,$$

where

$$\underset{\sim}{R}_i = \begin{bmatrix} R^{(1)}_{i11}, & \ldots, & R^{(1)}_{i1N_{i1}}, & R^{(1)}_{i21}, & \ldots, & R^{(1)}_{i2N_{i2}}, & \ldots, & R^{(1)}_{ip1}, & \ldots, & R^{(1)}_{ipN_{ip}} \\ R^{(2)}_{i11}, & \ldots, & R^{(2)}_{i1N_{i1}}, & R^{(2)}_{i21}, & \ldots, & R^{(2)}_{i2N_{i2}}, & \ldots, & R^{(2)}_{ip1}, & \ldots, & R^{(2)}_{ipN_{ip}} \\ \cdot & \ldots & \cdot & \cdot & \ldots & \cdot & \ldots & \cdot & \ldots & \cdot \\ R^{(c)}_{i11}, & \ldots, & R^{(c)}_{i1N_{i1}}, & R^{(c)}_{i21}, & \ldots, & R^{(c)}_{i2N_{i2}}, & \ldots, & R^{(c)}_{ip1}, & \ldots, & R^{(c)}_{ipN_{ip}} \end{bmatrix} \tag{3.2.4}$$

are conditionally equally probable, each one occurring with probability 1/2 for any value of i.

If given the set of matrices $\{\underset{\sim}{R}_1, \ldots, \underset{\sim}{R}_b\}$, we restrict the space to the set of 2^b points generated by g, then, as a consequence of the sign invariance under H_o, a uniform (conditional) law can be defined on that set. Let $\mathcal{P}_b$ designate that conditional law. Under $\mathcal{P}_b$,

$$E(R_{ij\ell}^{(k)}|\mathcal{P}_b) = \frac{R_{ij\ell}^{(k)} + (N_i+1 - R_{ij\ell}^{(k)})}{2} = \frac{N_i+1}{2} \tag{3.2.5}$$

$$E(R_{ij\ell}^{(k)}R_{i'j'\ell'}^{(k')}|\mathcal{P}_b) = \begin{cases} \dfrac{R_{ij\ell}^{(k)}R_{ij'\ell'}^{(k')} + (N_i+1-R_{ij\ell}^{(k)})(N_i+1 - R_{ij'\ell'})}{2} & \text{if } i = i', \\ \dfrac{(N_i+1)}{2} \cdot \dfrac{(N_i+1)}{2} & \text{if } i \neq i' \end{cases} \tag{3.2.6}$$

and

$$\operatorname{Cov}(R_{ij\ell}^{(k)},R_{i'j'\ell'}^{(k')}|\mathcal{P}_b) = \begin{cases} (R_{ij\ell}^{(k)} - \frac{N_i+1}{2})(R_{ij'\ell'}^{(k')} - \frac{N_i+1}{2}) & \text{if } i = i' \\ 0 & \text{if } i \neq i' \end{cases} \tag{3.2.7}$$

for $i, i' = 1, \ldots, b$; $j, j' = 1, \ldots, p$; $\ell = 1, \ldots, N_{ij}$, $\ell' = 1, \ldots, N_{i'j'}$ and $k, k' = 1, \ldots, c$.

The test statistic is based on the rank totals

$$T_{bj}^{(k)} = \sum_{i=1}^{b} \sum_{\ell=1}^{N_{ij}} R_{ij\ell}^{(k)} \qquad \begin{pmatrix} j = 1, \ldots, p \\ k = 1, \ldots, c \end{pmatrix}. \tag{3.2.8}$$

Conditionally, we have that

$$E(T_{b_j}^{(k)}|\mathcal{P}_b) = \sum_{i=1}^{b} N_{ij}(\frac{N_i+1}{2}) \tag{3.2.9}$$

$$\operatorname{Cov}(T_{b_j}^{(k)}, T_{b_{j'}}^{(k')}|\mathcal{P}_b) = \sum_{i=1}^{b}[\sum_{\ell=1}^{N_{ij}}(R_{ij\ell} - \frac{N_i+1}{2})][\sum_{\ell=1}^{N_{ij'}} (R_{ij'\ell} - \frac{N_i+1}{2})]. \tag{3.2.10}$$

Let us consider the vector $\underset{\sim}{T}_b$ as defined in (1.2 4). For each k,

$$\sum_{j=1}^{p} T_{bj}^{(k)} = \sum_{i=1}^{b} \frac{N_i(N_i+1)}{2} \quad (k = 1, 2, \ldots, c) \tag{3.2.11}$$

and also

$$\sum_{j'=1}^{p} \mathrm{Cov}(T_{bj}^{(k)}, T_{bj'}^{(k')} | \mathcal{P}_b) = 0, \tag{3.2.12}$$

for j = 1, 2, ..., p and any pair (k, k').

From (3.2.12) it can be seen that the rank of $\underset{\sim}{V}_b$ is at most c(p-1). If there are S non-compared sets, it follows from (1.2.4.4) that the rank of $\underset{\sim}{V}_b$ is at most c(p-S).

If the rank of $\underset{\sim}{V}_b$ is c(p-1), the test statistic is taken to be

$$W_b = [\underset{\sim}{T}_b - E(\underset{\sim}{T}_b|\mathcal{P}_b)]'\underset{\sim}{C}'[\underset{\sim}{C}\underset{\sim}{V}\underset{\sim}{C}']^{-1}\underset{\sim}{C}[\underset{\sim}{T}_b - E(\underset{\sim}{T}_b|\mathcal{P}_b)] \tag{3.2.13}$$

where $\underset{\sim}{C}'$ is a (p-1)c x pc matrix of the form $\underset{\sim}{C} = \mathrm{diag}(\underset{\sim}{D}, \underset{(c)}{\ldots}, \underset{\sim}{D})$, and $\underset{\sim}{D}$ is a (p-1) x p matrix of linearly independent contrasts. $\underset{\sim}{D}$ can be taken as $\underset{\sim}{C}$ in (2.2.2.14), and then,

$$W_b = [\underset{\sim}{T}_{b*} - E(\underset{\sim}{T}_{b*}|\mathcal{P}_b]'\underset{\sim}{V}_*^{-1}[\underset{\sim}{T}_{b*} - E(\underset{\sim}{T}_{b*}|\mathcal{P}_b)] \tag{3.2.14}$$

If there are S non-compared sets of treatments and the rank of each $\underset{\sim}{V}_{b(s)}$ is $c(p_s-1)$, a formula similar to (2.2.2.17) can be used, i.e.,

$$W_b = \sum_{s=1}^{S}[\underset{\sim}{T}_{b(s)}-E(\underset{\sim}{T}_{b(s)}|\mathcal{P}_b)]'\underset{\sim}{C}'_{(s)}[\underset{\sim}{C}_{(s)}\underset{\sim}{V}_{b(s)}\underset{\sim}{C}'_{(s)}]^{-1}\underset{\sim}{C}_{(s)}[\underset{\sim}{T}_{b(s)}-E(\underset{\sim}{T}_{b(s)}|\mathcal{P}_b)] \tag{3.2.15}$$

where $C_{(s)} = \mathrm{diag}\ (\underset{\sim}{D}_{(s)}, \underset{(c)}{\ldots}, \underset{\sim}{D}_{(s)})$.

The distribution of W_b is stated in the following theorem:

Under the hypothesis H_o and assumptions (3.1.2), (3.1.3) and A_1, if

i) $\lim\limits_{b\to\infty} ((\rho_{jj'}^{kk'})) = ((\rho_{jj'}^{kk'}))$ has rank $c(p-1)$

ii) The N_i's are uniformly bounded

iii) For each k and j, $Var(T_{bj}^{(k)}|\rho_b) > \epsilon > 0$ and $Var(T_{bj}^{(k)}|\rho_b)$ goes to infinity with b,

then, the permutation distribution of W_b, as defined in (3.2.13) is, asymptotically, the distribution of a χ^2 with $c(p-1)$ d.f.

The proof is given in theorems 7.1.1 and 7.1.2 of Chapter 7.

If the rank of $((\rho_{jj'}^{kk'}))$ is r, then the corresponding W_b is distributed as a χ^2 with r degrees of freedom.

As in the univariate case, it can be said with respect to assumption iii) that $Var(T_{bj}^{(k)}|\rho_b)$ will be positive for each j and k, if N_{ij} is taken to be equal to either 0 or 1, unless some functional relations among the coordinates of the blocks exist.

The asymptotic permutation test for H_o is given by the rule:

Reject H_o if $W_b \geq \chi^2_{r,\alpha}$

Accept H_o if $W_b < \chi^2_{r,\alpha}$

where r is the rank of $\underset{\sim}{V}_b$ and $\chi^2_{r,\alpha}$ is such that

$$P(\chi^2_r \leq \chi^2_{r,\alpha}) = 1-\alpha, \quad (0 < \alpha < 1).$$

3.3 Case MUII

The test is based on intrablock rankings. Ranking is carried out over the N_i observations within each block and done individually for each variable.

Let

$$R_{ij}^{(k)} = \text{rank}\{Y_{ij}^{(k)}: \ Y_{i11}^{(k)}, \ldots, Y_{i1N_{i1}}^{(k)}, \ldots, Y_{ip1}^{(k)}, \ldots, Y_{ipN_{ip}}^{(k)}\}$$

and

$$\underset{\sim}{R}_i = \begin{bmatrix} R_{i11}^{(1)}, & \ldots, & R_{i1N_{i1}}^{(1)}, & R_{i21}^{(1)}, & \ldots, & R_{i2N_{i2}}^{(1)}, \ldots, & R_{ip1}^{(1)}, & \ldots, & R_{ipN_{ip}}^{(1)} \\ R_{i11}^{(2)}, & \ldots, & R_{i1N_{i1}}^{(2)}, & R_{i21}^{(2)}, & \ldots, & R_{i2N_{i2}}^{(2)}, \ldots, & R_{ip1}^{(2)}, & \ldots, & R_{ipN_{ip}}^{(2)} \\ . & \ldots & . & . & \ldots & . \quad \ldots & . & \ldots & . \\ R_{i11}^{(c)}, & \ldots, & R_{i1N_{i1}}^{(c)}, & R_{i21}^{(c)}, & \ldots, & R_{i2N_{i2}}^{(c)}, \ldots, & R_{ip1}^{(c)}, & \ldots, & R_{ipN_{ip}}^{(c)} \end{bmatrix} .$$

$$(i = 1, \ldots, b) \qquad (3.3.1)$$

Because of A_2, all the matrices that can be obtained from $\underset{\sim}{Y}_i$ (3.3.1) by permutation of the N_i columns, are, under H_o, equally probable. The same is true with respect to $\underset{\sim}{R}_i$, so that the distribution of $\underset{\sim}{R}_i$ on the set of $N_i!$ matrices originated by the permutational invariance is $\frac{1}{(N_i)!}$. So,

$$P(\underset{\sim}{R}_1, \ldots, \underset{\sim}{R}_b) = \prod_{i=1}^{b} \frac{1}{(N_i)!}$$

since the rankings are independent for different blocks. Let us call $\mathcal{P}_b$ this uniform (conditional) law.

Under $\mathcal{P}_b$,

$$E(R_{ij\ell}^{(k)}|\mathcal{P}_b) = \frac{N_i+1}{2} \tag{3.3.2}$$

$$E(R_{ij\ell}^{(k)}R_{i'j'\ell'}^{(k')}|\mathcal{P}_b) = \begin{cases} \dfrac{1}{N_i}\sum\limits_{t=1}^{p}\sum\limits_{h=1}^{N_{it}} R_{ith}^{(k)}R_{ith}^{(k')} & \text{if } \begin{array}{l} i = i' \\ j = j' \\ \ell = \ell' \end{array} \\[2ex] \dfrac{-1}{N_i(N_i-1)}\left[\sum\limits_{t=1}^{p}\sum\limits_{h=1}^{N_{it}} R_{ith}^{(k)}R_{ith}^{(k')} - \left(\dfrac{N_i(N_i-1)}{2}\right)^2\right] & \\ \qquad \text{if } i = i' \text{ and } \begin{cases} j = j' \text{ and } \ell \neq \ell' \\ \text{or } j \neq j' \end{cases} & \\[2ex] \dfrac{N_i+1}{2}\,\dfrac{N_{i'}+1}{2} & \text{if } i \neq i' \end{cases} \tag{3.3.3}$$

for any $i, i' = 1, 2, \ldots, b$; $j, j' = 1, \ldots, p$; $\ell = 1, \ldots, N_{ij}$, $\ell' = 1, \ldots, N_{i'j'}$; $k, k' = 1, 2, \ldots, c$.

<u>Proof</u>

Under $\mathcal{P}_b$, the $N_i!$ orderings of $(\underset{\sim}{R}_{i11}, \ldots, \underset{\sim}{R}_{ipN_{ip}})$ are equally likely, and hence,

$$E(R_{ij\ell}^{(k)}|\mathcal{P}_b) = \frac{1}{N_i}\sum_{t=1}^{p}\sum_{h=1}^{N_{it}} R_{ith}^{(k)} = \frac{N_i(N_i+1)}{2N_i} = \frac{N_i+1}{2}\,.$$

Also, since $R_{ij\ell}^{(k)}$ and $R_{ij\ell}^{(k')}$ belong to the same vector $\underset{\sim}{R}_{ij\ell}$,

$$E(R_{ij\ell}^{(k)},\, R_{ij\ell}^{(k')}|\mathcal{P}_b) = \frac{1}{N_i}\sum_{t=1}^{p}\sum_{h=1}^{N_{it}} R_{ith}^{(k)}R_{ith}^{(k')}$$

and the result is true if k and k' are either equal or not.

If $i \neq i'$, the random matrices $\underset{\sim}{R}_i$ and $\underset{\sim}{R}_{i'}$ are independent, so

$$E(R_{ij\ell}^{(k)} R_{i'j'\ell'}^{(k')} | \rho_b) = E(R_{ij\ell}^{(k)} | \rho_b) E(R_{i'j'\ell'}^{(k')} | \rho_b) = \frac{N_i+1}{2} \quad \frac{N_{i'}+1}{2}$$

for any j, j', ℓ, ℓ', k and k'.

If $\underset{\sim}{R}_{ij\ell}$ and $\underset{\sim}{R}_{ij'\ell'}$ are different because either $j \neq j'$ or $j = j'$ and $\ell \neq \ell'$, but both vectors belong to the same block, they are not independent any longer and

$$E(R_{ij\ell}^{(k)} R_{ij'\ell'}^{(k')} | \rho_b) = \sum_{t=1}^{p} \sum_{h=1}^{N_{it}} E(R_{ij\ell}^{(k)} | R_{ith}^{(k')}) \frac{R_{ith}^{(k')}}{N_i}$$

$$= \sum_{t=1}^{p} \sum_{h=1}^{N_{it}} \left[\frac{\sum_{t'=1}^{p} \sum_{h'=1}^{N_{it'}} R_{it'h'}^{(k)} - R_{ith}^{(k)}}{(N_i-1)} \quad \frac{R_{ith}^{(k')}}{N_i} \right]$$

$$= \frac{1}{N_i(N_i-1)} [(\sum_{t=1}^{p} \sum_{h=1}^{N_{it}} R_{ith}^{(k)})(\sum_{t=1}^{p} \sum_{h=1}^{N_{it}} R_{ith}^{(k')})$$

$$- \sum_{t=1}^{p} \sum_{h=1}^{N_{it}} R_{ith}^{(k)} R_{ith}^{(k')}]$$

$$= \frac{-1}{N_i(N_i-1)} [\sum_{t=1}^{p} \sum_{h=1}^{N_{it}} R_{ith}^{(k)} R_{ith}^{(k')} - (\frac{N_i(N_i-1)}{2})^2].$$

The last equality follows from the fact that, for any k,

$$\sum_{t=1}^{p} \sum_{h=1}^{N_{it}} R_{ith}^{(k)} = \frac{N_i(N_i+1)}{2} .$$

The test will be based on the totals

$$T_{bj}^{(k)} = \sum_{i=1}^{b} \sum_{\ell=1}^{N_{ij}} R_{ij\ell}^{(k)} \cdot \quad \begin{pmatrix} j = 1, \ldots, p \\ k = 1, \ldots, c \end{pmatrix} . \qquad (3.3.4)$$

The first and second moments of the T'_s are, conditionally,

$$E(T_{bj}^{(k)} | \mathcal{P}_b) = \sum_{i=1}^{b} N_{ij} \left(\frac{N_i+1}{2}\right) \quad \begin{pmatrix} j = 1, \ldots, p \\ k = 1, \ldots, c \end{pmatrix} \tag{3.3.5}$$

$$\mathrm{Cov}(T_{bj}^{(k)}, T_{bj'}^{(k')} | \mathcal{P}_b) = \sum_{i=1}^{b} N_{ij} (\delta_{jj'} N_i - N_{ij'}) \mathcal{R}_i^{(kk')}, \tag{3.3.6}$$

where

$$\mathcal{R}_i^{(kk')} = \frac{\sum_{t=1}^{p} \sum_{h=1}^{N_{it}} (R_{ith}^{(k)} - \frac{N_i+1}{2})(R_{ith}^{(k')} - \frac{N_i+1}{2})}{N_i(N_i-1)} . \tag{3.3.7}$$

For each k,

$$\sum_{j=1}^{p} T_{bj}^{(k)} = \sum_{i=1}^{b} N_{ij} \left(\frac{N_i+1}{2}\right)$$

and

$$\sum_{j'=1}^{p} \mathrm{Cov}(T_{bj}^{(k)} T_{bj'}^{(k')} | \mathcal{P}_b) = 0$$

for $j = 1, 2, \ldots, p$ and any pair (k, k').

If $\underset{\sim}{T}_b$ is defined as in (1.2.4.1), it can be seen that the rank of its covariance matrix $\underset{\sim}{V}_b$ is at most $c(p-1)$. $\underset{\sim}{V}_b$ can be written as

$$\underset{\sim}{V}_b = \sum_{i=1}^{b} ((\eta_{ijj'})) \otimes ((\mathcal{R}_i^{(kk')})), \tag{3.3.8}$$

where

$$\mathfrak{N}_i = ((\eta_{ijj'})) = \begin{bmatrix} N_{i1}(N_i-N_{i1}) & \cdot & -N_{i1}N_{i2} & \cdot & -N_{i1}N_{ip} \\ -N_{i2}N_{i1} & \cdot & N_{i2}(N_i-N_{i2}) & \cdot & -N_{i2}N_{ip} \\ \cdot & & \cdot\ \cdot & & \cdot\ \cdot \\ -N_{ip}N_{i1} & \cdot & -N_{ip}N_{i2} & \cdot & N_{ip}(N_i-N_{ip}) \end{bmatrix} . \tag{3.3.9}$$

Let $\underset{\sim}{\eta}_{i*}$ be the matrix obtained from $\underset{\sim}{\eta}_i$ by omitting the last row and column. Then,

$$V_{b*} = \sum_{i=1}^{b} ((\eta_{ijj'*})) \otimes ((\mathfrak{R}_i^{(kk')})) \qquad (3.3.10)$$

is the variance-covariance matrix of $\underset{\sim}{T}_{b*}$.

If $((\eta_{ijj'*}))$ and $((\mathfrak{R}_i^{(kk')}))$ are non-singular for each i,

$$((\eta_{ijj'*})) \otimes ((\mathfrak{R}_i^{(kk')}))$$

is positive definite and so is $\underset{\sim}{V}_{b*}$.

Tawski's theorem can be applied to the matrix $\underset{\sim}{\eta}_{i*}$ and it can be concluded that:

> "If for any block i, $((\mathfrak{R}_i^{kk'}))$ is non-singular and the matrix $\underset{\sim}{\eta}_{i*}$ cannot be transformed into a matrix of the form
>
> $$\begin{pmatrix} \underset{\sim}{P} & \underset{\sim}{U} \\ \underset{\sim}{0} & \underset{\sim}{Q} \end{pmatrix}$$
>
> by the same permutation of the rows and columns, where $\underset{\sim}{P}$ and $\underset{\sim}{Q}$ are square matrices and $\underset{\sim}{0}$ consists of zeros only, then, V_{b*} is non-singular."

The condition can also be proved to be sufficient. If the frequencies N_{ij} are the same for each block, then

$$((\eta_{ijj'})) = ((\eta_{jj'}))$$

and

$$\underset{\sim}{V}_{b*} = ((\eta_{jj'*})) \otimes ((\sum_{i=1}^{b} \mathcal{R}_i^{(kk')})). \qquad (3.3.11)$$

If the rank of $\underset{\sim}{V}_b$ is c(p-1), the test statistic is taken to be

$$W_b = [\underset{\sim}{T}_b - E(\underset{\sim}{T}_b|\mathcal{P}_b)]'\underset{\sim}{C}'[\underset{\sim}{C}\underset{\sim}{V}_b\underset{\sim}{C}']^{-1}\underset{\sim}{C}[\underset{\sim}{T}_b - E(\underset{\sim}{T}_b|\mathcal{P}_b)], \qquad (3.3.12)$$

where $\underset{\sim}{C}$ is a (p-1)c x pc matrix of the form

$$\underset{\sim}{C} = \text{diag}\ (\underset{\sim}{D}, \underset{(c)}{\ldots}, \underset{\sim}{D})$$

and $\underset{\sim}{D}$ is a [(p-1)x p] matrix of linearly independent contrasts. If $\underset{\sim}{D}$ is defined as $\underset{\sim}{C}$ in (2.2.2.14), W_b takes the form

$$W_b = [\underset{\sim}{T}_{b*} - E(\underset{\sim}{T}_{b*}|\mathcal{P}_b)]'\underset{\sim}{V}_{b*}^{-1}[\underset{\sim}{T}_{b*} - E(\underset{\sim}{T}_{b*}|\mathcal{P}_b)] \qquad (3.3.13)$$

and when the number of non-compared sets is S > 1,

$$W_b = \sum_{s=1}^{S}[\underset{\sim}{T}_{b(s)} - E(\underset{\sim}{T}_{b(s)}|\mathcal{P}_b)]'\underset{\sim}{C}'_{(s)}[\underset{\sim}{C}_{(s)}\underset{\sim}{V}_{b(s)}\underset{\sim}{C}'_{(s)}]^{-1}\underset{\sim}{C}_{(s)}[\underset{\sim}{T}_{b(s)} - E(\underset{\sim}{T}_{b(s)}|\mathcal{P}_b)] \qquad (3.3.14)$$

under the same conditions stated in case MUI.

In the special case that

$$N_{ij} = N_j \ \forall\ j,$$

after some algebra, it can be found that, if the rank of $\underset{\sim}{V}_b$ is (p-1)c, (that implies that $N_j > 0$),

$$W_b = \sum_{k=1}^{c}\sum_{k'=1}^{c} \frac{1}{N} \mathcal{R}^{-kk'} \sum_{j=1}^{p} \frac{[T_{bj}^{(k)} - E(T_{bj}^{(k)}|\mathcal{P}_b)][T_{bj}^{(k')} - E(T_{bj}^{(k')}|\mathcal{P}_b)]}{N_j}, \qquad (3.3.15)$$

where $\mathcal{R}^{-kk'}$ is the general element of the inverse of the matrix

$$((\sum_{i}^{b} \mathcal{R}_i^{(kk')})).$$

It has to be recalled that, in this case as well as in the other two multivariate cases, if either the design is connected and the rank of $\underset{\sim}{V}_b$ is $r < (p-1)c$ or if there are S (> 1) non-compared sets of treatments and the rank is less than $(p-S)c$, the value of the statistic W_b is no longer invariant, under different choices of $\underset{\sim}{D}$(or $D_{(s)}$'s).

The next theorem states the distribution of the test statistic W_b.

Under the hypothesis H_o and assumptions (3.12), (3.13) and A_3, if

i) $\lim_{b\to\infty} ((\rho_{bjj'}^{kk'})) = ((\rho_{jj'}^{(kk')}))$ has rank $c(p-1)$

ii) The N_i's are uniformly bounded

iii) If b_j goes to infinity with b,

then the permutation distribution of W_b as defined in (2.3.3.12) is asympotically the distribution of a χ^2 variable with $c(p-1)$ d.f.

The proof is given in theorems 7.1.3 and 7.1.4 of Chapter 7.

The theorem extends to the case in which the rank of $((\rho_{jj'}^{kk'}))$ is r. From the previous theorem, follows that the asymptotic permutation test is given by the rule:

Reject H_o if $W_b \geq \chi^2_{r,\alpha}$

Accept H_o if $W_b < \chi^2_{r,\alpha}$

where r is the rank of $\underset{\sim}{V}_b$ and $\chi^2_{r,\alpha}$ is such that

$$P(\chi^2_r \leq \chi^2_{r,\alpha}) = 1-\alpha, \qquad (0 < \alpha < 1).$$

3.4 Case MUIV

Let us define the aligned (vector) yields and errors by

$$\underset{\sim}{Z}_{ij\ell} = \underset{\sim}{Y}_{ij\ell} - \bar{\underset{\sim}{Y}}_i \quad \text{where} \quad \bar{\underset{\sim}{Y}}_i = \frac{1}{N_i} \sum_{j=1}^{p} \sum_{\ell=1}^{N_{ij}} \underset{\sim}{Y}_{ij\ell} \qquad \begin{pmatrix} i = 1, \ldots, b \\ j = 1, \ldots, p \\ \ell = 1, \ldots, N_{ij} \end{pmatrix}$$

(3.4.1)

and

$$\underset{\sim}{\epsilon}_{ij\ell} = \underset{\sim}{e}_{ij\ell} - \bar{\underset{\sim}{e}}_i \quad \text{with} \quad \bar{\underset{\sim}{e}}_i = \frac{1}{N_i} \sum_{j=1}^{p} \sum_{\ell=1}^{N_{ij}} \underset{\sim}{e}_{ij\ell} \qquad (3.4.2)$$

The cummulative distribution function of

$$\underset{\sim}{\epsilon}_i = (\underset{\sim}{\epsilon}_{i11}, \ldots, \underset{\sim}{\epsilon}_{i1N_{i1}}, \ldots, \underset{\sim}{\epsilon}_{ip1}, \ldots, \underset{\sim}{\epsilon}_{ipN_{ip}})$$

remains the same under permutation of its N_i columns, because of assumption A_3.

It can be written that

$$\underset{\sim}{Z}_{ij\ell} = \underset{\sim}{\epsilon}_{ij\ell} \qquad \begin{pmatrix} i = 1, \ldots, b \\ j = 1, \ldots, p \\ \ell = 1, \ldots, N_{ij} \end{pmatrix} \qquad (3.4.3)$$

so that the hypothesis H_o reduces to testing the interchangeability of

$$(\underset{\sim}{Z}_{i11}, \ldots, \underset{\sim}{Z}_{i1N_{i1}}, \ldots, \underset{\sim}{Z}_{ipN_{ip}})$$

under shift alternatives.

Let

$$r_{ij\ell}^{(k)} = \text{Rank}\{Z_{ij\ell}^{(k)};\ Z_{111}^{(k)}, \ldots, Z_{bpN_{bp}}^{(k)}\} \quad \begin{pmatrix} i = 1, \ldots, b \\ j = 1, \ldots, p \\ \ell = 1, \ldots, N_{ij} \\ k = 1, \ldots, c \end{pmatrix} \tag{3.4.4}$$

and

$$\underset{\sim}{r}_i = \begin{bmatrix} r_{i11}^{(1)}, \ldots, r_{i1N_{i1}}^{(1)}, r_{i21}^{(1)}, \ldots, r_{i2N_{i2}}^{(1)}, \ldots, r_{ip1}^{(1)}, \ldots, r_{ipN_{ip}}^{(1)} \\ r_{i11}^{(2)}, \ldots, r_{i1N_{i1}}^{(2)}, r_{i21}^{(2)}, \ldots, r_{i2N_{i2}}^{(2)}, \ldots, r_{ip1}^{(2)}, \ldots, r_{ipN_{ip}}^{(2)} \\ . \quad \ldots \quad . \quad . \quad \ldots \quad . \quad \ldots . \quad \ldots \quad . \\ r_{i11}^{(c)}, \ldots, r_{i1N_{i1}}^{(c)}, r_{i21}^{(c)}, \ldots, r_{i2N_{i2}}^{(c)}, \ldots, r_{ip1}^{(c)}, \ldots, r_{ipN_{ip}}^{(c)} \end{bmatrix}$$

$$(i = 1, \ldots, b). \tag{3.4.5}$$

For each variable, the ranks in the blocks are a subset of N_i numbers out of the first N natural numbers.

The test is based on intra-block permutation of the columns of $\underset{\sim}{r}_i$. The number of configurations derived from a given set of b blocks is

$$\prod_{i=1}^{b} (N_i)!,$$

each configuration having (conditionally) the same probability.

Let us denote this uniform probability distribution over the

$$\prod_{i=1}^{b} N_i!$$

equally likely realizations by $\mathcal{P}_b$. Under $\mathcal{P}_b$,

$$E(r_{ij\ell}^{(k)}|\mathcal{P}_b) = \frac{\sum_{t=1}^{p}\sum_{h=1}^{N_{it}} r_{ith}^{(k)}}{N_i} = \bar{r}_i^{(k)} \tag{3.4.6}$$

$$E(r_{ij\ell}^{(k)} r_{i'j'\ell'}^{(k')}|\mathcal{P}_b) = \begin{cases} \frac{1}{N_i}\sum_{t=1}^{p}\sum_{h=1}^{N_{it}} r_{ith}^{(k)} r_{ith}^{(k')} & \text{if } \begin{array}{l} i = i' \\ j = j' \\ \ell = \ell' \end{array} \\[2ex] \frac{-1}{N_i(N_i-1)}\left[\sum_{t=1}^{p}\sum_{h=1}^{N_{it}} r_{ith}^{(k)} r_{ith}^{(k')} - \left(\sum_{t=1}^{p}\sum_{h=1}^{N_{it}} r_{ith}^{(k)}\right)\cdot\right. \\ \quad \left.\cdot\left(\sum_{t=1}^{b}\sum_{h=1}^{N_{it}} r_{ith}^{(k')}\right)\right] \quad \text{if } i = i' \\ \quad \text{and } \begin{cases} j = j' \text{ and } \ell \neq \ell' \\ \text{or } j \neq j' \end{cases} \\[2ex] \left(\sum_{t=1}^{p}\sum_{h=1}^{N_{it}} r_{ith}^{(k)}\right)\left(\sum_{t=1}^{p}\sum_{h=1}^{N_{it}} r_{ith}^{(k')}\right) \quad \text{if } i \neq i' \end{cases} \tag{3.4.7}$$

for any i, $i' = 1, 2, \ldots, b$; j, $j' = 1, \ldots, p$; $\ell = 1, \ldots, N_{ij}$, $\ell' = 1, \ldots, N_{i'j'}$; k, $k' = 1, 2, \ldots, c$.

The proofs are similar to the ones of the expressions (3.3.2) and (3.3.3) in the case MUII, since in both models $\mathcal{P}_b$ is originated by the invariance of the matrix of ranks under permutation of its columns. But in this case, oppositely the other,

$$\sum_j \sum_\ell r_{ij\ell}^{(k)}$$

is not equal to

$$\frac{N_i(N_i+1)}{2}$$

since the set of ranks in each row of the $i^{\underline{th}}$ block is not any longer the set of the first N_i natural numbers.

Let

$$T_{bj}^{(k)} = \sum_{i=1}^{b} \sum_{\ell=1}^{N_{ij}} r_{ij\ell}^{(k)} \qquad \begin{pmatrix} j = 1, \ldots, p \\ k = 1, \ldots, c \end{pmatrix} . \tag{3.4.8}$$

Conditionally,

$$E(T_{bj}^{(k)} | \mathcal{P}_b) = \sum_{i=1}^{b} N_{ij} \bar{r}_i^{(k)} \tag{3.4.9}$$

$$\mathrm{Cov}(T_{bj}^{(k)}, T_{bj'}^{(k')} | \mathcal{P}_b) = \sum_{i=1}^{b} N_{ij} (\partial_{jj'} N_i - N_{ij'}) \mathcal{R}_i^{(kk')}, \tag{3.4.10}$$

where

$$\mathcal{R}_i^{(kk')} = \frac{\sum_{t=1}^{p} \sum_{h=1}^{N_{it}} (r_{ith}^{(k)} - \bar{r}_i^{(k)})(r_{ith}^{(k')} - \bar{r}_i^{(k')})}{N_i(N_i - 1)} . \tag{3.4.11}$$

For each k,

$$\sum_{j=1}^{p} T_{bj}^{(k)} = \frac{N(N+1)}{2} \tag{3.4.12}$$

and

$$\sum_{j'=1}^{p} \mathrm{Cov}(T_{bj}^{(k)}, T_{bj'}^{(k')} | \mathcal{P}_b) = 0, \qquad \begin{pmatrix} j = 1, 2, \ldots, p \\ k, k' = 1, \ldots, c \end{pmatrix}, \tag{3.4.13}$$

as a consequence, the rank of the covariance matrix of $\underset{\sim}{T}_b$, (1.2.4.1), is at most $(p-1)c$ if the design is connected and is at most $(p-S)c$ if there are $S(\geq 1)$ non-compared sets of treatments.

The matrix $\underset{\sim}{V}_b$ can be written

$$\underset{\sim}{V}_b = \sum_{i=1}^{b} ((\eta_{ijj'})) \otimes ((\rho_i^{kk'})) \qquad (3.4.14)$$

with $((\eta_{ijj'}))$ defined as in (3.3.9).

Similarly to (3.3.10), the covariance matrix of $\underset{\sim}{T}_{b*}$ can be written as

$$\underset{\sim}{V}_{b*} = \sum_{i=1}^{b} ((\eta_{ijj'})) \otimes ((\rho_i^{kk'})). \qquad (3.4.15)$$

The sufficient condition (Section 3.3, page 83) based on Tawski's theorem for the non-singularity of $\underset{\sim}{V}_{b*}$ applies to this case as well.

If $N_{ij} = N_j \forall i$,

$$\underset{\sim}{V}_{b*} = ((\eta_{ijj'*})) \otimes ((\sum_{i=1}^{b} \rho_i^{(kk')})). \qquad (3.4.16)$$

If the rank of $\underset{\sim}{V}_b$ is $c(p-1)$, the test statistic is

$$W_b = [\underset{\sim}{T}_b - E(\underset{\sim}{T}_b|\rho_b)]'\underset{\sim}{C}'[\underset{\sim}{C}\underset{\sim}{V}_b\underset{\sim}{C}']^{-1}\underset{\sim}{C}[\underset{\sim}{T}_b - E(\underset{\sim}{T}_b|\rho_b)] \qquad (3.4.17)$$

where $\underset{\sim}{C}$ is defined as in (3.2.13).

Formulas (3.3.13), (3.3.14) and (3.3.15) apply to this case after obvious adaptations. Also, comments on the selection of $\underset{\sim}{C}$ when the rank of $\underset{\sim}{V}_b$ is $r < (p-S)c$ $(S \geq 1)$ apply.

The next theorem gives the distribution of W_b.

Under the hypothesis H_o and assumptions (3.1.2), (3.1.3), A_2 and A_3, if

i) $\lim_{b \to \infty} ((\rho_{bjj'}^{(kk')})) = ((\rho_{jj'}^{kk'}))$ has rank $c(p-1)$

ii) The N_i's are uniformly bounded

iii) $\lim_{b \to \infty} \frac{b_j}{b} = h \neq 0$ then,

the permutation distribution of W_b, as defined in (3.4.17) is asymptotically the distribution of a χ^2 with $c(p-1)$ d.f.

The proof is given in theorems 7.1.5 and 7.1.6 of Chapter 7.

The theorem can be extended to the case of $S(> 1)$ non-compared sets of treatments and rank of $\underset{\sim}{V}_b$ is $(p-S)c$, and to the situation in which there are $S(\geq 1)$ non-compared sets and rank $r \leq (p-S)c$. Of course, the definition of W_b has to be changed accordingly.

The asymptotic permutation test for H_o is given by the rule,

Reject H_o if $W_b \geq \chi^2_{r,\alpha}$

Accept H_o if $W_b < \chi^2_{r,\alpha}$

where r is the rank of $\underset{\sim}{V}_b$ and $\chi^2_{r,\alpha}$ is defined by the equation

$$P(\chi^2_r \leq \chi^2_{r,\alpha}) = 1-\alpha \qquad (0 < \alpha < 1).$$

4. MULTIVARIATE MIXED MODEL WITH SEVERAL (EVENTUALLY ONE OR ZERO) RESULTS PER TREATMENT AND INCOMPLETE RESPONSES - CASE IMUII

4.1 Introduction

Incomplete multiresponse designs are characterized by the fact that not all the variates are measured on each experimental unit. Such a situation arises when it is either physically impossible or otherwise inadvisable or inconvenient to measure each response on each unit. Even in the parametric theory, the problem of handling of multivariate data with observations missing on some or all the variables under study has only been solved in a very limited number of situations.

In the non-parametric framework almost no results have been presented on the subject.

Included in this chapter is a discussion of a permutation-rank test for the hypothesis of no treatment effects in an incomplete multiresponse mixed model. The experiment is of the kind that was discussed in Chapter 3, Section 3.3.

The set of coordinates is divided in two groups: one of primary interest and the other of secondary importance, and according to this classification, a variate-wise design is superimposed on the complete layout.

The denomination "primary interest" is used in a technical sense, and it is applied to that group of variables that is measured in all experimental units. The reason for doing it can be either that these variables are actually more important than the other variables, or, that other practical situations (cost, facilities at hand) are taken into consideration.

Without loss of generality, it can be assumed that the variables of primary interest are the ones identified by the set of superscripts $(1, 2, \ldots, q)$.

The model and the test are given in the next section.

4.2 The Model and The Test

As in the usual situation of several trials per treatment, it is assumed that the $i^{\underline{th}}$ subject is submitted to

$$N_i = \sum_j N_{ij}$$

trials. The trials are divided into two groups: one consisting of n_i ($\leq N_i$) trials and the other consisting of the remaining $(N_i - n_i)$ ones.

For the trials in the first group, the responses are vectors of c components while the responses of the trials in the second group have only q coordinates (the ones that correspond to the variables of less importance).

The number n_i is fixed, but which n_i trials (out of the N_i) are going to be complete and which are not, is determined at random.

The selection of the n_i trials is made in a way that gives to any set of n_i trials out of the N_i, equal chance of having a complete response. The selection disregards which treatments are involved in the n_i complete trials. For a given treatment, all the N_{ij} trials can be complete, or only part of them, or none.

The symbol n_{ij} represents the number of complete trials that correspond to treatment j in block i. n_{ij} is a random variable that varies from zero to N_{ij}. On the other hand, N_{ij} is considered to be fixed. The symbol

$$S_{n_i}$$

is used to indicate the set of indexes of the trials with complete responses;

$$S_{n_i}$$

can take

$$\binom{N_i}{n_i}$$

different forms. The symbol

$$b_j^{(k)}$$

indicates the number of blocks in which the k^{th} variable of the p^{th} treatment is measured. The outcome of the experiment can be represented by a set of incomplete matrices of the form

$$\underset{\sim}{Y}_i = [\underset{\sim}{Y}_{i11}, \ldots, \underset{\sim}{Y}_{i1N_{i1}}, \underset{\sim}{Y}_{i21}, \ldots, \underset{\sim}{Y}_{i2N_{i2}}, \ldots, \underset{\sim}{Y}_{ip1}, \ldots, \underset{\sim}{Y}_{ipN_{ip}}]$$

$$(i = 1, \ldots, b)$$

such that for each j, n_{ij} out of the N_{ij} vectors

$$\underset{\sim}{Y}_{ij\ell} \ (\ell = 1, \ldots, N_{ij})$$

have c coordinates while the other $(N_{ij} - n_{ij})$ vectors have only q $(< c)$. The set of q coordinates of primary importance does not change with either j or i.

Given any block, its entries

$$\{Y_{ij\ell}^{(k)}\}$$

can be thought of as arising from a two-stage sampling process. The first stage consists in the random selection of a set of n_i indexes, say

$$S_{n_i},$$

out of the N_i indexes that identify the trials of the block.

The selection is assumed to be made by giving equal probabilities to all the

$$\binom{N_i}{n_i}$$

possible sets that can be obtained from the block.

The procedure associates with each set

$$S_{n_i}$$

a particular $[c \times n_i + q \times (N_i - n_i)]$-variate cumulative distribution function,

$$F_{S_{n_i}}(\underset{\sim}{y}).$$

The

$$\binom{N_i}{n_i}$$

possible cumulative distribution functions associated with the sets

$$\left\{S_{n_i}\right\}$$

are the marginals obtained from a certain fixed $(c \times N_i)$-variate cumulative distribution function $F_i(\underset{\sim}{y})$, $\underset{\sim}{y}$ being defined as in (3.1.1), by integrating on the last $(c-q)$ variables of the columns whose indexes do not belong to the corresponding sets

$$S_{n_i}\text{'s}.$$

It is assumed that $F_i(\underset{\sim}{y})$ fulfills the conditions (3.1.2) and (3.1.3) of Section 3.1, i.e.,

$$F_i(\underset{\sim}{y}) = G_i(\underset{\sim}{y} - \underset{\sim}{m}_i) = G_i(\underset{\sim}{e}) \tag{4.2.1}$$

where $\underset{\sim}{m}_i$ is a $(c \times N_i)$ matrix with general entry

$$m_{ij\ell}^{(k)} = \tau_j^{(k)} + b_i^{(k)} \left(\begin{array}{l} i = 1, \ldots, b \\ j = 1, \ldots, p \\ \ell = 1, \ldots, N_{ij} \\ k = 1, \ldots, c. \end{array}\right).$$

It is also assumed that the N_i columns of the $(c \times N_i)$ matrix $\underset{\sim}{e}_i = \underset{\sim}{y}_i - \underset{\sim}{m}_i$ are interchangeable random variables.

This implies that if H_o is true, A_5: The marginals

$$F_{S_{n_i}}(\underset{\sim}{y})$$

obtained by integrating with respect to the last (c-q)-variables of any group of columns of $\underset{\sim}{Y}_i$ are invariant under the group of transformations G such that $g \in G$ is defined as a product, $g = g_1 g_2$, with g_1 belonging to the group of permutations of the complete columns and g_2 belonging to the group of permutations of the incomplete ones.

The hypothesis that is wanted to be tested is:

$$H_o: \; \tau_1^{(k)} = \tau_2^{(k)} = \ldots = \tau_p^{(k)} = 0$$

for $k = 1, \ldots, c$.

The test is based on intrablock ranking. The ranking is made independently for each variable. The ranks run from 1 to N_i for the first q variates and from 1 to n_i for the last (c-q).

Let $\underset{\sim}{R}_i$ be the incomplete matrix of ranks that corresponds to the (incomplete) matrix $\underset{\sim}{Y}_i$. Given a matrix $\underset{\sim}{R}_i$, let us consider all the forms that it could have taken if its entries would have been the result of other samples of indexes and all the treatments would have been equally effective.

Because of the irrelevance of the treatments and the random process of the first stage, the complete columns of $\underset{\sim}{R}_i$ could belong to any set of trials as well as the ones actually chosen. Also, under H_o and because of statement A_5 above, given any set of trials, say

$$S_{n_i},$$

all the matrices that could be obtained from $\underset{\sim}{R}_i$ by permuting the complete columns and the incomplete ones among themselves would be equally probable.

So, given the actual value of $\underset{\sim}{R}_i$, there are

$$N_i! = [n_i!(N_i - n_i)!]\binom{N_i}{n_i}$$

associated incomplete matrices that have, under H_o, equal probabilities. As a consequence, the

$$\prod_{i=1}^{b} N_i! = \prod_{i=1}^{b} n_i!(N_i - n_i)!\binom{N_i}{n_i}$$

potential outcomes associated with the set of incomplete matrices

$$\{\underset{\sim}{R}_1, \underset{\sim}{R}_2, \ldots, \underset{\sim}{R}_b\}$$

are, under H_o, equally probable as well.

In this cell of the space of possible results, a uniform probability law $\mathcal{P}_b$ can be defined (under H_o), and a conditional test can be constructed.

The test is based on the statistics:

$$T_{bj}^{(k)} = \begin{cases} \sum_{i=1}^{b} \sum_{\ell=1}^{N_{ij}} R_{ij\ell}^{(k)} & \text{if } 1 \le k \le q \\ \\ \sum_{i=1}^{b} \sum_{\ell \in S_{n_{ij}}} R_{ij\ell}^{(k)} & \text{if } (q+1) \le k \le c. \end{cases} \tag{4.2.3}$$

$$j = 1, \ldots, p.$$

$$S_{n_{ij}}$$

is the set of indexes that identify the complete trials of treatment j in the $i^{\underline{th}}$ block. It is considered that the N_i's and N_{ij}'s are such that $b_j \neq 0$ for any treatment and that the sample is large enough so that all the variables of every treatment are present in at least one block of the experiment, i.e., it is assumed that

$$T_{bj}^{(k)} > 0 \ \forall \ j,k.$$

Under $\wp_b$, the following results are obtained:

$$E(T_{bj}^{(k)}|\wp_b) = \begin{cases} \sum_{i=1}^{b} \left(\frac{N_i+1}{2}\right) & k = 1, \ldots, q \\ \\ \sum_{i=1}^{b} N_{ij} \frac{n_i}{N_i}\left(\frac{n_i+1}{2}\right) & k = q+1, \ldots, c. \end{cases} \tag{4.2.4}$$

$$\mathrm{Cov}(T_{bj}^{(k)}, T_{bj'}^{(k')}|\wp_b)$$

equals to

$$\sum_{i=1}^{b} \frac{(\partial_{jj'}N_i - N_{ij})N_{ij'}}{N_i(N_i-1)} \sum_{t=1}^{p} \sum_{h \in S_{n_{it}}} [R_{ith}^{(k)}R_{ith}^{(k')} - \frac{n_i}{N_i}\left(\frac{n_i+1}{2}\right)^2], \tag{4.2.5}$$

if

$$k = q+1, \ldots, c; \ k' = q+1, \ldots, c.$$

$$\sum_{i=1}^{b} \frac{(\partial_{jj'}N_i - N_{ij})N_{ij'}}{N_i(N_i-1)} \sum_{t=1}^{p} \sum_{h=1}^{N_{it}} [R_{ith}^{(k)}R_{ith}^{(k')} - \left(\frac{N_i+1}{2}\right)^2], \tag{4.2.6}$$

if

$$k = 1, \ldots, q; \; k' = 1, \ldots, q.$$

$$\sum_{i=1}^{b} \frac{(\partial_{jj'}N_i - N_{ij})N_{ij'}}{N_i(N_i-1)} \sum_{t=1}^{p} \sum_{h \in S_{n_{it}}} [R_{ith}^{(k)} R_{ith}^{(k')} - (\frac{n_i+1}{2})(\frac{N_i+1}{2})], \quad (4.2.7)$$

if

$$k = 1, 2, \ldots, q; \; k' = q+1, \ldots, c.$$

The treatment totals fulfill the relationships:

$$\sum_{j}^{p} T_{bj}^{(k)} = \begin{cases} \sum_{i=1}^{b} \frac{N_i(N_i+1)}{2} & \text{if } 1 \leq k \leq q \\ \sum_{i=1}^{b} \frac{n_i(n_i+1)}{2} & \text{if } q+1 \leq k \leq c. \end{cases} \quad (4.2.8)$$

Also,

$$\sum_{j'=1}^{p} \mathrm{Cov}(T_{bj}^{(k)}, T_{bj'}^{(k')} | \mathcal{P}_b) = 0 \quad (4.2.9)$$

for any j and any pair (k, k').

It follows, then, that:

$$\underset{\sim}{V}_b = ((\mathrm{Cov}(T_{bj}^{(k)}, T_{bj'}^{(k')} | \mathcal{P}_b))$$

has, at most, rank c(p-1). If the number of non-compared sets is S (> 1), then the rank is, at most, c(p-S).

Let us call

$$\frac{1}{N_i(N_i-1)} \sum_{t=1}^{p} \sum_{h \in S_{n_{it}}} [R_{ith}^{(k)} R_{ith}^{(k')} - \frac{n_i}{N_i}(\frac{n_i+1}{2})^2] = \mathcal{R}_{icc}^{(kk')} \quad (4.2.10)$$

$$\frac{1}{N_i(N_i-1)} \sum_{t=1}^{p} \sum_{h=1}^{N_{it}} [R_{ith}^{(k)} R_{ith}^{(k')} - (\frac{N_i+1}{2})^2] = \mathcal{R}_{iqq}^{(kk')} \qquad (4.2.11)$$

$$\frac{1}{N_i(N_i-1)} \sum_{t=1}^{p} \sum_{h \in S_{n_{it}}} [R_{ith}^{(k)} R_{ith}^{(k')} - (\frac{n_i+1}{2})(\frac{N_i+1}{2})] = \mathcal{R}_{icq}^{(kk')}. \qquad (4.2.12)$$

The matrix $\underset{\sim}{V}_b$ can be written as

$$\underset{\sim}{V}_b = \sum_{i=1}^{b} ((\eta_{ijj'})) \otimes \left(\begin{array}{c|c} \underset{\sim}{\mathcal{R}}_{iqq} & \underset{\sim}{\mathcal{R}}_{iqc} \\ \hline \underset{\sim}{\mathcal{R}}'_{iqc} & \underset{\sim}{\mathcal{R}}_{icc} \end{array}\right), \qquad (4.2.13)$$

where $((\eta_{ijj'}))$ is defined as in (2.3.3.9) and the matrices

$$\underset{\sim}{\mathcal{R}}_{iqq}, \underset{\sim}{\mathcal{R}}_{iqc}, \underset{\sim}{\mathcal{R}}_{icc},$$

with general entries as defined by (4.2.10), (4.2.11) and (4.2.12) respectively, have dimensions q x q, q x (c-q) and (c-q) c (c-q).

Also, as in (2.3.3.10), the covariance matrix of $\underset{\sim}{V}_{b*}$ can be written:

$$V_{b*} = \sum_{i=1}^{b} ((\eta_{ijj'*})) \otimes \left(\begin{array}{c|c} \underset{\sim}{\mathcal{R}}_{iqq} & \underset{\sim}{\mathcal{R}}_{iqc} \\ \hline \underset{\sim}{\mathcal{R}}'_{iqc} & \underset{\sim}{\mathcal{R}}_{icc} \end{array}\right).$$

In the same way that in the complete response case, it can be said that if

$$((\eta_{ijj'*}))$$

and

$$\left(\begin{array}{c|c} \underset{\sim}{\mathcal{R}}_{iqq} & \underset{\sim}{\mathcal{R}}_{iqc} \\ \hline \underset{\sim}{\mathcal{R}}'_{iqc} & \underset{\sim}{\mathcal{R}}_{icc} \end{array}\right)$$

are non-singular, so is the matrix $\underset{\sim}{V}_b$.

Also, a necessary and sufficient condition for $((\eta_{ijj'*}))$ to be non-singular is that it cannot be transformed in a matrix of the form

$$\begin{pmatrix} \underset{\sim}{P} & \underset{\sim}{U} \\ \underset{\sim}{O} & \underset{\sim}{Q} \end{pmatrix}$$

by the same permutation of the rows and columns, where $\underset{\sim}{P}$ and $\underset{\sim}{Q}$ are square matrices and $\underset{\sim}{O}$ has all its entries equal to zero.

If $N_{ij} = N_j$ for any block i,

$$V_{b*} = ((\eta_{jj'*})) \otimes \left(\begin{array}{c|c} \sum_i \underset{\sim}{\mathcal{R}}_{iqq} & \sum_i \underset{\sim}{\mathcal{R}}_{iqc} \\ \hline \sum_i \underset{\sim}{\mathcal{R}}'_{iqc} & \sum_i \underset{\sim}{\mathcal{R}}_{icc} \end{array}\right) . \qquad (4.2.15)$$

If all the $N_{ij} = 1$, the V_{b*} is positive definite if the second factor of the kronecker product is non-singular.

When the rank of V_b is c(p-1), the test statistic for testing H_o is

$$W_b = (\underset{\sim}{T}_b - E(\underset{\sim}{T}_b|\rho_b)]'\underset{\sim}{C}'[\underset{\sim}{C}\underset{\sim}{V}_b\underset{\sim}{C}']^{-1}\underset{\sim}{C}[\underset{\sim}{T}_b - E(\underset{\sim}{T}_b|\rho_b)] \qquad (4.2.16)$$

where $\underset{\sim}{C}$ is a (p-1) c x pc matrix of the form

$$\underset{\sim}{C} = \text{diag}(\underset{\sim}{D}, \underset{(c)}{\ldots}, \underset{\sim}{D}),$$

and $\underset{\sim}{D}$ is a (p-1) x p matrix of contrasts. With $\underset{\sim}{D}$ defined as $\underset{\sim}{C}$ in (2.2.2.14)

$$W_b = [T_{b*} - E(T_{b*}|\mathcal{P}_b)]' V_{b*}^{-1} [T_{b*} - E(T_{b*}|\mathcal{P}_b)]. \tag{4.2.17}$$

If the number of non-compared sets is $S > 1$, and $r(V_{b(s)}) = c(p_s - 1)$

$$W_b = \sum_{s=1}^{S} [T_{b(s)} - E(T_{b(s)}|\mathcal{P}_b)]' C'_{(s)} [C_{(s)} V_{b(s)} C'_{(s)}]^{-1} \underset{\sim}{C}_{(s)} [T_{b(s)}$$

$$- E(T_{b(s)}|\mathcal{P}_b)] \tag{4.2.18}$$

The remarks about the situations in which the rank of $\underset{\sim}{V}_b$ is $r < (p-S)c$ (where $S(\geq 1)$ is the number of non-compared sets) made for the analogous complete case are also valid in this case of incomplete responses.

The following theorem states the distribution of the test statistic W_b.

Under the hypothesis H_o and the assumption (3.2.1), (3.2.2) and A_5, if

i) $\lim_{b\to\infty} ((\rho_{bjj'}^{(kk')})) = ((\rho_{jj'}^{(kk')}))$ and $((\rho_{jj'}^{(kk')}))$ has rank $c(p-1)$

ii) the N_i's are uniformly bounded

iii) b_j goes to infinity with b,

the asymptotic (permutational) distribution of W_b as defined in (4.2.17) is the distribution of a χ^2 with c(p-1) d.f.

The proof is given in Chapter 7; theorems 7.1.11 and 7.1.12.

If there are S non-compared sets, W_b (defined in (4.2.18)) has a χ^2-distribution with c(p-S) d.f.

If there are $S(\geq 1)$ non-compared sets of treatments and the rank of V_b is less than c(p-S), say r, still W_b is distributed as a χ^2 variable with r degrees of freedom.

The asymptotic permutation test for H_o is given by the rule

$$\text{Reject } H_o \text{ if } W_b \geq \chi^2_{r,\alpha}$$

$$\text{Accept } H_o \text{ if } W_b < \chi^2_{r,\alpha},$$

where r is the rank of $\underset{\sim}{V}_b$ and $\chi^2_{r,\alpha}$ is such that

$$P(\chi^2_r \leq \chi^2_{r,\alpha}) = 1-\alpha, \quad (0 < \alpha < 1).$$

4.3 A Numerical Example

Some records of the memorizing of sonnets are presented in a paper by Gordon (1933). The data are the scores obtained for thirteen subjects submitted to the following trial:

Ten Shakespeare's sonnets were used as material to be memorized. The experimenter read sonnet "A" (Shakespeare's 59th) aloud once. The subject then wrote down all that he could recall. The experimenter read the same sonnet through again, and the subject, without referring to his first trial, wrote all that he could recall. This procedure continued until the sonnet had been heard five times and five reproductions had been written. These results were scored and the grades (of only five out of the ten sonnets) are shown on the following page.

The multivariate model MUII can be superimposed on these data, under which each score is represented by the equation

$$Y_{ij}^{(k)} = b_i^{(k)} + \tau_j^{(k)} + e_{ij}^{(k)} \quad \begin{pmatrix} i = 1, \ldots, 13 \\ j = 1, \ldots, 5 \\ k = 1, \ldots, 5 \end{pmatrix},$$

where $b_i^{(k)}$ is the subject effect associated with the $k^{\underline{th}}$ recall of the sonnet, $\tau_j^{(k)}$ is the sonnet effect (in the $k^{\underline{th}}$ recall) and $e_{ij}^{(k)}$ is the error term. The test statistic given by (3.3.12) was computed for the example and the result obtained was $W_b = 1092{,}56$ which indicates that the effects of the sonnets are significantly distinct.

The same data have been modified to illustrate the test proposed in this chapter.

Some treatments (sonnets) were omitted in some blocks according to a fixed treatment-wise design. Afterwards, random experiments were conducted in order to select the incomplete trials.

The blanks obtained by this procedure are shown in the table by the shadowed areas.

The total scores for each sonnet and trial are shown in the table on page 107.

The number in parenthesis express the number of blocks that has been added to give the total in each cell. The test statistic (4.2.16) was computed for the incomplete data, and the result was $W_b = 937.59$, a highly significant result, as it was the case with the complete design.

Treatment	A					B					C					D					E				
	TRIALS					TRIALS					TRIALS					TRIALS					TRIALS				
Subject	I	II	III	IV	V	I	II	III	IV	V	I	II	III	IV	V	I	II	III	IV	V	I	II	III	IV	V
Hey	5	15	10	27	50	15	20	34	37	43	10	22	29	36	48	10	25	28	39	42	10	21	31	33	36
DU	16	14	21	29	26	9	21	20	29	39	11	23	19	28	28	16	19	32	33	39	25	46	49	57	71
Ma	4	11	18	48	71	14	35	47	65	70	4	12	17	26	50	33	53	68	90	111	12	39	57	70	87
Gr	9	25	37	47	57	22	40	46	63	69	24	40	58	60	73	35	52	60	76	82	26	66	73	75	92
Wy	17	42	48	71	88	34	65	84	114	114	30	41	37	63	66	27	34	60	67	92	43	76	82	96	106
Sc	12	25	29	36	38	17	18	24	34	42	12	23	27	36	40	22	30	38	42	45	21	41	45	53	54
Wa	8	26	27	41	54	12	34	42	59	68	16	30	33	49	51	20	29	36	50	61	23	31	49	55	69
Mc	5	26	25	48	70	9	15	35	45	65	6	25	31	40	59	15	27	38	54	73	20	39	60	40	78
Sm	17	35	45	53	61	23	41	67	88	96	20	34	42	53	68	23	48	69	88	96	22	43	51	74	100
Ke	1	3	15	22	24	11	19	25	36	52	13	22	33	41	46	15	17	33	45	49	10	28	46	61	72
Ed	8	19	13	19	21	12	10	13	23	31	13	15	27	30	36	20	32	37	44	53	14	24	37	44	53
Go	11	21	20	25	44	6	19	24	31	41	22	37	26	45	56	27	31	46	50	62	30	33	47	51	63
Mi	8	9	22	39	60	17	41	49	77	79	50	81	103	108	108	25	43	60	73	87	11	40	53	76	82

Sonnet Trial	A	B	C	D	E
I	11.0(9)	18.0(9)	22.0(9)	29.0(9)	25.0(9)
II	14.0(9)	15.5(9)	23.5(9)	25.0(9)	27.0(9)
III	11.5(9)	15.5(9)	21.0(9)	28.5(9)	28.5(9)
IV	6.5(5)	8.5(5)	5.0(4)	6.0(3)	4.0(3)
V	8.0(5)	6.0(5)	5.0(4)	6.0(3)	5.0(3)

5. ON FURTHER APPLICATIONS OF MIXED MODELS, SUMMARY AND SUGGESTIONS FOR FUTURE RESEARCH

5.1 Introduction

The first section of the present chapter discusses the application of mixed models to the analysis of experimental situations other than the random block design type. The following section presents a summary of the principal points of this research and some suggestions for further research are made at the end.

5.2 Some Further Applications of Mixed Models

In the previous chapters, new mixed models have been proposed that are useful for studying the outcomes of several types of experiments.

This section is devoted to some comments about how the existing models can be adapted to make them suitable for the study of some experimental situations other than random blocks.

In applying rank tests, the ranking can be applied to the original observations as well as to functions of them. If the original model includes nuisance parameters, it may be possible "to clean" the model by a transformation of the basic variables. If this is the case, care has to be put in the interpretation of the error terms. Even if the original errors were assumed to be independent, the new ones will not, in general, be so. Mixed models offer a great deal of help in this kind of application. Their feasibility comes from the fact that they do not necessarily assume independence of the error terms associated with the same block.

As an illustration of these principles, two experimental situations and their models will be discussed in the next paragraphs.

5.2.1 Test of the Interaction Terms in a Factorial Design Laid in Complete Blocks with Unequal Number of Observations in the Cells

Let us consider the case of a factor A with p levels and another B with q levels.

Let $Y_{ij\ell h}$ represent the yield in the $h^{\underline{th}}$ trial of the combination of the levels (j,ℓ), (j being a level of A and ℓ a level of B), in the $i^{\underline{th}}$ block. The following model is assumed to describe the data:

$$Y_{ij\ell h} = b_i + \tau_j + \partial_\ell + \gamma_{j\ell} + e_{ij\ell h} \begin{pmatrix} i = 1, \ldots, b \\ j = 1, \ldots, p \\ \ell = 1, \ldots, q \\ h = 1, \ldots, N_{ij\ell} \end{pmatrix}$$

where b_i, τ_j and ∂_ℓ stand for the effects of the block, factor A and B respectively, and $\gamma_{j\ell}$ represent the interaction effects that are wanted to be tested. The e's are the error terms and it is assumed that the vectors

$$\underset{\sim}{e}_i' = (e_{i11}, \; e_{i11N_{i11}}, \ldots, e_{ipq1}, \ldots, e_{ipqN_{ipq}})$$

are independently distributed with a continuous cumulative distribution function G_i.

One restriction is imposed on the cell frequencies: $N_{ij\ell} = N_{i\ell}$ for each block and level of B. That means that the $i^{\underline{th}}$ block has a pattern that can be represented as follows:

B \ A	1	2	...	p
1	$\bar{Y}_{i11}$ (N_{i1})	$\bar{Y}_{i21}$ (N_{i1})	...	$\bar{Y}_{ip1}$ (N_{i1})
2	$\bar{Y}_{i21}$ (N_{i2})	$\bar{Y}_{i22}$ (N_{i2})	...	$\bar{Y}_{ip2}$ (N_{i2})
	...	...	...	...
q	$\bar{Y}_{i1q}$ (N_{iq})	$\bar{Y}_{i2q}$ (N_{iq})	...	$\bar{Y}_{ipq}$ (N_{iq})

where $\bar{Y}_{ij\ell}$ stands for the mean of the observations of the $(j\ell)$ cell and $N_{i\ell}$ is the frequency in all the cells of the same row "ℓ".

The hypothesis to be tested is

$$H_o: \underset{\sim}{\gamma} = ((\gamma_{j\ell})) = \underset{\sim}{0}$$

against alternatives $H_1: \underset{\sim}{\gamma} \neq \underset{\sim}{0}$.

Let us consider the transformed variables, $Z_{ij\ell}$,

$$Z_{ij\ell} = \bar{Y}_{ij\ell} - \bar{Y}_{ij.} - \bar{Y}_{i.\ell} + \bar{Y}_{i.} \quad \begin{pmatrix} i = 1, \ldots, b \\ j = 1, \ldots, p \\ \ell = 1, \ldots, q \end{pmatrix}$$

where $\bar{Y}_{ij.}$ and $\bar{Y}_{i.\ell}$ are the unweighted means of the $\bar{Y}_{ij\ell}$'s in the $j^{\underline{th}}$ row and $\ell^{\underline{th}}$ column respectively; $\bar{Y}_{i.}$ is the unweighted mean of the pq values of the matrix $((\bar{Y}_{ij\ell}))$, associated with the $i^{\underline{th}}$ block.

The $Z_{ij\ell}$'s are free from the nuisance parameters b, τ and ∂, and if the restrictions

$$\sum_{j=1}^{p} \gamma_{j\ell} = \sum_{\ell=1}^{q} \gamma_{j\ell} = \sum_{j=1}^{p} \sum_{\ell=1}^{q} \gamma_{j\ell} = 0$$

hold, they can be expressed as:

$$Z_{ij\ell} = \gamma_{j\ell} + \epsilon_{ij\ell},$$

where

$$\epsilon_{ij\ell} = \bar{e}_{ij\ell\cdot} - \bar{e}_{ij\cdot} - \bar{e}_{i\cdot\ell} + \bar{e}_{i\cdot\cdot}$$

and the $\bar{e}$'s are unweighted means.

The model assumes that the joint distribution of

$$\{ e_{ij\ell h}\colon j = 1, \ldots, p, \ \ell = 1, \ldots, q, \ h = 1, \ldots, N_{ij\ell} \}$$

is invariant under the group of permutations of the

$$N_i = \sum_{j=1}^{p} \sum_{\ell=1}^{q} N_{ij\ell}$$

subscripts of the e's.

At the same time, the joint distribution of the ϵ's is invariant under the group of permutations generated by interchanging the "j" subscripts of the ϵ's, for each block i.

As the subscripts "j" interchange, so do the vectors

$$\underset{\sim}{\epsilon}'_{ij} = (\epsilon_{ij1}, \epsilon_{ij2}, \ldots, \epsilon_{ijq}).$$

The model can be restated in terms of the Z's as follows:

Let each block to be formed by p vectors

$$(\underset{\sim}{Z}_{i1}, \underset{\sim}{Z}_{i2}, \ldots, \underset{\sim}{Z}_{ip})$$

where

$$\underset{\sim}{Z}'_{ij} = (Z_{ij1}, Z_{ij2}, \ldots, Z_{ijq})$$

and the Z's are such that

$$\underset{\sim}{Z}_{ij} = \underset{\sim}{\gamma}_j + \underset{\sim}{\epsilon}_{ij},$$

with

$$\gamma'_j = (\gamma_{j1}, \ldots, \gamma_{jq}).$$

Let the $\underset{\sim}{\epsilon}_i$'s have joint cumulative distribution function that is symmetric in its p arguments $(\underset{\sim}{\epsilon}_{i1}, \ldots, \underset{\sim}{\epsilon}_{ip})$.

The hypothesis H_o can be written

$$\underset{\sim}{\gamma}_j = \underset{\sim}{0} \quad \text{for} \quad j = 1, \ldots, p.$$

The test statistic to be used for testing H_o depends on further assumptions on the distribution of the errors and the pattern of frequencies.

i) If additivity of the blocks is assumed to hold and the same pattern of frequencies is used in each block, the mixed model MUIV can be used to describe the situation and the corresponding statistic can be used for the test of the nullity of the interactions.

ii) If additivity of the blocks does not hold and/or the pattern of frequencies varies with the blocks, mixed model MUII is the appropriate one and so is the related test statistic for the test of H_o.

5.2.2 One Analysis of Covariance-Type of Problem with Unequal Number of Observations Per Cell

A fictitious example will help to illustrate the problem.

Suppose that a study is performed about the effect on the success in learning produced by a change in some condition related to the method of teaching and, for this purpose, an experiment is conducted on twelve individuals. Each individual is submitted to a series of trials and the number of right answers in each trial is recorded. In the middle of the experiment, the condition under study is changed. The experiment continues under the modified condition until another series of tests have been performed by the same individuals.

Let us symbolize with $Y_{ij\ell}$ the score obtained in the $i^{\underline{th}}$ individual in the $\ell^{\underline{th}}$ test under condition j., and assume that the following model is considered to be appropriate for the situation:

$$Y_{ij\ell} = b_i + \tau_j + \beta_i(x_{i\ell} - \bar{x}) + e_{ij\ell} \quad \begin{pmatrix} i = 1, \ldots, b = 12 \\ j = 1, \ldots, p = 2 \\ \ell = 1, \ldots, N_{ij} \end{pmatrix}$$

where b_i represents the effect of the individual, τ_j is the effect of the condition, $x_{j\ell}$ represents the serial number of the trial and β_i is the average increment in score per trial due to learning under either conditions.

For the purpose of the study, b_i and $\beta_i x_{j\ell}$ are nuisance parameters that are wanted to be eliminated.

Let $Z_{ij\ell}$ be the "adjusted" value of $Y_{ij\ell}$.

$$Z_{ij\ell} = Y_{ij\ell} - \hat{\beta}_i(x_{j\ell} - \bar{x}),$$

with

$$\hat{\beta}_i = \frac{\sum_{j=1}^{p} \sum_{\ell=1}^{N_{ij}} (x_{j\ell} - \bar{x}_j) Y_{ij\ell}}{\sum_{j=1}^{p} \sum_{\ell=1}^{N_{ij}} (x_{j\ell} - \bar{x}_j)^2}$$

and

$$\bar{x}_j = \frac{\sum_{\ell=1}^{N_{ij}} x_{j\ell}}{N_{ij}},$$

whose location parameter is $(\tau_j + b_i)$, so that

$$Z_{ij\ell} = b_i + \tau_j + \epsilon_{ij\ell}.$$

The new error term ϵ_{ij} is a linear combination of the previous e's.

If the Y's are assumed to be diagonally symmetric, a test of $\tau_j = 0$ can be performed by computing the test statistic (2.2.2.13) on the adjusted values "Z".

5.3 Summary

Most of the current tests based on ranks are only applicable to simple experimental designs. Looking for a wider application of these procedures, this work has been addressed to the search for tests that are suitable for the analysis of experiments having incomplete and/or unbalanced designs, with univariate as well as multivariate responses. In relation to multivariate data, a test has been developed that allows for missing observations in some variables of the responses.

5.3.1 Underlying Designs and Models

The legitimate use of rank tests requires that a null hypothesis be specified and a model for certain functions of the observations be established such that the related null distribution of the test statistic to be used be applicable.

The models that underlie the tests proposed in this thesis can be considered as extensions of the non-parametric mixed models I, II and IV defined by Koch and Sen (1968). The designs to which they are applicable can be assimilated to the designs known by the names of

i) generalized randomized block design

ii) randomized block design with various results per cell

iii) incomplete randomized block design.

For mixed model experiments, each of b randomly selected subjects responds to each of p treatments exactly once. The outcome of the experiment constitutes an observation matrix composed of b vectors, each of length p. The hypothesis of interest is the equality of treatment effects.

In Chapter 2 (Section 2.2) non-parametric models have been extended to the situation in which several (or eventually only one or more) results are associated with each treatment, so that the vector of observations of a given subject, call it "i", can be symbolized as follows:

$$\underset{\sim}{Y}_i = (Y_{i11}, \ldots, Y_{i1N_{i1}}, Y_{i21}, \ldots, Y_{i2N_{i2}}, \ldots, Y_{ip1}, \ldots, Y_{ipN_{ip}})$$
$$i = 1, \ldots, b \qquad (5.3.1.1)$$

where $Y_{ij\ell}$ ($j = 1, \ldots, p$, $\ell = 1, \ldots, N_{ij}$) represents the $\ell^{\underline{th}}$ observation of the $j^{\underline{th}}$ treatment; N_{ij} is the number of observations corresponding to the $j^{\underline{th}}$ treatment in the block and

$$N_i = \sum_{j=1}^{p} N_{ij}$$

is the length of the $i^{\underline{th}}$ block.

The phrase "results per treatment" has been deliberately chosen to be ambiguous. It may indicate either "results of replicated trials of a given treatment" or "replicated results of a unique trail of the treatment". In the first case, the design in the background is a type of generalized block experiment; in the second case, it is a kind of randomized block design with several observations per cell. In the particular situation in which the numbers $\{N_{ij}\}$ are only ones or zeros, the design is called incomplete block design.

Assumptions analogous to the ones that originate models I and IV have been used to define two models, called mixed model UI and UIV, that can be looked at as their extensions. (The extension of case II is in the work by Benard and van Elteren (1953)).

Case UI can be applied to any type of design mentioned in the previous paragraph, while case UIV (as well as the model underlying the test of Benard and van Elteren) is suitable only for either generalized randomized block designs or incomplete block designs.

Also, a new model is introduced (Section 2.3) which is especially useful for the description of experiments with several observations per cell, that can be thought of as another type of extension of case II. It has been identified as case UII_o.

The distinctive characteristics of models UII and UII_o become apparent writing down the covariance matrix of the vector $\underset{\sim}{Y}_i$ (under H_o) that is implicit in both models. Let N_{ij} be constant, with respect to treatments and blocks, and let $\underset{\sim}{\Sigma}_i$ symbolize the covariance matrix of $\underset{\sim}{Y}_i$. Then, for model UII;

$$\underset{\sim}{\Sigma}_i = \begin{bmatrix} \sigma^2 & \rho\sigma^2, & \ldots, & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2, & \ldots, & \rho\sigma^2 \\ \cdot & \cdot & \cdots & \cdot \\ \rho\sigma^2 & \rho\sigma^2 & \ldots, & \sigma^2 \end{bmatrix},$$

while for model UII_o,

$$\underset{\sim}{\Sigma}_i = \begin{bmatrix} \sigma_1^2, & \ldots, & \rho_1\sigma_1^2, & \ldots, & \rho\sigma^2, & \ldots, & \rho\sigma^2 \\ \cdot & \cdots & \cdot & \cdots & \cdot & \cdots & \cdot \\ \rho_1\sigma_1^2, & \ldots, & \sigma_1^2, & \ldots, & \rho\sigma^2, & \ldots, & \rho\sigma^2 \\ \cdot & \cdots & \cdot & \cdots & \cdot & \cdots & \cdot \\ \rho\sigma^2, & \ldots, & \rho\sigma^2, & \ldots, & \sigma_1^2, & \ldots, & \rho_1\sigma_1^2 \\ \cdot & \cdots & \cdot & \cdots & \cdot & \cdots & \cdot \\ \rho\sigma^2, & \ldots, & \rho\sigma^2, & \ldots, & \rho_1\sigma_1^2, & \ldots, & \sigma_1^2 \end{bmatrix}.$$

In Chapter 3 a further extension of cases UI, UII and UIV had been made to include multiresponses. The observations corresponding to the $i^{\underline{th}}$ subject can be symbolized as in (5.3.1.1), but each $Y_{ij\ell}$ has to be interpreted as a vector of "c" coordinates instead of as a number.

Three models have been defined and identified by the symbols MUI, MUII, MUIV, respectively, a denomination that makes clear the type of univariate model to which they are related.

In Chapter 4 a model has been created in order to cope with the problem of incomplete responses in a multivariate general incomplete block design.

The model has been identified by IMUII and is an extension of case MUII. In relation to model IMUII, a vector response is said to be incomplete if it includes only a subset of the "c" variables that the research takes into account. It is assumed in the model that the $i^{\underline{th}}$ block, has n_i complete responses and $(N_i - n_i)$ incomplete ones. The set of variables that form the incomplete responses varies with neither the block in the experiment nor the trial in the block.

Models UII_o and IMUII have a common characteristic that has to be remarked. In both cases, the actual design of the experiment has been conceived as a realized sample from a larger population of designs. In case UII_o, the allocations of the frequencies to the treatments of each block are the result of random experiments and this is also the case with the selection of the complete trials of each block in the mixed model IMUII.

5.3.2 The Tests

The test statistics that are proposed in this work are extensions of the ones used by Wilcoxon, Kruskal and Friedman. The underlying theory for these test procedures is discussed in Chatterjee and Sen (1966), Puri and Sen (1966), Koch and Sen (1968), Sen (1968), Gerig (1969) and Koch (1969).

The tests are based on some function of the observations. After the experiment has been performed, the actual values taken by these functions are ranked. Cases I and II use intrablock ranking and, in case IV, a general ranking on the whole set of blocks is performed.

The space of all possible sets of ranks that can be obtained by the indicated procedures is considered. A partition of this space is chosen in such a way that, under the null hypothesis, all the points lying in the same cell are equally probable. Given an actual result, say R_o, the test of the hypothesis is performed in the cell, $S(R_o)$, to which the point belongs.

All the (conditional) tests presented in this thesis are based on criterion statistics that are quadratic forms that can be symbolically written as:

$$W = [\underset{\sim}{T} - E(\underset{\sim}{T})]\underset{\sim}{C}'[\underset{\sim}{CVC}]^{-1}\underset{\sim}{C}[\underset{\sim}{T} - E(\underset{\sim}{T})],$$

where $\underset{\sim}{T}$ is a vector whose components are the sums (over the whole experiment) of the ranks that correspond to each treatment and variable separately. Only for one of the tests, related to model UII_o, there is a different definition for $\underset{\sim}{T}$. In that case, $\underset{\sim}{T}$ is the unweighted sum of the p averages of the ranks for treatments 1, 2, ..., p. $E(\underset{\sim}{T})$ and $\underset{\sim}{V} = V(\underset{\sim}{T})$ are the (conditional) expectation vector and covariance matrix of $\underset{\sim}{T}$, $\underset{\sim}{C}$ is a convenient matrix of linearly independent vectors that breaks the singularity of $\underset{\sim}{V}$.

Restricted to $S(R_o)$, the conditional distribution function of W can be found. Nevertheless, since in the case of unbalanced designs, many parameters are involved, only asymptotic distribution of W has been considered. In all the cases, the resulting asymptotic (conditional) laws are of the χ^2-type, and are independent of the probabilistic structure of the cell, so they can be used also as an unconditional approximation.

In cases UII_o and IMUII, because of the random device that has been used for selecting the design, the space of possible results is enlarged.

Accordingly, given the actual value R_o, the cell $S(R_o)$ is enlarged, associating with R_o the results that would have been produced if other patterns had been selected in the first stage. Under H_o, a uniform (conditional) law is defined in the cell and the test is carried out as in the fixed design situation.

5.3.3 Examples

Two numerical examples have been presented; the first is designed as an incomplete block experiment, with several observations per cell.

Three mixed models (cases I, UI, UII_o) have been superimposed on the data and the corresponding tests statistics computed. The most significant results were obtained for the mixed model UII_o, but as the number of blocks do not justify a large sample approximation, the result has to be considered with caution.

The data of the second example refers to a psychological test performed on thirteen subjects. Each subject was submitted to five different treatments once, and five variables were measured on each trial. The design was balanced and complete ($N_{ij} = 1$ for any i and j). A mixed model MUII was assumed to be appropriate for describing the data and the corresponding test statistic was computed. Later on, some treatments and variables were deleted to make the data conform to an unequal frequency multivariate block design of the type that can be described by a mixed model IMUII. The new set of data was analyzed by the corresponding method and a result highly coincidental with that given for the complete data was found.

Two other (non-numerical) examples are given at the beginning of this chapter to illustrate the feasibility of mixed models in the description of different practical situations.

5.4 Suggestions for Further Research

i) The models presented in this work can be classified according to two criterions, i.e.,

a) Assumption on the distribution of the errors in ~~the~~ same block (diagonal symmetry, compound symmetry, additivety, fragmented compound symmetry).

b) Nature of the responses (univariate, complete and incomplete multivariate).

The crossed classification of the models originate twelve cases, but only seven out of them have been considered. The remaining five cases need to be developed.

ii) Other patterns of incomplete responses can be conceived of and, accordingly, new mixed models introduced.

iii) The idea of choosing incomplete designs at random can be used in relation to other non-parametric tests.

iv) The power properties of the new tests has to be studied. Power comparisons of tests with complete and incomplete responses has to be done.

v) New models can be developed for experiments with a complex design in the blocks.

6. LIST OF REFERENCES

Abbreviations of Journals

AMS = Annals of Mathematical Statistics

JASA = Journal of the Americal Statistical Association

JRSS = Journal of the Royal Statistics Society

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7. A P P E N D I X

7. APPENDIX

In many of the proofs of the previous chapter, the following theorem, due to Tausski, has been mentioned.

> "Let $((a_{ik}))$ be an n x n matrix with complex elements such that:
>
> $$|a_{ii}| \geq \sum_{k \neq i}^{n} |a_{ik}| \qquad (i = 1, \ldots, n)$$
>
> with equality in at most (n-1) cases. Assume further that the matrix cannot be transformed into a matrix of the form
>
> $$\begin{pmatrix} \underset{\sim}{P} & \underset{\sim}{U} \\ \underset{\sim}{0} & \underset{\sim}{Q} \end{pmatrix}$$
>
> by the same permutation of the rows and columns, where $\underset{\sim}{P}$ and $\underset{\sim}{Q}$ are square matrices and $\underset{\sim}{0}$ consists of zero only. It follows that $\det(a_{ik}) \neq 0$."

<u>Definition</u>

A random vector $\underset{\sim}{Y}'_n = (Y_{1n}, Y_{2n}, \ldots, Y_{qn})$ is said to be asymptotically q-variate normal with mean μ_n and dispersion $\underset{\sim}{\Sigma}_n$, if

$$\frac{\underset{\sim}{a}'(\underset{\sim}{Y}_n - \mu_n)}{\underset{\sim}{a}' \underset{\sim}{\Sigma}_n \underset{\sim}{a}}$$

is asymptotically N(0, 1) for every $\underset{\sim}{a}' = (a_1, a_2, \ldots, a_q)$ in the vector space X_q. If X_q can have at most rank r, then the rank of the asymptotic distribution is r.

7.1 Limit Theorems

Several theorems that refer to the asymptotic null distribution of the test criteria proposed in this thesis are given in this section of the Appendix. Before considering each of them separately, some remarks will be made that apply to all theorems.

Since the conditional distribution of W_b is constructed from the actual data of the experiment, it can be symbolized by $F_b(w, \mathcal{R}_b)$ where $\mathcal{R}_b$ stands for the set of matrices

$$\{\underset{\sim}{R}_1, \underset{\sim}{R}_2, \ldots, \underset{\sim}{R}_b\} .$$

The expression asymptotic distribution of W actually means limit of $F_b(w, \mathcal{R}_b)$ when b goes to infinity. In all models, this convergence is studied for selected types of sequences $\{\mathcal{R}_b\}$ that fulfill specific conditions (for instance, the one quoted in the theorems under (i)). As will be seen later on, the asymptotic results do not depend on which of the selected sequences $\mathcal{R}_b$ has been taken in consideration, so they are also valid unconditionally. In fact, the unconditional cumulative distribution function of W_b is

$$F_b(w) = E[F_b(w, \mathcal{R}_b)]$$

and the proof that $F_b(w)$ goes to $F_{\chi^2}(w)$ can be done in the same way that Corollary 3 to Theorem 3.2.2 in the paper by Chatterjee and Sen (1964).

7.1.1 Theorem

If the hypothesis H_o and the assumptions that correspond to case MUI hold, and

i) $\lim_{b\to\infty} ((\rho^{kk'}_{bjj'})) = ((\rho^{kk'}_{jj'}))$ and $((\rho^{(kk')}_{jj'}))$ has rank $c(p-1)$

ii) the N_i's are uniformly bounded

iii) for any j and k, $\lim_{b\to\infty} \mathrm{Var}(T^{(k)}_{bj}|\mathcal{P}_b) = \infty$, then

the joint asymptotic permutation distribution of

$$\{t^{(k)}_{bj}: \quad j = 1, \ldots, p; \; k = 1, \ldots, c\}$$

with

$$t^{(k)}_{bj} = \frac{T^{(k)}_{bj} - E(T^{(k)}_{bj}|\mathcal{P}_b)}{[\mathrm{Var}(T^{(k)}_{bj}|\mathcal{P}_b)]^{\frac{1}{2}}} \tag{7.1.1.1}$$

and $T^{(k)}_{bj}$, as defined in (3.2.8), is a pc-variate normal of rank c(p-1) with mean vector zero and dispersion matrix

$$((\rho^{kk'}_{bjj'}))$$

or

$$((\rho^{kk'}_{jj'})).$$

Proof

Let

$$\{a^{(k)}_j: \quad j = 1, \ldots, p; \; k = 1, \ldots, c\}$$

be arbitrary constants such that

$$\sum_{j=1}^{p} a_j^{(k)} = 0$$

for each k, and b to be large enough so that

$$\sigma_{bj}^{(k)^2} > \epsilon$$

for any j and k. Let

$$\mathfrak{m}_b = \frac{\sum_{j=1}^{p} \sum_{k=1}^{c} a_j^{(k)} t_{bj}^{(k)}}{[\mathrm{Var}(\sum_{j=1}^{p} a_j^{(k)} t_{bj}^{(k)})]^{\frac{1}{2}}} = \frac{\sum_{k=1}^{c} \sum_{j=1}^{p} a_j^{(k)} t_{bj}^{(k)}}{[\sum_{k=1}^{c} \sum_{k'=1}^{c} \sum_{j=1}^{p} \sum_{j'=1}^{p} a_j^{(k)} a_{j'}^{(k')} \rho_{bjj'}^{(kk')}]^{\frac{1}{2}}} .$$

Setting

$$\sum_k \sum_{k'} \sum_j \sum_{j'} a_j^{(k)} a_{j'}^{(k')} \rho_{bjj'}^{(kk')} = \Delta_b ,$$

$\mathfrak{m}_b$ can be written as

$$\mathfrak{m}_b = \frac{\sum_{k=1}^{c} \sum_{j=1}^{p} \frac{a_j^{(k)}}{\sigma_{bj}^{(k)}} \sum_{i=1}^{b} \sum_{\ell=1}^{N_{ij}} (R_{ij\ell} - \frac{N_i+1}{2})}{\Delta_b^{\frac{1}{2}}} = \sum_{i=1}^{b} g_{bi},$$

with

$$g_{bi} = \frac{\sum_{k=1}^{c} \sum_{j=1}^{p} \frac{a_j^{(k)}}{\sigma_{bj}^{(k)}} \sum_{\ell=1}^{N_{ij}} (R_{ij\ell} - \frac{N_i+1}{2})}{\Delta_b^{\frac{1}{2}}} .$$

The random variables

$$\{g_{bi}: i = 1, \ldots, b\}$$

are independent and

$$E(g_{bi}|\mathcal{P}_b) = 0$$

$$\mathrm{Var}(g_{bi}|\mathcal{P}_b) = \frac{1}{\Delta_b} \sum_{k=1}^{c} \sum_{k'=1}^{c} \sum_{j=1}^{p} \sum_{j'=1}^{p} \frac{a_j^{(k)}}{\sigma_{bj}^{(k)}} \frac{a_{j'}^{(k')}}{\sigma_{bj'}^{(k')}} [\sum_{\ell=1}^{N_{ij}} (R_{ij\ell}^{(k)} - \frac{N_i+1}{2})] \times$$

$$\times [\sum_{\ell'=1}^{N_{ij'}} (R_{ij'\ell'}^{(k')} - \frac{N_i+1}{2})],$$

so that

$$\sum_{i=1}^{b} \mathrm{Var}(g_{bi}|\mathcal{P}_b) = 1 . \qquad (7.1.1.2)$$

Since N_i (and so forth, the ranks in the $i^{\underline{th}}$ block) are bounded, as well as the $a_j^{(k)}$'s, a constant M can be found such that for any configuration of ranks and any i, j and k,

$$M \geq |a_j^{(k)} \sum_{\ell=1}^{N_{ij}} (R_{ij\ell}^{(k)} - \frac{N_i+1}{2})| .$$

Then, for any i,

$$|g_{bi}| \leq \sum_{k=1}^{c} \sum_{j=1}^{p} \frac{M}{\sigma_{bj}^{(k)}} \frac{1}{\Delta_b^{\frac{1}{2}}} = K_b . \qquad (7.1.1.3)$$

Let us call $\mathcal{L}$ the Lyapunoff quotient. From (7.1.1.2) and (7.1.1.3), it follows that

$$\mathcal{L} = \frac{\sum_{i=1}^{b} E(|g_{bi}|^3|\mathcal{P}_b)}{[\sum_{i=1}^{b} \mathrm{Var}(g_{bi}|\mathcal{P}_b)]^{3/2}} \leq \frac{K_b \sum_{i=1}^{b} E(|g_{bi}|^2|\mathcal{P}_b)}{[\sum_{i=1}^{b} \mathrm{Var}(g_{bi}|\mathcal{P}_b)]^{3/2}} = K_b ;$$

from (7.1.1.3),

$$\lim_{b\to\infty} K_b = \sum_{k=1}^{c} \sum_{j=1}^{p} \frac{M}{\lim_{b\to\infty} \sigma_{bj}^{(k)} \cdot \lim_{b\to\infty} \Delta_b^{\frac{1}{2}}} .$$

It is known that

$$\lim_{b\to\infty} \Delta_b^{\frac{1}{2}} > 0$$

since, in the limit, Δ_b is a positive definite quadratic form by i). Also, by iii),

$$\mathrm{Var}(T_{bj}^{(k)} | \mathcal{P}_b) = \sum_{i=1}^{b} [\sum_{\ell=1}^{N_{ij}} (R_{ij\ell} - \frac{N_i+1}{2})]^2$$

goes to ∞ with b; then

$$\lim_{b\to\infty} K_b = 0 ,$$

what implies that the Lyapunoff condition for the asymptotic normality of $\mathfrak{m}_b$ is fulfilled. From this result, theorem 7.1.1 is proved.

7.1.2 Theorem

Under the conditions stated in theorem 7.1.1, the asymptotic permutation distribution of W_b defined in (3.2.13) is the distribution of a χ^2 variable with c(p-1) d.f.

Proof

Since W_b can be written as a continuous function of the vector

$$\underset{\sim}{t}_b' = (t_{b1}^{(1)}, \ldots, t_{bp}^{(1)}, t_{b1}^{(2)}, \ldots, t_{bp}^{(2)}, \ldots, t_{b1}^{(c)}, \ldots, t_{bp}^{(c)}),$$

theorem 7.1.2 follows by applying the results of Sverdrup (1952) to the one stated in theorem 7.1.1.

7.1.3 Theorem

If the hypothesis H_o and the assumptions that correspond to case MUII hold, and the following conditions are satisfied,

i) $\lim\limits_{b\to\infty}((\rho_{bjj'}^{(kk')})) = ((\rho_{jj'}^{(kk')}))$, where $((\rho_{jj'}^{(kk')}))$ has rank $c(p-1)$,

ii) the N_i's are uniformly bounded,

iii) $\lim\limits_{b\to\infty}\frac{b_j}{b} = h \neq 0$ for some constant h,

then the joint asymptotic (permutational) distribution of

$$\{t_{bj}^{(k)}: \ j = 1, 2, \ldots, p; \ k = 1, 2, \ldots, c\},$$

with

$$t_{bj}^{(k)} = \frac{T_{bj}^{(k)} - E(T_{bj}^{(k)}|\mathcal{P}_b)}{[\mathrm{Var}(T_{bj}^{(k)}|\mathcal{P}_b)]^{\frac{1}{2}}}$$

and $T_{bj}^{(k)}$ defined by (3.3.4), is the one of a pc-variate normal variable, of rank c(p-1), with vector zero mean and dispersion matrix

$$((\rho_{bjj'}^{(kk')}))$$

or

$$((\rho_{jj'}^{(kk')})).$$

Proof.

First, note that in this case, ${\sigma_{bj}^{(k)}}^2 > 0$ for any j and k with probability one.

Let

$$\{a_j^{(k)}: \ j = 1, \ldots, p; \ k = 1, \ldots, c\}$$

be arbitrary constants such that

$$\sum_{j=1}^{p} a_j^{(k)} = 0$$

for each k. Let

$$\mathfrak{M}_b = \frac{\sum_{j=1}^{p} \sum_{k=1}^{c} a_j^{(k)} t_{bj}^{(k)}}{[\mathrm{Var}(\sum_{j=1}^{p} \sum_{k=1}^{c} a_j^{(k)} t_{bj}^{(k)})]^{\frac{1}{2}}} = \frac{\sum_{j=1}^{p} \sum_{k=1}^{c} a_j^{(k)} t_{bj}^{(k)}}{[\sum_{k=1}^{c} \sum_{k'=1}^{c} \sum_{j=1}^{p} \sum_{j'=1}^{p} a_j^{(k)} a_{j'}^{(k')} \rho_{bjj'}^{(kk')}]^{\frac{1}{2}}} .$$

Defining

$$\sum_{k=1}^{c} \sum_{k'=1}^{c} \sum_{j=1}^{p} \sum_{j'=1}^{p} a_j^{(k)} a_{j'}^{(k')} \rho_{bjj'}^{(kk')} = \Delta_b,$$

$\mathfrak{M}_b$ can be written as

$$\sum_{i=1}^{b} g_{bi}$$

with

$$g_{bi} = \frac{\sum_{k=1}^{c} \sum_{j=1}^{p} \frac{a_j^{(k)}}{\sigma_{bj}^{(k)}} \sum_{\ell=1}^{N_{ij}} (R_{ij\ell}^{(k)} - \frac{N_i+1}{2})}{\Delta_b^{\frac{1}{2}}} , \quad i = 1, \ldots, b.$$

The g_{bi}'s are independent random variables and under the conditional law $\mathcal{P}_b$,

$$E(g_{bi}|\mathcal{P}_b) = 0$$

$$\mathrm{Var}(g_{bi}|\mathcal{P}_b) = \frac{1}{\Delta_b} \sum_{k=1}^{c} \sum_{k'=1}^{c} \sum_{j=1}^{p} \sum_{j'=1}^{p} \frac{a_j^{(k)}}{\sigma_{bj}^{(k)}} \frac{a_{j'}^{(k')}}{\sigma_{bj'}^{(k')}} N_{ij'} \times$$

$$\times (\partial_{jj'} N_i - N_{ij}) \mathcal{R}_i^{(kk')}$$

$$\sum_{i=1}^{b} \mathrm{Var}(g_{bi}|\mathcal{P}_b) = 1. \qquad (7.1.3.1)$$

As a consequence of condition ii), there exists a constant M such that for any configuration of ranks and any i, j and k,

$$|a_j^{(k)} \sum_{\ell=1}^{N_{ij}} (R_{ij\ell} - \frac{N_i+1}{2})| \leq M.$$

Then, for any i,

$$|g_{bi}| \leq \sum_{\ell=1}^{c} \sum_{j=1}^{k} \frac{M}{\sigma_{bj}^{(k)}} \frac{1}{\Delta_b^{\frac{1}{2}}} = K_b. \qquad (7.1.3.2)$$

If $\mathcal{L}$ is the Lyapunoff quotient, we have, because of (7.1.3.1) and (7.1.3.2),

$$\mathcal{L} = \frac{\sum_{i=1}^{b} E(|g_{bi}|^3|\mathcal{P}_b)}{[\sum_{i=1}^{b} \mathrm{Var}(g_{bi}|\mathcal{P}_b)]^{3/2}} \leq \frac{K_b \sum_{i=1}^{b} E(|g_{bi}|^3|\mathcal{P}_b)}{[\sum_{i=1}^{b} \mathrm{Var}(g_{bi}|\mathcal{P}_b)]^{3/2}} \leq K_b. \qquad (7.1.3.3)$$

Also,

$$\sigma_{bj}^{(k)2} = \sum_{i=1}^{b} \frac{N_{ij}(N_i-N_{ij}) \sum_{t=1}^{p} \sum_{h=1}^{N_{it}} (R_{ith} - \frac{N_i+1}{2})^2}{N_i(N_i-1)}$$

$$\geq \sum_{i\in S_{b_j}} \frac{\sum_{t=1}^{p} \sum_{h=1}^{N_{it}} (R_{ith}^{(k)} - \frac{N_i+1}{2})^2}{N_i}$$

$$\geq b_j K \text{ for some } K \neq 0. \qquad (7.1.3.4)$$

The first inequality follows from the fact that, for a given N_i, the minimum value that

$$\frac{N_{ij}(N_i-N_{ij})}{(N_i-1)}$$

can attain occurs when the N_{ij}'s are either ones or zeros. The second follows from the result:

$$\frac{\sum_{t=1}^{p}\sum_{h=1}^{N_{it}}\left(R_{ith}^{(k)}-\frac{N_i+1}{2}\right)^2}{N_i} > \partial > 0$$

because the R's are different integers. Even if ties exist, it can be proven that

$$\frac{\sum_{t=1}^{p}\sum_{h=1}^{N_{it}}\left(R_{ith}^{(k)}-\frac{N_i+1}{2}\right)^2}{N_i(N_i-1)} \geq \frac{1}{4}.$$

From (7.1.3.4), it follows that if b_j goes to infinity with b, $\sigma_{bj}^{(k)^2}$ goes to infinity as well (no matter the values of j and k); then $\mathfrak{L}$ goes to zero and Lyapunoff condition for the normality of $\mathfrak{m}_b$ if fulfilled.

7.1.4 Theorem

Under the conditions stated in theorem 7.1.2, the asymptotic permutation distribution of W_b defined in (3.3.10) is the distribution of a χ^2-variable with c(p-1) d.f.

Proof

As given in theorem 7.1.2.

7.1.5 Theorem

If the hypothesis H_o and the assumptions that correspond to case MUIV hold, and

i) $\lim_{b \to \infty} ((\rho_{bjj'}^{(kk')})) = ((\rho_{jj'}^{kk'}))$ where $((\rho_{jj'}^{(kk')}))$ has rank $c(p-1)$

ii) the N_i's are uniformly bounded

iii) $\lim_{b \to \infty} \frac{b_j}{b} = h \neq 0$ for some constant h,

the joint asymptotic (permutational) distribution of

$$\{t_{bj}^{(k)}: \quad j = 1, \ldots, p, \; k = 1, \ldots, s\},$$

where

$$t_{bj} = \frac{T_{bj}^{(k)} - E(T_{bj}^{(k)} | \mathcal{P}_b)}{[\mathrm{Var}(T_{bj}^{(k)} | \mathcal{P}_b)]^{\frac{1}{2}}} \tag{7.1.5.1}$$

with $T_{bj}^{(k)}$ defined by (3.4.8), is the one of a pc-variate normal variable of rank c(p-1), with mean vector zero and dispersion matrix

$$((\rho_{bjj'}^{(kk')}))$$

or

$$((\rho_{jj'}^{(kk')})).$$

Proof

Let

$$\{a_j^{(k)}: \quad j = 1, \ldots, p; \; k = 1, \ldots, c\}$$

be arbitrary constants such that

$$\sum_{j=1}^{p} a_j^{(k)} = 0$$

for each k.

Let

$$\mathfrak{m}_b = \frac{\sum_{j=1}^{p} \sum_{k=1}^{c} a_j^{(k)} t_{bj}^{(k)}}{[\mathrm{Var}(\sum_{j=1}^{p} \sum_{k=1}^{c} a_j^{(k)} t_{bj}^{(k)})]^{\frac{1}{2}}} = \frac{\sum_{j=1}^{p} \sum_{k=1}^{c} a_j^{(k)} t_{bj}^{(k)}}{[\sum_{k=1}^{c} \sum_{k'=1}^{c} \sum_{j=1}^{p} \sum_{j'=1}^{p} a_j^{(k)} a_{j'}^{(k')} \rho_{bjj'}^{(kk')}]^{\frac{1}{2}}} .$$

If we call

$$\sum_{k=1}^{c} \sum_{k'=1}^{c} \sum_{j=1}^{p} \sum_{j'=1}^{p} a_j^{(k)} a_{j'}^{(k')} \rho_{bjj'}^{(kk')} = \Delta_b,$$

$\mathfrak{m}_b$ can be written

$$\mathfrak{m}_b = \sum_{i=1}^{b} g_{bi},$$

with

$$g_{bi} = \frac{\sum_{k=1}^{c} \sum_{j=1}^{p} \frac{a_j^{(k)}}{\sigma_{bj}^{(k)}} \sum_{\ell=1}^{N_{ij}} (r_{ij\ell}^{(k)} - \bar{r}_i^{(k)})}{\Delta_b^{\frac{1}{2}}} \qquad i = 1, \ldots, b.$$

The g_{bi}'s are independent random variables and conditionally to $\mathcal{P}_b$,

$$E(g_{bi} | \mathcal{P}_b) = 0$$

$$\mathrm{Var}(g_{bi} | \mathcal{P}_b) = \frac{1}{\Delta_b} \sum_{k=1}^{c} \sum_{k'=1}^{c} \sum_{j=1}^{p} \sum_{j'=1}^{p} \frac{a_j^{(k)}}{\sigma_{bj}^{(k)}} \frac{a_{j'}^{(k')}}{\sigma_{bj}^{(k')}} N_{ij'} (\partial_{jj'} N_i - N_{ij}) \mathcal{R}_i^{(kk')}, \tag{7.1.5.2}$$

so that

$$\sum_{i=1}^{b} \mathrm{Var}(g_{bi} | \mathcal{P}_b) = 1.$$

The Lyapunoff condition for the asymptotic normality of $\mathfrak{M}_b$ is

$$\lim_{b\to\infty} \mathfrak{L} = 0,$$

where

$$\mathfrak{L} = \frac{\sum_{i=1}^{b} E(|g_{bi}|^3|\rho_b)}{[\sum_{i=1}^{b} \mathrm{Var}(g_i|\rho_b)]^{3/2}} = \sum_{i=1}^{b} E(|g_{bi}|^3|\rho_b). \qquad (7.1.5.3)$$

Let us consider the inequality

$$\sum_{i=1}^{b} E(|g_{bi}|^3|\rho_b) = \sum_{i=1}^{b} \frac{1}{\Delta_b^{3/2}} E(|\sum_{j=1}^{p} \sum_{k=1}^{c} \frac{a_j^{(k)}}{\sigma_{bj}^{(k)}} \sum_{\ell=1}^{N_{ij}} (r_{ij\ell}^{(k)} - \bar{r}_i^{(k)})|^3|\rho_b)$$

$$\leq \frac{b \max|\sum_{j=1}^{p} \sum_{k=1}^{c} a_j^{(k)} \sum_{\ell=1}^{N_{ij}} (r_{ij\ell}^{(k)} - \bar{r}_i^{(k)})|^3}{\Delta_b^{3/2} \min[\sigma_{bj}^{(k)}]^3}. \qquad (7.1.5.4)$$

Because of relation (7.1.5.4), it is seen that the proof of (7.1.5.3) can be made in two parts, by showing that in the set of all possible configuration of ranks

a) $\qquad \max|\sum_{j=1}^{p} \sum_{k=1}^{c} a_j^{(k)} \sum_{\ell=1}^{N_{ij}} (r_{ij\ell}^{(k)} - \bar{r}_i^{(k)})|^3 \leq \alpha\, b^3.$

for some positive constant α (for any i), and

b) $\qquad \min|\sigma_{bj}^{(k)^2}| \geq \delta\, b^3$

for some positive constant δ (for any j and k).

Statement a) follows almost immediately since the maximum value of

$$(r_{ij}^{(k)} - \bar{r}_i^{(k)})$$

can be taken to be less than N. As a consequence,

$$\max\left|\sum_{j=1}^{p}\sum_{k=1}^{c} a_j^{(k)} \sum_{\ell=1}^{N_{ij}} (r_{ij\ell}^{(k)} - \bar{r}_i^{(k)})\right| \leq m\,N \tag{7.1.5.5}$$

where m is a fixed constant depending on the $a_j^{(k)}$'s and the N_{ij}'s.

Besides this, by assumption ii) there exists a constant, say n, such that $N_i \leq n \forall i$, what implies that $N \leq nb$. This inequality together with (7.1.5.5) proves statement a).

According to (3.4.10),

$$\sigma_{bj}^{(k)^2} = \sum_{i=1}^{b} \frac{N_{ij}(N_i - N_{ij})}{(N_i - 1)} \frac{\sum_{t=1}^{p}\sum_{h=1}^{N_{it}}(r_{ith}^{(k)} - \bar{r}_i^{(k)})^2}{N_i} .$$

Since the ranks are all different numbers, $\sigma_{bj}^{(k)^2}$ is positive (with probability one). For a fixed N_i, the two minimum values that

$$\frac{N_{ij}(N_i - N_{ij})}{(N_i - 1)}$$

(≥ 0) can attain are 0 (that happens when either $N_{ij} = 0$ or $N_{ij} = N_i$) and 1 (that occurs if either $N_{ij} = 1$ or $N_{ij} = N_i - 1$). Let us put 1 in place of N_{ij} in the expression of

$$\sigma_{bj}^{(k)^2},$$

if $N_{ij} > 0$. Then, accordingly,

$$\sigma_{bj}^{(k)^2} \geq \sum_{i=1}^{b} \frac{\sum_{t=1}^{p}\sum_{h=1}^{N_{it}}(r_{ith}^{(k)} - \bar{r}_i^{(k)})^2}{N_i}$$

and

$$\min \sigma_{bj}^{(k)^2} \geq \min \sum_{i=1}^{b} \frac{\sum_{t=1}^{p} \sum_{h=1}^{N_{it}} (r_{ith}^{(k)} - \bar{r}_i^{(k)})^2}{N_i}$$

$$\geq \sum_{i=1}^{b} \min \frac{\sum_{t=1}^{p} \sum_{h=1}^{N_{it}} (r_{ith}^{(k)} - \bar{r}^{(k)})^2}{N_i} . \qquad (7.1.5.6)$$

On the other hand, because the aligment is made around the mean, in each block at least one deviation lies to each side of the origin. Let $r_o^{(k)}$ be the rank of the origin if the origin is in the set

$\{(Y_{ij\ell}^{(k)} - \bar{Y}_i^{(k)}): i = 1, \ldots, b; j = 1, \ldots, p; \ell = 1, \ldots, N_{ij}\}$.

If it is not so, let $r_o^{(k)}$ be the average of the ranks of the two

$$(Y_{ij\ell}^{(k)} - \bar{Y}_i^{(k)})\text{'s}$$

inbetween which the origin stands. Then, for any variable,

$$\frac{\sum_{t=1}^{p} \sum_{h=1}^{N_{it}} (r_{ith}^{(k)} - \bar{r}_i^{(k)})^2}{N_i} \geq \frac{1}{N_i} \min_{t,h} (r_{ith}^{(k)} - r_o^{(k)})^2$$

$$\geq \frac{1}{n} \min_{t,h} (r_{ith}^{(k)} - r_o^{(k)})^2 . \qquad (7.1.5.7)$$

The first inequality follows from the fact that in each block at least one $r_{ith}^{(k)}$ has to lie on each side of $r_o^{(k)}$, what implies that at least one $r_{ith}^{(k)}$ must be closer to $r_o^{(k)}$ than to $\bar{r}_i^{(k)}$. The second inequality follows from assumption ii).

It also follows from the definition of $r_o^{(k)}$ that since in each block at least one rank lies to the right and one lies to the left of

$r_o^{(k)}$, there are at least b_j ranks bigger than $r_o^{(k)}$ and equal number smaller than $r_o^{(k)}$ in the whole set of ranks for each variable.

On the other part, the N ranks $\{r_{ij\ell}^{(k)}\}$ are assumed to be (wpl) distinct integers, so that if $r_o^{(k)}$ belongs to the set

$$\{r_{ith}: \ i = 1, \ldots, b; \ t = 1, \ldots, p; \ h = 1, \ldots, N_{it}\}$$

and b_j is odd

$$\sum_{i \in S_{bj}} \min_{t,h} (r_{ith}^{(k)} - r_o^{(k)})^2 \geq 2(1^2 + 2^2 + \ldots + (\frac{b_j - 1}{2})^2) \qquad (7.1.5.8)$$

and this sum is of the order of b_j^3.

The other three cases (that arise from the circumstance that $r_o^{(k)}$ may or may not belong to the sets of ranks and b_j can be either even or odd) lead to similar statements.

It has been assumed (assumption iii) that b_j goes linearly to infinity with b. Then, combining results (7.1.5.6), (7.1.5.7) and (7.1.5.8), it follows that

$$\min \sigma_{bj}^{(k)2} \geq \sum_{i \in S_{bj}} \min \frac{\sum_{t=1}^{p} \sum_{h=1}^{N_{it}} (r_{ith}^{(k)} - \bar{r}_i^{(k)})^2}{N_i} \geq \partial \, b^3$$

for some constant δ, and statement b) is proved. Note that the proof does not depend on which k and j are involved in $\sigma_{bj}^{(k)}$.

It follows from the proof of statements a) and b) that

$$\mathcal{L} \leq K \frac{b^4}{b^{9/2}} ,$$

for some constant K, and this shows that

$$\lim_{b \to \infty} \mathcal{L} = 0.$$

Then m_b is asymptotically normally distributed and theorem 7.1.5 is proved.

7.1.6 Theorem

Under the assumptions stated in theorem 7.1.5, the asymptotic permutation distribution of W_b defined in (3.4.16) is the distribution of a χ^2 variation with (p-1) d.f.

Proof

As in theorem 7.1.2.

7.1.7 Theorem

Under the model described in Section 2.3, and the hypothesis H_o, and the assumptions

i) $\lim_{b\to\infty} ((\rho_{bjj'})) = ((\rho_{jj'}))$ where $((\rho_{jj'}))$ has rank (p-1)

ii) the N_i's are uniformly bounded

iii) $\lim_{b\to\infty} \sigma^2_{bj} = \infty$ for any j (j = 1, ..., p),

the joint asymptotic (permutational) distribution of

$$\{t_{bj}: \ j = 1, 2, \ldots, p\},$$

with

$$t_{bj} = \frac{T_{bj} - E(T_{bj}|\mathcal{P}_b)}{[\mathrm{Var}(T_{bj}|\mathcal{P}_b)]^{\frac{1}{2}}}, \tag{7.1.7.1}$$

and T_{bj} has been defined in (2.3.7), is a p-variate normal distribution of rank (p-1) with mean vector zero and dispersion matrix

$$((\rho_{bjj'}))$$

or

$$((\rho_{jj'})).$$

Proof

Let b be large enough so that

$$\sigma_{bj}^2 > 0$$

for any j and let

$$\{a_j : j = 1, 2, \ldots, p\}$$

be arbitrary constants such that

$$\sum_{j=1}^{p} a_j = 0.$$

Define

$$\mathfrak{m}_b = \frac{\sum_{j=1}^{p} a_j t_{bj}}{[\mathrm{Var}(\sum_{j=1}^{p} a_j t_{bj})]^{\frac{1}{2}}} = \frac{\sum_{j=1}^{p} a_j t_{bj}}{[\sum_{j}\sum_{j'} a_j a_{j'} \rho_{bjj'}]^{\frac{1}{2}}}.$$

Letting

$$\sum_{j=1}^{p}\sum_{j'=1}^{p} a_j\, a_{j'}\, \rho_{bjj'} = \Delta_b,$$

$\mathfrak{m}_b$ can be written as

$$\mathfrak{m}_b = \sum_{i=1}^{b} g_{bi}$$

with

$$g_{bi} = \frac{\sum_{j=1}^{p} \frac{a_j}{\sigma_{bj}} (R_{ij.} - \frac{N_i}{p} (\frac{N_i+1}{2}))}{\Delta_b^{\frac{1}{2}}} .$$

The g_{bi}'s are independent random variables and

$$E(g_{bi} | \wp_b) = 0$$

$$\mathrm{Var}(g_{bi} | \wp_b) = E(g_{bi}^2 | \wp_b)$$

$$\sum_j \sum_{j'} \frac{a_j}{\sigma_j} \frac{a_{j'}}{\sigma_{j'}} E \frac{(R_{ij.} - \frac{N_i}{p}(\frac{N_i+1}{2}))(R_{ij'.} - \frac{N_i}{p}(\frac{N_i+1}{2}))}{\Delta_b} .$$

Adding over the blocks,

$$\sum_{i=1}^{p} \mathrm{Var}(g_{bi} | \wp_b) = 1. \qquad (7.1.7.2)$$

Let us consider the Lyapunoff quotient

$$\mathcal{L} = \frac{\sum_{i=1}^{b} E(|g_{bi}|^3 | \wp_b)}{[\sum_{i=1}^{b} \mathrm{Var}(g_{bi} | \wp_b)]^{3/2}} .$$

Since the N_i's (and so the ranks in the blocks) are bounded, as well as are the a_j's, a constant M can be found such that for any configuration of ranks and any block and treatment

$$M > |a_j (R_{ij.} - \frac{N_i}{p} (\frac{N_i+1}{2}))|.$$

This implies that

$$|g_{bi}| \leq \sum_{j=1}^{p} \frac{M}{\sigma_{bj}} \frac{a_j}{\Delta_b^{\frac{1}{2}}} = K_b.$$

Then,

$$\lim_{b\to\infty} \mathcal{L} \leq \lim_{b\to\infty} K_b \frac{\sum_{i=1}^{b} E|g_{bi}^2|}{[\mathrm{Var}(g_{bi}|\rho_b)]^{3/2}} = \lim_{b\to\infty} K_b.$$

By assumption i)

$$\lim_{b\to\infty} \Delta_b$$

is finite; and this result, together with assumptions iii), implies that

$$\lim_{b\to\infty} K_b = 0.$$

The theorem is proved.

7.1.8 Theorem

Under the conditions stated in theorem 7.1.7, the asymptotic permutation distribution of W_b defined in (2.3.15) is the distribution of a χ^2-variable with (p-1) d.f.

Proof

Same as in theorem 7.1.2.

7.1.9 Theorem

Under the model described in Section 2.4 and the hypothesis H_o, and the assumptions

i) $\lim_{b\to\infty} ((\rho_{bjj'})) = ((\rho_{jj'}))$ where $((\rho_{jj'}))$ has rank (p-1)

ii) the N_i's are uniformly bounded

iii) $\mathrm{Var}(T_{b\bar{R}j}|\mathcal{P}_b) > \epsilon > 0$ for any j (j - 1, ..., p).

the joint asymptotic (permutational) distribution of

$$\{t_{b\bar{R}j}: j = 1, 2, \ldots, p\},$$

with

$$t_{b\bar{R}j} = \frac{T_{b\bar{R}j} - E(T_{b\bar{R}j}|\mathcal{P}_b)}{[\mathrm{Var}(T_{b\bar{R}j}|\mathcal{P}_b)]^{\frac{1}{2}}},$$

where $T_{b\bar{R}j}$ has been defined in (2.3.8), is a p-variate normal distribution with mean vector zero and dispersion matrix

$$((\rho_{bjj'}))$$

or

$$((\rho_{jj'})).$$

7.1.10 Theorem

Under the conditions stated in theorem 7.1.8, the asymptotic permutation distribution of $W_{b\bar{R}}$ defined in (2.3.16) is the distribution of a χ^2-variable with (p-1) d.f.

Proof

The proofs of the last theorems follow the pattern of those of theorems 7.1.6 and 7.1.7, and are omitted.

7.1.11 Theorem

When the hypothesis H_o is true, under the model described in Chapter 4 (with conditions 4.2.1, 4.2.2 and A_5 holding), and if

i) $\lim_{b\to\infty} ((\rho_{bjj'}^{(kk')})) = ((\rho_{jj'}^{(kk')})$ with $((\rho_{jj'}^{(kk')}))$ of rank $c(p-1)$

ii) the N_i's are uniformly bounded

iii) $\lim_{b\to\infty} \frac{b_j}{b} = h \neq 0$ for some constant h,

the joint asymptotic (permutational) distribution of

$$\{t_{bj}^{(k)}: \ j = 1, \ldots, p; \ k = 1, 2, \ldots, c\},$$

with

$$t_{bj}^{(k)} = \frac{T_{bj}^{(k)} - E(T_{bj}^{(k)}|\rho_b)}{[\mathrm{Var}(T_{bj}^{(k)}|\rho_b)]^{\frac{1}{2}}}, \qquad (7.1.11.1)$$

$T_{bj}^{(k)}$ defined in (4.2.8), is a pc-variate normal of rank c(p-1) with mean vector zero and dispersion matrix

$$((\rho_{bjj'}^{(kk')}))$$

or

$$((\rho_{jj'}^{(kk')})).$$

Proof

Let

$$\{a_j^{(k)}: j = 1, 2, \ldots, p; \ k = 1, \ldots, c\}$$

be arbitrary constants such that

$$\sum_{j=1}^{p} a_j^{(k)} = 0$$

for each k. Let

$$\mathfrak{m}_b = \frac{\sum_{j=1}^{p} \sum_{k=1}^{c} a_j^{(k)} t_{bj}^{(k)}}{[\mathrm{Var}(\sum_{k=1}^{c} \sum_{j=1}^{p} a_j^{(k)} t_{bj}^{(k)})]^{\frac{1}{2}}}$$

$$= \frac{\sum_{k=1}^{c} \sum_{j=1}^{p} a_j^{(k)} t_{bj}^{(k)}}{[\sum_{k=1}^{c} \sum_{k'=1}^{c} \sum_{j=1}^{p} \sum_{j'=1}^{p} a_j^{(k)} a_{j'}^{(k')} \rho_{bjj'}^{(kk')}]^{\frac{1}{2}}} \ .$$

Calling

$$\sum_{k=1}^{c} \sum_{k'=1}^{c} \sum_{j=1}^{p} \sum_{j'=1}^{p} a_j^{(k)} a_{j'}^{(k')} \rho_{bjj'}^{(kk')} = \Delta_b ,$$

$\mathfrak{m}_b$ can be written as the sum of b variables:

$$\mathfrak{m}_b = \sum_{i=1}^{b} g_{bi},$$

with

$$g_{bi} = \{ \sum_{j=1}^{p} \sum_{k=1}^{q} \frac{a_j^{(k)}}{\sigma_j^{(k)}} [\sum_{\ell=1}^{N_{ij}} R_{ij\ell}^{(k)} - N_{ij}(\frac{N_i+1}{2})] +$$

$$+ \sum_{j=1}^{p} \sum_{k=q+1}^{c} \frac{a_j^{(k)}}{\sigma_j^{(k)}} [\sum_{h \in Sn_{ij}} R_{ij\ell}^{(k)} - N_{ij} \frac{n_i}{N_i}(\frac{n_i+1}{2})] \} \frac{1}{\Delta_b^{\frac{1}{2}}} \ .$$

It can be shown that under the two stage law $\mathcal{P}_b$

$$E(g_{bi}|\mathcal{P}_b) = 0$$

$$\sum_{i=1}^{b} \mathrm{Var}(g_{bi}|\mathcal{P}_b) = 1. \qquad (7.1.11.2)$$

Because of assumption ii), there exists a constant M such that for any configuration of ranks and any i, j and k, satisfy the two inequalities:

$$|a_j^{(k)} [\sum_{\ell=1}^{N_{ij}} R_{ij\ell}^{(k)} - N_{ij} (\frac{N_i+1}{2})]| \leq M,$$

and

$$|a_j^{(k)}|\sum_{\ell \epsilon Sn_{ij}} R_{ij\ell} - N_{ij} (\frac{n_i}{N_i})(\frac{n_i+1}{2})| \leq M.$$

Then, since the variances $\sigma_{bj}^{(k)}$ are positive w.p.1., it can be written that

$$|g_{bi}| \leq \sum_{j=1}^{b} \sum_{k=1}^{q} \frac{M}{\sigma_j^{(k)} \Delta_b^{\frac{1}{2}}} + \sum_{j=1}^{p} \sum_{k=q+1}^{c} \frac{M}{\sigma_j^{(k)} \Delta_b^{\frac{1}{2}}} \cdot \qquad (7.1.11.3)$$

Let us designate the Lyapunoff quotient with $\mathfrak{L}$. From (7.1.11.2) and (7.1.11.3), it follows that

$$\mathfrak{L} = \frac{\sum_{i=1}^{b} E(|g_{bi}|^3 | \mathcal{P}_b)}{[\sum_{i=1}^{b} Var(g_{bi} | \mathcal{P}_b)]^{3/2}} \leq \sum_{j=1}^{p} \sum_{k=1}^{q} \frac{M}{\sigma_j^{(k)} \Delta_b^{\frac{1}{2}}} + \sum_{j=1}^{p} \sum_{k=q+1}^{c} \frac{M}{\sigma_j^{(k)} \Delta_b^{\frac{1}{2}}} \cdot$$

The statement about the positiveness of $\sigma_j^{(k)^2}$ is easy to prove.

If $k = q + 1, \ldots, c$

$$\sigma_{bj}^{(k)^2} = \sum_{i=1}^{b} \frac{N_{ij}(N_i - N_{ij})}{N_i(N_i - 1)} \sum_{t=1}^{p} \sum_{h \epsilon Sn_{it}} [R_{ith}^{(k)^2} - \frac{n_i}{N_i}(\frac{n_i+1}{2})^2]$$

$$\geq \sum_{i=1}^{b} \frac{N_{ij}(N_i - N_{ij})}{N_i(N_i - 1)} \frac{n_i}{N_i}(\frac{n_i+1}{2}) \geq \sum_{i=1}^{b_j^{(k)}} \frac{n_i}{N_i^2}(\frac{n_i+1}{2})$$

$$\geq b_j^{(k)} m$$

for some constant $m \neq 0$. $b_j^{(k)}$ was defined on page 93 .

For

$$k = 1, 2, \ldots, q,$$

$\sigma_{bj}^{(k)^2}$ coincides with the variance defined in (3.3.6) with $j = j'$ for the complete multiresponse design.

If

$$k = (q+1), \ldots, c,$$

$\sigma_{bj}^{(k)^2}$ goes to infinity with b, since by the random selection of the complete columns of the design $b_j^{(k)}$ goes to infinity with b_j, in probability, and b_j goes to infinity with b by assumption iii).

The proof that $\sigma_{bj}^{(k)^2}$ goes to infinity with b for the variables

$$k = 1, 2, \ldots, q$$

follows by the same arguments given when proving the same statement about the variance of the complete multiresponse case.

Then for any j and k,

$$\lim_{b \to \infty} \sigma_j^{(k)^2} = \infty .$$

As the limit of Δ_b is finite (see assumption i)),

$$\lim_{b \to \infty} \mathcal{L} = 0$$

and the theorem is proven.

7.1.12 Theorem

Under the conditions stated in theorem 7.1.11, the asymptotic permutation distribution of W_b (given in 4.2.17) is the distribution of a χ^2-variate with c(p-1) d.f.

The theorems of this appendix could have been stated in a more general form considering that the rank of

$$((\rho_{jj'}^{(kk')})),$$

was a number $r < c(p-1)$.

In such cases, the arbitrary constants should have been chosen in the subspace X_r spanned by

$$\underset{\sim}{C}\,\underset{\sim}{a}_j^{(k)}$$

where $\underset{\sim}{C}$ is an r x pc matrix such that $\underset{\sim}{CVC'}$ is non-singular.

7.2 Computation of Terms that Enter in the Formulas for the Expected Values, Variances and Covariances of Section 2.3

In this part of the Appendix, some conditional expectations that are key terms in the development of the expressions 2.3.10, 2.3.11, 2.3.13, and 2.3.14 will be calculated.

It has to be noticed that according to the model of Section 2.3, for any block i, N_{ij} is a positive random variable with values in the set

$$\{M_1^i, M_2^i, \ldots, M_{U_i}^i\}$$

and such that the event

$$(N_{ij} = M_u^i)$$

occurs if and only if the fixed $j^{\underline{th}}$ treatment belongs to the random set S_u^i of π_i, i.e.,

$$P(N_{ij} = M_u^i) = \frac{p_u^i}{p} .$$

Therefore

$$E(\sum_{\ell=1}^{N_{ij}} R_{ij\ell}) = \sum_{u=1}^{U_i} E(\sum_{\ell=1}^{N_{ij}} R_{ij\ell} | N_{ij}=M_u^i) P(N_{ij}=M_u^i)$$

and

$$E(R_{ij\ell} | N_{ij}=M_u^i) = \sum_{t \in S_u^i} \frac{\sum_{h=1}^{M_u^i} R_{ith}}{p_u^i M_u^i},$$

i.e., it is the average of all the ranks associated with the set S_u^i of treatments. Let us call

$$R_{it\cdot} = \sum_{h=1}^{N_{it}} R_{ith},$$

then,

$$E(\sum_{\ell=1}^{N_{ij}} R_{ij\ell}) = \sum_{u=1}^{U_i} M_u^i (\sum_{t \in S_u^i} \frac{R_{it\cdot}}{p_u^i M_u^i}) \frac{p_u^i}{p} = \sum_{u=1}^{U_i} \sum_{t \in S_u^i} \frac{R_{it\cdot}}{p} .$$

Since

$$\{S_u^i, u = 1, 2, \ldots, U_i\}$$

is a partition of the p treatments involved in the experiment,

$$\sum_{u=1}^{U_i} \sum_{t \in S_u^i} \frac{R_{it\cdot}}{p} = \sum_{t=1}^{p} \frac{R_{it\cdot}}{p} = \frac{N_i(N_i+1)}{2p} ,$$

so,

$$\boxed{E\left(\sum_{\ell=1}^{N_{ij}} R_{ij\ell}\right) = \frac{N_i}{p}\left(\frac{N_i+1}{2}\right)}.$$

Furthermore,

$$E\left(\sum_{\ell=1}^{N_{ij}} R_{ij\ell}\right)^2 = E\left(\sum_{\ell=1}^{N_{ij}} R_{ij\ell}^2\right) + E\left(\sum_{\substack{\ell=1 \\ \ell \neq \ell'}}^{N_{ij}} \sum_{\ell'=1}^{N_{ij}} R_{ij\ell}R_{ij\ell'}\right) \qquad (7.2.1)$$

$$E\left(\sum_{\ell=1}^{N_{ij}} R_{ij\ell}^2\right) = \sum_{u=1}^{U_i} E\left(\sum_{\ell=1}^{N_{ij}} R_{ij\ell}^2 \middle| N_{ij}=M_u^i\right) P(N_{ij}=M_u^i)$$

$$= \sum_{u=1}^{U_i} M_{ui} \sum_{t \in S_u^i} \frac{\sum_{h=1}^{M_u^i} R_{ith}^2}{p_u^i M_u^i} \frac{p_u^i}{p}$$

$$= \frac{\sum_{t=1}^{p} \sum_{h=1}^{N_{it}} R_{ith}^2}{p},$$

$$E\left(\sum_{\substack{\ell=1 \\ \ell\neq\ell'}}^{N_{ij}} \sum_{\ell'=1}^{N_{ij}} R_{ij\ell}R_{ij\ell'}\right) = \sum_{u=1}^{U_i} E\left(\sum_{\substack{\ell=1 \\ \ell\neq\ell'}}^{N_{ij}} \sum_{\ell'=1}^{N_{ij}} R_{ij\ell}R_{ij\ell'} \middle| N_{ij}=M_u^i\right) P(N_{ij}=M_u^i)$$

$$= \sum_{u=1}^{U_i} M_u^i(M_u^i-1) E(R_{ij\ell}R_{ij\ell'} | N_{ij}=M_u^i) \frac{p_u^i}{p}$$

$$= \sum_{u=1}^{U_i} M_u^i(M_u^i-1) \sum_{t \in S_u^i} \frac{\sum_{h=1}^{M_u^i} R_{ith}(R_{it\cdot}-R_{ith})}{p_u^i M_u^i (M_u^i-1)} \frac{p_u^i}{p}$$

$$= \sum_{j=1}^{p} \sum_{h=1}^{N_{ij}} \frac{R_{ith}(R_{it\cdot}-R_{ith})}{p}$$

$$= \frac{\sum_{t=1}^{p} R_{it\cdot}^2 - \sum_{t=1}^{p} \sum_{h=1}^{N_{ij}} R_{ith}^2}{p}.$$

Adding the results obtained for the two expectations on the right side of equation (7.2.1)

$$\boxed{E\left(\sum_{\ell=1}^{N_{ij}} R_{ij\ell}\right)^2 = \sum_{t=1}^{p} \frac{R_{it\cdot}^2}{p}}.$$

Also, if $j \neq j'$,

$$E\left[\left(\sum_{\ell=1}^{N_{ij}} R_{ij\ell}\right)\left(\sum_{\ell'=1}^{N_{ij'}} R_{ij'\ell'}\right)\right] =$$

$$= \sum_{u=1}^{U_i} E\left(\sum_{\ell=1}^{N_{ij}} \sum_{\ell'=1}^{N_{ij'}} R_{ij\ell} R_{ij'\ell'} \middle| N_{ij}=M_u^i,\ N_{ij'}=M_u^i\right) P(N_{ij}=M_u^i,\ N_{ij'}=M_u^i) +$$

$$+ \sum_{\substack{u=1 \\ u' \neq u}}^{U_i} \sum_{u'=1}^{U_i} E\left(\sum_{\ell=1}^{N_{ij}} \sum_{\ell'=1}^{N_{ij'}} R_{ij\ell} R_{ij'\ell'} \middle| N_{ij}=M_u^i,\ N_{ij'}=M_{u'}^i\right) P(N_{ij}=M_u^i,\ N_{ij'}=M_{u'}^i)$$

$$= \sum_{u=1}^{U_i} \frac{\left[\sum_{t\epsilon S_u^i} \sum_{h=1}^{M_u^i} R_{ith}\right]^2 - \sum_{t\epsilon S_u^i}\left(\sum_{h=1}^{M_u^i} R_{ith}\right)^2}{p_u^i M_u^i (p_u M_u^i - M_u^i)} \frac{p_u^i}{p} \frac{p_u^i - 1}{p} +$$

$$+ \sum_{\substack{u'=1 \\ u' \neq u}}^{U_i} \sum_{u=1}^{U_i} \frac{\left(\sum_{t\epsilon S_u^i} \sum_{h=1}^{M_u^i} R_{ith}\right)\left(\sum_{t\epsilon S_{u'}^i} \sum_{h=1}^{M_{u'}^i} R_{ith}\right)}{p_u^i M_u^i p_{u'}^i M_{u'}^i} \frac{p_u^i}{p} \frac{p_{u'}^i}{p}$$

$$= \frac{\left[\sum_{u=1}^{U_i} \sum_{t\epsilon S_u^i} R_{it\cdot}\right]^2 - \sum_{u=1}^{U_i} \sum_{t\epsilon S_u^i} R_{it\cdot}^2}{p(p-1)}.$$

Then,

$$\boxed{E\left[\left(\sum_{\ell=1}^{N_{ij}} R_{ij\ell}\right)\left(\sum_{\ell'=1}^{N_{ij'}} R_{ij'\ell'}\right)\right] = \frac{\left[\frac{N_i(N_i+1)}{2}\right]^2 - \sum_{t=1}^{p} R_{it\cdot}^2}{p(p-1)}}\ .$$

7.3 Computation of Terms that Enter in the Formulas for the Expected Values, Variances and Covariances of Section 4.2

Most of these terms involve expectations of sums of products of ranks such that one factor is related to one variable, say k, and the other factor to a second variable k′ (k and k′ can be equal). The procedures that have to be used depend on which group of variables (the group of primary interest or that of secondary importance) k and k′ belong to. Three cases have to be distinguished.

Case A: k and k′ belong to the set

$$\{q + 1, \ldots, c\}$$

of secondary interest.

Case B: k and k′ belong to the set

$$\{1, \ldots, q\}$$

of primary interest.

Case C: k belongs to the set

$$\{1, \ldots, q\}$$

and k′ to

$$\{q + 1, \ldots, c\}.$$

In finding the expectations, the variables n_{ij} play an important role. From the sampling scheme used for selecting the n_i complete trials, it follows that the n_{ij}'s have the hypergeometric distribution, i.e.,

$$P(n_{ij}=n_{ij}^o) = f_{N_i,n_i}^{N_{ij}}(n_{ij}^o) = \frac{\binom{N_{ij}}{n_{ij}^o}\binom{N_i-N_{ij}}{n_i-n_{ij}^o}}{\binom{N_i}{n_i}} .$$

If $i \neq i'$, n_{ij} and $n_{i'j'}$ are independent but if $i' = i$ (and $j \neq j'$), the joint probability distribution is given as follows:

$$P(n_{ij}=n^o_{ij}, n_{ij}=n^o_{ij'}) = f^{N_{ij}N_{ij'}}_{N_i, n_i}(n^o_{ij}, n^o_{ij'}) = \frac{\binom{N_{ij}}{n^o_{ij}}\binom{N_{ij'}}{n^o_{ij'}}\binom{N_i-N_{ij}-N_{ij'}}{n_i-n^o_{ij}-n^o_{ij'}}}{\binom{N_i}{n_i}} .$$

The symbols

$$S_{n_{ij}}$$

and

$$S_{N_{ij}-n_{ij}}$$

will represent the set of ℓ indexes in block "i" and treatment "j" for which the trials are complete and incomplete respectively.

Also,

$$S_{n_{ij}}$$

and

$$S_{N_{ij}-n_{ij}}$$

are disjoint sets, such that

$$S_{n_{ij}} \cup S_{N_{ij}-n_{ij}} = S_{N_{ij}},$$

the set of all the trials for the combination (i,j).

The expectations are computed under the law $\mathcal{P}_b$.

7.3.1 Case A

$$E\left(\sum_{\ell \in S_{n_{ij}}} R_{ij\ell}^{(k)}\right) = \sum_{n_{ij}=0}^{N_{ij}} E\left(\sum_{\ell \in S_{n_{ij}}} R_{ij\ell}^{(k)} \middle| n_{ij}\right) f_{N_i n_i}^{N_{ij}}(n_{ij})$$

$$= \sum_{n_{ij}=0}^{N_{ij}} n_{ij}\left(\frac{n_i+1}{2}\right) f_{N_i,n_i}^{N_{ij}}(n_{ij}).$$

After rearrangement of factors and a change of variables, the last expression can be written as:

$$N_{ij}\left(\frac{n_i+1}{2}\right)\frac{n_i}{N_i} \sum_{n_{ij}=0}^{N_{ij}-1} f_{N_i-1,n_i-1}^{N_{ij}-1}(n_{ij}).$$

Since for any parameters

$$\sum_{n_{ij}} f(n_{ij}) = 1,$$

it follows that

$$\boxed{E\left(\sum_{\ell \in S_{n_{ij}}} R_{ij\ell}^{(k)}\right) = N_{ij}\left(\frac{n_i+1}{2}\right)\frac{n_i}{N_i}}\ .$$

By similar arguments, it can be proved that:

$$\boxed{E\left(\sum_{\ell \in S_{n_{ij}}} R_{ij\ell}^{(k)} R_{ij\ell}^{(k')}\right) = \frac{\sum_{t=1}^{p} \sum_{h \in S_{n_{it}}} R_{ijh}^{(k)} R_{ijh}^{(k')}}{n_i} N_{ij} \frac{n_i}{N_i}}\ .$$

Let $j \neq j'$, then

$$E[(\sum_{\ell \in S_{n_{ij}}} R^{(k)}_{ij\ell})(\sum_{\ell' \in S_{n_{ij'}}} R^{(k')}_{ij'\ell'})] =$$

$$= \sum_{n_{ij}=0}^{N_{ij}} \sum_{n_{ij'}=0}^{N_{ij'}} E[(\sum_{\ell \in S_{n_{ij}}} R^{(k)}_{ij\ell})(\sum_{\ell' \in S_{n_{ij'}}} R^{(k')}_{ij'\ell'}) | n_{ij}, n_{ij'}] f^{N_{ij}, N_{ij'}}_{N_i, n_i}(n_{ij}, n_{ij'})$$

$$= \sum_{n_{ij}=0}^{N_{ij}} \sum_{n_{ij'}=0}^{N_{ij'}} E[\sum_{\ell \in S_{n_{ij}}} \sum_{\ell' \in S_{n_{ij'}}} R^{(k)}_{ij\ell} R^{(k')}_{ij'\ell'} | n_{ij}, n_{ij'}] f^{N_{ij}, N_{ij'}}_{N_i, n_i}(n_{ij}, n_{ij'}).$$

Once that n_{ij} and $n_{ij'}$ are fixed, the expectation can be obtained by using the same arguments used in the case of the analogous unconditional expectation for the complete model MUII. Then,

$$[\sum_{n_{ij}=0}^{N_{ij}} \sum_{n_{ij'}=0}^{N_{ij'}} \frac{n_{ij} n_{ij'}}{n_i(n_i-1)}[(\frac{n_i(n_i+1)}{2})^2 - \sum_{t=1}^{p} \sum_{h \in S_{n_{it}}} R^{(k)}_{ith} R^{(k')}_{ith}] \times$$

$$\times f^{N_{ij}, N_{ij'}}_{N_i, n_i}(n_{ij}, n_{ij'})$$

$$= \frac{N_{ij} N_{ij'}}{N_i(N_i-1)}[(\frac{n_i(n_i+1)}{2})^2 -$$

$$- \sum_{t=1}^{p} \sum_{h \in S_{n_{it}}} R^{(k)}_{ith} R^{(k')}_{ith}] \times$$

$$\times \sum_{n_{ij}=0}^{N_{ij}-1} \sum_{n_{ij'}=0}^{N_{ij'}-1} f^{(N_{ij}-1)(N_{ij'}-1)}_{(N_i-2)(n_i-2)}$$

The double sum adds to one. Then,

$$\boxed{E[(\sum_{\ell\in S_{n_{ij}}} R_{ij\ell}^{(k)})(\sum_{\ell'\in S_{n_{ij'}}} R_{ij'\ell'}^{(k')})] = \frac{N_{ij}N_{ij'}}{N_i(N_i-1)}[(\frac{n_i(n_i+1)}{2})^2 - \sum_{t=1}^{p}\sum_{h\in S_{n_{it}}} R_{ith}^{(k)}R_{ith}^{(k')}]}\ .$$

Also

$$E[(\sum_{\ell\in S_{n_{ij}}} R_{ij\ell}^{(k)})(\sum_{\ell\in S_{n_{ij}}} R_{ij\ell}^{(k')})] =$$

$$= \sum_{n_{ij}=0}^{N_{ij}} E[(\sum_{\ell\in S_{n_{ij}}} R_{ij\ell}^{(k)})(\sum_{\ell\in S_{n_{ij}}} R_{ij\ell}^{(k')})|n_{ij}]f_{N_i,n_i}^{N_{ij}}(n_{ij})$$

$$= \sum_{n_{ij}=0}^{N_{ij}} E(\sum_{\ell\in S_{n_{ij}}} R_{ij\ell}^{(k)}R_{ij\ell}^{(k')} +$$

$$+ \sum_{\substack{\ell,\ell'\in S_{n_{ij}} \\ \ell\neq\ell'}} R_{ij\ell}^{(k)}R_{ij\ell'}^{(k')}|n_{ij})f_{N_i,n_i}^{N_{ij}}(n_{ij}).$$

For fixed n_{ij}, it can be shown, as in the complete case, that,

$$E(R_{ij\ell}^{(k)}R_{ij\ell}^{(k')}|n_{ij}) = \sum_{t=1}^{p}\sum_{h\in S_{n_{it}}} \frac{R_{ith}^{(k)}R_{ith}^{(k')}}{n_i}$$

and

$$E(R_{ij\ell}^{(k)}R_{ij\ell'}^{(k')}|n_{ij}) = \frac{1}{n_i(n_i-1)}[(\frac{n_i(n_i+1)}{2})^2 - \sum_{t=1}^{p}\sum_{h\in S_{n_{it}}} R_{ith}^{(k)}R_{ith}^{(k')}].$$

So, the expectation above is equal to:

$$\sum_{n_{ij}=0}^{N_{ij}} \{ \frac{n_{ij}}{n_i} \sum_{t=1}^{p} \sum_{h \in S_{n_{it}}} R_{ith}^{(k)} R_{ith}^{(k')} + \frac{n_{ij}(n_{ij}-1)}{n_i(n_i-1)} [(\frac{n_i(n_i+1)}{2})^2$$

$$- \sum_{t=1}^{p} \sum_{h \in S_{n_{it}}} R_{ith}^{(k)} R_{ith}^{(k')}] \} f_{N_i, n_i}^{N_{ij}} (n_{ij})$$

$$= N_{ij} \frac{n_i}{N_i} \sum_{t=1}^{p} \sum_{h \in S_{n_{it}}} \frac{R_{ith}^{(k)} R_{ith}^{(k')}}{n_i} \sum_{n_{ij}=0}^{N_{ij}-1} f_{N_i-1, n_i-1}^{N_{ij}-1} (n_{ij})$$

$$+ \frac{N_{ij}(N_{ij}-1)}{N_i(N_i-1)} [(\frac{n_i(n_i+1)}{2})^2$$

$$- \sum_{t=1}^{p} \sum_{h \in S_{n_{it}}} R_{ith}^{(k)} R_{ith}^{(k')}] \times$$

$$\times \sum_{n_{ij}=0}^{N_{ij}-2} f_{n_i-2, n_i-2}^{N_{ij}-2} (n_{ij}).$$

Then,

$$\boxed{\begin{aligned} E[(\sum_{\ell \in S_{n_{ij}}} R_{ij\ell}^{(k)}) (\sum_{\ell \in S_{n_{ij}}} R_{ij\ell}^{(k')})] = {} & N_{ij} \frac{n_i}{N_i} \sum_{t=1}^{p} \sum_{h \in S_{n_{it}}} \frac{R_{ith}^{(k)} R_{ith}^{(k')}}{n_i} \\ & + \frac{N_{ij}(N_{ij}-1)}{N_i(N_i-1)} [(\frac{n_i(n_i+1)}{2})^2 \\ & - \sum_{t=1}^{p} \sum_{h \in S_{n_{it}}} R_{ith}^{(k)} R_{ith}^{(k')}] \end{aligned}}$$

7.3.2 Case B

$$E\left(\sum_{\ell=1}^{N_{ij}} R_{ij\ell}^{(k)}\right) = \sum_{n_{ij}=0}^{N_{ij}} E\left[\left(\sum_{\ell \in S_{n_{ij}}} R_{ij\ell}^{(k)} + \sum_{\ell \in S_{(N_{ij}-n_{ij})}} R_{ij\ell}\right) \Big| n_{ij}\right] f_{N_i,n_i}^{N_{ij}}(n_{ij})$$

$$E(R_{ij\ell}^{(k)} | n_{ij}) = \frac{\sum_{t=1}^{p} \sum_{h \in S_{(N_{it}-n_{it})}} R_{ith}^{(k)}}{N_i - n_i}$$

if the trial is in

$$S_{(N_{ij}-n_{ij})}$$

and

$$E(R_{ij\ell}^{(k)} | n_{ij}) = \frac{\sum_{t=1}^{p} \sum_{h \in S_{n_{it}}} R_{ith}^{(k)}}{n_i}$$

if the trial is in

$$S_{n_{ij}}.$$

$$E\left(\sum_{\ell=1}^{N_{ij}} R_{ij\ell}^{(k)}\right) = N_{ij} \frac{n_i}{N_i} \frac{\sum_{t=1}^{p} \sum_{h \in S_{n_{it}}} R_{ith}^{(k)}}{n_i} \sum_{n_{ij}=0}^{(N_{ij}-1)} f_{N_i-1,n_i-1}^{N_{ij}-1}(n_{ij})$$

$$+ \frac{N_{ij}(N_i - n_i)}{N_i(N_i - n_i)} \sum_{t=1}^{p} \sum_{h \in S_{(N_{it}-n_{it})}} R_{ith}^{(k)} \sum_{n_{ij}=0}^{N_{ij}-1} f_{N_i-1,n_i}^{N_{ij}-1}(n_{ij}).$$

So,

$$\boxed{E\left(\sum_{\ell=1}^{N_{ij}} R_{ij\ell}^{(k)}\right) = \frac{N_{ij}}{N_i}\left(\sum_{t=1}^{p} \sum_{h=1}^{N_{it}} R_{ith}^{(k)}\right) = N_{ij}\left(\frac{N_i+1}{2}\right)} \quad .$$

By similar arguments, it can be proved that,

$$\boxed{E\left(\sum_{\ell=1}^{N_{ij}} R_{ij\ell}^{(k)} R_{ij\ell}^{(k')}\right) = \frac{N_{ij}}{N_i} \sum_{t=1}^{p} \sum_{h=1}^{N_{it}} R_{ith}^{(k)} R_{ith}^{(k')}}.$$

Let $j \neq j'$, then

$$E\left[\left(\sum_{\ell=1}^{N_{ij}} R_{ij\ell}^{(k)}\right)\left(\sum_{\ell'=1}^{N_{ij'}} R_{ij'\ell'}\right)\right] = E\left(\sum_{\ell\in S_{n_{ij}}} \sum_{\ell'\in S_{n_{ij'}}} R_{ij\ell}^{(k)} R_{ij'\ell'}^{(k')}\right) +$$

$$+ E\left(\sum_{\ell\in S_{n_{ij}}} \sum_{\ell'\in S_{(N_{ij'}-n_{ij'})}} R_{ij\ell}^{(k)} R_{ij'\ell'}^{(k')}\right) +$$

$$+ E\left(\sum_{\ell\in S_{(N_{ij}-n_{ij})}} \sum_{\ell'\in S_{n_{ij'}}} R_{ij\ell}^{(k)} R_{ij'\ell'}^{(k')}\right) +$$

$$+ E\left(\sum_{\ell\in S_{(N_{ij}-n_{ij})}} \sum_{\ell'\in S_{(N_{ij'}-n_{ij'})}} R_{ij\ell}^{(k)} R_{ij'\ell'}^{(k')}\right).$$

Let us call I, II, III and IV the four terms on the right hand of the previous equality

$$I = \sum_{n_{ij}=0}^{N_{ij}} E\left(\sum_{\ell\in S_{n_{ij}}} \sum_{\ell'\in S_{n_{ij'}}} R_{ij\ell}^{(k)} R_{ij'\ell'}^{(k')} \Big| n_{ij} n_{ij'}\right) f_{N_i,n_i}^{N_{ij},N_{ij'}}(n_{ij},n_{ij'}),$$

$$E(R_{ij\ell} R_{ij\ell'} | n_{ij}, n_{ij'}) =$$

$$= \frac{\left(\sum_{t=1}^{p} \sum_{h\in S_{n_{it}}} R_{ith}^{(k)}\right)\left(\sum_{t=1}^{p} \sum_{h\in S_{n_{it}}} R_{ith}^{(k')}\right) - \sum_{t=1}^{p} \sum_{h\in S_{n_{it}}} R_{ith}^{(k)} R_{ith}^{(k')}}{n_i(n_i-1)}.$$

(The argument by which this inequality is obtained is similar to the one used in the complete case, except that in that case $n_i = N_i$, so that

$$\sum_{t=1}^{p}\sum_{h=1}^{N_{it}} R_{ith}^{(k)} = \frac{N_i(N_i+1)}{2} .$$

$$I = \frac{\left(\sum_{t=1}^{p}\sum_{h\in S_{n_{it}}} R_{ith}^{(k)}\right)\left(\sum_{t=1}^{p}\sum_{h\in S_{n_{it}}} R_{ith}^{(k')}\right) - \sum_{t=1}^{p}\sum_{h\in S_{n_{it}}} R_{ith}^{(k)}R_{ith}^{(k')}}{n_i(n_i-1)} \times$$

$$\times \sum_{n_{ij}=0}^{N_{ij}} \sum_{n_{ij'}=0}^{N_{ij'}} n_{ij}n_{ij'} f_{N_i,n_i}^{N_{ij}N_{ij'}}(n_{ij},n_{ij'}) =$$

$$= \frac{N_{ij}N_{ij'}}{N_i(N_i-1)}\Big[\left(\sum_{t=1}^{p}\sum_{h\in S_{n_{it}}} R_{ith}^{(k)}\right)\left(\sum_{t=1}^{p}\sum_{h\in S_{n_{it}}} R_{ith}^{(k')}\right)$$

$$- \sum_{t=1}^{p}\sum_{h\in S_{n_{it}}} R_{ith}^{(k)}R_{ith}^{(k')}\Big] \sum_{n_{ij}=0}^{N_{ij}-1} \sum_{n_{ij'}=0}^{N_{ij'}-1} f_{(N_i-2),(n_i-2)}^{(N_{ij}-1),(N_{ij'}-1)}(n_{ij},n_{ij'}) .$$

Then,

$$I = \frac{N_{ij}N_{ij'}}{N_i(N_i-1)}\Big[\left(\sum_{t=1}^{p}\sum_{h\in S_{n_{it}}} R_{ith}^{(k)}\right)\left(\sum_{t=1}^{p}\sum_{h\in S_{n_{it}}} R_{ith}^{(k')}\right) - \sum_{t=1}^{p}\sum_{h\in S_{n_{it}}} R_{ith}^{(k)}R_{ith}^{(k')}\Big].$$

For fixed values of n_{ij}, $n_{ij'}$ $(j, j' = 1, \ldots, p)$, $R_{ij\ell}$ and $R_{ij'\ell'}$ with

$$\ell \in S_{n_{ij}}$$

and

$$\ell' \in S_{(N_{ij'}-n_{ij'})} ,$$

are independent variables since complete and incomplete trials permute each other independently. Then,

$$II = \sum_{n_{ij}=0}^{N_{ij}} \sum_{n_{ij'}=0}^{N_{ij'}} E\left(\sum_{\ell \in S_{n_{ij}}} \sum_{\ell' \in S_{(N_{ij'}-n_{ij'})}} R_{ij\ell}^{(k)} R_{ij'\ell'}^{(k')} \,\middle|\, n_{ij}, n_{ij'}\right) \times$$

$$\times f_{n_i, n_i}^{N_{ij}, N_{ij'}}(n_{ij}, n_{ij'})$$

with

$$E(R_{ij\ell}^{(k)} R_{ij'\ell'}^{(k')} | n_{ij}, n_{ij'}) =$$

$$= \frac{\sum_{t=1}^{p} \sum_{h \in S_{n_{it}}} R_{ith}}{n_i} \frac{\sum_{t=1}^{p} \sum_{h \in S_{(N_{it}-n_{it})}} R_{ith}^{(k')}}{(N_i - n_i)} .$$

So,

$$II = \frac{N_{ij} N_{ij'}}{N_i (N_i - 1)} \sum_{t=1}^{p} \sum_{h \in S_{n_{it}}} R_{ith}^{(k)} \sum_{t=1}^{p} \sum_{h \in S_{(N_{it}-n_{it})}} R_{ith}^{(k')} \times$$

$$\times \sum_{n_{ij}=0}^{N_{ij}-1} \sum_{n_{ij'}=0}^{N_{ij'}-1} f_{(N_i-2)(n_i-2)}^{(N_{ij}-1)(N_{ij'}-1)} (n_{ij}, n_{ij'}),$$

or,

$$II = \frac{N_{ij} N_{ij'}}{N_i (N_i - 1)} \left(\sum_{t=1}^{p} \sum_{h \in S_{n_{it}}} R_{ith}^{(k)}\right)\left(\sum_{t=1}^{p} \sum_{h \in S_{(N_{it}-n_{it})}} R_{ith}^{(k')}\right) .$$

III can be solved as term II, while IV is the analogous of I for the case in which both trials $(j\ell)$ and $(j'\ell')$ belong to the set of incomplete columns. It follows that

$$III = \frac{N_{ij} N_{ij'}}{N_i (N_i - 1)} \left(\sum_{t=1}^{p} \sum_{h \in S_{(N_{it}-n_{it})}} R_{ith}^{(k)}\right)\left(\sum_{t=1}^{p} \sum_{h \in S_{n_{it}}} R_{ith}^{(k')}\right),$$

$$IV = \frac{N_{ij}N_{ij'}}{N_i(N_i-1)}[(\sum_{t=1}^{p}\sum_{h\in S_{(N_{it}-n_{it})}} R_{ith}^{(k)})(\sum_{t=1}^{p}\sum_{h\in S_{(N_{it}-n_{it})}} R_{ith}^{(k')}) -$$

$$- \sum_{t=1}^{p}\sum_{h\in S_{(N_{it}-n_{it})}} R_{ith}^{(k)}R_{ith}^{(k')}] .$$

Adding the four terms,

$$\boxed{E[(\sum_{\ell=1}^{N_{ij}} R_{ij\ell}^{(k)})(\sum_{\ell'=1}^{N_{ij'}} R_{ij'\ell'}^{(k')})] = \frac{N_{ij}N_{ij'}}{N_i(N_i-1)}[(\frac{N_i(N_i+1)}{2})^2 - \sum_{t=1}^{p}\sum_{h=1}^{N_{it}} R_{ith}^{(k)}R_{ith}^{(k')}].}$$

If j = j′, it follows by similar arguments than the ones used for the analogous expectation in case A that

$$\boxed{\begin{aligned} E[(\sum_{\ell=1}^{N_{ij}} R_{ij\ell})(\sum_{\ell'=1}^{N_{ij'}} R_{ij'\ell'})] &= N_{ij}(\frac{N_i+1}{2}) \\ &+ \frac{N_{ij}(N_{ij}-1)}{N_i(N_i-1)}[(\frac{N_i(N_i+1)}{2})^2 \\ &- \sum_{t=1}^{p}\sum_{h=1}^{N_{it}} R_{ith}^{(k)}R_{ith}^{(k')}] \end{aligned}}.$$

7.3.3 Case C

Let $j \neq j'$, then

$$E[(\sum_{\ell=1}^{N_{ij}} R_{ij\ell}^{(k)})(\sum_{\ell'\in S_{n_{ij'}}} R_{ij'\ell'}^{(k')})] = E[(\sum_{\ell\in S_{n_{ij}}} R_{ij\ell}^{(k)})(\sum_{\ell'\in S_{n_{ij'}}} R_{ij'\ell'}^{(k')})] +$$

$$+ E[(\sum_{\ell\in S_{(N_{ij}-n_{ij})}} R_{ij\ell}^{(k)})(\sum_{\ell'\in S_{n_{ij'}}} R_{ij'\ell'}^{(k')})].$$

Let us call I and II the two terms on the right hand of the equality above,

$$\mathrm{I} = \sum_{n_{ij}=0}^{N_{ij}} \sum_{n_{ij'}=0}^{N_{ij'}} E[(\sum_{\ell\in S_{n_{ij}}} R_{ij\ell}^{(k)})(\sum_{\ell'\in S_{n_{ij'}}} R_{ij'\ell'}^{(k')})|n_{ij},n_{ij'}] \times$$

$$\times\ f_{N_i,n_i}^{N_{ij},N_{ij'}}(n_{ij},n_{ij'}) = \frac{N_{ij}N_{ij'}}{N_i(N_i-1)}[\frac{n_i(n_i+1)}{2} \sum_{t=1}^{p} \sum_{h\in S_{n_{it}}} R_{ith}^{(k)} -$$

$$- \sum_{t=1}^{p} \sum_{h\in S_{n_{it}}} R_{ith}^{(k)} R_{ith}^{(k')}]$$

$$\mathrm{II} = \sum_{n_{ij}=0}^{N_{ij}} \sum_{n_{ij'}=0}^{N_{ij'}} E[(\sum_{\ell\in S_{(N_{ij}-n_{ij})}} R_{ij\ell}^{(k)})(\sum_{\ell'\in S_{n_{ij'}}} R_{ij'\ell'}^{(k')})|n_{ij},n_{ij'}] \times$$

$$\times\ f_{N_i,n_i}^{N_{ij},N_{ij'}}(n_{ij},n_{ij'})$$

$$= \frac{N_{ij}N_{ij'}}{N_i(N_i-1)} n_i(\frac{n_i+1}{2}) \sum_{t=1}^{p} \sum_{\ell\in S_{(N_{it}-n_{it})}} R_{ith}^{(k)} .$$

Then, from I and II,

$$\boxed{E[(\sum_{\ell=1}^{N_{ij}} R_{ij\ell}^{(k)})(\sum_{\ell'\in S_{n_{ij'}}} R_{ij'\ell'}^{(k')})] = \frac{N_{ij}N_{ij'}}{N_i(N_i-1)}[n_i(\frac{n_i+1}{2})N_i(\frac{N_i+1}{2}) - \sum_{t=1}^{p} \sum_{h\in S_{n_{it}}} R_{ith}^{(k)} R_{ith}^{(k')}]}$$

$$E[(\sum_{\ell=1}^{N_{ij}} R_{ij\ell}^{(k)})(\sum_{\ell\in S_{n_{ij}}} R_{ij\ell}^{(k')})] = E[(\sum_{\ell\in S_{n_{ij}}} R_{ij\ell}^{(k)})(\sum_{\ell\in S_{n_{ij}}} R_{ij\ell}^{(k')})] +$$

$$+ E[(\sum_{\ell\in S_{(N_{ij}-n_{ij})}} R_{ij\ell}^{(k)})(\sum_{\ell'\in S_{n_{ij}}} R_{ij\ell'}^{(k')})].$$

The first term on the right hand is:

$$I = \sum_{n_{ij}=0}^{N_{ij}} E[(\sum_{\ell\in S_{n_{ij}}} R_{ij\ell}^{(k)})(\sum_{\ell\in S_{n_{ij}}} R_{ij\ell}^{(k')})|n_{ij}] f_{N_i,n_i}^{N_{ij}}(n_{ij}) =$$

$$= \frac{N_{ij}}{N_i(N_i-1)}[(N_i-N_{ij})\sum_{t=1}^{p}\sum_{\ell\in S_{n_{it}}} R_{ith}^{(k)} R_{ith}^{(k')} +$$

$$+ (N_{ij}-1)n_i(\frac{n_i+1}{2})\sum_{t=1}^{p}\sum_{h\in S_{n_{it}}} R_{ith}^{(k)}],$$

and the second term is,

$$II = \sum_{n_{ij}=0}^{N_{ij}} E[(\sum_{\ell\in S_{(N_{ij}-n_{ij})}} R_{ij\ell}^{(k)})(\sum_{\ell'\in S_{n_{ij}}} R_{ij\ell'}^{(k')})|n_{ij}] f_{N_i,n_i}^{N_{ij}}(n_{ij})$$

$$= \frac{N_{ij}(N_{ij}-1)}{N_i(N_i-1)} n_i(\frac{n_i+1}{2})\sum_{t=1}^{p}\sum_{h\in S_{(N_{it}-n_{it})}} R_{ith}^{(k)}.$$

Then,

$$\boxed{\begin{gathered} E[(\sum_{\ell=1}^{N_{ij}} R_{ij\ell}^{(k)})(\sum_{\ell\in S_{n_{ij}}} R_{ij\ell}^{(k')})] = \\ \frac{N_{ij}}{N_i(N_i-1)}[(N_i-N_{ij})\sum_{t=1}^{p}\sum_{h\in S_{n_{it}}} R_{ith}^{(k)} R_{ith}^{(k')} + (N_{ij}-1)n_i(\frac{n_i+1}{2})\frac{N_i(N_i+1)}{2}] \end{gathered}}.$$

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