Abstract

DAI, JIN. Stochastic Volatility Corrections for Interest Rate Models.

(Under the direction of Dr. Jean-Pierre Fouque)

This paper is mainly focused on how to price the interest rate derivatives by stochastic volatility models. We will use CIR model and introduce a new Ito process to the model with fast mean-reverting stochastic volatility to compute the corrections of interest rate derivatives. There is a significant difference of the shape of yield curves between the corrected model and original CIR model. It can also be used to price interest rate derivatives such as bond options.

Stochastic Volatility Corrections

for Interest Rate Models

by

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Biography

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TABLE OF CONTENTS

List of Figures

I. Introduction

In the financial market, we will meet lots of interest rate derivatives, such as Treasure bonds, Commercial papers, Repurchase agreement, European/American bond options. How to price these derivatives is a big project attracting researchers who have tons of interests in developing the pricing skills.

The issue of pricing interest-rate derivatives has been addressed by the financial literature in a number of different ways. One of the most popular approaches is based on modeling the evolution of the instantaneous spot interest rate (shortly referred to as "short rate") . For example, Vasicek model by Vasicek, O. (1977), Cox-Ingersoll-Stephen (CIR) model by John C. Cox, Jonathan E. Ingersoll, Jr., Stephen A. Ross (1985). In this paper, I will introduce these two models and the corrected CIR models to price interest rate derivatives. The zero coupon bond with maturity date T , which is also called a T-bond, a contract guaranteeing the holder 1 dollar to be paid on the date T , will be priced mainly by the models (and also the European bond option). The price at time t of a bond with maturity date T is denoted by $P(t,T)$.

Stochastic volatility models are one of the most popular models in financial mathematics. These models will be helpful and efficient in solving the problem such as determining the prices of interest rate derivatives. The short rate, under the objective probability measure P , has the solution of the form as

(1) $$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t$$

where W_t is a Wiener process, μ and σ are given deterministic functions. The function $\sigma(t, r_t)$ is known as the volatility of r_t, while $\mu(t, r_t)$ is mean of rate here.

Some well-known standard models are

- **Vasicek** model

$$dr_t = a(r^* - r_t)dt + \sigma dW_t$$

which has the property of being mean reverting and an explicit formula for European bond options;

- **CIR** (Cox-Ingersoll-Ross) model

$$dr_t = a(r^* - r_t)dt + \sigma\sqrt{r_t}\,dW_t^*$$

where the model is based on an ATS(Affine Term Structure) which extremely simplifies the analytical and computational method and is efficiently used now.

Note: The parameters in these models will be interpreted in the later section.

CIR (Cox-Ingersoll-Ross) model was presented by John C. Cox, Jonathan E. Ingersoll, and Stephen A. Ross in 1986. In this model, the term structure theorems have all been taken consideration. Many of the factors traditionally mentioned as influencing the term structure are thus included in a way which is fully consistent with maximizing behavior and rational expectations. From those specific formulas for interest rates and bond prices, CIR model is well suited for empirical testing.

We will simply introduce Vasicek model first in next section, which is pricing the bonds and bond options explicitly. Stochastic volatility is described into CIR model in the later section and the bond and bond option price will be corrected in the new two-factor model.

II. Vasicek Model

Vasicek model in this chapter will just be introduced as non-corrected model, which is one-factor and no stochastic process adding to this model. It has constant volatility.
In this model, there is the probability space $\left(\Omega, F, P^*\right)$ equipped with an increasing filtration$\left(F_t\right)_{t\geq 0}$, where P^* is an equivalent martingale (pricing) measure, and, the mean-reverting Gaussian stochastic process $\left(r_t\right)_{t\geq 0}$ is the corresponding short-rate on this risk neutral probability space.

It will follow the model

(2) $$dr_t = a\left(r^* - r_t\right)dt + \sigma dW_t^*,$$

where $\left(W_t^*\right)$ is a standard P^*-Brownian Motion and σ is the constant volatility. In other words, in the risk-neutral world P^*, the short rate $\left(r_t\right)$ is an Ornstein-Uhlenbeck process possessing a stationary distribution and fluctuating around its mean level r^* with a rate of non-negative mean-reversion $a > 0$. It is a Markov process with normally distributed increments. Because the volatility of Vasicek model is constant, it is undisputedly regarded as one of the simplest model in a large family of stochastic models under the affine term structure.

1. Bond Prices

Under the no-arbitrage circumstance, the bond price at time t of zero-coupon bond maturing at time T is given by

(3) $$\Lambda(t,T) = E^*\left\{e^{-\int_t^T r_s ds} \middle| F_t\right\}.$$

At time 0, we can discount from t to 0 also:

$$\Lambda(0,T) = e^{\int_0^t r_s ds}\Lambda(t,T) = E^*\left\{e^{-\int_0^T r_s ds} \middle| F_t\right\}.$$

Suppose

(4) $$\Lambda(t,T) = P(t, r_t; T), \quad r_t = x \text{ is the current short rate,}$$

According to the Feynman-Kac formula, by the stochastic differential equation (2), $P(t,x;T)$ should be the solution of the partial differential equation

(5) $$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial x^2} + a\left(r^* - x\right)\frac{\partial P}{\partial x} - xp = 0\,,$$

with the terminal condition $P(T,x;T) = 1$.

With the replacement of spot time t by time-to-maturity $\tau = T - t$, because of ATS, we have the solution of the form

(6) $$P(T-\tau, x; T) = A(\tau)e^{-B(\tau)x}$$

The terminal condition is transferred to initial condition $A(0)=1$ and $B(0)=0$. Then the differential equation (5) can be solvedexplicitly

$$
(7)\qquad
\begin{aligned}
B(\tau) &= \frac{1-e^{-a\tau}}{a} \\
A(\tau) &= \exp\left\{-\left[R_\infty \tau - R_\infty \frac{1-e^{-a\tau}}{a} + \frac{\sigma^2}{4a^3}\left(1-e^{-a\tau}\right)^2\right]\right\}
\end{aligned}
$$

where $R_\infty = r^* - \dfrac{\sigma^2}{2a^2}$ is the interest rate when T goes to infinity.

The zero-coupon bond price is $\Lambda(t,T) = P(t,r_t;T)$ with

$$
(8)\qquad P(t,x;T) = \exp\left\{-\left[R_\infty (T-t) - (R_\infty - x)\frac{1-e^{-a(T-t)}}{a} + \frac{\sigma^2}{4a^3}\left(1-e^{-a(T-t)}\right)^2\right]\right\},
$$

The yield, defined by

$$
R(t,T) = -\frac{1}{T-t}\log\left(\Lambda(t,T)\right),
$$

is correspondingly solved as

$$
(9)\qquad R(t,T-t) = R_\infty - (R_\infty - x)\frac{1-e^{-a(T-t)}}{a(T-t)} + \frac{\sigma^2}{4a^3(T-t)}\left(1-e^{-a(T-t)}\right)^2,
$$

This is the explicit solution for bond price and yield, so we can obtain the figure of them directly.

[Figure 1 about Here]

2. Bond Option Price

We continue to denote the maturity of the bond by T and the maturity of a European option written on that bond is denoted by T_0 with $t \le T_0 < T$. The payoff function is defined by $h(\Lambda(T_0, T))$, which is related with the bond price at the expiration time of the option. We write the payoff as $h(\Lambda(T_0, T)) = (\Lambda(T_0, T) - K)^+$, and it is also the function of r_{T_0}, which can be presented as $\tilde{h}(r_{T_0})$. Using the Markovian structure, we denote by $Q(t, x; T, T_0)$ the price of the option at time t for an observed interest rate $r_t = x$ and we obtain

$$Q(t, x; T, T_0) = E^* \left\{ e^{-\int_t^{T_0} r_s ds} h(\Lambda(T_0, T)) \middle| r_t = x \right\} = E^* \left\{ e^{-\int_t^{T_0} r_s ds} \tilde{h}(r_{T_0}) \middle| r_t = x \right\}.$$

By Feynman-kac formula, it is also the solution of the partial differential equation (5) with terminal condition $Q(T_0, x; T, T_0) = \tilde{h}(x) = h(P(T_0, x; T))$ where $P(T_0, x; T)$ is figured out by (8) at $t = T_0$.

Under the no-arbitrage world, the random variable pair $\left(r_{T_0}, \int_t^{T_0} r_s ds \right)$ is normally distributed and we can reduce the bond option price computation to a Gaussian integral.

In the particular case, for a European call option, denoting the price of the option by $C(t,x)$, we obtain

(10) $$C(t,x) = P(t,x;T)N(h_1) - KP(t,x;T_0)N(h_2)$$

where N is the $\mathrm{N}(0,1)$ cumulative distribution function, and $h_{1,2}$ are given explicitly by

$$h_{1,2} = \frac{\log\left(\frac{P(t,x;T)}{P(t,x;T_0)}\right) - \log K \pm \frac{1}{2}\upsilon^2}{\upsilon} \quad .$$

$$\upsilon^2 = \frac{\sigma^2}{2a^3}\left(1 - e^{-2a(T_0 - t)}\right)\left(1 - e^{-a(T-t)}\right)^2$$

Note: More details about the corrected and uncorrected Vasicek model can be referred by [2]

III. CIR Model

CIR model is first presented by J. C. Cox, J. E. Ingersoll, Jr., and S. A. Ross in 1985. It is an equilibrium asset pricing model to study the term structure of interest rates. In this model, anticipations, risk aversion, investment alternatives, and preferences about the timing of consumption play roles in determining bond prices. And this model leads to specific formulas for bond prices, which are well suited for empirical testing.

In this section, I will give the correction form of CIR model, which will fit better for the empirical data. The idea of this correction is just introducing a new ergodic Markov diffusion process with a unique invariant distribution. The interest rates will change definitely because of the stochastic variable.

Now, first of all, the CIR model is described as followed,

(11) $$dr_t = a\left(r^* - r_t\right)dt + \sigma\sqrt{r_t}\,dW_t^*.$$

Here, $a > 0$ determines the rate of mean-reversion adjustment, r^* is long-term mean level of (r_t), and, (W_t^*) is a standard P^*-Brownian motion defined on $[0, \infty)$.

The interest rate behavior implied by the structure has the following empirically relevant properties:

- Negative interest rates are precluded.
- If the interest rate reaches zero, it can subsequently become positive
- The absolute variance of the interest rate increases when the interest rate itself increases.
- There is a steady state distribution for the interest rate.

Also (r_t) cannot reach zero if $ar^* \geq \sigma^2/2$. The upward drift is sufficiently large to make the origin inaccessible. In this case, the singularity of the diffusion coefficient at the origin implies that an initially nonnegative interest rate can never subsequently become negative. This is shown by the proposition presented.

Proposition Define $\tau_t^x = \inf \{t \geq 0 | r_t^x = 0\}$, the SDE has the form

$$dr_t = a(r^* - r_t)dt + \sigma\sqrt{r_t}\,dW_t,$$

where (W_t) is a standard Brownian motion defined on $[0,\infty)$, $r_0 = x > 0$.

For all $x > 0$, if $ar^* \geq \sigma^2/2$, then $P(\tau_0^x = \infty) = 1$ and the solution r_t is strictly positive for all $t > 0$.

Proof: (Outline)

For $x, M > 0$, we note τ_M^x the stopping time defined by $\tau_M^x = \inf \{t \geq 0 | r_t^x = M\}$.

Let s be the function defined on $[0,\infty)$ by

$$s(x)=\int_1^x e^{2ay/\sigma^2}\, y^{-2ar^*/\sigma^2}\, dy\,,$$

And we can prove it satisfies

$$\frac{\sigma^2}{2}x\frac{d^2s}{dx^2}+a\left(r^*-x\right)\frac{ds}{dx}=0\,.$$

For $\varepsilon<x<M$, we set $\tau^x_{\varepsilon,M}=\tau^x_\varepsilon\wedge\tau^x_M$, for any $t>0$, we have

$$s\left(r^x_{t\wedge\tau^x_{\varepsilon,M}}\right)=s(x)+\int_0^{t\wedge\tau^x_{\varepsilon,M}} s'\left(r^x_s\right)\sigma\sqrt{r^x_s}\,dW_s\ .$$

One thing need to deduce, taking the variance on both sides and using the fact that s' is bounded from below on the interval $[\varepsilon,M]$, that $E\left(\tau^x_{\varepsilon,M}\right)<\infty$, which implies that $\tau^x_{\varepsilon,M}$ is finite a.s..

Then if $\varepsilon<x<M$, $s(x)=s(\varepsilon)P\left(\tau^x_\varepsilon<\tau^x_M\right)+s(M)P\left(\tau^x_\varepsilon>\tau^x_M\right)$.

We assume $ar^*\geq\sigma^2/2$, and have $\lim_{x\to0}s(x)=-\infty$. Deduce that $P\left(\tau^x_0<\tau^x_M\right)=0$ for all $M>0$, then that $P\left(\tau^x_0<\infty\right)=0$.

Now we assume that $0\leq ar^*<\sigma^2/2$ and we set $s(0)=\lim_{x\to0}s(x)$, for all $M>x$, we have $s(x)=s(0)P\left(\tau^x_0<\tau^x_M\right)+s(M)P\left(\tau^x_0>\tau^x_M\right)$ and that completes the proof.

1. Bond Prices

The value of no-arbitrage price at time t of a zero-coupon bond maturing at time T, is similar as the bond price in Vasicek model

$$\Lambda(t,T) = E^{*}\left\{ e^{-\int_t^T r_s ds} \middle| F_t \right\},$$

and as the function of the current short rate r_t, we similarly denote this bond pricing function by $P(t,x;T)$ with the current rate $r_t = x$. It becomes

$$\Lambda(t,T) = P(t,r_t;T).$$

From the Feynman-Kac formula, $P(t,x;T)$ is the solution of the partial differential equation

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 x \frac{\partial^2 P}{\partial x^2} + a\left(r^{*} - x\right)\frac{\partial P}{\partial x} - xp = 0 \tag{12}$$

with terminal condition $P(T,x;T) = 1$.

Let $t = T - \tau$, we assume that the bond price has the form with initial condition

$$\begin{cases} P(T-\tau,x;T) = A(\tau)e^{-B(\tau)x} \\ A(0) = 1 \\ B(0) = 0 \end{cases}. \tag{13}$$

The partial differential equation (12) will be transformed to the new equation

$$(14)\qquad -\frac{\partial P}{\partial \tau}+\frac{1}{2}\sigma^2 x\frac{\partial^2 P}{\partial x^2}+a\left(r^*-x\right)\frac{\partial P}{\partial x}-xp=0\ .$$

Apply (13) into (14), we can get the following PDEs

$$\begin{cases} A'(\tau)+ar^*A(\tau)B(\tau)=0 \\ A(\tau)B'(\tau)+\frac{1}{2}\sigma^2 A(\tau)B^2(\tau)+aA(\tau)B(\tau)-A(\tau)=0 \end{cases}$$

By some computation for this PDE, $A(\tau)$ and $B(\tau)$ are obtained by elementary method of solving equations

$$(15)\qquad \begin{cases} A(\tau)=\left[\dfrac{2\theta e^{(\theta+a)\tau/2}}{(\theta+a)\left(e^{\theta\tau}-1\right)+2\theta}\right]^{2ar^*/\sigma^2} \\ B(\tau)=\dfrac{2\left(e^{\theta\tau}-1\right)}{(\theta+a)\left(e^{\theta\tau}-1\right)+2\theta} \end{cases},\qquad \theta=\sqrt{a^2+2\sigma^2}\ .$$

Then we have the bond price

$$(16)\qquad P(t,x;T)=\left[\frac{2\theta e^{(\theta+a)\tau/2}}{(\theta+a)\left(e^{\theta\tau}-1\right)+2\theta}\right]^{2ar^*/\sigma^2} e^{\frac{-2\left(e^{\theta\tau}-1\right)}{(\theta+a)\left(e^{\theta\tau}-1\right)+2\theta}x}$$

Corresponding to the bond price, the yield can be presented by,

$$
(17)\quad \begin{aligned} R(x,t;T) &= \frac{1}{T-t}\cdot\frac{2\left(e^{\theta(T-t)}-1\right)x}{(\theta+a)\left(e^{\theta(T-t)}-1\right)+2\theta} \\ &\quad -\frac{2ar^*}{(T-t)\sigma^2}\cdot\left\{\log 2\theta+\frac{(\theta+a)(T-t)}{2}-\log\left[(\theta+a)\left(e^{\theta(T-t)}-1\right)+2\theta\right]\right\} \end{aligned}.
$$

As we consider longer and longer maturities, the yield of bonds, which is given by $R(x,t;\infty)=\dfrac{2ar^*}{\theta+a}$, approaches a limit independently of the current interest rate. Figure 2 shows the price function and the yield curve of zero-coupon bonds under CIR model.

[Figure 2 about here.]

2. Bond Option Prices

Use the same valuation framework of bond prices, we can easily apply to other securities whose payoffs depend on interest rates, such as bond options. Continuously, We will denote the maturity of the bond by T and the maturity of a European option written on that bond by T_0 with $t \le T_0 < T$.

The payoff of the bond option at time T_0 is a function $h(\Lambda(T_0,T))$, which is the bond price at expiration moment of the relative option. We will discuss a call option of the bond with striking price K and expiration time T_0 as payoff

$$h(\Lambda(T_0,T)) = (\Lambda(T_0,T) - K)^+ .$$

The no-arbitrage price of the bond option is obtained by evaluating the expectation of discount value of payoff function that is under the measure P^*. Since the bond function $\Lambda(T_0,T)$ can be regarded as the function of rate r_{T_0} from the bond price formula (16), the payoff function is just $\tilde{h}(r_{T_0})$ of the rate at time T_0. By this structure, we use $Q(t,x;T,T_0)$ to express the bond option price from an observed short rate $r_t = x$,

$$\begin{aligned} Q(t,x;T,T_0) &= E^*\left\{ e^{-\int_t^{T_0} r_s ds} h(\Lambda(T_0,T)) \middle| r_t = x \right\} \\ &= E^*\left\{ e^{-\int_t^{T_0} r_s ds} \tilde{h}(r_{T_0}) \middle| r_t = x \right\} \end{aligned} .$$

In the particular case of a call option, we can present the bond option price by $C(t,x)$. Here we must give that the probability density of the interest rate at time s, conditional on its value at the current time t, is provided by

$$f(r_s, s; r_t, t) = c \cdot e^{-u-v}\left(\frac{v}{u}\right)^{q/2} I_q\left(2(uv)^{1/2}\right),$$

where

$$\begin{aligned} c &= \frac{2\alpha}{\sigma^2\left(1-e^{-\alpha(s-t)}\right)}, \\ u &= cr_t e^{-\alpha(s-t)}, \\ v &= cr_s, \\ q &= \frac{2\alpha r^*}{\sigma^2} - 1, \end{aligned}$$

and $I_q(\cdot)$ is the modified Bessel function of the first kind of order q. The distribution function is the noncentral chi-square, $\chi^2[2cr_s; 2q+2, 2u]$, with $2q+2$ degrees of freedom and non-central parameter $2u$

Now, take the relevant expectations, we deduce the following formula for the bond option price

(18) $$\begin{aligned} C(t,x) = P(t,x;T)\cdot\chi^2&\left(2\bar{r}[\phi+\psi+B(T_0)]; \frac{4ar^*}{\sigma^2}, \frac{2\phi^2 x e^{\theta(T-t)}}{\phi+\psi+B(T_0)}\right) \\ - KP(t,x;T_0)\cdot\chi^2&\left(2\bar{r}[\phi+\psi]; \frac{4ar^*}{\sigma^2}, \frac{2\phi^2 x e^{\theta(T-t)}}{\phi+\psi}\right), \end{aligned}$$

where

$$
\begin{aligned}
\theta &= \sqrt{a^2 + 2\sigma^2} \\
\phi &= \frac{2\theta}{\sigma^2\left(e^{\theta(T-t)} - 1\right)} \\
\psi &= (a+\theta)/\sigma^2 \\
\bar{r} &= \left[\log\left(\frac{A(T_0)}{K}\right)\right] \Big/ B(T_0)
\end{aligned}
\qquad (19)
$$

$\bar{r}$ is the critical interest rate below which exercise will occur; i.e. $K = P(\bar{r}, T_0; T)$.

IV. Stochastic Volatility CIR model

In this section, we will introduce a new stochastic process into the CIR model for bond and bond option prices and correct the CIR formula to include some uncertain volatility. The assumption of the volatility is fast mean-reverting with respect to the long time-scale of derivative contracts.

1. Two Factor Model

In the real world, the yield curve is not simply a smooth curve and the real interest rate is vibrated. Therefore, let us introduce an ergodic Ito diffusion process (Y_t) with a unique invariant distribution $N\left(m, \frac{b^2}{2a}\right)$, and replace the diffusion of r_t by the stochastic volatility $\sigma(r_t, Y_t)$.

Assume that under the risk-neutral world P^*, we have the two-factor model, which the independent Brownian motions have the same square-root-of-affine coefficients in front of them since we want to get the affine yield curve solution.

(20)
$$\begin{cases} dr_t = a\left(r^* - r_t\right)dt + f\left(Y_t\right)\sqrt{r_t}\,dW_t^* \\ dY_t = \alpha r_t\left(m - Y_t\right)dt + \beta\sqrt{r_t}\left(\rho dW_t^* + \sqrt{1-\rho^2}\,dZ_t^*\right) \end{cases},$$

where $\left(W_t^*\right)$ and $\left(Z_t^*\right)$ are two independent standard P^*-Brownian Motion, $f\left(Y_t\right)$ is a positive function regarding as the stochastic volatility with $0 < c_1 \le f \le c_2 < \infty$ for some constants c_1 and c_2. And the proposition in page 10 guarantees existence of a strong solution for $\left(r_t\right)$ if $f^2\left(y\right) \le 2ar^*$.

The parameter ρ with $|\rho| < 1$ allows a correlation between the Brownian motion $\left(W_t^*\right)$ driving the short rate and its volatility. Especially, we could assume $\rho > 0$ as rising volatility trends to push bond prices down, and correspondingly, yields up.

The reason we need to introduce the square-root-of-affine coefficients in the diffusion term is that we hope the derivative pricing PDE has linear coefficients, which admits an affine yield curve solution. However, yield curves in our CIR model are affine surely in the short-rate level $x = r_t$, but not of course in $y = Y_t$. This allows a closed-form (up to solution of ODEs) for the stochastic volatility correction.

2. Correction for Bond Prices

By the previous section about bond price with respect to CIR model, we can learn that the bond pricing function $P(t, x, y; T)$ is measured under $x = r_t$ and $y = Y_t$, which is

(21) $$P(t, x, y; T) = E^* \left\{ e^{-\int_t^T r_s ds} \middle| r_t = x, Y_t = y \right\}$$

where the expectation E^* under the risk-neutral world is related with the distribution of (r_t, Y_t) which is solved from the CIR model starting at time t from (x, y).

Using the same framework, by two-dimensional Feynman-Kac formula, we have the partial differential equation

(22) $$\begin{aligned} &\frac{\partial P}{\partial t} + \left[a\left(r^* - x\right)\frac{\partial P}{\partial x} + \alpha x(m - y)\frac{\partial P}{\partial y} \right] \\ &\quad + \left[\frac{1}{2} f^2(y) x \frac{\partial^2 P}{\partial x^2} + \beta\rho f(y) x \frac{\partial^2 P}{\partial x \partial y} + \frac{1}{2}\beta^2 x \frac{\partial^2 P}{\partial y^2} \right] - xP = 0 \end{aligned}$$

with terminal condition $P(T, x, y; T) = 1, \forall x, y$.

Under the fast mean-reverting stochastic volatility, the rate of mean-reversion α is large and $\nu = \beta / \sqrt{2\alpha}$ keeps order one. The asymptotic method will be figured out as $1/\alpha$ trends to small and we define:

(23)
$$\varepsilon = 1/\alpha,$$
$$\beta = \frac{\sqrt{2}\nu}{\sqrt{\varepsilon}},$$
$$P^{\varepsilon}(t,x,y;T) = P(t,x,y;T),$$
$$L^{\varepsilon} = \frac{1}{\varepsilon}L_0 + \frac{1}{\sqrt{\varepsilon}}L_1 + L_2,$$

where

(24)
$$L_0 = x\left(\nu^2 \frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y}\right)$$

is the infinitesimal generator of the Ito diffusion process (Y_t) scaled by x/α,

(25)
$$L_1 = \sqrt{2}\nu\rho x f(y)\frac{\partial^2}{\partial x \partial y},$$

and

(26)
$$L_2 = \frac{\partial}{\partial t} + \frac{1}{2}f^2(y)x\frac{\partial^2}{\partial x^2} + a(r^* - x)\frac{\partial}{\partial x} - x1$$

is the CIR operator with volatility $f(y)$ and long run mean r^*.

The partial differential equation can be written as

(27)
$$L^{\varepsilon}P^{\varepsilon} = 0,$$

with the terminal condition $P^{\varepsilon}(T,x,y;T) = 1$.

Now we want to find an asymptotic solution of the form

(28)
$$P^{\varepsilon}(t,x,y;T) = P_0(t,x,y;T) + \sqrt{\varepsilon}P_1(t,x,y;T) + \varepsilon P_2(t,x,y;T) + \cdots,$$

with the terminal conditions $P_0(T,x,y;T) = 1$ and $P_1(T,x,y;T) = 0$.

Note: Using this probabilistic representation, which the equation should be satisfied with the error and the regularity of the approximation, we will apply the theorem in Appendix in this section.

Zero-order term

Look at the equations at each order starting with the highest order $O(\varepsilon^{-1})$ which easily obtain from the expansion

$$L_0 P_0 = 0 .$$

Then we can deduce

$$P_0 = P_0(t,x;T) \text{ which does not depend on } y .$$

Note: This conclusion cannot be deduced directly from the generator operating on the variable y only. It should be obtained from the original part of expansion

$$L_0 P_0 = \nu^2 \frac{\partial^2 P_0}{\partial y^2} + (m-y)\frac{\partial P_0}{\partial y} = 0 ,$$

The solution of this ODE is

$$P_0(t,x;T) = c_1(t,x;T)\int_{-\infty}^{y} e^{\frac{(m-u)^2}{\nu^2}} du + c_2(t,x;T).$$

For the bond price $P_0(t,x;T)$, it is impossible that P_0 goes to infinite. Thus, we can understand the term of $c_1(t,x;T)\int_{-\infty}^{y} e^{\frac{(m-u)^2}{\nu^2}}\,du$ should be zero, i.e., $c_1(t,x;T)=0$.

Therefore, we have $P_0 = P_0(t,x;T)$.

The $O\left(1/\sqrt{\varepsilon}\right)$ terms give

$$L_0 P_1 + L_1 P_0 = 0$$

$$\Rightarrow \; P_1 = P_1(t,x;T)$$

This is because P_0 does not depend on y from the previous conclusion and L_1 takes derivatives with respect to y. So, we obtain that $P_1 = P_1(t,x;T)$ also does not depend on y. This result is very important because the present value $Y_t = y$ of diffusion (Y_t) is no need for the first correction to P_0.

The $O(1)$ terms show us

$$L_2 P_0 + L_1 P_1 + L_0 P_2 = 0 \Rightarrow L_2 P_0 + L_0 P_2 = 0$$

since P_1 does not depend on y.

This is a Poisson equation for P_2, which cannot be solved only when $L_0 P_2$ is centered with respect to the Gaussian density, i.e. the solvability condition is $\langle L_2 P_0 \rangle = 0$.

Since P_0 does not depend on y, it can be out of the brackets

$$\begin{aligned} \langle L_2 P_0 \rangle &= \langle L_2 \rangle P_0 \\ &= \left(\frac{\partial}{\partial t} + \frac{1}{2}\bar{\sigma}^2 x \frac{\partial^2}{\partial x^2} + a\left(r^* - x\right)\frac{\partial}{\partial x} - x \right) P_0 = 0 \end{aligned} \tag{29}$$

with $P_0(T, x) = 1$ and $\bar{\sigma}^2 = \langle f^2 \rangle$.

Note: $\langle \cdot \rangle$ means the averaging with invariant distribution of (Y_t).

Equation (29) is similar to the equation by one-factor affine CIR model, changing the variable $\tau = T - t$ for convenience, we get

$$\begin{aligned} P_0(T - \tau, x; T) &= A(\tau) e^{-B(\tau)x} \\ A(\tau) &= \left(\frac{2\theta e^{(\theta + a)\tau/2}}{(\theta + a)\left(e^{\theta\tau} - 1\right) + 2\theta} \right)^{2ar^*/\bar{\sigma}^2} \\ B(\tau) &= \frac{2\left(e^{\theta\tau} - 1\right)}{(\theta + a)\left(e^{\theta\tau} - 1\right) + 2\theta} \\ \theta &= \sqrt{a^2 + 2\bar{\sigma}^2} \end{aligned} . \tag{30}$$

Note: Actually, $P_0(T - \tau, x)$ is the original pricing bond function without correction.

First-order term

The $O\left(\sqrt{\varepsilon}\right)$ terms

(31) $$L_0 P_3 + L_2 P_1 + L_1 P_2 = 0,$$

which is also a Poisson equation in P_3 with solvability condition $\left\langle L_2 P_1 + L_1 P_2 \right\rangle = 0$

By previous description $L_0 P_2 + L_2 P_0 = 0$ and $\left\langle L_2 \right\rangle P_0 = 0$, we have

$$P_2 = -L_0^{-1}\left(L_2 P_0 - \left\langle L_2 \right\rangle P_0\right) + k(t,x),$$

where $k(t,x)$ does not depend on y. Then we can rewrite equation

$$\begin{aligned}\left\langle L_2 \right\rangle \tilde{P}_1 &= \sqrt{\varepsilon}\left\langle L_1 L_0^{-1}\left(L_2 - \left\langle L_2 \right\rangle\right)\right\rangle P_0 \\ &\doteq \hat{\lambda} P_0\end{aligned},$$

with zero terminal condition at $t = T$.

The operator $L_2 - \left\langle L_2 \right\rangle$ is given by

$$L_2 - \left\langle L_2 \right\rangle = \frac{1}{2}\left(f(y)^2 - \left\langle f^2 \right\rangle\right)\frac{\partial^2}{\partial x^2}$$

Now introduce a centered function Φ, such that $L_0\Phi = \left(f(y)^2 - \left\langle f^2 \right\rangle\right)x$

Then,

(32)
$$\begin{aligned} \tilde{\lambda} &= \sqrt{\varepsilon}\left\langle L_1 L_0^{-1}\left(L_2 - \left\langle L_2 \right\rangle\right)\right\rangle \\ &= \frac{\nu}{\sqrt{2\alpha}}\rho\left\langle f(y)\frac{\partial\Phi}{\partial y}\right\rangle x\frac{\partial^3}{\partial x^3} \doteq V \cdot x\frac{\partial^3}{\partial x^3} \end{aligned}$$

where $V = \dfrac{\nu}{\sqrt{2\alpha}}\rho\left\langle f\Phi' \right\rangle$.

So we have

$$\left\langle L_2 \right\rangle \tilde{P}_1 = \tilde{\lambda} P_0 = Vx\frac{\partial^3 P_0}{\partial x^3}.$$

Note: If no skew $(\rho = 0)$, it implies $V = 0$, i.e., $\left\langle L_2 \right\rangle \tilde{P}_1 = 0$. Then the correction is just equal to P_0 itself.

Change the variable $\tau = T - t$, the equation becomes

(33)
$$-\frac{\partial\tilde{P}_1}{\partial\tau} + \frac{1}{2}\bar{\sigma}^2 x\frac{\partial^2\tilde{P}_1}{\partial x^2} + a\left(r^* - x\right)\frac{\partial\tilde{P}_1}{\partial x} - x\tilde{P}_1 = -Vx\frac{\partial^3}{\partial x^3}\left[A(\tau)e^{-B(\tau)x}\right]$$

with the initial condition $\tilde{P}_1(T-0, x; T) = 0$.

We now try to find a solution of the form

(34)
$$\begin{aligned} \tilde{P}_1(T-t, x; T) &= \left(D_1(t)x + D_2(t)\right)A(t)e^{-B(t)x} \\ &= \left(D_1 x + D_2\right)P_0 \end{aligned}.$$

Apply (34) into PDE (33), we can deduce to the following equation,

$$\left(D_1{}'x + D_2{}'\right) = -\overline{\sigma}^2 D_1 Bx + a\left(r^* - x\right)D_1 + V \cdot B^3 x \,,$$

which imply the ODEs

(34) $$\begin{cases} D_1' = VB^3 - \left(\overline{\sigma}^2 B + a\right)D_1 \\ D_2' = ar^* D_1 \end{cases},$$

with zero initial condition.

3. Numerical Solutions

However, we cannot find an explicit solution of (34) directly, because B is a complicated function of τ, even by some efficient solving software such as Mathematica 4.0.

Alternatively, we use Matlab to obtain the asymptotic solution for ODEs (34).
Matlab 6.0 has developed bunch of solvers to figure out the initial value problems for ordinary differential equations, which is mainly ode45, ode23, ode113, ode15s and ode23s.

We should also observe that (34) is a stiff problem, so we pick up the ode15s and ode23s, which are developed to solve the stiff ordinary differential equations.

ode23s is based on a modified Rosenbrock formula of order 2. Because it is a one-step solver, it may be more efficient than ode15s at crude tolerances. It can solve some kinds of stiff problems for which ode15s is not effective.

Then, we decide to choose Matlab ODE solver **ode23s** to solve D_1 & D_2 . The solver code is

```
[t,D]=ode23s(odefun,tspan,D0,options,parameters).
```

Note: odefun---a function that evaluates the right-hand side of the differential equations;

tspan---a vector specifying the interval of integration;

D0---a vector of initial conditions;

options---optional integration argument created using the odeset function to adjust the integration parameters of the ODE solvers;

parameters---optional parameters to be passed to odefun.

By this asymptotic method, the bond correction price is

$$P = (1 + D_1 x + D_2)P_0 ,$$

and the yield is

$$R = -(\ln P)/\tau ,$$

which could be obtained directly from the numerical solutions D_1 & D_2 .

In the Figure 2 attached, we can compare the correction bond prices and yield curves with those non-corrections' (just from the one-factor model).

[Figure 3 about here]

We should observe that, in Figure 3, the fast mean-reverting rate of Y_t , $\alpha = 5$, which means if the factor comes back to the average level a little bit slow, it will affect the yield curve a lot. It becomes more clear if we give out the bond price and yield curve at $\alpha = 50$ and $\alpha = 500$.

When $\alpha = 50$, comparing to that of $\alpha = 5$, the shape of corrected yield curve has had some tendency to decrease a little before increase, though the difference of corrected bond price and the original one is trivial, whatever the value of α. We can learn from the Figure 4.

[Figure 4 about here]

When $\alpha = 500$, the shape of corrected yield curve is similar with the uncorrected one, because if α goes to infinity, it means the volatility of r_t becomes constant, i.e., the uncorrected model is the same as corrected model. The Figure 5 will also show this.

[Figure 5 about here]

4. Corrections for Bond Option Prices

Now, we focus on the bond option prices problem in terms of this two-factor model, which introduced in previous section. The short rate process (r_t) and the diffusion (Y_t) driving the volatility are notified by (20). We still consider the zero-coupon bond option as described above with payoff function $h(\Lambda(T_0, T))$. Under the martingale measure P^*, and given the current rate $r_t = x$ and current volatility level $Y_t = y$, the price of zero-coupon bond option is

$$Q(t,x,y;T,T_0) = E^*\left\{ e^{-\int_t^{T_0} r_s ds} h(\Lambda(T_0,T)) \middle| r_t = x, Y_t = y \right\}.$$

Where the payoff function is coming from the bond price at time T_0,

$$\Lambda(T_0,T) = P(T_0, r_{T_0}, Y_{T_0}; T) = E^*\left\{ e^{-\int_{T_0}^{T} r_s ds} \middle| r_{T_0}, Y_{T_0} \right\}.$$

Again, by Feynman-Kac formula, the price of bond option $Q(t,x,y;T_0,T)$ is still the solution of partial differential equation

$$\begin{aligned} &\frac{\partial Q}{\partial t} + \left[a(r^* - x)\frac{\partial Q}{\partial x} + \alpha x(m-y)\frac{\partial Q}{\partial y} \right] \\ &\quad + \left[\frac{1}{2} f^2(y) x \frac{\partial^2 Q}{\partial x^2} + \beta\rho f(y) x \frac{\partial^2 Q}{\partial x \partial y} + \frac{1}{2}\beta^2 x \frac{\partial^2 Q}{\partial y^2} \right] - xQ = 0 \end{aligned}$$

which is the same as (22), only has the difference of terminal condition at time T_0

(35)
$$\begin{aligned} Q(T_0, x, y; T, T_0) &= h\left(E^*\left\{ e^{-\int_{T_0}^{T} r_s ds} \middle| r_{T_0} = x, Y_{T_0} = y \right\} \right) \\ &= h(P(T_0, x, y; T)) \end{aligned}$$

where $P(T_0, x, y; T)$ is the price of bond with current short rate x and volatility level y.

The framework for bond option price expansion is just similar to that of bond price correction. We have the bond price $P^{\varepsilon}(t, x, y; T)$ expanding as in notation (28),

$$P^{\varepsilon}(t, x, y; T) = P_0(t, x, y; T) + \sqrt{\varepsilon} P_1(t, x, y; T) + \cdots$$

which could be proved not to depend on y in the later part of that section.

So, we can expand the bond option price $Q^{\varepsilon}(t, x, y; T_0, T)$ as that of bond price $P^{\varepsilon}(t, x, y; T)$

(36)
$$Q^{\varepsilon}(t, x, y) = Q_0(t, x, y) + \sqrt{\varepsilon} Q_1(t, x, y) + \cdots.$$

Assuming that the payoff function h is smooth, by Taylor formula, the terminal condition (35) can be expanded as

$$Q^{\varepsilon}(T_0, x, y) = h(P_0(T_0, x; T)) + \sqrt{\varepsilon} P_1(T_0, x; T) h'(P_0(T_0, x; T)) + \cdots$$

Observe that, for zero-order and first-order ($O(\sqrt{\varepsilon})$), the terminal condition $Q^{\varepsilon}(T_0, x, y)$ also doesn't depend on y.

Zero-order term

Expand the equation $L^{\varepsilon}Q^{\varepsilon} = 0$, where L^{ε} and correspondingly L_1, L_2 and L_3 is denoted as (24)-(26).

The $O\left(\frac{1}{\varepsilon}\right)$ term is $L_0 Q_0 = 0$, which implies that $Q_0(t,x,y) = Q_0(t,x)$ doesn't depend on y.

The $O\left(1/\sqrt{\varepsilon}\right)$ term is $L_0 Q_1 + L_1 Q_0 = 0$. From Q_0 denoted as above and definition of L_1, it is just $L_0 Q_1 = 0$, which implies that $Q_1(t,x,y) = Q_1(t,x)$ also doesn't depend on y.

The $O(1)$ term is $L_0 Q_2 + L_1 Q_1 + L_2 Q_0 = 0$, which reduce to $L_0 Q_2 + L_2 Q_0 = 0$. This is a Poisson equation for Q_2 with the solvability solution $\langle L_2 Q_0 \rangle = \langle L_2 \rangle Q_0 = 0$. (Here Q_0 can be moved out of blanket because of its independence on y. Therefore, $Q_0(t,x)$ is the solution of equation

$$\frac{\partial Q_0}{\partial t} + \frac{1}{2}\left\langle f^2(y)\right\rangle x \frac{\partial^2 Q_0}{\partial x^2} + a\left(r^* - x\right)\frac{\partial Q_0}{\partial x} - xQ_0 = 0$$

with terminal condition $Q_0(T_0,x) = h(P_0(T_0,x;T))$ at time T_0.

If we set $\bar{\sigma}^2 = \left\langle f^2(y)\right\rangle$, this is just equation (29) with Q_0 instead of P_0.

Consequently, $Q_0(t,x) = Q(t,x;T,T_0)$ represents the bond option price of CIR model. It can be written as

$$
\begin{aligned}
Q_0(t,x) &= E^*\left\{ e^{-\int_t^{T_0} r_s ds} h\left(P_0\left(T_0, r_{T_0}; T\right)\right) \middle| r_t = x \right\} \\
&= E^*\left\{ e^{-\int_t^{T_0} r_s ds} h\left(A(T-T_0) e^{-B(T-T_0) r_{T_0}}\right) \middle| r_t = x \right\}
\end{aligned}
$$

where A and B are denoted by (30), and (r_t) is the risk-neutral one-factor process with the expectation under the one-factor martingale measure P^*.

First-order Term

The $O(\sqrt{\varepsilon})$ term:

$$L_2 Q_1 + L_1 Q_2 + L_0 Q_3 = 0$$

which is also a Poisson equation in Q_3 with the solvability condition of

$$\langle L_2 Q_1 + L_1 Q_2 \rangle = 0 .$$

We can get the similar deduction about this equation from the first-order term of bond price correction with the same notation $\tilde{\lambda} = V \cdot x \frac{\partial^3}{\partial x^3}$, and $V = \frac{\nu}{\sqrt{2\alpha}} \rho \langle f \Phi' \rangle$,

$L_0 \Phi = \left(f(y)^2 - \langle f^2 \rangle\right) x$ introducing Φ as an increasing function.

Here, we define $\tilde{Q}_1(t,x) = \sqrt{\varepsilon} Q_1(t,x)$, the solvability condition becomes

$$\langle L_2 \rangle \tilde{Q}_1 = \tilde{\lambda} Q_0 .$$

Now we write the terminal condition at time T_0,

$$\tilde{Q}_1(T_0,x) = \tilde{P}_1(T_0,x;T) h'(P_0(T_0,x;T)).$$

This is a combined stochastic Dirichlet and Poisson problem, by the corollary (p167) in [1]. We can represent the solution as the following probabilistic form

$$\begin{aligned}\tilde{Q}_1(t,x) = E^* \Big\{ & e^{-\int_t^{T_0} r_s ds} \tilde{P}_1(T_0, r_{T_0}; T) h'(P_0(T_0, r_{T_0}; T)) \\ & - \int_t^{T_0} e^{-\int_t^u r_s ds} \tilde{\lambda} Q_0(u, r_u) du \Big| r_t = x \Big\} \end{aligned},$$

Then by (36)

$$\begin{aligned}\tilde{Q}(t,x) &\approx Q_0(t,x) + \tilde{Q}_1(t,x) \\ &= E^* \left\{ e^{-\int_t^{T_0} r_s ds} \left[h(P_0(T_0, r_{T_0}; T)) + \tilde{P}_1(T_0, r_{T_0}; T) h'(P_0(T_0, r_{T_0}; T)) \right] \Big| r_t = x \right\} \\ &\quad - E^* \left\{ - \int_t^{T_0} e^{-\int_t^u r_s ds} \tilde{\lambda} Q_0(u, r_u) du \Big| r_t = x \right\}\end{aligned}$$

Again, in particular case of European call bond option, we have the first term of bond option price as $Q_0(t,x) = C(t,x)$ which can be provided explicitly by (18) with the notation (19). The payoff function is $h(P_0(T_0, r_{T_0}; T)) = (P_0(T_0, r_{T_0}; T) - K)^+$, which can be known that $h'\left(P_0\left(T_0, r_{T_0}; T\right)\right) = 1_{\left\{P_0\left(T_0, r_{T_0}; T\right) > K\right\}}$. And the correction part of bond option price can also obtain from the expectation under risk-neutral world P^*,

$$\begin{aligned}\tilde{Q}_1(t,x) = {} & E^*\left\{ e^{-\int_t^{T_0} r_s ds} \tilde{P}_1\left(T_0, r_{T_0}; T\right) \cdot 1_{\left\{P_0\left(T_0, r_{T_0}; T\right) > K\right\}} \Big| r_t = x \right\} \\ & - E^*\left\{ \int_t^{T_0} e^{-\int_t^u r_s ds} \tilde{\lambda} Q_0(u, r_u) du \Big| r_t = x \right\}\end{aligned}.$$

From the function D_1 and D_2 numerically calculated by (34), and the formula of (18) for $\overline{C}(t,x)$, the corrected call bond price becomes

(37)

$$\begin{aligned}\overline{C}(t,x) = {} & C(t,x) + \left(D_1(T-T_0)x + D_2(T-T_0)\right) P_0(t,x;T) \cdot \chi^2\left(2\bar{r}[\phi+\psi+B(T_0)]; \frac{4ar^*}{\sigma^2}, \frac{2\phi^2 x e^{\theta(T-t)}}{\phi+\psi+B(T_0)}\right) \\ & - \int_t^{T_0} E^*\left\{ e^{-\int_t^u r_s ds} \tilde{\lambda} Q_0(u, r_u) \Big| r_t = x \right\} du \\ \doteq {} & I_1 - I_2\end{aligned}$$

where θ, ψ, ϕ, $\bar{r}$ are followed by (19).

For the second term I_2, it need to be presented carefully,

(38)

$$\begin{aligned}I_2 &= \int_t^{T_0} E^*\left\{ e^{-\int_t^u r_s ds} \tilde{\lambda} Q_0(u, r_u) \Big| r_t = x \right\} du \\ &= \int_t^{T_0} E^*\left\{ e^{-\int_t^u r_s ds} V \cdot r_u \frac{\partial^3}{\partial^3 r_u} Q_0(u, r_u) \Big| r_t = x \right\} du \\ &\doteq \int_t^{T_0} E^*\left\{ G\left(u; r_u, \int_t^u r_s ds\right) \Big| r_t = x \right\} du\end{aligned}$$

Here $G(u;r,R) = e^{-R}V \cdot r\frac{\partial^3}{\partial^3 r}Q_0(u;r,R)$ with the variable pair $\left(r_u, \int_t^u r_s\,ds\right)$, which has the complicated distribution. However, we can still get this distribution by Inverse Laplace Transform.

First we give out the a proposition which will help us to solve the distribution

Proposition *For any non-negative* λ *and* μ*, we have*

(39) $$E\left(e^{-\lambda r_u} e^{-\mu\int_t^u r_s\,ds} \middle| r_t = x\right) = \exp\left(-ar^*\xi_{\lambda,\mu}(u-t)\right)\exp\left(-b\zeta_{\lambda,\mu}(u-t)\right)$$

where the function $\xi_{\lambda,\mu}$ *and* $\zeta_{\lambda,\mu}$ *are given by*

(40) $$\xi_{\lambda,\mu}(v) = -\frac{2}{\sigma^2}\log\left(\frac{2\omega e^{\frac{v(\omega+a)}{2}}}{\sigma^2\lambda\left(e^{\omega v}-1\right)+\omega-a+e^{\omega v}(\omega+a)}\right)$$

and

(41) $$\zeta_{\lambda,\mu}(v) = \frac{\lambda\left(\omega+a+e^{\omega v}(\omega-a)\right)+2\mu\left(e^{\omega v}-1\right)}{\sigma^2\lambda\left(e^{\omega v}-1\right)+\omega-a+e^{\omega v}(\omega+a)}$$

with $\omega = \sqrt{a^2+2\sigma^2\mu}$.

Note: Proof can be obtained from [8], p130-131.

By the 2-dimension Laplace Transform, we have the formula of

(42) $$E\left(e^{-lr_u} e^{-m\int_t^u r_s\,ds} \middle| r_t = x\right) = \int_0^\infty\int_0^\infty e^{-lr} e^{-mR} f_u(r,R)\,dr\,dR$$
$$= \mathbf{L}\left(f_u(r,R)\right)$$

If we combine (39) and (42) and use inverse Laplace transform, we can get the $f_u(r,R)$, which is just the joint density function of the pair $\left(r_u, \int_t^u r_s ds\right)$,

(43) $$f_{u,l,m}(r,R) = \mathbf{L}^{-1}\left(\exp\left(-ar^* x_{l,m}(u-t)\right)\exp\left(-xz_{l,m}(u-t)\right)\right),$$

where $\xi_{\lambda,\mu}$ *and* $\zeta_{\lambda,\mu}$ follow (40), (41).

Now we have the joint density function for $\left(r_u, \int_t^u r_s ds\right)$, then, it is very easy to get its conditional expectation,

$$E^*\left\{G\left(u; r_u, \int_t^u r_s ds\right) \middle| r_t = x\right\} = \int_0^\infty \int_0^\infty G(u;r,R) \cdot f_{u,l,m}(r,R)\,drdR.$$

If we return to (38), we can obtain I_2,

(44) $$\begin{aligned} I_2 &= \int_t^{T_0} E^*\left\{G\left(u; r_u, \int_t^u r_s ds\right) \middle| r_t = x\right\} du \\ &= \int_t^{T_0} \int_0^\infty \int_0^\infty G(u;r,R) \cdot f_{u,l,m}(r,R)\,drdRdu \end{aligned},$$

where $f_{u,l,m}(r,R)$ is given by (43).

Finally, we can combine the (37) and (44) together and get the formula for $C(t,x)$,

$$\begin{aligned} \overline{C}(t,x) = C(t,x) + \left(D_1(T-T_0)x + D_2(T-T_0)\right)P_0(t,x;T) \cdot \chi^2\left(2\bar{r}[\phi+\psi+B(T_0)]; \frac{4ar^*}{\sigma^2}, \frac{2\phi^2 x e^{\theta(T-t)}}{\phi+\psi+B(T_0)}\right) \\ - \int_t^{T_0} \int_0^\infty \int_0^\infty G(u;r,R) \cdot f_{u,\lambda,\mu}(r,R)\,drdRdu \end{aligned}$$

Remark: There are various methods to solve the bond option prices. One of the easiest way is to use the distribution function in Mathematica 4.0,

NoncentralChiSquareDistribution[n, lambda],

which is the noncentral chi-square distribution function with **n** degrees of freedom and noncentrality parameter **lambda**.

Also, we can use the **Integrate[f, x]** to calculate the multiple integral in the results.

Then combine the numerical solution of D_1 and D_2 together, we can directly get the final answer of bond option price.

5. Appendix

Theorem *Suppose* P_0, P_1, P_2 *and* P_3 *is given by above correspondingly. Define the error term* Z^{ε} *by*

$$Z^{\varepsilon} = P_0 + \tilde{P}_1 + \varepsilon P_2 + \varepsilon^{3/2} P_3 - P^{\varepsilon}$$

where P^{ε} *is the model's bond pricing function defined above. Then for any fixed* $t < T$, $x, y \in \Re$,

$$Z^{\varepsilon}(t, x, y) = \mathrm{O}(\varepsilon).$$

Proof: Because of the smoothness and boundness of the payoff function 1 and the boundness assumption on $f(y)$, $\lambda(y)$ and $\gamma(y)$.

This follows from the standard theory of linear parabolic partial differential equations. (See Friedman 1964.) Further, solutions of the Poisson equations in $O(1)$ term and in $O(\sqrt{\varepsilon})$ term are at most linearly growing in $|y|$, so do P_2 and P_3.

At the terminal time T we have

$$Z^{\varepsilon}(T, x, y) = \varepsilon\left(P_2(T, x, y) + \sqrt{\varepsilon} P_3(T, x, y)\right),$$

with the terminal conditions $P^{\varepsilon}(T, x, y) = P_0(T, x, y) = 1$ and $\tilde{P}_1(T, x, y) = 0$, i.e. $Z^{\varepsilon}(T, x, y) = O(\varepsilon)$.

Next, by the properties of $\left(P^{\varepsilon}, P_0, P_1, P_2, P_3\right)$,

$$\begin{aligned} L^{\varepsilon} Z^{\varepsilon} &= L^{\varepsilon}\left(P_0 + \sqrt{\varepsilon} P_1 + \varepsilon P_2 + \varepsilon\sqrt{\varepsilon} P_3 - P_{\varepsilon}\right) \\ &= \varepsilon\left(L_1 P_3 + L_2 P_2\right) + \varepsilon^{3/2} L_2 P_3 \\ &= O(\varepsilon) \end{aligned},$$

We denote $L_{\varepsilon} Z_{\varepsilon} = \varepsilon F^{\varepsilon}, Z^{\varepsilon}(T, x, y) = \varepsilon G^{\varepsilon}$ where

$$\begin{aligned} F^{\varepsilon} &= L_1 P_3 + L_2 P_2 + \sqrt{\varepsilon} L_2 P_3 \\ G^{\varepsilon} &= P_2 + \sqrt{\varepsilon} P_3 \end{aligned},$$

This is the combined Dirichlet and Poisson problem, we can get the probabilistic representation of Z^{ε}

$$\begin{aligned} Z^{\varepsilon}(t, x, y) = \varepsilon E^{*}\Bigg\{ & e^{-\int_t^T r_s ds} G^{\varepsilon}\left(r_T, Y_T\right) \\ & - \int_t^T e^{-\int_t^s r_u du} F^{\varepsilon}\left(s, r_s, Y_s\right) ds \,\Big|\, r_t = x, Y_t = y \Bigg\} \end{aligned}.$$

Consequently

$$P^{\varepsilon}(t, x, y) = \left(P_0(t, x) + \tilde{P}_1(t, x)\right) + O(\varepsilon)$$

which prove the results of our theorem.

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[10] Marco Avellaneda, Peter Laurence, *Quantitative Modeling of Derivative Securities*, Chapman & Hall/CRC

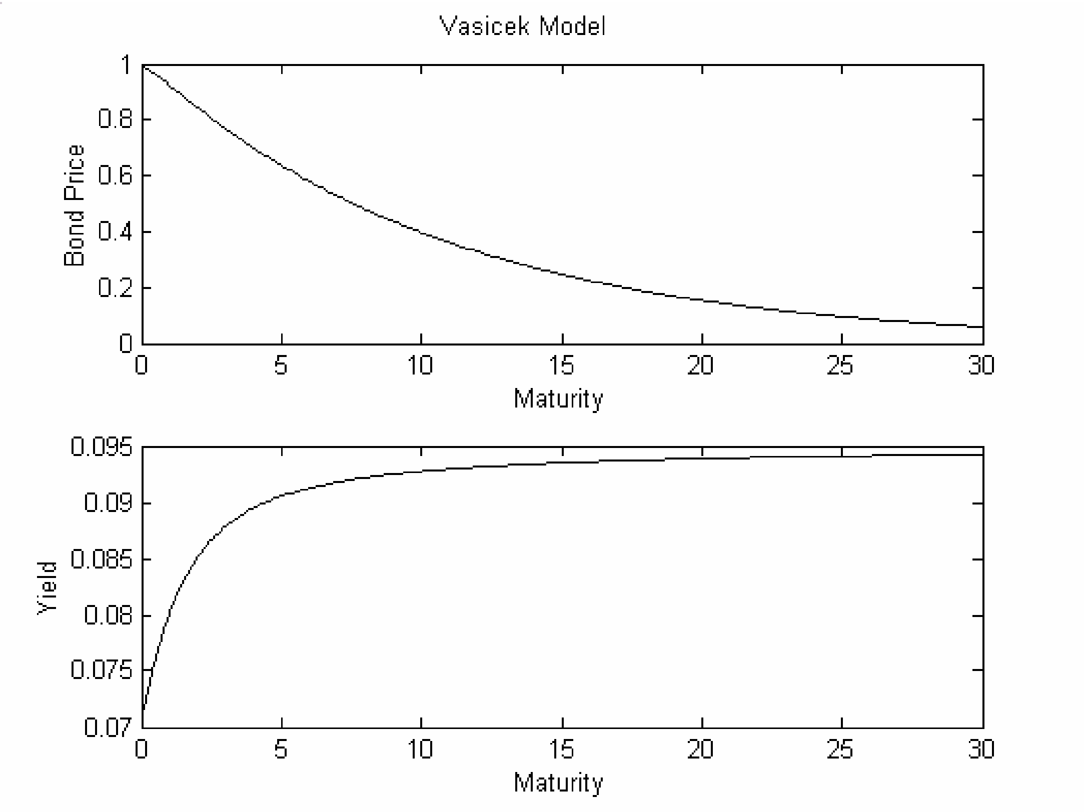

Figure 1: *Bond prices (top) and Yield curve (bottom) in the Vasicek model with* $a = 1$, $r^* = 0.1$ *and* $\sigma = 0.1$. *Maturity* τ *runs from 0 to 30 years.* $R_\infty = 0.095$ *and the initial rate is* $x = 0.07$.

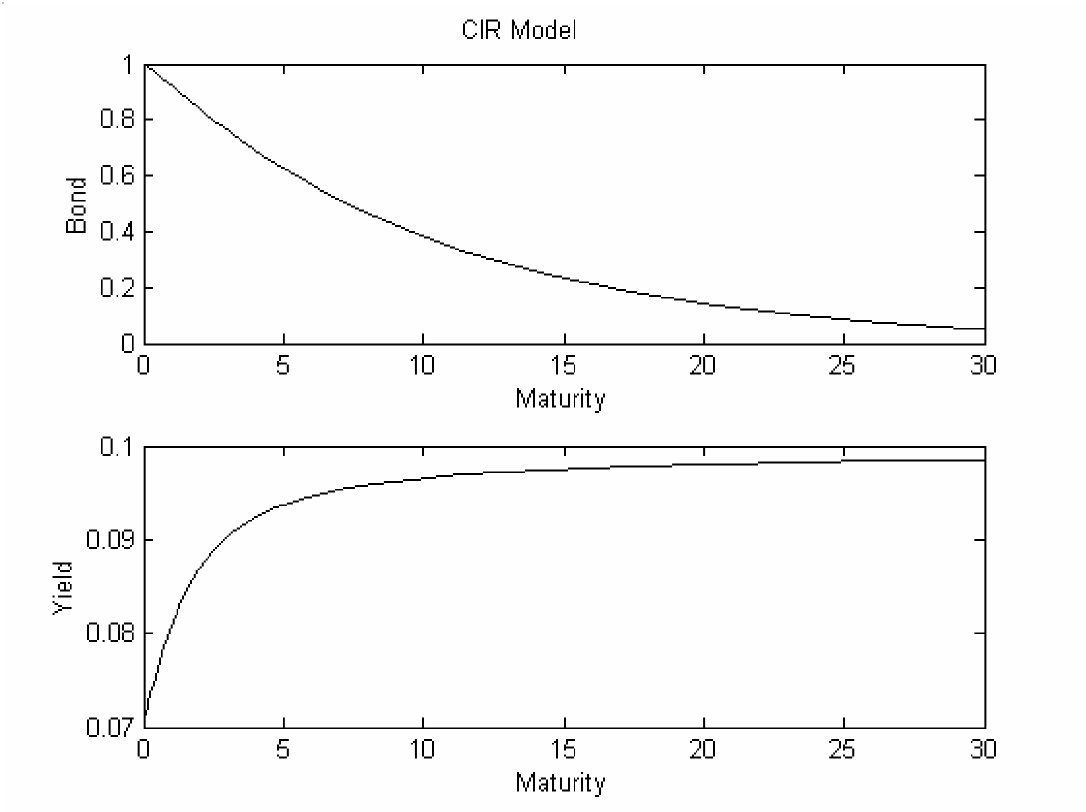

Figure 2: *Bond prices (top) and Yield curve (bottom) in the CIR model with* $a = 1$, $r^* = 0.1$ *and* $\sigma = 0.1$. *Maturity* τ *runs from 0 to 30 years. And the initial rate is* $x = 0.07$.

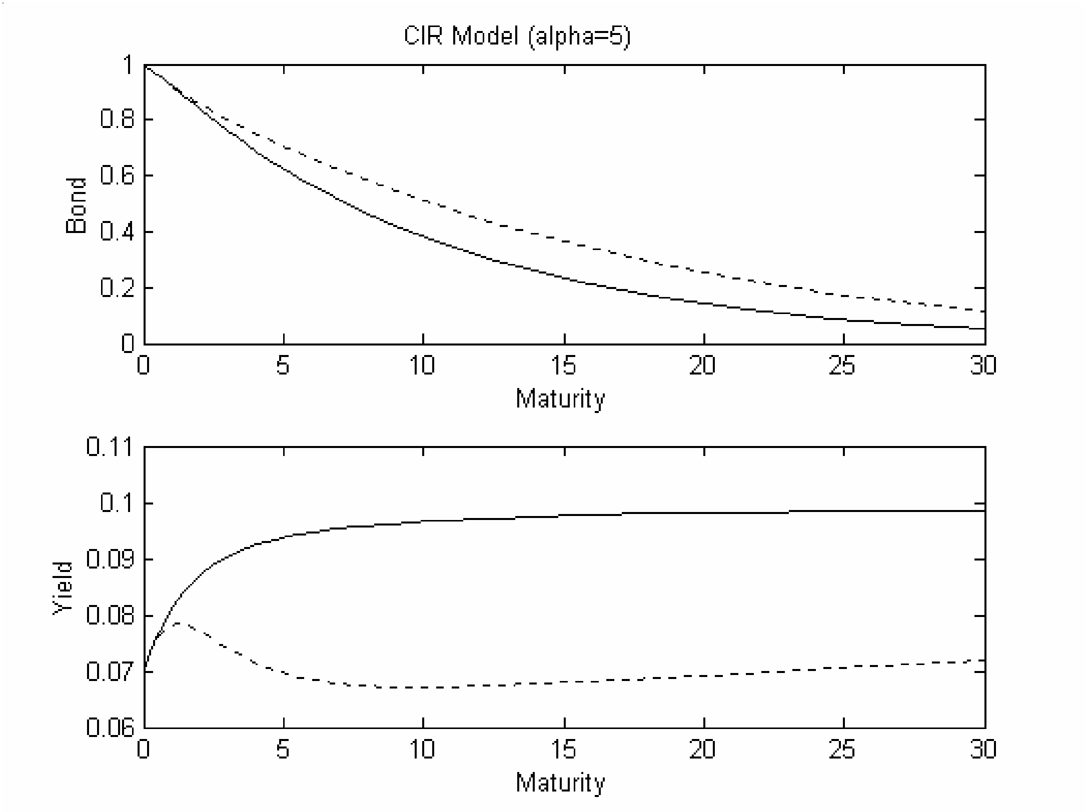

Figure 3: *Top: bond prices (solid curve) and corrected bond prices (dotted curve). Bottom: yield curve (solid curve) and corrected yield curve (dotted curve) in the simulated CIR model (constant and stochastic volatility) with* $a = 1$, $r^* = 0.1$ *and* $\sigma = 0.1$ *as in Figure 1. Concerning the correction we have used test parameter values* $V = 1/\sqrt{\alpha}$ *which assumes a nonzero skew,* $\alpha = 5$ *and maturity* τ *funs from 0 to 30 years and the initial rate is* $x = 0.07$.

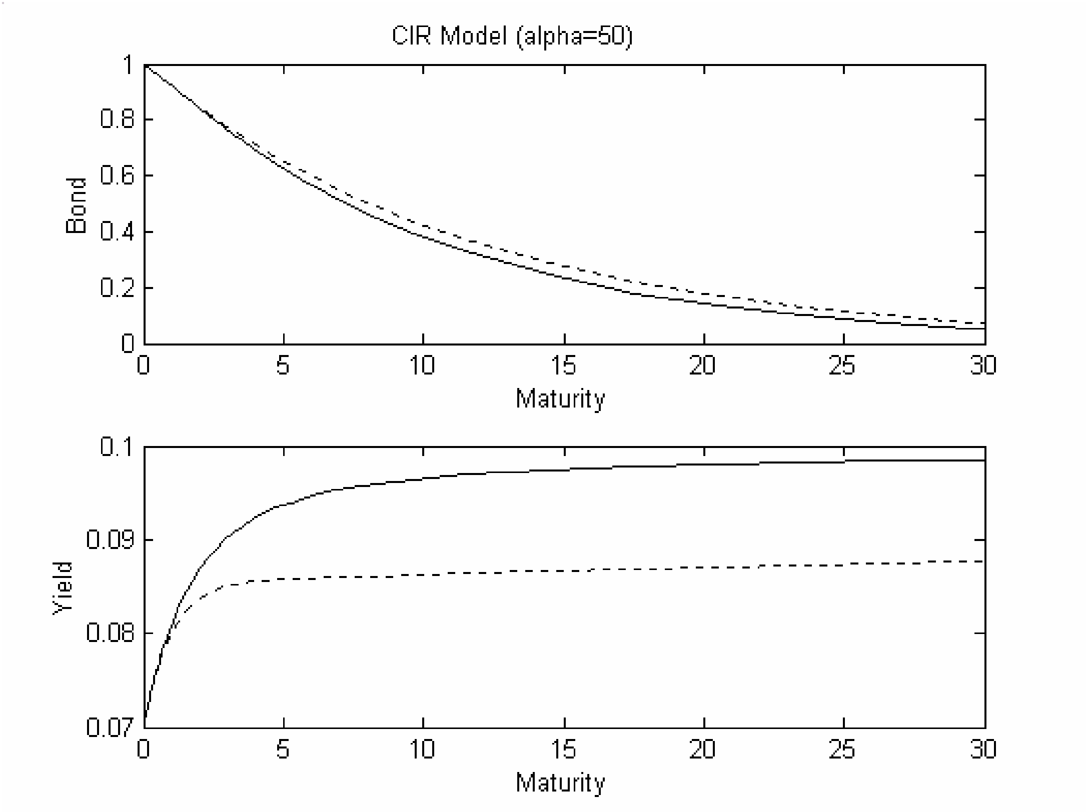

Figure 4: *Top: bond prices (solid curve) and corrected bond prices (dotted curve). Bottom: yield curve (solid curve) and corrected yield curve (dotted curve) in the simulated CIR model (constant and stochastic volatility) with* $a = 1$, $r^* = 0.1$ *and* $\sigma = 0.1$ *as in Figure 1. Concerning the correction we have used test parameter values* $V = 1/\sqrt{\alpha}$ *which assumes a nonzero skew,* $\alpha = 50$ *and maturity* τ *funs from 0 to 30 years and the initial rate is* $x = 0.07$.

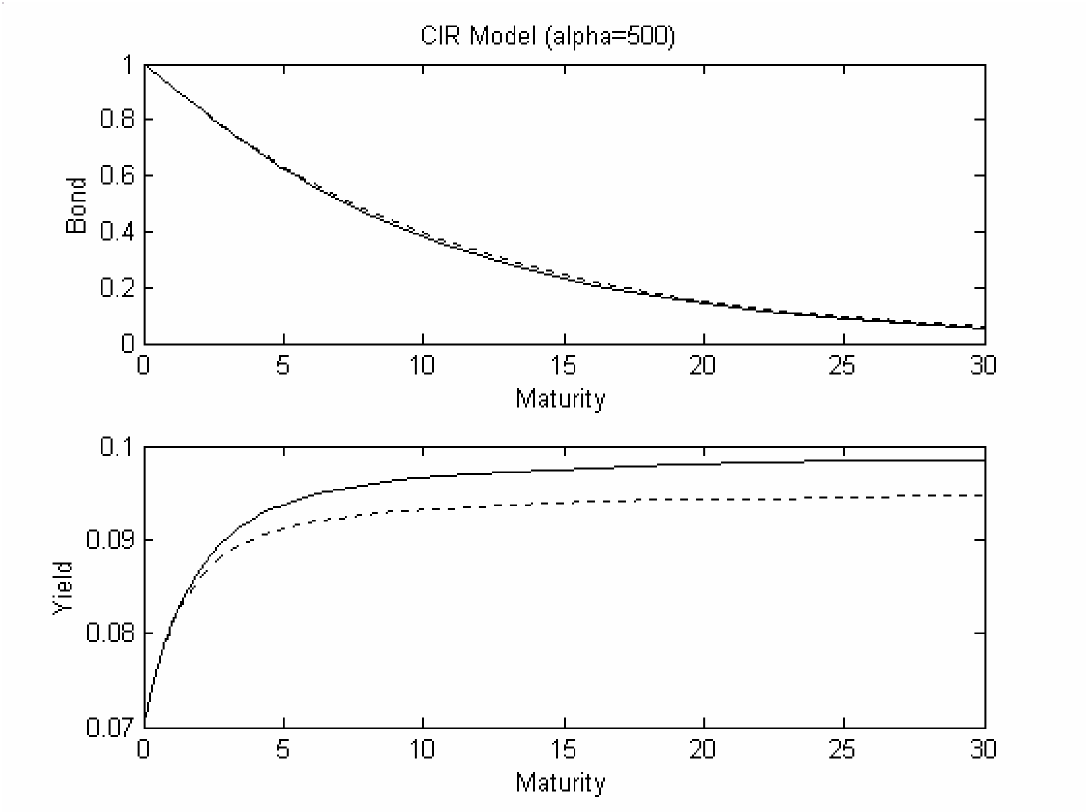

Figure 5: *Top: bond prices (solid curve) and corrected bond prices (dotted curve). Bottom: yield curve (solid curve) and corrected yield curve (dotted curve) in the simulated CIR model (constant and stochastic volatility) with* $a = 1$, $r^* = 0.1$ *and* $\sigma = 0.1$ *as in Figure 1. Concerning the correction we have used test parameter values* $V = 1/\sqrt{\alpha}$ *which assumes a nonzero skew,* $\alpha = 500$ *and maturity* τ *funs from 0 to 30 years and the initial rate is* $x = 0.07$.