GENERALIZED LEAST SQUARES ANALYSIS OF THE SPLIT-PLOT MODEL USING AN ESTIMATED VARIANCE-COVARIANCE MATRIX

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ABSTRACT

LANDOIS, LUIS LEON. Generalized Least Squares Analysis of the Split-Plot Model Using An Estimated Variance-Covariance Matrix. (Under the direction of FRANCIS GIESBRECHT.)

The variance-covariance matrix of the observations obtained from a split-plot type of experiment depends on two variance components. In the unbalanced case the best linear unbiased estimates of the treatment effects depends upon these unknown components. In this thesis the use of generalized least squares to estimate the treatment effects using an estimated variance-covariance matrix is considered. C. R. Rao's (1979) minimum norm quadratic estimation (MINQE) and invariance with respect to translation of the fixed effects are principles used to obtain estimates of the variance components. It is shown that invariant unbiased MINQE of the two variance components converge in probability to their true values. This is used to establish asymptotic properties of the generalized least squares estimators. Alternate estimators are obtained by using an iterated invariant MINQUE and by imposing the side condition that the estimate of the split-plot component be non-negative. A method for estimating the degrees of freedom associated with estimated variances of linear constraints among the generalized least squares estimates of the fixed effects is developed. This permits the construction of tests of hypotheses about constraints among the fixed effects. A small simulation is included to check on the performance of the procedure.

BIOGRAPHY

Luis Leon Landois-Palencia was born August 27, 1943, in Los Herreras Nuevo Leon, México. He received his elementary education in Torreón Coahuila, Mexico, his secondary and preparatory education in Monterrey Nuevo Leon, Mexico.

He received the degree of Ingeniera-Agrónomo from the Facultad-de-Agronomia in the Universidad de Nuevo Leon in December 1967. He worked as an assistant professor for six months in the Facultad-de-Agronomia. He entered in 1968 to the Centro-de-Estadística-y-Cálculo of the Colegio-de-Postgraduados at Chapingo, México, where he finished his courses in statistics in 1970, and he received the Master of Science in Statistics in May 1976.

While he was pursuing his master's degree, he was an assistant professor for the Biometry course at the undergraduate level, a position that he kept until 1970. He then became a professor of the course and held this position until 1976.

In 1970, he began to work as a consultant in Statistics and Statistical Computing and as a professor in the Computer Department at the same center. From 1972 until December 1976 he was the Head of the Computer Department and at the same time he was a part-time professor at the graduate level for the Computer Science Department.

In January 1977, the author enrolled in the Department of Statistics at North Carolina State University at Raleigh on a full-time basis until May 1981. He has now returned to his duty at the Centro-de-Estadística-y-Cálculo at Chapingo, Mexico.

The author has been married to Guadalupe Alvarez-Icaza de Landois since 1970. His wife has studied Economics at the Universidad Nacional Autónoma de México.

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1. INTRODUCTION

The split-plot experiment comes naturally when the researcher has data that arise from one of the following cases:

Case A. The random selection of whole units from which several measures (or subunits) are made.

Case B. The random selection of whole units followed by the selection of several subunits at random within each of the whole units.

Case C. A factorial arrangement of the factors A and B with p and q levels respectively in which the factor A with p-1 degrees of freedom is completely confounded with whole plots and B is confounded with subdivisions of the plots, i.e., split plots.

Case D. A large number of treatments assigned to plots in a randomly selected group of blocks, i.e., the incomplete block design.

In each one of these cases the structure of the errors in the model consists of two random elements, one associated with the whole plots (or whole units or blocks in case of the incomplete block design) and the second associated with the split plots (or split units or plots in the incomplete block). All of these random elements are assumed to be independently distributed with mean zero. Those in the first group have variance σ_1^2, the second group σ_2^2.

In general, the model associated with the split-plot experiment can be written as

$$Y = X\beta + \varepsilon$$

where Y is an r-vector of observations, x is a r x m matrix of known constants, β is an m-vector of parameters and ε has structure $U_1\varepsilon_1 + V_2\varepsilon_2$ where U_1 and U_2 are r x m_i matrices and ε_1 and ε_2 are m_i-vectors of random errors, such that $E(\varepsilon_i) = 0$ and variance-covariance matrix $D(\varepsilon_i) = I_{m_i}\sigma_i^2$, i = 1,2.

Analyses are straightforward when the number of split plots is constant for all whole plots. The object of this study is to consider cases when this is not true. In particular, this thesis is a report of the study of the behavior of the generalized least squares estimator of the parameters β, say $\hat{\beta} = (X'V^{-1}X)^{-}X'V^{-1}Y$, using an estimated variance-covariance matrix $\hat{V}$, i.e., to observe the behavior of $\tilde{\beta} = (X'\hat{V}^{-1}X)^{-}X'\hat{V}^{-1}Y$ when σ_1^2 and σ_2^2 are estimated using the MINQUE method.

A second problem to be considered here is to obtain a general formula for computing the degrees of freedom associated with a t-test for testing a linear function of the parameters β of the form

$$H_o: \lambda'\beta = \lambda'\beta_o$$

$$\text{against } H_a: \quad \neq \; \geq \; <$$

when the variance-covariance matrix V in the generalized least squares estimator $\hat{\beta}$ is replaced by an estimate $\hat{V}$ that comes from the MINQUE procedure.

2. REVIEW OF LITERATURE

The general Gauss Markoff linear model can be written as

$$Y = X\beta + \varepsilon \qquad (2.1)$$

where: Y is the n x 1 response vector, X is an n x p matrix with rank $q < p$, β is a p x 1 vector of parameters and ε is an n x 1 vector of random errors. It is common to assume that ε has $E[\varepsilon] = 0$ and variance-covariance matrix $D[\varepsilon] = I\sigma^2$, σ^2 unknown. Techniques for estimating under these conditions are well known.

Under certain conditions it is much more reasonable to assume that $D[\varepsilon] = V$. If V is known, or at least known up to a constant multiplier, then again the analysis is well known, though in general the computing may present some difficulties.

The object of this thesis is to examine several methods for estimating β when V is unknown but has some structure; i.e., depends on 2 unknown parameters σ_1^2 and σ_2^2. In particular it is assumed that the data (and model) are from an unbalanced split plot experiment and that σ_1^2 and σ_2^2 can be estimated.

2.1 Estimation of the Parameters β in A General Linear Model

One of the first persons to work with the weighted least squares principle was Aitken (1943). He showed that the minimum variance linear unbiased estimator of β, under the linear model (2.1), is $\hat{\beta} = (X'V^{-1}X)^{-}X'V^{-1}Y$. The obvious difficulty with this approach is that

V is often unknown. A possible alternative is to ignore V and use ordinary least squares.

Magness and McGuire (1962) were able to express the differences between variances of estimated regression coefficients under ordinary least squares and generalized least squares as functions of the variance-covariance matrix V.

Following along similar lines, McElroy (1967) found that ordinary least squares estimators also are best linear unbiased, when all errors have common variance and common non-negative coefficient of correlation between all pairs. Williams (1967) extended this to show that ordinary least squares estimators were best linear unbiased when the rows of X were a full-rank linear combination of the characteristic vectors of V.

Zyskind (1969) gives several conditions on the form of the covariance matrix such that simultaneously for all models with common specified systematic part every ordinary least square estimator is also best linear unbiased. He shows that models with such a covariance structure may be viewed as possessing just one error term. Linear models for many complex experiments, like the split-plot design, with an induced covariance structure under which all linear ordinary least square estimators are also best linear unbiased estimators, often possess several natural error terms.

Bement and Williams (1969) studied the regression model with independent but not homogeneous errors. They examined the performance of weighted least squares estimators with estimated weights. Their conclusion was that the weighted least squares estimators had smaller variance than ordinary least squares estimators if the variances could be estimated with at least 10 degrees of freedom each.

J. N. Rao and Subrahmaniam (1971) studied the following two problems:

(i) Combining k independent estimators $\bar{y}_i$, $i = 1,2,\ldots,k$ of a parameter μ, where $\bar{y}_i$ is the mean of n_i observations being normally and independently distributed with mean μ and variance σ_i^2.

(ii) Estimating the parameters α and β in a regression model $y_{ij} = \alpha + \beta x_i + \varepsilon_{ij}$, $j = 1,2,\ldots,n_i$, $i = 1,2,\ldots,k$, where the x_i are known constants and the ε_{ij} are normally and independently distributed with mean zero and variance σ_i^2.

Their approach differed from that of the previous authors in that they used the MInimum Norm Quadratic Unbiased Estimation (MINQUE) principle introduced by C. R. Rao (1970) to estimate the unknown variances.

They found explicit formulas for the MINQUE estimates of the variances in (i), given by

$$\sigma_i^2 = n(n_i(n-2))^{-1} \sum_{j=1}^{n_i} (y_{ij}-\bar{y})^2 - (n-2)^{-1}s^2$$

where $n = \sum_i n_i$ and $s^2 = (n-1)^{-1}\Sigma_i\Sigma_j(y_{ij}-\bar{y})^2$.

These estimators are not necessarily positive and must be modified to provide satisfactory weights. On the basis of a simulation study, they concluded that weighted least squares using the modified MINQUE values is more efficient than the weighted least squares using s_i^2 when n is small and k is large. Another conclusion was that MINQUE may not lead to substantial gain in efficiency when $n_i = m \geq 8$, specially for small k.

Fuller and J. N. K. Rao (1978) considered the problem of estimating the parameter β in the linear model with heteroscedastic variances, $Y = X\beta + \varepsilon$ where ε is the n-vector of random variables with mean zero and dispersion matrix V = block diagonal $(\sigma_1^2 I_{n_i}, \ldots, \sigma^2 I_{n_k})$, $\{\sigma_i^2\}$ are unknown variances and I_{n_i} is an $n_i \times n_i$ identity matrix. The ordinary least squares estimator of β is $\hat{\beta} = (X'X)^{-1}X'Y$. They defined the two step weighted least squares estimator of β by $\tilde{\beta} = (X'\hat{V}^{-1}X)^{-}X'\hat{V}^{-1}Y$ where $\hat{V}$ = block diagonal $\{\hat{\sigma}_1^2 I_{n_i}, \ldots, \hat{\sigma}_k^2 I_{n_k}\}$ and $\hat{\sigma}_i^2$ is an estimator of σ_i^2. The first step was to compute estimator $\hat{V}$ and the second to compute $\tilde{\beta}$. They studied the class of two step estimators of β given by $\tilde{\beta}_w$ = $(X'\hat{V}^{-1}WX)^{-1}X'\hat{V}^{-1}WY$ where $\hat{\varepsilon}' = (Y - X\tilde{\beta})'$, $\hat{\sigma}_i^2 = n_i^{-1}\Sigma\hat{\varepsilon}_{ij}^2$, W = block diagonal $\{w_1 I_{n_1}, \ldots w_k I_{n_k}\}$ and $w_i = g(n_i)$ for some $g(\cdot)$ such that $0 < \gamma_1 < w_i < \gamma_2 < \infty$ for all i and γ_1 and γ_2 being constants. When they replicated the model, the $\tilde{\beta}_w$ reduced to $\tilde{\beta} = (X'\hat{V}^{-1}X)^{-}X'\hat{V}^{-1}Y$. They used the two step estimator $\tilde{\beta}_w$ to construct a new estimator of σ_i^2 and inserted these estimated variances into $\tilde{\beta}_w$ to obtain the three step estimator of β. They viewed the maximum likelihood estimator as the limit of an iterated process with W = I. They investigated the special case of the two step estimators of a common mean and found that the two step estimator was superior to the maximum likelihood estimator for a considerable range of parameter values.

Fuller and Battese (1973) considered a linear model with a nested error structure. They demonstrated that it is possible to make a relatively simple transformation and then compute the generalized least squares estimators of the fixed parameters. The transformation requires estimates of the variance components. These they estimated using the fitting constants method of Henderson's method III described in Searle (1968). They estimated the variance-covariance matrix, say $\hat{V}$, and obtained the value of the generalized least squares estimator $\tilde{\beta}$ = $(X'\hat{V}^{-1}X)^{-}X'\hat{V}^{-1}Y$ by the ordinary least squares regression of $\hat{V}^{-1/2}Y$ on

$\hat{V}^{-1/2}X$. They gave sufficient conditions under which the estimated generalized least squares estimator is unbiased and asymptotically equivalent to the generalized least squares estimator.

Williams (1975) studied convergence rates of weighted least squares to best linear unbiased estimators. He wrote the model as follows:

$$Y = X\beta_o + V_o^{1/2}Z$$

where Y is a n x 1 response vector, X is a n x p matrix of full rank p, β_o is a p x 1 vector of parameters and $V_o^{1/2}$ is a symmetric square root of the dispersion matrix V_o of the vector Y, and the respective mean vector and dispersion matrix of Z was 0_n and I_n. His study concentrated on situations where the matrix V_o could be written as $V(\Theta_o)$, i.e., a function of a relatively limited number of parameters which remained constant as n increased. Under these conditions he showed that for sufficient large n a weighted least squares estimator $\hat{\beta}_w = (X'\hat{V}^{-1}X)^{-}X'\hat{V}^{-1}Y$ can be constructed and regarded either as an estimator of β_o or of the best linear unbiased estimator $\hat{\beta} = (X'V_o^{-1}X)^{-}X'V_o^{-1}Y$.

2.2 Missing Values and Unbalanced Data

In a general experiment if the number of observations belonging to the subclass is the same for all subclasses, then the experiment is said to have balanced data or no missing observations. In contrast, if there is an unequal number of observations in the subclasses or if some subclasses contain no observations at all, the experiment is called unbalanced or with missing data.

One of the oldest papers that describes methods to estimate the yield of a missing plot is due to Allan and Wishart (1930). They provided formulas for a single missing plot. Yates (1933) extended that method to estimate several missing plots, and provided an iterative procedure to estimate the missing plots. Anderson (1946) derived formulas for one missing plot in a split-plot experiment. He used the covariance method for the derivation in both cases when there is one subplot missing and when there is one whole plot missing. In principle the covariance analysis method can be extended to more missing values. However, this quickly becomes impractical and one is forced to use methods appropriate for the general unbalanced case.

Hocking and Speed (1974) compared a number of methods for handling unbalanced data sets.

Speed, Hocking and Hackney (1978) reviewed the existing methods for analyzing experimental design models with unbalanced data and related them with existing computer programs. The methods are distinguished by the hypotheses associated with the sum of squares which are generated. Their claim is that the choice of the method should be based on the appropriateness of the hypotheses rather than on computational convenience or the orthogonality of the quadratic form.

One possible alternative for the experimenter faced with missing cells is provided by Hocking, Speed and Coleman (1980). This alternative is based on the assumption that the experimenter would test the usual hypotheses if the experiment had been balanced. The hypotheses suggested are derived from the balanced case. They also provided a criterion for choosing the hypotheses to be tested in the unbalanced case and in addition also provided an algorithm for testing the desired hypotheses.

The salient feature in the Hocking and Speed (1975), Speed, Hocking and Hackney (1978) and Hocking, Speed and Coleman (1980) papers is that only the simple error structure with independent identically distributed errors is considered.

This thesis deals with unbalanced data and a more complex error structure. Emphasis will be on the error structure and the questions concerning hypotheses to be tested will not be addressed.

2.3 Estimation of Variance Components

The large body of literature dealing with variance component estimation has recently been reviewed by Harville (1977). Subsequent to that, Rao and Kleffe (1979) described a series of modifications of the MInimum Norm Quadratic (MINQ) estimation principle. In particular, they discussed MINQ-unbiased, MINQ-invariant and MINQ-unbiased, invariant estimators.

3. MINQUE ESTIMATORS FOR THE SPLIT-PLOT VARIANCE COMPONENTS

3.1 The Split-Plot Model

The linear statistical model appropriate for the basic split-plot design in which observations are taken on n_i split-plots in the ith whole-plot can be written as:

$$y_{ij} = \sum_{k=1}^{m} x_{ijk}\beta_k + u_{ij} \qquad j = 1,2,\ldots,n_i,\ i = 1,2,\ldots,a$$

where the $\{y_{ij}\}$ are the observed response values, the $\{x_{ijk}\}$ are the m different control variables, the $\{\beta_k\}$ are the m fixed unknown parameters, and the $\{u_{ij}\}$ are the unobservable random errors. These errors consist of two components, a random element associated with the ith whole-plot, say v_i, and a second independent random element associated with the jth subplot in the ith whole-plot, say e_{ij}; i.e., $u_{ij} = v_i + e_{ij}$. The $\{v_i\}$ and $\{e_{ij}\}$ are assumed to be independently distributed with zero mean and variances $\sigma_v^2 \geq 0$, $\sigma_e^2 > 0$ respectively. Also, the $\{v_i\}$ and $\{e_{ij}\}$ are independent of each other. These assumptions imply

$$E[u_{ij}u_{i'j'}] = \begin{cases} \sigma_v^2 + \sigma_e^2 & \text{if} \quad i = i' \text{ and } j = j' \\ \sigma_v^2 & \text{if} \quad i = i' \text{ and } j \neq j' \\ 0 & \text{if} \quad i \neq i' \end{cases}$$

Alternatively, in matrix notation, the statistical model for the split-plot becomes a general Gauss Markoff model

$$Y = X\beta + \varepsilon \tag{3.1}$$

where Y is an r-vector of observations, X is an r x m matrix, $\beta \in R^m$ is an m-vector of parameters, and ε is an r-vector of random errors.

For the split-plot design, ε has a special structure. It can be written as $\varepsilon = U_1\varepsilon_1 + U_2\varepsilon_2$ where U_1 and U_2 are $r \times m_1$ and $r \times m_2$ matrices of known constants and ε_1 and ε_2 are m_1 and m_2-vectors of random errors. Also, $E[\varepsilon_\ell] = 0$, $E[\varepsilon_\ell\varepsilon_\ell'] = \sigma^2 I_{m_\ell}$ for $\ell = 1,2$, $E[\varepsilon_1\varepsilon_2'] = 0$ where I_p is the p x p identity matrix.

It follows that the variance-covariance matrix of ε (or Y) is given by $D(\varepsilon) = V(\sigma) = V_1\sigma_1^2 + V_2\sigma_2^2$ where σ_1^2 and σ_2^2 are the whole and split plot error variances respectively. Note that if the elements of Y are arranged with $Y' = (y_{11}, y_{12}, \ldots, y_{a_{na}})$, then V_1 is a block diagonal matrix, where each block contains unitary elements and is of order $n_i \times n_i$ and V_2 is the identity matrix of order r x r.

The remainder of this chapter will be devoted to obtaining estimates for σ_1^2 and σ_2^2.

3.2 The MInimum Norm Quadratic Unbiased Estimator (MINQUE)

The purpose of this section is to present the MINQUE theory for the variance components as initially presented in C. R. Rao (1970, 1972) and C. R. Rao and J. Kleffe (1979).

Let $Y = X\beta + \varepsilon$ be a linear model such that Y is an r-vector of observations, X is an r x m matrix, $\beta \in R^m$ is an m-vector of parameters, and ε is an r-vector of random errors, such that the structure of ε is given by $\varepsilon = U_1\varepsilon_1 + U_2\varepsilon_2 + \ldots + U_p\varepsilon_p$ where $U_1, \ldots, U_p$ are $r \times m_\ell$, $\ell = 1,2, \ldots, p$ matrices of known constants and $\varepsilon_1, \ldots, \varepsilon_p$ are m_ℓ-vectors of random errors. Also, $E[\varepsilon_\ell] = 0$, $E[\varepsilon_\ell\varepsilon_\ell'] = \sigma_\ell^2 I_{m_\ell}$, where I_{m_ℓ} is the $m_\ell \times m_\ell$ identity matrix for $\ell = 1,2, \ldots, p$ and $E[\varepsilon_\ell\varepsilon_{\ell'}'] = 0$ $\ell \neq \ell'$.

It follows that the variance-covariance matrix of ε (or Y) is given by $D(\varepsilon) = V(\sigma) = V_1\sigma_1^2 + \ldots + V_p\sigma_p^2$, where $V_\ell = U_\ell U_\ell'$ $\ell = 1,2, \ldots, p$.

In general, the above papers deal with the problem of obtaining estimators for the linear parametric function

$$\gamma(\beta,\sigma^*) = c'\beta + f'\sigma^*$$

where $c \in R^m$, $f \in R^p$, $\beta \in R^m$, $\sigma^* \in \tau$ and $\tau \subset R^p$ is an open set. Note that the model contains m fixed effects and p variance components.

The estimators of the linear parametric function $\gamma(\beta,\sigma)$ are functions of the observed values and have the form $g(Y) = a'Y + Y'AY$.

Before proceeding to the estimation technique itself, a number of preliminary concepts must be established.

3.2.1 Identifiability

Definition 3.2.1

A parametric function $\gamma(\beta,\sigma)$ is said to be identifiable iff $V(\sigma_1) = V(\sigma_2)$ and $X\beta_1 = X\beta_2$ implies $\gamma(\beta_1,\sigma_1) = \gamma(\beta_2,\sigma_2)$.

The following lemma establishes the necessary and sufficient conditions to have identifiability of a parametric function.

Lemma 3.2.1

The parametric function $\gamma(\beta,\sigma) = c'\beta + f'\sigma$ is identifiable iff $c \in \delta(X')$ and $f \in \delta(W)$ where $\delta(A)$ is the vector space generated by the columns of A, W is the matrix with elements w_{ij} equal to $tr(V_iV_j)$, and tr represents the trace of a matrix.

The proof is given in C. R. Rao and J. Kleffe (1979).

3.2.2 Unbiasedness

Suitable conditions to establish unbiasedness are given by the following theorem.

Theorem 3.2.1

The estimator $g(Y) = a'Y + Y'AY$ is an unbiased estimator of the parametric function $\gamma(\beta,\sigma) = c'\beta + f'\sigma$ if $c' = a'X$, $X'AX = 0$ and $tr(AV_\ell) = f_\ell$, $\ell = 1,2, \ldots, p$ where A is a symmetric matrix.

The proof of this theorem is given in C. R. Rao and J. Kleffe (1979). These authors also prove the following theorem concerning the existence of an unbiased estimator.

Theorem 3.2.2

There exists an unbiased estimator g(Y) if $c \in \delta(X')$ and $f \in \delta(Q)$ where $Q = (q_{\ell\ell'})$ is a matrix with $q_{\ell\ell'} = tr(V_\ell V_{\ell'} - P_X V_\ell P_X V_{\ell'})$ and where P_X is the projection operator of X onto $\delta(X')$.

3.2.3 An Invariance Principle

If the vector of parameters β is replaced by

$$\beta_\alpha = \beta - \beta_o$$

where β_o is arbitrary, then the model (3.1) becomes

$$Y - X\beta_o = X(\beta-\beta_o) + \varepsilon.$$

Letting $Y_d = Y - X\beta_o$ leads to

$$Y_d = X\beta_d + \varepsilon$$

and the quadratic estimator corresponding to $Y'AY$ becomes $Y_d' AY_d$. The quadratic estimator $Y'AY$ is said to be invariant with respect to X if $Y_d'AY_d$ and $Y'AY$ are equal for all β_o. It is clear from the expression

$$\begin{aligned} Y_d'AY_d &= (Y - X\beta_o)'A(Y - X\beta_o) \\ &= Y'AY - 2\beta_o'X'AY + \beta_o'X'AX\beta_o \\ &= Y'AY \end{aligned}$$

that $AX = 0$ is both necessary and sufficient for invariance to hold.

3.2.4 Minimum Norm Principle

If the $\{\varepsilon_\ell\}$ were known, then a natural invariant estimator of the parametric function $f'\sigma$ is

$$\frac{f_1}{m_1}\varepsilon_1'\varepsilon_1 + \frac{f_2}{m_2}\varepsilon_2'\varepsilon_2 + \dots + \frac{f_p}{m_p}\varepsilon_p'\varepsilon_p .$$

This can be written as

$$(\varepsilon_1', \varepsilon_2', \dots, \varepsilon_p')\,\Delta \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_p \end{bmatrix} \tag{3.2.4.1}$$

where

$$\Delta = \begin{bmatrix} \frac{f_1}{m_1} I_{m_1} & & & 0 \\ & \frac{f_2}{m_2} I_{m_2} & & \\ & & \ddots & \\ 0 & & & \frac{f_p}{m_p} I_{m_p} \end{bmatrix} .$$

In practice, the $\{\varepsilon_\ell\}$ are not observable and the natural estimator of $f'\sigma$ is

$$g(Y) = Y'AY.$$

Imposing the condition for invariance leads to

$$\begin{aligned} g(Y) &= (X\beta+\varepsilon)' A (X\beta+\varepsilon) \\ &= \varepsilon' A \varepsilon \end{aligned}$$

and since ε has structure $U_1\varepsilon_1 + \ldots + U_p\varepsilon_p$, then

$$\begin{aligned} g(Y) &= (U_1\varepsilon_1 + \ldots + U_p\varepsilon_p)' A (U_1\varepsilon_2 + \ldots + U_p\varepsilon_p) \\ &= (\varepsilon_1'U_1' + \ldots + \varepsilon_p'U_p') A (U_1\varepsilon_2 + \ldots + U_p\varepsilon_p) \\ &= (\varepsilon_1', \varepsilon_2', \ldots, \varepsilon_p') \begin{bmatrix} U_1' \\ U_2' \\ \vdots \\ U_p' \end{bmatrix} A (U_1 : U_2 : \ \ldots \ : U_p) \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_p \end{bmatrix} \\ &= \varepsilon_*' U' A U \varepsilon_* \end{aligned} \qquad (3.2.4.2)$$

where

$$\varepsilon_* = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_p \end{bmatrix} \qquad \text{and} \qquad U = (U_1 : U_2 : \ \ldots \ : U_p)$$

A reasonable strategy is to make the difference between (3.2.4.1) and (3.2.4.2), $\varepsilon_*'(U'AU - \Delta)\varepsilon_*$ small.

The MINQUE principle is to minimize $||U'AV - \Delta||$, where $||H||$ denotes the Euclidean norm $\sqrt{tr(H\,H')}$ of a matrix.

This leads to the formal definition of the MINQUE of a parametric function:

Definition 3.2.2

The quadratic form $g(Y) = Y'AY = \varepsilon_*'U'AV\varepsilon_*$ is said to be a MINQUE of the parametric function $f'\sigma = \sum_{k=1}^{p} f_\ell \sigma_\ell^2$ if the matrix A is determined such that $||U'AU - \Delta||$ is a minimum, subject to the conditions $AX = 0$ and $tr(AV_\ell) = f_\ell$, $\ell = 1,2, \ldots, p$.

In order to obtain an explicit solution, it is convenient to minimize the square of the Euclidean norm, $||U'AU - \Delta||^2 = tr((U'AU - \Delta)(U'AU - \Delta))$. This can be simplified somewhat by using the following result.

Lemma 3.2.2

Let A be a real symmetric matrix subject to the conditions $AX = 0$ and $tr(AV_\ell) = f_\ell$, $\ell = 1, \ldots, p$. Then $tr(U'AU\Delta) = tr(\Delta\Delta)$.

Proof

$$tr(U'AU\Delta) = tr((U_1: \ldots :U_p)'A(U_1: \ldots :U_p) \begin{bmatrix} \frac{f_1}{m_1} I_{m_1} & & 0 \\ & \ddots & \\ 0 & & \frac{f_p}{m_p} I_{m_p} \end{bmatrix}$$

$$= tr((U_1: \ldots :U_p)'A(\frac{f_1}{m_1} U_1: \ldots :\frac{f_p}{m_p} U_p))$$

$$= tr(\begin{bmatrix} U_1'AU_1 \frac{f_1}{m_1} & U_1'AU_2 \frac{f_2}{m_2} & \cdots & U_1'AU_p \frac{f_p}{m_p} \\ U_2'AU_2 \frac{f_1}{m_1} & U_2'AU_2 \frac{f_2}{m_2} & \cdots & U_2'AU_p \frac{f_p}{m_p} \\ \vdots & & & \\ U_p'AU_1 \frac{f_1}{m_1} & U_p'AU_2 \frac{f_2}{m_2} & \cdots & U_p'AU_p \frac{f_p}{m_p} \end{bmatrix})$$

$$= \mathrm{tr}(U_1'AU_1 \frac{f_1}{m_1}) + \mathrm{tr}(U_2'AU_2 \frac{f_2}{m_2}) + \ldots + \mathrm{tr}(U_p'AU_p \frac{f_p}{m_p})$$

$$= \frac{f_1}{m_1} \mathrm{tr}(AU_1U_1') + \frac{f_2}{m_2} \mathrm{tr}(AU_2U_2') + \ldots + \frac{f_p}{m_p} \mathrm{tr}(AU_pU_p')$$

$$= \sum_\ell \frac{f_\ell}{m_\ell} \mathrm{tr}(AV_\ell) \qquad \text{where } V_\ell = U_\ell U_\ell'$$

$$= \sum_\ell \frac{f_\ell^2}{m_\ell} \quad , \qquad \text{because } \mathrm{tr}(AV_\ell) = f_\ell .$$

$$\mathrm{tr}(\Delta\Delta) = \mathrm{tr}\left(\begin{bmatrix} \frac{f_1}{m_1} I_{m_1} & & 0 \\ & \ddots & \\ 0 & & \frac{f_p}{m_p} I_{m_p} \end{bmatrix} \begin{bmatrix} \frac{f_1}{m_1} I_{m_1} & & 0 \\ & \ddots & \\ 0 & & \frac{f_p}{m_p} I_{m_p} \end{bmatrix} \right)$$

$$= \mathrm{tr}\left(\begin{bmatrix} \frac{f_1^2}{m_1^2} I_{m_1} & & \\ & \ddots & \\ & & \frac{f_p^2}{m_p^2} I_{m_p} \end{bmatrix} \right)$$

$$= \Sigma_\ell \frac{f_\ell^2}{m_\ell^2} \mathrm{tr}(I_{m_\ell})$$

$$= \Sigma_\ell \frac{f_\ell^2}{m_\ell} .$$

It follows that

$$\mathrm{tr}((U'AU - \Delta)(U'AU - \Delta)) = \mathrm{tr}(U'AUU'AU) - 2\mathrm{tr}(U'AU\Delta) + \mathrm{tr}(\Delta\Delta)$$

$$= tr(U'AUU'AU) - tr(\Delta\Delta)$$

$$= tr(AVAV) - tr(\Delta\Delta), \text{ where } V = UU'.$$

Since $tr(\Delta\Delta)$ does not involve A, the problem of MINQUE is reduced to minimizing $tr(AVAV)$ subject to the conditions $AX = 0$ and $tr(AV_\ell) = f_\ell$, $\ell = 1,2, \ldots, p$.

Also, since the ε_ℓ in the model (3.1) may have different standard deviations, it is reasonable to rewrite the difference $\varepsilon_*'(U'AU - \Delta)\varepsilon_*$ in terms of standardized variables $\xi_\ell = \varepsilon_\ell/\sigma_\ell$. This leads to the quadratic form

$$\xi' \begin{bmatrix} \sigma_1 I_{m_1} & & 0 \\ & \ddots & \\ 0 & & \sigma_p I_{m_p} \end{bmatrix} (U'AU - \Delta) \begin{bmatrix} \sigma_1 I_{m_1} & & 0 \\ & \ddots & \\ 0 & & \sigma_p I_{m_p} \end{bmatrix} \xi .$$

The problem is now to minimize $tr(AV_*AV_*)$ under the conditions $AX = 0$ and $trAV_\ell = f_\ell$, $\ell = 1,2, \ldots, p$ and $V_* = V_1\sigma_1^2 + \ldots + V_p\sigma_p^2$. In practice the $\{\sigma_i^2\}$ will be unknown and will have to be replaced by the best value available from prior knowledge. Note that knowledge of the ratios of the variance components is sufficient.

The required solution is obtained via the following theorem.

Theorem 3.2.3

Let P by the projector operator onto the space generated by the columns of X, using inner product $(X,Y) = X'V^{-1}Y$, where V is a positive definite matrix, and defined by $P = X(X'V^{-1}X)^-X'V^{-1}$. The minimum of $tr(AVAV)$ subject to the conditions $AX = 0$ and $tr(AV_\ell) = f_\ell$, $\ell = 1, \ldots, p$ is attained at $A_* = \sum_\ell \lambda_\ell Q_V V_\ell Q_V$, where $Q_V = V^{-1}(I-P)$ and $\lambda' = (\lambda_1, \ldots, \lambda_p)$

is determined from the equations $S\lambda = f$ where $f' = (f_1, \ldots, f_p)$, and $S = (S_{\ell\ell'})$ and $S_{\ell\ell'} = tr(Q_V V_\ell Q_V V_{\ell'})$.

The proof of this theorem is given in C. R. Rao (1972).

It follows that the MINQUE of $\Sigma_\ell f_\ell \sigma_\ell^2$ is

$$\begin{aligned} g(Y) &= Y'AY \\ &= Y'(\Sigma_\ell \lambda_\ell Q_V V\ Q_V)Y \\ &= \lambda' q \end{aligned}$$

where $q' = (Y'Q_V V_1 Q_V Y, Y'Q_V V_2 Q_V Y, \ldots, Y'Q_V V_p Q_V Y)$ and λ is a solution of $\Sigma_\ell \lambda_\ell tr(Q_V V_\ell Q_V V_{\ell'}) = f_{\ell'}$, $\ell' = 1,2, \ldots, p$, that is $S\lambda = f$.

In this case $\lambda = S^{-1}f$ is a solution to $S\lambda = f$, and provided the inverse of S exists.

Also

$$\begin{aligned} \lambda' q &= (S^{-1}f)'q \\ &= f'S^{-1}q \\ &= f'\hat{\sigma} \end{aligned}$$

where $\hat{\sigma}$ is a solution to $S\hat{\sigma} = q$.

3.3 Positive Definite MINQUE Estimators

A criticism of the estimators defined in section 3.2 is that the estimates may be negative.

Rao and Kleffe (1979) derived a procedure to obtain non-negative definite (n.n.d.) estimators of the variance components.

They showed that if the following hold:

(i) $V_\ell \geq 0$ for $\ell = 1,2, \ldots, p$ ($V_\ell \geq 0$ means positive semidefinite)

(ii) $V = \Sigma_\ell V$

(iii) $V_{(\ell)} = V - V_\ell$

(iv) and $B_\ell = X\ 'V_\ell X$, where X is any matrix in the space orthogonal to $\delta(X')$, then there exists an n.n.d quadratic unbiased estimator of σ_ℓ^2 iff $\delta(B_\ell) \not\subset \delta(B_{(\ell)})$.

Rao and Kleffe (1979) also showed that $\delta(B_\ell) \not\subset \delta(B_{(\ell)})$ is equivalent to $(I-P_G)V_\ell(I-P_G) \neq 0$, where P_G is the projector operator onto the space generated by the columns of the compound matrix

$$G = [X:V_1: \ldots :V_{\ell-1}: \ldots :V_p]$$

3.3.1 Positive Semi-definite Estimator for the Split-Plot Error Variance

It is informative to apply the above result to the model for the split-plot experiment.

Clearly, $V_2 = I$, V_1 is a block diagonal matrix and $G = [X:V_1]$. Also $(I-P_G)V_2(I-P_G) = (I-P_G)(I-P_G)$

$= I-P_G \neq 0$ because P_G is not the identity matrix.

Consequently there exists a non-negative definite quadratic estimator for σ_2^2.

The non-negative quadratic unbiased estimator $\hat{\sigma}_2^2$ of σ_2^2 is given by

$$\hat{\sigma}_2^2 = Y'(I-P_G)Y/tr(I-P_G).$$

Unbiasedness follows from

$$\begin{aligned} E(\hat{\sigma}_2^2) &= E(Y'(I-P_G)Y/tr(I-P_G)) \\ &= (tr(I-P_G))^{-1}tr(E(Y'(I-P_G)Y)) \\ &= (tr(I-P_G))^{-1}tr((I-P_G)E(YY')) \\ &= (tr(I-P_G))^{-1}tr((I-P_G)V) \\ &= (tr(I-P_G))^{-1}[tr((I-P_G)V_1)\sigma_1^2 + tr((I-P_G)V_2)\sigma_2^2] \end{aligned}$$

$$= (tr(I-P_G))^{-1}(tr(I-P_G))\sigma_2^2$$
$$= \sigma_2^2 .$$

It is to be noted that this unbiased estimator for σ_2^2 agrees with the one obtained by Henderson's method III.

Observe that in the notation of S. R. Searle (1968) the model (3.1) can be rewritten as

$$Y = X_a\beta_a + X_b\beta_b + \varepsilon$$

where $X_a = X$, $X_b = U_1$, $\beta_2 = \beta$, $\beta_b = \varepsilon_1$ and $\varepsilon = U_2\varepsilon_2$.

Using the notation $X_{a,b} = [X_a:X_b]$, write

$$P_{X_{a,b}} = [X_a:X_b] \begin{bmatrix} X_a'X_a & X_a'X_b \\ X_b'X_a & X_b'X_b \end{bmatrix}^{-} [X_a:X_b]' .$$

Also let $R(\beta_a, \beta_b) = Y'P_{X_{a,b}}Y$.

It follows that $E[Y'Y - R(\beta_a, \beta_b)]$ involves only σ_2^2 and $Y'Y - R(\beta_a,\beta_b) = Y'(I-P_{X_{a,b}})Y$.

Now note that $\delta(X:V_1) = \delta(X:U_1)$, since $V_1 = U_1U_1'$. It follows that the estimator of σ_2^2 obtained from Henderson's method III agrees with the n.n.d estimator of Rao and Kleffe.

3.4 The Variance Component Estimators for the Split-Plot Model

3.4.1 Preliminary Definitions

In order to construct explicit forms for the estimators for the variance components of the split plot experiment, the following sequence of definitions is needed.

Definition 3.4.1

A submatrix or block matrix is a matrix that is obtained from the original matrix by deleting certain rows and columns.

Definition 3.4.2

A partitioned matrix is a matrix whose elements are block matrices.

Example:

$$A = \left[\begin{array}{cc:ccc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ \hdashline a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} .$$

It is convenient to denote a partition matrix

$$A = \text{PMATX}(A_{kk'}; \; k = 1,2, \ldots, a, \; k' = 1,2, \ldots, b)$$

or simply $A = \text{PMATX}\ (A_{kk'})$, if there is no confusion in the order of the indices.

Definition 3.4.3

The matrix A is a block diagonal matrix if A contains block matrices in the diagonal and zeros elsewhere, and is written as

$$A = \text{BMATX}\ (A_k; \; k = 1,2, \ldots, a)$$

or more simply $A = \text{BDMATX}\ (A_k)$.

Definition 3.4.4

The matrix A is a column matrix if A is written as

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_a \end{bmatrix}$$

where the $\{A_k\}$ are block matrices having the same number of columns. The notation

$$A = \text{CMATX } (A_k;\ k = 1,2,\ \ldots,\ a)$$

or $\quad A = \text{CMATX } (A_k)\quad$ will be used.

Definition 3.4.5

Define $\underline{a}$ as a column vector if $\underline{a}$ can be written as

$$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}$$

where the $\{a_i\}$ are real numbers. Also write

$$\underline{a} = \text{CVECTOR } (a_i;\ i = 1,2,\ \ldots,\ k)$$

or $\quad \underline{a} = \text{CVECTOR } (a_i).$

Definition 3.4.6

The matrix A is a row matrix if A is written as $A = [A_1 : A_2 : \ \ldots \ : A_2]$ where the $\{A_k\}$ are block matrices having the same number of rows. This will also be written as $A = \text{RMATX } (A_k;\ k = 1,2,\ \ldots,\ a)$ or $A = \text{RMATX } (A_k)$.

Definition 3.4.7

The symbol $\underline{a}'$ will denote a row vector if $\underline{a}' = [a_1, a_2, \ldots, a_k]$ where the $\{a_i\}$ are real numbers. Alternatively

$$\underline{a}' = \text{RVECTOR } (a_i;\ i = 1,2, \ldots, k)$$

or $$\underline{a}' = \text{RVECTOR } (a_i).$$

Using this notation one can write

$$\underline{y}_i' = [y_{i1}, y_{i2}, \ldots, y_{in_i}],$$
$$Y' = \text{RMATX } (\underline{y}_i'),$$
$$\underline{x}_{ij}' = [x_{ij1}\ x_{ij2}, \ldots, x_{ijm}],$$

and $$X_i = \text{CMATX } (\underline{x}_{ij}';\ j = 1,2, \ldots, n_i).$$

3.4.2 The MINQUE Equations

Recall from section 3.1 the error vector ε has the structure

$$\varepsilon = U_1\varepsilon_1 + U_2\varepsilon_2, \text{ and}$$
$$D(\varepsilon) = U_1U_1'\sigma_1^2 + U_2U_2'\sigma_2^2$$
$$= V_1\sigma_1^2 + V_2\sigma_2^2$$

where V_1 is a block diagonal matrix, each block an n_i x n_i matrix of ones.

It is convenient to let J_{n_i} or simply J_i represent an n_i x n_i matrix of ones for $i = 1,2, \ldots, a$. It follows that

$$V_1 = \text{BDMATX } (J_i)$$

Now assume that prior information indicates that α_1^2 and α_2^2 are good guesses for σ_1^2 and σ_2^2 respectively. Use this to write V as

$V_1\alpha_1^2 + V_2\alpha_2^2$, or equivalently

$$V = \text{BDMATX } (J_i\alpha_1^2 + I_i\alpha_2^2).$$

Lemma 3.4.1

If $A_i = J_i\alpha_1^2 + I_i\alpha_2^2$ where $\alpha_1^2 \geq 0$ and $\alpha_2^2 > 0$, A_i of order $n_i \times {}_n i$, then A_i^{-1} is given by $\alpha_2^{-2}I_i - \alpha_1^2\alpha_2^{-2}\ (n_i\alpha_1^2 + \alpha_2^2)^{-1}J_i$.

It follows that $V^{-1} = \text{BDMATX } (a_oI_i + a_iJ_i)$ where $a_0 = \alpha_2^{-2}$ and $a_i = -\alpha_1^2\alpha_2^{-2}\ (n_i\alpha_1^2 + \alpha_2^2)^{-1}$.

Observe that $V = \text{BDMATX } (J_i)$ can be written as $V_1^bV_1^b$ where $V_1^b = \text{BDMATX } (n_i^{-1/2}J_i)$.

In section 3.2 it was shown that the MINQUE of $\Sigma_\ell f_\ell\sigma_\ell^2$ is $g(Y) = \lambda'q$, where $q' = (Y'Q_VV_1Q_VY, \ldots, Y'Q_VQ_PQ_VY)$ and λ is a solution for $\Sigma_\ell\lambda_\ell \text{tr}(Q_VV_\ell Q_VV_{\ell'}) = f_{\ell'}$, $\ell' = 1, \ldots, p$.

Also, it was shown that $g(Y) = f'\hat{\sigma}$ where $\hat{\sigma}$ is a solution to $\Sigma_\ell\sigma_\ell^2\text{tr}(Q_VV_\ell Q_VV_{\ell'}) = Y'Q_VV_{\ell'}Q_VY$, $\ell' = 1, \ldots, p$.

Therefore, letting $\ell, \ell' = 1,2$, the estimators for the split-plot variance component will be completely determined by solving the following system of equations:

$$\begin{bmatrix} \text{tr}(Q_VV_1Q_VV_1) & \text{tr}(Q_VV_1Q_VV_2) \\ \text{tr}(Q_VV_2Q_VV_1) & \text{tr}(Q_VV_2Q_VV_2) \end{bmatrix} \begin{bmatrix} \hat{\sigma}_1^2 \\ \hat{\sigma}_2^2 \end{bmatrix} = \begin{bmatrix} Y'Q_VV_1Q_VY \\ Y'Q_VV_2Q_VY \end{bmatrix} \qquad (3.4.2.1)$$

In order to obtain an explicit equation for the estimators of the variance components, it is necessary to compute the terms involved in (3.4.2.1).

3.4.2.1 Computing $Y'Q_VV_1Q_VY$

Note first of all that $Y'Q_VV_1Q_VY = Y'Q_VV_1^bV_1^bQ_VY$.

Now obtain $V_1^bQ_VY$ as follows:

$$V_1^bQ_VY = V_1^b(V^{-1} - V^{-1}P_V)Y$$

$$= V_1^bV^{-1}(Y - X\hat{\beta})$$

where $P_V = X(X'V^{-1}X)^-X'V^{-1}$, $\hat{\beta} = CX'V^{-1}Y$ and $C = (X'V^{-1}X)^-$ and $(X'V^{-1}X)^-$ is the generalized inverse of $X'V^{-1}X$.

Observe that

$$V_1^bV^{-1} = \text{BDMATX } (n_i^{-1/2}J_i) \cdot \text{BDMATX } (a_oI_i + a_iJ_i)$$

$$= \text{BDMATX } (n_i^{-1/2}(a_o + a_in_i)J_i).$$

Also $X\hat{\beta} = \text{CVECTOR } (\sum_k x_{ijk}\hat{\beta}_k;\ j = 1, \ldots, n_i,\ i = 1, \ldots, a)$

and $\hat{\beta}_k = \sum_t c_{kt}(a_o\sum_{ij}\sum x_{ijt}y_{ij} + \sum_i a_ix_{i+t}y_{i+}),\ k = 1,2, \ldots, m$

where c_{kt} are the elements of C and $x_{i+t} = \sum_j x_{ijt}$ and $y_{i+} = \sum_j y_{ij}$.

Observe that

$$Y - X\hat{\beta} = \text{CVECTOR } (y_{ij} - \sum_j x_{ijk}\hat{\beta}_k)$$

and $V_1^bV^{-1}(Y - X\hat{\beta}) = \text{BDMATX } (n_i^{-1/2}(a_o + a_in_i)J_i) \cdot \text{CVECTOR } (y_{ij} - \sum_k x_{ijk}\hat{\beta}_k)$

$$= \text{CMATX } (n_i^{-1/2}(a_o + a_in_i)(y_{i+} - \sum_k x_{i+k}\hat{\beta}_k)\underline{1}_i).$$

It follows that

$$Y'Q_VV_1Q_VY = \sum_i n_i^{-1}(a_o + a_in_i)^2(y_{i+} - \sum_k x_{i+k}\hat{\beta}_k)^2\underline{1}_i'\underline{1}_i$$

$$= \sum_i (a_o + a_in_i)^2(y_{i+} - \sum_k x_{i+k}\hat{\beta}_k)^2. \qquad (3.4.2.1)$$

3.4.2.2 Computing $Y'Q_VQ_VY$

The first step is to compute

$$Q_VY = V^{-1}(I-P_V)Y$$

$$= V^{-1}(Y - X\hat{\beta})$$

$$= \text{BDMATX}(a_oI_i + a_iJ_i)\text{CVECTOR}(y_{ij} - \Sigma_k x_{ijk}\hat{\beta}_k)$$

$$= \text{CVECTOR}(a_oy_{ij} - a_o\Sigma_k x_{ijk}\hat{\beta}_k + a_iy_{i+} - a_i \sum_k x_{i+k}\hat{\beta}_k).$$

Consequently,

$$Y'Q_VQ_VY = \Sigma_i\Sigma_j(a_oy_{ij} - a_0\Sigma_k x_{ijk}\hat{\beta}_k + a_iy_{i+} - a_i\Sigma_k x_{i+k}\hat{\beta}_k)^2$$

$$= \Sigma_i\Sigma_j[(a_oy_{ij} - a_o\Sigma_k x_{ijk}\hat{\beta}_k)^2 + (a_iy_{it} - a_i\Sigma_k x_{i+k}\hat{\beta}_k)^2$$

$$+ 2\ a_oa_i(y_{ij} - \Sigma_k x_{ijk}\hat{\beta}_k)(y_{it} - \Sigma_k x_{i+k}\hat{\beta}_k)]$$

$$= \Sigma_i\Sigma_j a_o^2(y_{ij} - \Sigma_k x_{ijk}\hat{\beta}_k)^2 + \Sigma_i a_i^2 n_i(y_{it} - \Sigma_k x_{i+k}\hat{\beta}_k)^2$$

$$+ 2\ \Sigma_i a_oa_i(y_{i+} - \Sigma_k x_{i+k}\hat{\beta}_k)^2$$

$$= \Sigma_i\Sigma_j a_o^2(y_{ij} - \Sigma_k x_{ijk}\hat{\beta}_k)^2 + \Sigma_i a_i(2a_o + a_in_i)(y_{i+} - \sum_k x_{i+k}\hat{\beta}_k)^2. \qquad (3.4.2.2)$$

3.4.2.3 Computing $tr(Q_VV_1Q_VV_1)$, $tr(Q_VV_1Q_V)$ and $tr(Q_VQ_V)$

As a first step observe that $tr(V_1Q_VVQ_V) = tr(V_1Q_V)$. Also observe that $tr(Q_VV_1Q_VV) = tr(Q_VV_1Q_VV_1)\alpha^2 + tr(Q_VV_1Q_V)\alpha_2^2$. This implies $tr(Q_VV_1Q_VV_1) = \alpha_1^{-2}[tr(V_1Q_V) - \alpha_2^2 tr(Q_VV_1Q_V)]$.

Also observe that $tr(Q_VVQ_V) = tr(Q_V)$.

Then $$tr(Q_VQ_VV) = tr(Q_VQ_VV_1)\alpha_1^2 + tr(Q_VQ_V)\alpha_2^2$$

implies $$tr(Q_VV_1Q_VV_1) = \alpha_1^{-2}[tr(V_1Q_V) - \alpha_1^{-2}\alpha_2^2 tr(Q_V) + \alpha_1^{-2}\alpha_2^4 tr(Q_VQ_V)].$$

It follows that only $tr(Q_VV_1)$, $tr(Q_V)$ and $tr(Q_VQ_V)$ need to be evaluated.

To evaluate $tr(Q_VV_1)$, write $tr(Q_VV) = tr(Q_VV_1)\alpha_1^2 + tr(Q_V)\alpha_2^2$.

Now $$\begin{aligned} Q_VV &= (V^{-1} - V^{-1}XCX'V^{-1})V \\ &= (I - V^{-1}XCX') \end{aligned}$$

and $$\begin{aligned} tr(Q_VV) &= tr(I - V^{-1}XCX') \\ &= tr(I) - tr(V^{-1}XCX'). \end{aligned}$$

Note that $V^{-1}XCX'$ is idempotent. Using the property, that the trace of an idempotent matrix is equal to its rank, and $R(X) = R(XCX'V^{-1}X) \leq R(XCX'V^{-1}) \leq R(X) = q$ where $R(A)$ means the rank of A, it follows that

$$tr(Q_VV) = n - q. \qquad (3.4.2.4)$$

Now $$\begin{aligned} tr(Q_V) &= tr(V^{-1} - V^{-1}XCX'V^{-1}) \\ &= tr(V^{-1}) - tr(V^{-1}XCX'V^{-1}) \\ &= tr(V^{-1}) - tr(CX'V^{-1}V^{-1}X). \end{aligned}$$

Observe that $\mathrm{tr}(V^{-1}) = \Sigma_i \mathrm{tr}(a_o I_i + a_i J_i)$

$$= \Sigma_i (a_o + a_i) n_i, \tag{3.4.2.5}$$

$$V^{-1}V^{-1} = \mathrm{BDMATX}(a_o^2 I_i + (2a_o a_i + a_i^2 n_i) J_i), \tag{3.4.2.6}$$

$$X'V^{-1}V^{-1}X = [X_1' : X_2' : \ \dots \ : X_a'] V^{-1}V^{-1} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_a \end{bmatrix}$$

$$= \Sigma_i X_i' (a_o^2 I_i + (2a_o a_i + a_i^2 n_i) J_i) X_i,$$

$$CX'V^{-2}X = \Sigma_i CX_i' (a_o^2 I_i + (2a_o a_i + a_i^2 n_i) J_i) X_i$$

$$= a_o^2 \Sigma_i CX_i' X_i + \Sigma_i (2a_o a_i + a_i^2 n_i) CX_i' J_i X_i,$$

and $\mathrm{tr}(CX'V^{-1}V^{-1}X) = a_o^2 \Sigma_i \mathrm{tr}(CX_i' X_i) + \Sigma_i (2a_o a_i + a_i^2 n_i) \mathrm{tr}(CX_i' J_i X_i).$

(3.4.2.7)

Recall that $\mathrm{tr}(CX'V^{-1}X) = q$

$$= \mathrm{tr}(CX' \cdot \mathrm{BDMATX}(a_o I_i + a_i J_i) X)$$

$$= a_o \Sigma_i \mathrm{tr}(CX_i' X_i) + \Sigma_i a_i \mathrm{tr}(CX_i' J_i X_i)$$

which can be rewritten as

$$a_o \Sigma_i \mathrm{tr}(CX_i' X_i) = q - \Sigma_i a_i \mathrm{tr}(CX_i' J_i X_i). \tag{3.4.2.8}$$

Substituting (3.4.2.8) in (3.4.2.7) gives

$$\mathrm{tr}(CX'V^{-1}V^{-1}X) = a_o q + \Sigma_i (a_o a_i + a_i^2 n_i) \mathrm{tr}(CX_i' J_i X_i)$$

$$= a_o q + \Sigma_i a_i (a_o + a_i n_i) t_i \tag{3.4.2.9}$$

where $$t_i = tr(CX_i' J_i X_i). \tag{3.4.2.10}$$

From (3.4.2.5) and (3.4.2.9),

$$tr(Q_V) = \Sigma_i (a_o + a_i) n_i - a_o q - \Sigma_i a_i (a_o + a_i n_i) t_i . \tag{3.4.2.11}$$

From (3.4.2.4) and (3.4.2.11),

$$tr(Q_V V_1) = \alpha_1^{-2} [(n-q) - \alpha_2^2 (\Sigma_i (a_o + a_i) n_i - a_o q - \Sigma_i a_i (a_o + a_i n_i) t_i)]$$

$$= \alpha_1^{-2}(n-q) - \alpha_1^{-2} \alpha_2^2 (\Sigma_i (a_o + a_i) n_i - a_o q - \Sigma_i a_i (a_o + a_i n_i) t_i). \tag{3.4.2.12}$$

To compute $tr(Q_V Q_V)$, observe that

$$tr(Q_V Q_V) = tr((V^{-1} - V^{-1}XCX'V^{-1})(V^{-1} - V^{-1}XCX'V^{-1}))$$

$$= tr(V^{-1}V^{-1}) - 2tr(CX'V^{-1}V^{-1}V^{-1}X) + tr(CX'V^{-1}V^{-1}XCX'V^{-1}V^{-1}X). \tag{3.4.2.13}$$

From (3.4.2.6) it follows immediately that

$$tr(V^{-1}V^{-1}) = tr(BDMATX(a_o^2 I_i + (2a_o a_i + a_i^2 n_i) J_i))$$

$$= \Sigma_i (a_o^2 tr(I_i) + (2a_o a_i + a_i^2 n_i) tr(J_i))$$

$$= \Sigma_i (a_o^2 + 2a_o a_i + a_i^2 n_i) n_i . \tag{3.4.2.14}$$

Note also that

$$V^{-1}V^{-1}V^{-1} = BDMATX(a_o^3 I_i + (3a_o^2 a_i + 3a_o a_i^2 n_i + a_i^3 n_i^2) J_i).$$

It follows that

$$\mathrm{tr}(CX'V^{-3}X) = a_o^3\mathrm{tr}(CX_i'X_i) + \Sigma_i(3a_o^2a_i + 3a_oa_i^2n_i + a_i^3n_i^2)\mathrm{tr}(CX_i'J_iX_i)$$

$$= a_o^2(q - \Sigma_i a_i \mathrm{tr}(CX_i'J_iX_i)) + \Sigma_i(3a_o^2a_i + 3a_oa_i^2$$

$$+ a_i^3n_i^2)\mathrm{tr}(CX_i'J_iX_i)$$

$$= a_o^2q + \Sigma_i(2a_o^2a_i + 3a_oa_i^2n_i + a_i^3n_i^2)\mathrm{tr}(CX_i'J_iX_i)$$

$$= a_o^2q + \Sigma_i(2a_o^2a_i + 3a_oa_i^2n_i + a_i^3n_i^2)t_i . \qquad (3.4.2.15)$$

Recalling $CX'V^{-2}X = a_o\Sigma_iCX_i'X_i + \Sigma_i(2a_oa_i + a_i^2n_i)CX_i'J_iX_i$ leads to

$$\mathrm{tr}(CX'V^{-2}XCX'V^{-1}X) = \mathrm{tr}((a_o\Sigma_iCX_i'X_i + \Sigma_i(2a_oa_i + a_i^2n_i)CX_i'J_iX_i)^2)$$

$$= a_o^2\Sigma_i\Sigma_{i'}t_{1ii'}$$

$$+ 2a_o\Sigma_i\Sigma_{i'}(2a_oa_{i'} + a_{i'}^2n_{i'})t_{2ii'}$$

$$+ \Sigma_i\Sigma_{i'}(2a_oa_i + a_i^2n_i)(2a_oa_{i'} + a_{i'}^2n_{i'})t_{3ii'}$$

(3.4.2.16)

where $t_{1ii'} = \mathrm{tr}(CX_i'X_iCX_{i'}'X_{i'})$, (3.4.2.17)

$$t_{2ii'} = \mathrm{tr}(CX_i'X_iCX_{i'}'J_{i'}X_{i'}) \qquad (3.4.2.18)$$

and

$$t_{3ii'} = \mathrm{tr}(CX_i'J_iX_iCX_{i'}'J_{i'}X_{i'}). \qquad (3.4.2.19)$$

Expressions (3.4.2.14), (3.4.2.15), and (3.4.2.16) lead to

$$\mathrm{tr}(Q_VQ_V) = \Sigma_i(a_o^2 + 2a_oa_i + a_i^2n_i)n_i - 2(a_o^2q + \Sigma_i(2a_o^2a_i$$

$$+ 3a_oa_i^2n_i + a_i^3n_i^2)t_i)$$

$$+ a_o^2\Sigma_i\Sigma_{i'}t_{1ii'} + 2a_o\Sigma_i\Sigma_{i'}(2a_oa_{i'} + a_{i'}^2n_{i'})t_{2ii'}$$

$$+ \Sigma_i\Sigma_{i'}(2a_oa_i + a_i^2n_i)(2a_oa_{i'} + a_{i'}^2n_{i'})t_{3ii'}. \quad (3.4.2.20)$$

Combining expressions (3.4.2.3), (3.4.2.4), (3.4.2.11), (3.4.2.12), and (3.4.2.20) yields:

$$\begin{aligned}
tr(Q_VV_1Q_VV_1) &= \alpha_1^{-2}[tr(Q_VV_1) - \alpha_1^{-2}\alpha_2^2tr(Q_V) + \alpha_1^{-2}\alpha_2^4tr(Q_VQ_V)] \\
&= \alpha_1^{-2}[\alpha_1^{-2}(tr(Q_VV) - \alpha_2^2tr(Q_V)) - \alpha_1^{-2}\alpha_2^2tr(Q_V) \\
&\quad + \alpha_1^{-2}\alpha_2^4tr(Q_VQ_V)] \\
&= \alpha_1^{-4}tr(Q_VV) - 2\alpha_1^{-4}\alpha_2^2tr(Q_V) + \alpha_1^{-4}\alpha_2^4tr(Q_VQ_V) \\
&= \alpha_1^{-4}(n-q) - 2\alpha_1^{-4}\alpha_2^2(\sum_i(a_o + a_i)n_i - a_oq \\
&\quad - \sum_i a_i(a_o + a_in_i)t_i) \\
&\quad + \alpha_1^{-4}\alpha_2^4(\sum_i(a_o^2 + 2a_oa_i + a_i^2n_i)n_i \\
&\quad - 2(a_o^2q + \sum_i(2a_o^2a_i + 3a_oa_i^2n_i + a_i^3n_i^2)t_i) \\
&\quad + \sum_{ii'}\sum(a_o^2t_{1ii'} + 2a_o(2a_oa_{i'} + a_{i'}^2n_{i'})t_{2ii'} \\
&\quad + (2a_oa_i + a_i^2n_i)(2a_oa_{i'} + a_{i'}^2n_{i'})t_{3ii'})).
\end{aligned}$$

(3.4.2.21)

Similarly, $tr(Q_VV_1Q_V)$ can be evaluated as

$$\begin{aligned}
tr(Q_VV_1Q_V) &= \alpha_1^{-2}[tr(Q_V) - \alpha_2^2tr(Q_VQ_V)] \\
&= \alpha_1^{-2}tr(Q_V) - \alpha_1^{-2}\alpha_2^2tr(Q_VQ_V)
\end{aligned}$$

$$= \alpha_1^{-2}(\sum_i (a_o^2 + a_i)n_i - a_o q - \sum_i a_i(a_o + a_i n_i)t_i)$$

$$- \alpha_1^{-2}\alpha_2^2(\sum_i (a_o^2 + 2a_o a_i + a_i^2 n_i)n_i$$

$$- 2(a_o^2 q + \sum_i (2a_o^2 a_i + 3a_o a_i^2 n_i + a_i^3 n_i^2)t_i)$$

$$+ \sum\sum_{ii'} (a_o^2 t_{1ii'} + 2a_o(2a_o a_{i'} + a_{i'}^2 n_{i'})t_{2ii'}$$

$$+ (2a_o a_i + a_i^2 n_i)(2aa_{i'} + a_{i'}^2 n_{i'})t_{3ii'})). \qquad (3.4.2.22)$$

3.5 Seely's Method to Obtain the Estimators of the Variance Components under the Invariance Condition

An alternative approach to variance components estimation was developed by Seely (1970a, 1970b). This technique will now be used to obtain estimators for the variance components in the split plot subject to the invariance condition.

First, recall the model (3.1) $Y = X\beta + U_1^*\varepsilon_1 + U_2^*\varepsilon_2$, in which it was assumed that $E[\varepsilon_i] = 0$, $E[\varepsilon_i\varepsilon_i'] = I_{m_i}\sigma_i^2$ $E[\varepsilon_i\varepsilon_j'] = 0$ $i \neq j$, and $D(Y) = V_1\sigma_1^2 + V_2\sigma_2^2$, where $V_1 = U_1^*U_1^{*'}$ and $V_2 = I = U_2^*U_2^{*'}$.

Second, let α_1^2 and α_2^2 represent prior values or guesses for σ_1^2 and σ_2^2. Use these to write $V = V_1\alpha_1^2 + V_2\alpha_2^2$, where V is positive definite. Let $V^{1/2}$ be the square root matrix of V, that is $V = V^{1/2}V^{1/2}$.

Third, obtain the generalized least squares estimator of the parameters β under the transformed model

$$V^{-1/2}Y = V^{-1/2}X\beta + V^{-1/2}U_1^*\varepsilon_1 + V^{-1/2}U_2^*\varepsilon_2.$$

This is $\tilde{\beta} = (X'V^{-1}X)^{-}X'V'Y$.

Fourth, obtain the residuals

$$Z = V^{-1/2}Y - V^{-1/2}X\tilde{\beta}$$

$$= V^{-1/2}(I - X(X'V^{-1}X)^{-}X'V^{-1})Y$$

$$= V^{-1/2}(I - X(X'V^{-1}X)^{-}X'V^{-1})(X\beta + U_1^*{}_1 + U_2^*{}_2)$$

$$= U_1\varepsilon_1 + U_2\varepsilon_2$$

where $U_1 = V^{-1/2}(I - X(X'V^{-1}X)^{-}X'V^{-1})U_1^*$

$U_2 = V^{-1/2}(I - X(X'V^{-1}X)^{-}X'V^{-1})U_2^*$

Fifth, let $Z \times Z = \begin{bmatrix} Z_1Z_1 \\ Z_1Z_2 \\ \vdots \\ Z_1Z_r \\ \vdots \\ Z_rZ_r \end{bmatrix}$ where x denotes the kronecker product of matrices

and let $U_i = \begin{bmatrix} \underline{u}'_{1i} \\ \underline{u}'_{2i} \\ \vdots \\ \underline{u}'_{ri} \end{bmatrix}$, i = 1,2, where $\underline{u}'_{ji} = [\underline{u}_{j1i}, \ldots, \underline{u}_{jm_i i}]'$.

Observe that

$$E[Z_jZ_{j'}] = E[(\underline{u}'_{j1}\underline{\varepsilon}_1 + u'_{j2}\underline{\varepsilon}_2)(\underline{u}'_{j'1}\underline{\varepsilon}_1 + \underline{u}'_{j'2}\underline{\varepsilon}_2)]$$

$$= E[\underline{u}'_{j1}\underline{\varepsilon}_1)(\underline{u}'_{j'1}\underline{\varepsilon}_1) + 2(\underline{u}'_{j1}\underline{\varepsilon}_1)(\underline{u}'_{j'2}\underline{\varepsilon}_2) + (\underline{u}'_{j2}\underline{\varepsilon}_2)(\underline{u}'_{j'2}\underline{\varepsilon}_2)]$$

$$= E[\sum_{\ell\ell'}\sum u_{j\ell 1}j_{j'\ell'1}\varepsilon_{\ell 1}\varepsilon_{\ell'1} + 2\sum_{\ell\ell'}\sum u_{j\ell 1}u_{j'\ell'2}\varepsilon_{\ell 1}\varepsilon_{\ell'2}$$

$$+ \sum_{\ell\ell'}\sum u_{j\ell 2}u_{j'\ell'2}\varepsilon_{\ell 2}\varepsilon_{\ell'2}]$$

$$= \sum_{\ell} u_{j\ell 1}u_{j'\ell 1}\sigma_1^2 + \sum_{\ell} u_{j\ell 2}u_{j'\ell' 2}\sigma_2^2$$

$$= \underline{u}_{j'1}'\underline{u}_{j1}\sigma_1^2 + \underline{u}_{j'2}'\underline{u}_{j2}\sigma_2^2$$

$$= [\underline{u}_{j'1}'\underline{u}_{j1},\ \underline{u}_{j'2}'\underline{u}_{j2}] \begin{bmatrix} \sigma_1^2 \\ \hat{\sigma}_1^2 \end{bmatrix}$$

The expectation of the vector Z x Z is

$$E[Z \times Z] = H\underline{\Theta}$$

where $\underline{\Theta} = (\sigma_1^2, \sigma_2^2)'$

and

$$H = \begin{bmatrix} \underline{u}_{11}'\underline{u}_{11} & \underline{u}_{12}'\underline{u}_{12}' \\ \underline{u}_{21}'\underline{u}_{11} & \underline{u}_{22}'\underline{u}_{12} \\ \vdots & \\ \underline{u}_{r1}'\underline{u}_{11} & \underline{u}_{r2}'\underline{u}_{12} \\ \vdots & \\ \underline{u}_{11}'\underline{u}_{r1} & \underline{u}_{12}'\underline{u}_{r2} \\ \underline{u}_{21}'\underline{u}_{r1} & \underline{u}_{22}'\underline{u}_{r2} \\ \vdots & \\ \underline{u}_{r1}'\underline{u}_{r1} & \underline{u}_{r2}'\underline{u}_{r2} \end{bmatrix} \qquad (3.5.1)$$

Observe that

$$\begin{bmatrix} \underline{u}_{11}'\underline{u}_{11} \\ \underline{u}_{21}'\underline{u}_{11} \\ \vdots \\ \underline{u}_{r1}'\underline{u}_{11} \end{bmatrix} = U_1\underline{u}_{11}$$

and that

$$\begin{bmatrix} U_1\underline{u}_{11} \\ U_1\underline{u}_{21} \\ \vdots \\ U_1\underline{u}_{r1} \end{bmatrix} = (I \times U_1) \begin{bmatrix} \underline{u}_{11} \\ \underline{u}_{21} \\ \vdots \\ \underline{u}_{r1} \end{bmatrix} \qquad (3.5.2)$$

Definition 3.5.1

Given a matrix A of order m x n, the vector of A, say vec(A), is the mn-vector that is obtained by writing vertically all the elements of A starting with the first element of A and proceeding in the lexicographical order.

Under the latter definition the right hand side of (3.5.2) becomes

$$(I \otimes U_1) \text{ vec } (U_1). \qquad (3.5.3)$$

From (3.5.3) and (3.5.2)

$$H = [(I \otimes U_1) \text{ vec } (U_1), (I \otimes U_2) \text{ vec } (U_2)].$$

Finally, the Seely's estimators of the variance components is obtained through the model

$$Z \otimes Z = H\Theta + \text{error terms}$$

by using least squares solutions, say

$$H'H\Theta = H'Z \otimes Z. \qquad (3.5.4)$$

The following two lemmas can be found in Rao and Kleffe (1979).

Lemma 3.5.1

Let A, B and C be m x n, n x k and k x s matrices. Then

i) $tr(AB) = \text{vec}(A)' \text{ vec}(B')$,

ii) $\text{vec}(AB) = (A \otimes I_k)\,\text{vec}(B)$,

iii) $\text{vec}(AB) = (I_m \otimes B')\,\text{vec}(A)$,

iv) $\text{vec}(ABC) = (A \otimes C')\,\text{vec}(B)$.

Lemma 3.5.2

Let A, B, C and D be m x n, k x s, n x q and s x t matrices. Then

$$(A \otimes B)(C \otimes D) = AC \otimes BD.$$

Using the latter lemmas the elements of the matrix H'H say H'H(i,j) are:

$$H'H(1,1) = [(I \otimes U_1)\,\text{vec}(U_1)]'[I \otimes U_1)\,\text{vec}(U_1)]$$

$$= [\text{vec}(U_1U_1')]'[\text{vec}(U_1U_1')]$$

$$= \text{tr}(U_1U_1'U_1U_1'),$$

$$H'H(1,2) = \text{tr}(U_1U_1'\,U_2U_2')$$

$$= H'H(2,1),$$

$$H'H(2,2) = \text{tr}(U_2U_2'\,U_2U_2')$$

and the elements of H'Z x Z, say H'Z x Z(i) are

$$H'Z \times Z(1) = [(I \otimes U_1)\,\text{vec}(U_1)]'\,Z \times Z$$

$$= \text{vec}(U_1)'(I \otimes U_1')\,Z \times Z$$

$$= \text{vec}(U_1)'(Z \otimes U_1'Z)$$

$$= [Z' \otimes Z'U_1) \text{ vec } (U_1)]'$$

$$= \text{vec } (Z'U_1U_1'Z)$$

$$= Z'U_1U_1'Z \;.$$

$$H'Z \otimes Z(2) = Z'U_2U_2'Z.$$

Recall that $U_i = V^{-1/2}(I - X(X'V^{-1}X)^{-}X'V^{-1})U_i^*$, then replacing the value of U_1 and U_2,

$$tr(U_1U_1'U_1U_1') = tr(V^{-1/2}(I-X(X'V^{-1}X)^{-}X'V^{-1})U_1^*U_1'(I-V^{-1}X(X'V^{-1}X)^{-}X')V^{-1/2}$$

$$\cdot\; V^{-1/2}(I-X(X'V^{-1}X)^{-}X'V^{-1})U_1^*U_1^*(I-V^{-1}X(X'V^{-1}X)^{-}X')V^{-1/2})$$

$$= tr(Q_VV_1Q_VV_1),$$

$$tr(U_1U_1'U_2U_2') = tr(Q_VV_1Q_VV_2),$$

$$tr(U_2U_2'U_2U_2') = tr(Q_VV_2Q_VV_2),$$

$$Z'U_1U_1'Z = Z'V^{-1/2}(I-X(X'V^{-1}X)^{-}X'V^{-1})U_1^*U_1^*(I-V^{-1}X(X'V^{-1}X)^{-}X')V^{-1/2}Z$$

$$= Y'Q_VV_1Q_VY$$

and

$$Z'U_2U_2'Z = Y'Q_VV_2Q_VY.$$

Therefore (3.5.4) becomes

$$\begin{bmatrix} tr(Q_V V_1 Q_V V_1) & tr(Q_V V_1 Q_V V_2) \\ tr(Q_V V_1 Q_V V_2) & tr(Q_V V_2 Q_V V_2) \end{bmatrix} \begin{bmatrix} \sigma_1^2 \\ \sigma_2^2 \end{bmatrix} = \begin{bmatrix} Y' Q_V V_1 Q_V Y \\ Y' Q_V V_2 Q_V Y \end{bmatrix}.$$

Observe that this is the MINQUE estimator of the variance components. Therefore, under the invariant condition, Seely's estimators of the variance components are also MINQUE estimators.

4. ASYMPTOTIC PROPERTIES

This chapter consists of two major parts. Section 4.1 deals with the asymptotic properties of the MINQUE of the variance components as the number of observations increases. Section 4.2 proceeds to the asymptotic properties of the generalized least squares estimate $\tilde{\beta}$ obtained by replacing the elements of the unknown variance-covariance matrix by functions of the MINQUE of the variance components.

4.1 Properties of the MINQUE Estimators of the Variance Components

Consider the model (3.1), $Y = X\beta + \varepsilon$ where Y ix $t \times 1$, and let t increase to infinity in such a way that Y permits a partition into n r-vectors, each one representing a replica for the unbalanced split-plot model in the study.

Rewrite model (3.1) as

$$Y_t = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} X\beta_1 \\ X\beta_2 \\ \vdots \\ X\beta_n \end{bmatrix} + \begin{bmatrix} \varepsilon_1^* \\ \varepsilon_2^* \\ \vdots \\ \varepsilon_n^* \end{bmatrix}$$

where ε_i^* has structure $\varepsilon_i^* = U_1^*\varepsilon_{i1} + U_2^*\varepsilon_{i2}$

$$= [U_1^* : U_2^*] \begin{bmatrix} \varepsilon_{i1} \\ \varepsilon_{i2} \end{bmatrix}$$

$$= U^*\varepsilon_i .$$

which can also be written as

$$Y_t = (I_n \otimes X)\beta_t + (I_n \otimes U^*)\varepsilon_t$$

where I_n is the n x n identity matrix, $\beta_t' = (\beta_1', \ldots, \beta_n')$ and $\varepsilon_t' = (\varepsilon_1', \ldots, \varepsilon_n')$.

It is assumed that $Y_1, Y_2, \ldots, Y_n$ are iid random variables such that $E(Y_i) = X\beta_i$ and $D(Y_i) = V$, where $V = V_1\sigma_1^2 + V_2\sigma_2^2$, is positive definite. Or equivalently $E(\varepsilon_i^*) = 0$ and $D(\varepsilon_i^*) = V$.

Observe that $E[I_n \otimes U^*)\varepsilon_t] = 0$, and

$$\begin{aligned} D[I_n \otimes U^*)\varepsilon_t] &= (I_n \otimes U^*)D[\varepsilon_t](I_n \otimes U^{*\prime}) \\ &= (I_n \otimes u^*)(I_n \otimes D[\varepsilon_i])(I_n \otimes U^{*\prime}) \\ &= (I_n \otimes U^*D[\varepsilon_i]U^{*\prime}) \\ &= (I_n \otimes V) . \end{aligned}$$

The goal is to estimate a linear parametric function $\Sigma_i q_i \sigma_i^2$ of the variance components by the quadratic function $Y_t'A_tY_t$ subject to the invariance, unbiasedness and minimum norm conditions of the MINQUE procedure.

Let α_1^2 and α_2^2 be prior values for σ_1^2 and σ_2^2, and use this to write $V = V_1\alpha_1^2 + V_2\alpha_2^2$, where V is positive definite. Also let $V^{1/2}$ be the square root matrix of V, such $V = V^{1/2}V^{1/2}$ and $V^{1/2} = V^{1/2\prime}$.

For each $1 \leq i \leq n$ let $\tilde{\beta}_i$ be the generalized least square estimator of the parameters β_i, under the transformed model $V^{-1/2}Y_i$, and let

$$Z_i = V^{-1/2}(Y_i - X\tilde{\beta}_i).$$

Let

$$Z_t = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{bmatrix} = \begin{bmatrix} V^{-1/2}(Y_1 - X\tilde{\beta}_1) \\ V^{-1/2}(Y_2 - X\tilde{\beta}_2) \\ \vdots \\ V^{-1/2}(Y_n - X\tilde{\beta}_n) \end{bmatrix}$$

$$= \begin{bmatrix} V^{-1/2}(I - P_V)Y_1 \\ V^{-1/2}(I - P_V)Y_2 \\ \vdots \\ V^{-1/2}(I - P_V)Y_n \end{bmatrix} \quad \text{where } P_V = X(X'V^{-1}X)^{-}X'V^{-1}$$

$$= (I_n \otimes V^{-1/2}(I - P_V)) \begin{bmatrix} X\beta_1 + U_1^*\varepsilon_{11} + U_2^*\varepsilon_{12} \\ X\beta_2 + U_1^*\varepsilon_{21} + U_2^*\varepsilon_{22} \\ \vdots \\ X\beta_n + U_1^*\varepsilon_{n1} + U_2^*\varepsilon_{n2} \end{bmatrix}$$

$$= (I_n \otimes V^{-1/2}(I - P_V))(I_n \otimes U_1^*) \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{21} \\ \vdots \\ \varepsilon_{n1} \end{bmatrix}$$

$$+ (I_n \otimes V^{-1/2}(I - P_V)(I_n \otimes U_2^*) \begin{bmatrix} \varepsilon_{12} \\ \varepsilon_{22} \\ \vdots \\ \varepsilon_{n2} \end{bmatrix}$$

$$= U_1\varepsilon_{t1} + U_2\varepsilon_{t2}$$

where

$$U_1 = (I_n \otimes V^{-1/2}(I - P_V))(I_n \otimes U_1^*)$$

$$U_2 = (I_n \otimes V^{-1/2}(I - P_V))(I_n \otimes U_2^*)$$

and $\varepsilon_{t1} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{21} \\ \vdots \\ \varepsilon_{n1} \end{bmatrix}$ and $\varepsilon_{t2} = \begin{bmatrix} \varepsilon_{12} \\ \varepsilon_{22} \\ \vdots \\ \varepsilon_{n2} \end{bmatrix}$

Then $Z_t = U_1 \varepsilon_{t1} + U_2 \varepsilon_{t2}$.

Now apply Seely's method of section 3.5. The elements of the matrix H'H are

$$H'H(1,1) = tr(U_1 U_1' U_1 U_1')$$

$$= tr[((I_n \otimes V^{-1/2}(I - P_V))(I_n \otimes U_1^*))$$
$$((I_n \otimes V^{-1/2}(I - P_V))(I_n \otimes U_1^*))'$$
$$((I_n \otimes V^{-1/2}(I - P_V))(I_n \otimes U_1^*))$$
$$((I_n \otimes V^{-1/2}(I - P_V))(I_n \otimes U_1^*))']$$

$$= tr[(I_n \otimes V^{-1}(I - P_V)U_1^* U_1^{*\prime} V^{-1}(I - P_V)U_1^* U_1^{*\prime}]$$

$$= tr(I_n \otimes Q_V V_1 Q_V V_1)$$

$$= ntr(Q_V V_1 Q_V V_1)$$

$$H'H(1,2) = ntr(Q_V V_1 Q_V V_2)$$

$$= H'H(2,1)$$

$$H'H(2,2) = ntr(Q_V V_2 Q_V V_2) .$$

Also the elements of the vector $H'Z_t \otimes Z_t$ are

$$H'Z_t \otimes Z_t(1) = Z_t'U_1U_1'Z_t$$

$$= Z_t'((I_n \otimes V^{-1/2}(I - P_V))(I_n \otimes U_1^*))$$
$$((I_n \otimes V^{-1/2}(I - P_V))(I_n \otimes U_1^*))'Z_t$$

$$= Y_t'(I_n \otimes (I - P_V')V^{-1/2})(I_n \otimes V^{-1/2}(I - P_V))(I_n \otimes U_1^*)$$
$$((I_n \otimes V^{-1/2})(I_n \otimes U_1^*))'(I_n \otimes V^{-1/2}(I - P_V))Y_t$$

$$= Y_t'(I_n \otimes Q_V V_1 Q_V)Y_t$$

$$= \sum_i \underline{Y}_i' Q_V V_1 Q_V Y_i$$

and

$$H'Z_t \otimes Z_t(2) = \sum_i \underline{Y}_i' Q_V V_2 Q_V \underline{Y}_i.$$

Therefore, the Seely's estimators of the variance components are solutions to the system of equations

$$H'H\sigma = H'Z_t \otimes Z_t \qquad (4.1.1)$$

where

$$H'H = \begin{bmatrix} n\,tr(Q_V V_1 Q_V V_1) & n\,tr(Q_V V_1 Q_V V_2) \\ n\,tr(Q_V V_1 Q_V V_2) & n\,tr(Q_V V_2 Q_V V_2) \end{bmatrix}$$

$$\sigma = \begin{bmatrix} \sigma_1^2 \\ \sigma_2^2 \end{bmatrix}$$

and

$$H'Z_t \otimes Z_t = \begin{bmatrix} \Sigma_i \underline{Y}_i' Q_V V_1 Q_V \underline{Y}_i \\ \\ \Sigma_i \underline{Y}_i' Q_V V_2 Q_V \underline{Y}_i \end{bmatrix}$$

Observe that (4.1.1) are the MINQUE equations for the variance components, and that the matrix A_t that minimizes the quadratic form $g(Y_t) = Y_t'A_tY_t$ with respect to the parametric function $\Sigma_j q_j {}_j^2$ is

$$A_t = \Sigma_j \lambda_j Q_{(I_n \otimes V)} (I_n \otimes V_j) Q_{(I_n \otimes V)} \qquad (4.1.2)$$

where λ_j is a solution to the equations

$$\sum_j \lambda_j \mathrm{tr}(Q_{(I_n \otimes V)} (I_n \otimes V_j) Q_{(I_n \otimes V)} (I_n \otimes V_{j'})) = q_{j'}, \quad j' = 1,2 \qquad (4.1.3)$$

and

$$Q_{(I_n \otimes V)} = I_n \otimes Q_V, \quad Q_V = V^{-1}(I - P_V), \quad P_V = X(X'V^{-1}X)^- X'V^{-1}$$

and

$$q_{j'} = \mathrm{tr}(A_t(I_n \otimes V_{j'})), \quad j' = 1,2.$$

Using a generalized version of section 3.2.3, it is observed that a necessary and sufficient condition for $Y_t'A_tY_t$ to be invariant with respect to the translation of β_t is that $(I_n \otimes X')A_t = 0$.

The following lemma shows that the quadratic form $g(Y_t) = Y_t'A_tY_t$ is unbiased with respect to the parametric function $\Sigma_j q_j \sigma_j^2$.

Lemma 4.1.1

The quadratic form $g(Y_t) = Y_t'A_tY_t$ is unbiased with respect to the parametric function $\Sigma_j q_j \sigma_j^2$ iff $q_j = \mathrm{tr}(A_t(I_n \otimes V_j))$.

Proof:

$$E[Y_t'A_tY_t] = tr(E[Y_t'A_tY_t])$$

$$= tr(A_tE[Y_tY_t'])$$

$$= tr(A_t(I_n \otimes V))$$

$$= tr(A_t(I_n \otimes \Sigma_j V_j \sigma_j^2))$$

$$= \sum_j tr(A_t(I_n \otimes V_j))\sigma_j^2$$

$$= \sum_j q_j \sigma_j^2 .$$

From (4.1.3) it is observed that

$$H'H\lambda = q$$

and a solution is given by

$$\hat{\lambda} = (H'H)^{-1}q .$$

From (4.1.2)

$$g(Y_t) = Y_t'A_tY_t$$

$$= \lambda'(H'Z_t \otimes Z_t)$$

$$= q'(H'H)^{-1}(H'Z_t \otimes Z_t) . \qquad (4.1.4)$$

Let Y_1, Y_2, ..., Y_n be independent and identical distributed normal random variables with mean $X\beta_i$ and variance-covariance matrix V.

It is known that if B and D are symmetric matrices, then under the latter condition

$$\text{cov}(Y_i'BY_i,\ Y_i'DY_i) = 2tr(BVDV) + 4\beta_i'XBVDX\beta_i$$
$$= 2tr(BVDV) \text{ under invariance.}$$

Observe that

$$\begin{aligned}\text{var}(Y_t'(I_n \otimes Q_V V_1 Q_V)Y_t,\ Y_t(I_n \otimes Q_V V_1 Q_V)Y_t) &= 2tr(I_n \otimes Q_V V_1 Q_V V Q_V V_1 Q_V V) \\ &= 2ntr(Q_V V_1 Q_V V Q_V V_1 Q_V V) \\ &= 2ntr(Q_V V_1 Q_V V_1 Q_V V) \\ &= 2ntr(Q_V V_1 Q_V V_1)\end{aligned}$$

$$\text{cov}(Y_t'(I_n \otimes Q_V V_1 Q_V)Y_t,\ Y_t'(I_n \otimes Q_V V_2 Q_V)Y_t) = 2ntr(Q_V V_1 Q_V V_2)$$

$$\text{var}(Y_t'(I_n \otimes Q_V V_2 Q_V)Y_t,\ Y_t'(I_n \otimes Q_V V_2 Q_V)Y_t) = 2ntr(Q_V V_2 Q_V V_2)$$

Then
$$\begin{aligned}\text{var}(g(Y_t)) &= q'(H'H)^{-1}D(H'Z_t \otimes Z_t)(H'H)^{-1}q \\ &= q'(H'H)^{-1}D\left[\begin{pmatrix} Y_t'(I_n \otimes Q_V V_1 Q_V)Y_t \\ Y_t'(I_n \otimes Q_V V_2 Q_V)Y_t \end{pmatrix}\right](H'H)^{-1}q \\ &= q'(H'H)^{-1}\ 2\ (H'H)\ (H'H)^{-1}q \\ &= 2q'(H'H)^{-1}q\ .\end{aligned}$$

Using Chebyshev's inequality

$$P\{|g(Y_t) - \Sigma_i q_i \sigma_i^2| > \varepsilon\} \leq \frac{D(g(Y_t))}{\varepsilon^2}\ .$$

Taking the limit when $n \to \infty$

$$\lim_{n\to\infty} P\{|g(Y_t) - \Sigma_i q_i \sigma_i^2| > \varepsilon\} = 0,$$

establishing that the MINQUE estimators are consistent.

4.2 Behavior of the Vector of Parameters $\tilde{\beta}$ When the Variance Components Are Replaced By Those Obtained By the MINQUE Method

It is not possible to guarantee that the MINQUE of the variance components will be positive or even non-negative. It follows that it is possible that the estimated matrix $\hat{V}$ may not be positive definite. Consequently, care must be exercised in using $\hat{V}$ to compute generalized least squares estimates of β. Three possible strategies come to mind.

4.2.1 Behavior of $\tilde{\beta}$ When $\hat{V}$ Is Computed with The MINQUE Estimators of σ_1^2 and σ_2^2

The justification of the procedure is rather weak, because the estimators $\hat{\sigma}_1^2$ could have negative values and yield undesirable results. It is also possible that $\hat{V}^{-1}$ does not exist and the method fails. But since it was shown that $\hat{\sigma}_i^2$ converge in probability to σ_i^2 where the latter always takes positive values, then the probability of obtaining positive estimates increases for large experiments and correspondingly the probability of obtaining a positive definite $\hat{V}$ increases. A small simulation study reported in Chapter 7 shows that this is indeed true and $\tilde{\beta}$ appears to be asymptotically normal.

4.2.2 Behavior of $\tilde{\beta}$ when $\hat{V}$ Is Computed with The MINQUE Estimators of σ_1^2 and σ_2^2, But The Estimators Are Restricted to Be Positive

The procedure is defined as follows:

$$\text{Let} \quad \hat{\hat{\sigma}}_1^2 = \begin{cases} 0 & \text{if } \hat{\sigma}_1^2 \leq 0 \\ \\ \sigma_1^2 & \text{otherwise} \end{cases}$$

$$\hat{\hat{\sigma}}_2^2 = \begin{cases} 0.01 & \text{if } \hat{\sigma}_2^2 \le 0 \\ \hat{\sigma}_2^2 & \text{otherwise.} \end{cases}$$

The justification of the procedure is based in the fact that since $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ converge in probability to σ_1^2 and σ_2^2 where the latter always are positive, so $\hat{\hat{\sigma}}_1^2$ and $\hat{\hat{\sigma}}_2^2$ also converge in probability to σ_1^2 and σ_2^2.

Moreover, the procedure guarantees that $\hat{V}$ is positive definite for all n.

Note that the vectors $Y_1, Y_2, \ldots, Y_n$ have bounded fourth moments. This assumption guarantees that $Y_n'A_nY_n$ is $0_p(n^{-\delta})$, $\delta > 0$.

Assumption 1

The elements of $I \otimes V$ are functions of a 2-dimensional vector of parameters σ such that the elements of the matrix $G_{ni}(\sigma) = \frac{\partial}{\partial \sigma_i^2} I \otimes V^{-1}$, $i = 1,2$, are continuous functions of an open sphere δ of $\sigma°$, where $\sigma°$ is the true value of the parameter vector σ.

Assumption 2

The sequence of matrices $\{I_n \otimes X\}$ and $\{I_n \otimes V\}$ are such that $\lim_{n\to\infty} n^{-1}(I_n \otimes X')(I_n \otimes V)^{-1}(I_n \otimes X) = M(\sigma)$ where $M(\sigma)$ is a p x p matrix of fixed constants such that $M^{-1}(\sigma)$ exist for all $\sigma \varepsilon \delta$ and $\lim_{n\to\infty} n^{-1}(I_n \otimes x)'G_{ni}(I_n \otimes X) = H_r(\sigma)$ where $H_r(\sigma)$ is a matrix whose elements are continuous functions of σ_i^2, $i = 1,2$.

Assumption 3

An estimator $I_n \otimes \hat{V} = V_n(\hat{\sigma})$ for $V_n = V_n(\sigma)$ is available, such that $V_n^{-1}(\hat{\sigma})$ exist for all n and $\hat{\sigma}$ satisfies the condition $\hat{\sigma}^2 = \sigma^2 + 0_p(n^{-\delta})$, $\delta > 0$.

Theorem 4.2.2.1 (Fuller and Battese, 1973)

Assumptions 1, 2, 3 are sufficient conditions for the estimator $\tilde{\beta}_n = (I \otimes (X'\hat{V}^{-1}X)^{-}X'\hat{V}^{-1})Y_n$ to have the same asymptotic distribution as the estimator $\hat{\beta} = (I \otimes (X'V^{-1}X)^{-}X'V^{-1})Y_n$ under the model $Y_n = (I \otimes X)\beta_n + (I \otimes U^*)\varepsilon_n$.

Applying the Fuller and Battese theorem, it follows that $\tilde{\beta} = (X'\hat{V}^{-1}X)^{-}X'\hat{V}^{-1}Y$ has the same asymptotic distribution as $\hat{\beta} = (X'V^{-1}X)^{-}X'V^{-1}Y$.

A small simulation study of the behavior of $\tilde{\beta}$ when $\hat{V}$ is computed with the restricted estimators $\hat{\hat{\sigma}}_1^2$ and $\hat{\hat{\sigma}}_2^2$ is given in Chapter 7. The distribution of $\tilde{\beta}$ appeared to be normal.

4.2.3 Behavior of $\tilde{\beta}$ When $\hat{V}$ Is Computed with A Positive Definite Estimator of σ_2^2 and A Restricted Positive Estimator of σ_1^2

This procedure also restricts the estimators of the variance components σ_1^2 and σ_2^2 to be non-negative.

$$\text{Let } \hat{\hat{\sigma}}^2 = (Y'Q_VV_1Q_VY - \text{tr}(Q_VV_1Q_V)\hat{\sigma}_{2H}^2)/\text{tr}(Q_VV_1Q_VV_1)$$

where $\hat{\sigma}_{2H}^2 = Y'(I - P_{[X:U_1]})Y/\text{tr}(I - P_{[X:U_1]})$, and $P_{[X:U_1]}$ is the projector operator onto the space of $[X:U_1]$.

$$\text{Also let } \hat{\hat{\sigma}}_1^2 = \begin{cases} 0 & \text{if } \hat{\hat{\sigma}}^2 \le 0 \\ \hat{\hat{\sigma}}^2 & \text{otherwise} \end{cases}$$

$$\hat{\hat{\sigma}}_2^2 = \hat{\sigma}_{2H}^2 .$$

The justification for this procedure is that $\hat{\sigma}^2_{1H}$ is the MINQUE positive definite estimator defined by Rao and Kleffe (1979). Also, it is the Analysis of Variance estimator. Moreover, Rao and Kleffe (1979) show that the distribution of $\hat{\sigma}^2_{2H}$ has a scaled $\chi^{2'}$ distribution. Note that $\hat{\hat{\sigma}}^2$ is unbiased for σ^2_1.

Chapter 7 gives the results of a small simulation study of the properties of $\tilde{\beta}$, where $\hat{V}$ is computed using the estimators defined in this section.

5. TESTING HYPOTHESES

Chapter 5 will be devoted to testing an estimable linear parametric function of the parameters β, say $\lambda'\beta$, that is,

$$H_o: \quad \lambda'\beta = \lambda'\beta_o$$
$$\text{against } H_a: \quad \neq, <, >$$

In Chapter 4 it was established that in large samples $\tilde{\beta}$ has a normal distribution with mean β and variance-covariance matrix $(X'V^{-1}X)^-$ where the latter can be approximated by $(X'\hat{V}^{-1}X)^-$.

The present chapter deals with testing hypotheses with small samples. The procedure is to test a linear combination of the parameters β using an approximate t-test. The method is to compute

$$t = \frac{\lambda'\tilde{\beta} - \lambda'\beta_o}{\sqrt{\hat{\text{var}}(\lambda'\tilde{\beta})}}$$

where $\tilde{\beta} = (X'\hat{V}^{-1}X)^- X'\hat{V}^{-1}Y$ and compare with tabulated percentage points of the t-distribution. The difficult question is to determine the degrees of freedom associated with the above t-test.

5.1 Procedure to Test the Degrees of Freedom Associated with a t-test for Testing a Linear Combination of the Vector of Parameters β

A survey of the literature reveals a general solution to some similar problems. Specifically, Satterthwaite (1946), B. L. Welch (1947), G. S. James (1951) and G. E. P. Box (1954) were concerned with the problem of approximating the distribution of a quadratic form. Their general method will be applied to the problem at hand.

Let $\lambda'\beta$ be an estimable linear function of the vector of parameters β, and

$$Z = f(\hat{\sigma}_1^2, \hat{\sigma}_2^2)$$

$$= \lambda'(X'\hat{V}^{-1}X)^{-}\lambda$$

where $\hat{V} = V_1\hat{\sigma}_1^2 + I\hat{\sigma}_2^2$.

The distribution of Z will be approximated by a gamma distribution with density

$$p(t)dt = \Gamma\left(\frac{f}{2}\right)^{-1} \left(\frac{t}{2g}\right)^{\frac{1}{2}f-1} \exp\left(-\frac{t}{2g}\right) d\left(\frac{t}{2g}\right) \qquad (5.1.1)$$

with parameters $\alpha = \frac{1}{2}f$ and $\beta = 2g$, f and g being constants.

In Section 5.2 the justification for this assumption will be examined in some detail.

The values f and g are chosen by equating the first two moments of Z and the approximating distribution.

Using Taylor's theorem one can write

$$Z = f(\hat{\sigma}_1^2, \hat{\sigma}_2^2)$$

$$= f(\sigma_1^2, \sigma_2^2) + (\hat{\sigma}_1^2 - \sigma_1^2)f_1(\sigma_1^2, \sigma_2^2) + (\hat{\sigma}_2^2 - \sigma_2^2)f_2(\sigma_1^2, \sigma_2^2)$$

$$+ \text{ an error term.}$$

Given that the error term is sufficiently small, one can write

$$E(Z) \doteq f(\sigma_1^2, \sigma_2^2)$$

and $$\mathrm{var}(Z) \doteq \mathrm{var}(\hat{\sigma}_1^2) f_1(\sigma_1^2, \sigma_2^2) + \mathrm{var}(\hat{\sigma}_2^2) f_2(\sigma_1^2, \sigma_2^2) + 2 \,\mathrm{cov}\,(\hat{\sigma}_1^2, \hat{\sigma}_2^2) f_1(\sigma_1^2, \sigma_2^2) f_2(\sigma_1^2, \sigma_2^2)$$

Note that

$$f_1(\sigma_1^2, \sigma_2^2) = \frac{\partial}{\partial\hat{\sigma}_1^2} \lambda'(X'\hat{V}^{-1}X)^{-}\lambda \Big|_{(\hat{\sigma}_1^2, \hat{\sigma}_2^2) = (\sigma_1^2, \sigma_2^2)}$$

$$= \lambda'\left(\frac{\partial}{\partial\hat{\sigma}_1^2}(X'\hat{V}^{-1}X)^{-}\right)\lambda$$

$$= -\lambda'(X'\hat{V}^{-1}X)^{-} \frac{\partial}{\partial\hat{\sigma}_1^2}(X'\hat{V}^{-1}X)(X'\hat{V}^{-1}X)^{-}\lambda$$

$$= -\lambda'(X'\hat{V}^{-1}X)^{-}X'\left(\frac{\partial}{\partial\hat{\sigma}_1^2}(\hat{V}^{-1})\right)X(X'\hat{V}^{-1}X)^{-}\lambda$$

$$=-\lambda'(X'\hat{V}^{-1}X)^{-}X'\hat{V}^{-1}\left(\frac{\partial}{\partial\hat{\sigma}_1^2}(\hat{V})\right)\hat{V}^{-1}X(X'\hat{V}^{-1}X)^{-}\lambda$$

$$=-\lambda'(X'\hat{V}^{-1}X)^{-}X'\hat{V}^{-1}V_1\hat{V}^{-1}X(X'\hat{V}^{-1}X)^{-}\lambda \Big|_{(\hat{\sigma}_1^2, \hat{\sigma}_2^2) = (\sigma_1^2, \sigma_2^2)}$$

$$=-\lambda'(X'V^{-1}X)^{-}X'V^{-1}V_1V^{-1}X(X'V^{-1}X)^{-}\lambda$$

$$=-q'V_1q$$

where $$q = \lambda'(X'V^{-1}X)^{-}X'V^{-1}.$$

Similar manipulations yield

$$f_2(\sigma_1^2, \sigma_2^2) = -q'q.$$

Consequently,

$$\mathrm{var}(Z) \doteq \mathrm{var}(\hat{\sigma}_1^2)(q_1V_1q_1)^2 + \mathrm{var}(\hat{\sigma}_2^2)(q'q)^2 + 2\,\mathrm{cov}(\hat{\sigma}_1^2, \hat{\sigma}_2^2)q'V_1q\cdot q'q$$

Observe that if $\lambda'\beta$ is estimable, then $\lambda' = q'X$, $q' = \lambda'(X'V^{-1}X)^{-}X'V^{-1}$

and
$$\begin{aligned} E(Z) &\doteq \lambda'(X'V^{-1}X)^{-}\lambda \\ &= \lambda'(X'V^{-1}X)^{-}X'V^{-1}Vq \\ &= q'Vq \\ &= q'V_1q\sigma_1^2 + q'q\sigma_2^2 . \end{aligned} \tag{5.1.2}$$

Now, equating these two moments of Z and the first two moments of the approximating distribution yields the following system of equations,

$$\begin{aligned} E(Z) &= q'V_1q\sigma_1^2 + q'q\sigma_2^2 \\ &= fg \end{aligned}$$

and
$$\begin{aligned} \text{var}(Z) &= \text{var}(\hat{\sigma}_1^2)(q'V_1q)^2 + \text{var}(\hat{\sigma}_2^2)(q'q)^2 + 2\,\text{cov}(\hat{\sigma}_1^2, \hat{\sigma}_2^2)q'V_1q\cdot q'q \\ &= 2fg^2. \end{aligned}$$

Solving this system of equations leads to

$$f = \frac{2(q'V_1q\sigma_1^2 + q'q\sigma_2^2)^2}{\text{var}(\hat{\sigma}_1^2)(q'V_1q)^2 + \text{var}(\hat{\sigma}_2^2)(\underline{q}'\underline{q})^2 + 2\,\text{cov}(\hat{\sigma}_1^2, \hat{\sigma}_2^2)q'V_1q\cdot q'q} \tag{5.1.3}$$

as the "degrees of freedom" parameter for the approximating gamma distribution and consequently the "t" statistic for the test of hypothesis.

Notice that the formula (5.1.3) for the degrees of freedom is a function of the true parameters σ_1^2 and σ_2^2.

Two alternate suggestions are available to compute the degrees of freedom.

i) Since the MINQUE are based on prior information, a reasonable thing to do is to use such information in the formula (5.1.3) to compute the degrees of freedom.

ii) An alternative suggestion is to iterate the MINQUE procedure several times. Experience shows that the process tends to converge rapidly and hence provides unique values to compute the degrees of freedom by the formula (5.1.3).

5.2 The Accuracy of the Formula for Approximating the Degrees of Freedom

A check on the adequacy of the foregoing procedure was performed with a small simulation study.

Step 1. Generate $Y \sim N(0,1)$

Step 2. Assume prior information and compute the MINQUE for the components, σ_1^2 and σ_2^2.

Step 3. Select two different designs and different degrees of unbalanced data.

Design I. A split-plot design arranged in randomized complete blocks, with two blocks, four whole plots, two split plots and five different cases of unbalance:

(i) no missing data

(ii) missing two observations say y_{111} and y_{242}

(iii) missing two observations say y_{112} and y_{212}

(iv) missing two observations say y_{221} and y_{222}

(v) missing y_{222}

Design II. A split-plot design arranged in randomized complete blocks, with four blocks, three whole plots, two split plots and four different cases of unbalance:

(i) no missing observations

(ii) missing two observations say y_{111} and y_{112}

(iii) missing four observations say y_{111}, y_{222}, y_{322} and y_{411}

(iv) missing twelve observations say y_{111}, y_{112}, y_{221}, y_{222}, y_{231}, y_{232}, y_{311}, y_{312}, y_{421}, y_{422}, y_{431} and y_{432}.

Step 4. Select three different contrasts:

(i) testing the difference between blocks, say λ_1

(ii) testing the difference between whole plots, say λ_2

(iii) testing the difference between split plots, say λ_3

Combining the different cases of unbalance in Designs I and II with the contrasts for testing hypotheses yields a total of twenty-seven cases. The notation used to identify the twenty-seven cases is given by (design, unbalance case, contrast), i.e., (I, (i), λ_1) corresponds to Design I, no missing observations, and a contrast for testing differences between blocks.

Step 5. For the twenty-seven cases, the degrees of freedom approximated by formula (5.1.3) were computed.

Step 6. For the twenty-seven cases, the estimates of the variance components were computed 100 times, and with these estimates the function $Z = f(\hat{\sigma}_1^2, \hat{\sigma}_2^2) = \lambda'(X'\hat{V}^{-1}X)^{-}\lambda$ evaluated. The mean and the variance of the distributions were computed and histograms constructed. The degrees of freedom of the distributions were computed, and compared with the one obtained in step 5.

For the first three cases (I, (i), $\underline{\lambda}_1$), (I, (i), $\underline{\lambda}_2$) and (I, (i), $\underline{\lambda}_3$), the function Z was evaluated 200 times. The purpose of this was to obtain a better approximation for the distribution of Z.

The results obtained from the simulation are summarized in Table 5.2.1.

Table 5.2.1. Degrees of freedom associated with different degrees of unbalanced designs

I, (i) Anova		I, (ii) Anova		I, (iii) Anova		I, (iv) Anova		I, (v) Anova	
Source	d.f.	Source	d.f.	Source	d.f.	Source	d.f.	Source	d.f.
B	1	B	1	B	1	B	1	B	1
W	3	W	3	W	3	W	3	W	3
Error (a)	3	Error (a)	3	Error (a)	3	Error (a)	2	Error (a)	3
S	1	S	1	S	1	S	1	S	1
SW	3	SW	3	SW	3	SW	3	SW	3
Error (b)	4	Error (b)	2	Error (b)	2	Error (b)	3	Error (b)	3
Total (Cov)	15	Total (Cov)	13	Total (Cov)	13	Total (Cov)	13	Total (Cov)	14

II, (i) Anova		II, (ii) Anova		II, (iii) Anova		II, (iv) Anova	
Source	d.f.	Source	d.f.	Source	d.f.	Source	d.f.
B	3	B	3	B	3	B	3
W	2	W	2	W	2	W	2
Error (a)	6	Error (a)	5	Error (a)	6	Error (a)	0
S	1	S	1	S	1	S	1
WS	2	WS	2	WS	2	WS	2
Error (b)	9	Error (b)	7	Error (b)	5	Error (b)	3
Total (Cov)	23	Total (Cov)	21	Total (Cov)	19	(total (Cov)	11

B: = Blocks or repetitions
W: = Whole plot
S: = Split plots
WS: = Interaction between whole and split plots

5.2.1 Results Obtained from the Simulation of Case (I, (i), λ_1)

The degrees of freedom associated with the t-test for testing the difference between blocks given by formula (5.1.3) was 3.00.

In Table (5.2.1) it is observed that error (a) which would normally be used to test block effects has 3 degrees of freedom. Observe the closeness between the degrees of freedom obtained from formula (5.1.3) and the one from Table (5.2.1).

The frequency histogram from the distribution of Z, which was based on 100 values, gives the values of 0.746206 and 0.416883 for the mean and variance respectively. The degrees of freedom associated with this distribution was 2.67.

A second frequency histogram based on 200 values was constructed, and gave the values of 0.685476 and 0.327729 for the mean and variance respectively. The degrees of freedom associated with this distribution was 2.86.

The graph of the latter frequency distribution is shown in Figure 5.2.1. Observe that the graph has the shape of a gamma distribution.

5.2.2 Results Obtained from the Simulation of Case (I, (i), λ_2)

The degrees of freedom associated with the t-test for testing the difference between whole plots, given by formula (5.1.3), was 3.00.

In Table (5.2.1) it is observed that error (a) which would normally be used to test whole plot effects has 3 degrees of freedom. Observe the closeness between the degrees of freedom obtained from formula (5.1.3) and the one from Table (5.2.1).

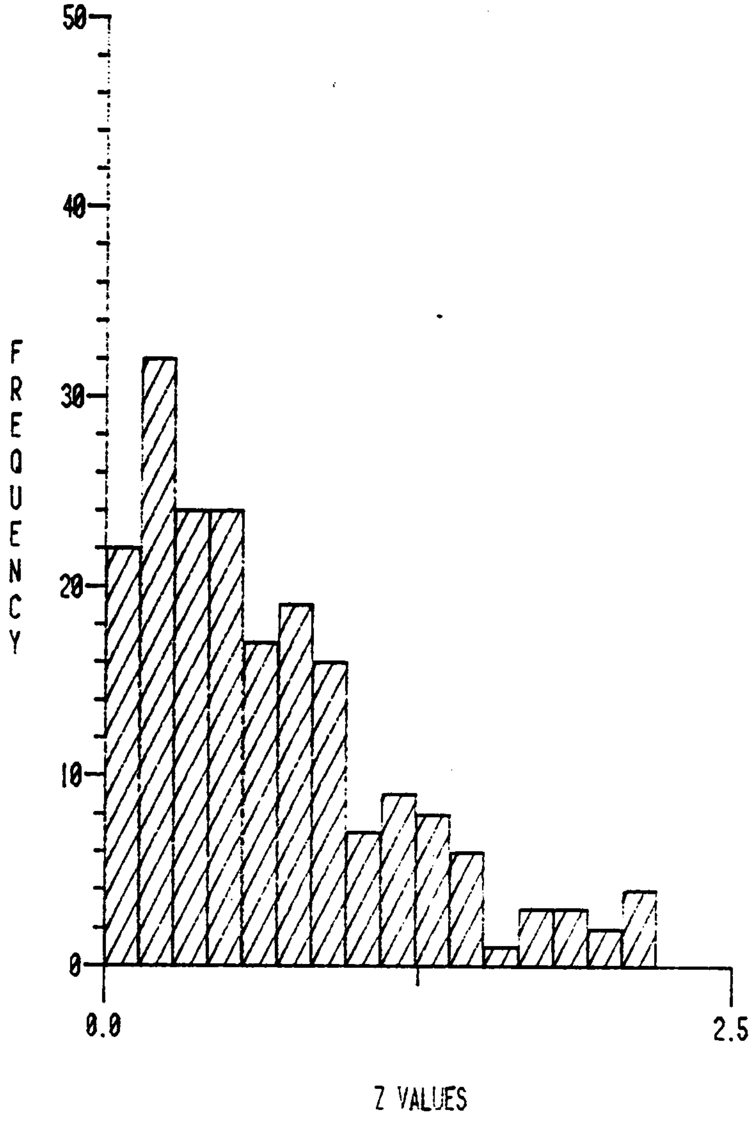

FIGURE 5.2.1 DISTRIBUTION OF Z
FOR TESTING A LINEAR CONTRAST BETWEEN BLOCKS
FOR A SPLIT PLOT MODEL WITH NO MISSING OBSERVATIONS

The frequency histogram from the distribution of Z, which was based on 100 values, gives the values of 1.3265 and 1.3175 for the mean and variance respectively. The degrees of freedom associated with the distribution was 2.67.

A second frequency histogram based on 200 values was constructed and gave the values of 1.2186 and 1.0357 for the mean and variance respectively. The degrees of freedom associated with this distribution was 2.86.

The graph of the latter frequency distribution is shown in Figure (5.2.2). Observe that the graph has the shape of a gamma distribution.

5.2.3 Results Obtained from the Simulation of Case (I, (i), λ_3)

The degrees of freedom associated with the t-test for testing the difference between split plots, given by formula (5.1.3) was 4.00.

In Table (5.2.1) it is observed that error (b) which would normally be used to test split-plot effects has 4 degrees of freedom. Observe the closeness between the degrees of freedom obtained from formula (5.1.3) and the one from Table (5.2.1).

The frequency histogram from the distribution of Z, which was based on 100 values, gives the values of .174674 and .014510 for the mean and variance respectively. The degrees of freedom associated with the distribution was 4.18.

A second frequency histogram based on 200 values was constructed and gave the values of .162663 and .014226 for the mean and the variance respectively. The degrees of freedom associated with this distribution was 3.71.

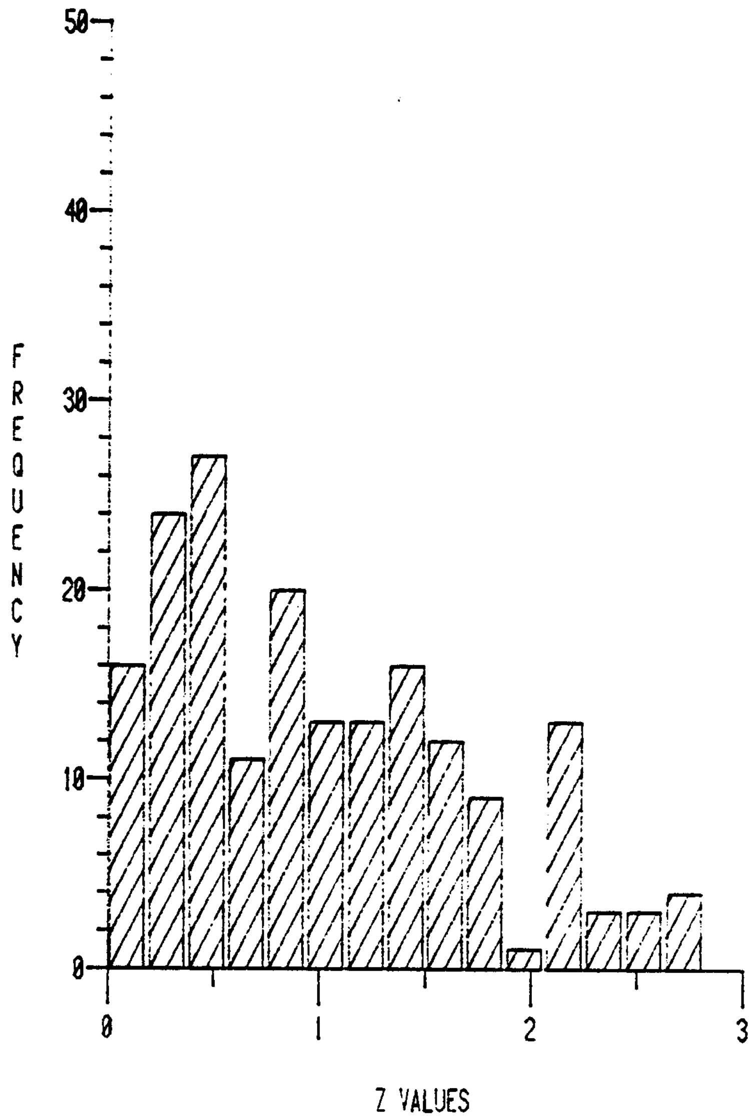

FIGURE 5.2.2 DISTRIBUTION OF Z
FOR TESTING A CONTRAST BETWEEN WHOLE PLOT EFFECTS
FOR A SPLIT PLOT MODEL WITH NO MISSING OBSERVATIONS

Table 5.2.2. Mean, variance, degrees of freedom of the distribution of $Z = f(\sigma_1^2, \sigma_2^2)$, (d.f.Z) and degrees of freedom associated with Formula (5.1.3) (d.f.F_o) for the 27 cases considered

Case	Mean	Variance	d.f.Z	d.f.F_o
(I, (i), λ_1)	.746206	.416883	2.67	3.00
(I, (i), λ_1)*	.685476	.327729	2.86	3.00
(I, (i), λ_2)	1.326589	1.317556	2.67	3.00
(I, (i), λ_2)*	1.218625	1.03578	2.86	3.00
(I, (i), λ_3)	.174674	.01451	4.18	4.00
(I, (i), λ_3)*	.162663	.014226	3.71	4.00
(I, (ii), λ_1)	.694981	.632815	2.94	2.88
(I, (ii), λ_2)	.785614	.471430	2.61	2.25
(I, (ii), λ_3)	.433751	.094709	3.97	2.25
(I, (iii), λ_1)	.882843	.552669	2.82	2.92
(I, (iii), λ_2)	2.266146	3.82542	2.68	2.63
(I, (iii), λ_3)	.321029	.060799	3.39	3.36
(I, (iv), λ_1)	1.02667	1.113545	1.89	2.00
(I, (iv), λ_2)	2.053346	4.454139	1.89	2.00
(I, (iv), λ_3)	.357754	.091751	2.78	3.00
(I, (v), λ_1)	.814711	.748071	1.77	2.95
(I, (v), λ_2)	2.07997	5.23834	1.65	2.63
(I, (v), λ_3)	.304351	.079917	2.31	3.25
(II, (i), λ_1)	2.09957	1.22603	7.19	6.00
(II, (i), λ_2)	1.04979	.306500	7.19	6.00
(II, (i), λ_3)	.096352	.001733	10.71	9.00
(II, (ii), λ_1)	2.63044	2.38098	5.81	5.00

Table 5.2.1. Continued

Case	Mean	Variance	d.f.Z	d.f.F_o
(II, (ii), λ_2)	0.292271	.029393	5.81	5.00
(II, (ii), λ_3)	.104107	.002057	10.53	8.00
(II, (iii), λ_1)	2.161932	1.618287	5.77	5.41
(II, (iii), λ_2)	.427687	.046884	7.80	7.37
(II, (iii), λ_3)	.150781	.007013	6.48	5.51
(II, (iv), λ_1)	3.719712	29.287296	.94	1.00
(II, (iv), λ_2)	.543057	.658062	.89	1.00
(II, (iv), λ_3)	.197671	.022929	3.40	3.00

The graph of the latter frequency distribution is shown in Figure (5.2.3). Observe that the graph has the shape of a gamma distribution.

5.2.4 Results Obtained from the Simulation of the Remaining Cases

The remaining cases are summarized in Table (5.2.2). It is clear that the degrees of freedom computed by formula (5.1.3) are satisfactory for approximating the distribution of Z.

5.3 Distribution of $Z = f(\hat{\sigma}_1^2, \hat{\sigma}_2^2) = \lambda'(X'\hat{V}^{-1}X)^-\lambda$ When $\hat{V}$ Is Computed Using the MINQUE Estimators of the Variance Components

In Chapter 4 it was shown that the parametric function $\Sigma_i q_i \sigma_i^2$ has a MINQUE of the form Y'AY, where A satisfies the conditions X'A = 0 and $tr(AV_i) = q_i$. In specific terms the equations

$$\begin{bmatrix} \hat{\sigma}_1^2 \\ \hat{\sigma}_2^2 \end{bmatrix} = \begin{bmatrix} tr(Q_V V_1 Q_V V_1) & tr(Q_V V_1 Q_V) \\ tr(Q_V V_1 Q_V) & tr\ (Q_V Q_V) \end{bmatrix}^{-1} \begin{bmatrix} Y'Q_V V_1 Q_V Y \\ Y'Q_V Q_V Y \end{bmatrix}$$

$$\begin{bmatrix} \hat{\sigma}_1^2 \\ \hat{\sigma}_2^2 \end{bmatrix} = S^{-1} \begin{bmatrix} Y'Q_V V_1 Q_V Y \\ Y'Q_V Q_V Y \end{bmatrix}$$

were obtained.

Let $\{S^{ij}\}$ denote the elements of S^{-1}. It follows that

$$S^{11} = tr(Q_V Q_V)/Det(S)$$
$$S^{21} = S^{12} = -\ tr(Q_V V_1 Q_V)/Det(S)$$
$$S^{22} = tr(Q_V V_1 Q_V)/Det(S)$$

where Det(S) is the determinant of the matrix S.

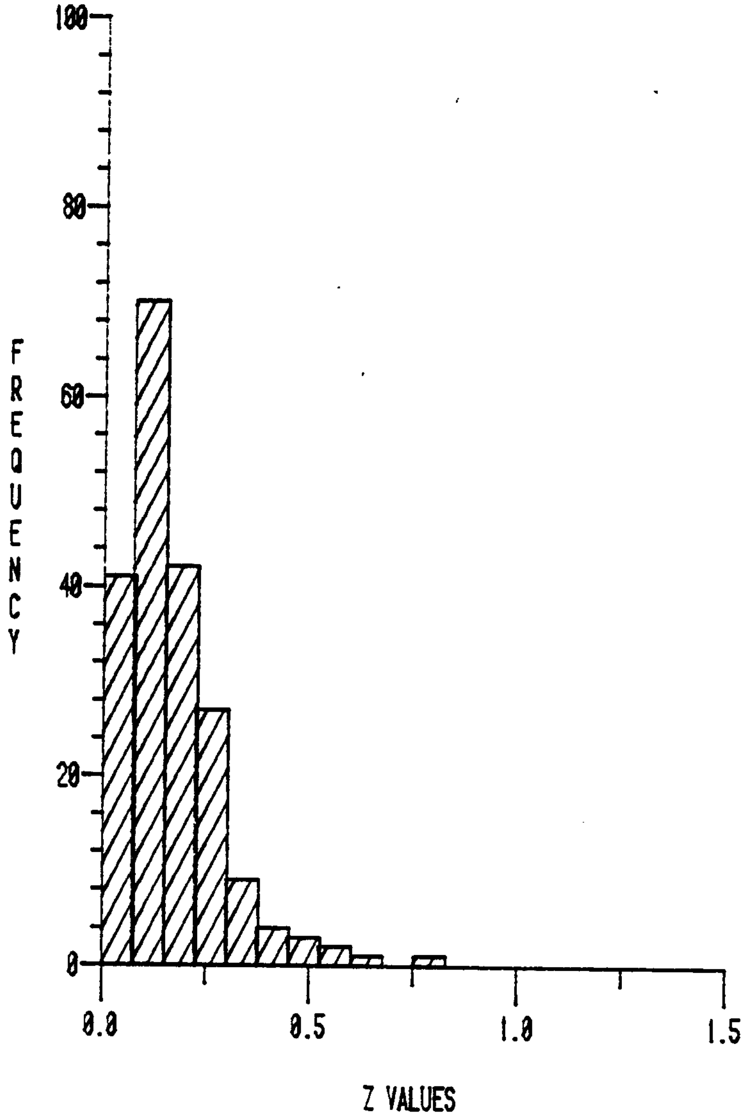

FIGURE 5.2.3 DISTRIBUTION OF Z
FOR TESTING A LINEAR CONTRAST BETWEEN SPLIT PLOT EFFECTS
FOR THE SPLIT PLOT MODEL WITH NO MISSING OBSERVATIONS

Now define $A_1 = Q_V V_1 Q_V S^{11} + Q_V Q_V S^{12}$

$$A_2 = Q_V V_1 Q_V S^{21} + Q_V Q_V S^{22}.$$

It follows that $\hat{\sigma}_1^2 = Y'A_1Y$, and $\hat{\sigma}_2^2 = Y'A_2Y$. Note that A_1 and A_2 are symmetric but not idempotent.

Scheffe (1970) found that under the assumption of normality of Y with mean μ and positive definite variance matrix V, the quadratic form $Q = Y'AY$ can be written as $Q = \sum_i \lambda_i \chi^{2'}_{h_i,\delta_i^2}$ where $\{\lambda_i\}$ are the characteristic roots of AV, $\chi^{2'}_{h_i,\delta_i^2}$ is a chi-square random variable with h_i degrees of freedom and non-centrality parameter δ_i^2. However, since MINQUE are not necessarily positive, a more general statement is needed.

J. P. Imhof (1961) worked with the quadratic form $Q = Y'AY = \Sigma_i \lambda_i \chi^{2'}_{h_i,\delta_i^2}$ where the $\chi^{2'}_{h_i,\delta_i^2}$ are chi-square random variables with h_i degrees of freedom and non-centrality parameters δ_i^2, and where $\lambda_1 > \lambda_2 > \ldots > \lambda_p > 0 > \lambda_{p+1} > \ldots > \lambda_m$ are the characteristic roots of AV. He found that under the assumption of normality of Y, the distribution of the quadratic form Q can be accurately approximated by

$$P\,\{Q > \chi\} \simeq P\,\{\chi^2_{h'} > y\}$$

where $h' = c_2^3/c_3^2$, $y = (x-c_1)(h'/c_2)^{1/2} + h'$ and $c_j = \Sigma_r \lambda_r^j (h_r + j\delta_r^2)$, $j = 1,2,3$.

The invariance condition of the estimators, $AX = 0$ leads to the simplification $C_j = \Sigma_r \lambda_r^j h_r$ for $j = 1,2,3$. It follows that $Y'A_1Y$ and $Y'A_2Y$ have approximate chi-squared distributions, $\chi^2_{h_1'}$ and $\chi^2_{h_2'}$ respectively where $h_i' = (C_2{}^{(i)})^3/(C_3{}^{(i)})^2$, $C_j^{(i)} = \Sigma_r (\lambda_r^{(i)})^j h_r{}^{(i)}$, $j = 1,2,3$ and $\lambda_1^{(i)} > \lambda_2^{(i)} > \ldots$ are the characteristic roots of A_iV for $i = 1,2$.

5.4 Simulation Study to Evaluate Adequacy of Approximation

In this section a simulation study to evaluate the adequacy of the approximation to the distribution of $Z = f(\hat{\sigma}_1^2, \hat{\sigma}_2^2) = \lambda'(X'\hat{V}^{-1}X)^{-}\lambda$ where $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ are MINQUE will be investigated.

5.4.1 Simulation of the Distribution of $Z = f(\hat{\sigma}_1^2, \hat{\sigma}_2^2)$ When the Prior Values Are Used to Compute the Parameters α and β of the Approximate Gamma Distribution

The simulation that is proposed is given as follows:

Step 1. Generate $Y \sim N(0,1)$

Step 2. Assume prior values and compute the MINQUE for the variance components, σ_1^2 and σ_2^2.

Step 3. Select a balance design with two blocks, four whole plots, two split plots and a linear parametric function for testing split plot effects.

Step 4. Compute the approximate parameters of the gamma distribution given by:

$$\alpha \doteq \frac{1}{2} \left(\frac{(V_1\sigma_1^2 + V_2\sigma_2^2)^2}{\mathrm{var}(\hat{\sigma}_1^2)(q'V_1q)^2 + \mathrm{var}(\hat{\sigma}_2^2)(q'V_2q)^2 + 2\mathrm{cov}(\hat{\sigma}_1^2, \hat{\sigma}_2^2)q'V_1q\ q'V_2q} \right) \tag{5.4.1.1}$$

$$\beta \doteq 2 \left(\frac{\mathrm{var}(\hat{\sigma}_1^2)(q'V_1q)^2 + \mathrm{var}(\hat{\sigma}_2^2)(q'V_2q)^2 + 2\mathrm{cov}(\hat{\sigma}_1^2, \hat{\sigma}_2^2)q'V_1q\ q'V_2q}{(V_1\sigma_1^2 + V_2\sigma_2^2)} \right) \tag{5.4.1.2}$$

by using prior values instead of the variance components.

Step 5. Generate the estimates of the variance components 100 times, and with these estimates evaluate the function $Z = f'(\hat{\sigma}_1^2, \hat{\sigma}_2^2) = \lambda'(X'\hat{V}^{-1}X)^{-}\lambda$. Use α, β and Z to generate a gamma distribution by using the function $u = \text{CDGAMA}(Z,\alpha,\beta)$.

Step 6. Test the hypothesis

$$H_o: \; u \in \tau$$

against $H_a: \; u \notin \tau$, where τ is the set of gamma distributions.

Three different competing test procedures will be used. The first is based on Watson's U^2 statistic given by

$$U^2 = \frac{1}{12N} + \sum_{i=1}^{N} \{\frac{(2i-1)}{2N} - u_{(i)}\}^2 - N(\bar{u} - 0.5)^2$$

where $u_i = P(Z \leq z)$ is the cumulative distribution function of Z and $u_{(i)}$ are the ordered u_i values, $\bar{u}$ is the mean, and N is the number of observations. The second procedure is based on $P_4^2 = \frac{1}{N}(t_1^2+t_2^2+t_3^2+t_4^2)$ where $t_r = \sum_j \Pi_r(u_j)$ for r = 1,2,3,4 and $\Pi_1(u) = \sqrt{12}\,(u - 1/2)$, $\Pi_2(u) = \sqrt{5}\,(6(u - 1/2)^2 - 1/2)$, $\Pi_3(u) = \sqrt{7}\,(20(u - 1/2)^3 - 3(u - 1/2))$ and $\Pi_4(u) = 210(u - 1/2)^4 - 45(u - 1/2)^2 + 9/8$. The third procedure is a χ^2 text based on the fact that one tenth of the $\{u_i\}$ values should fall in each of the intervals [0, .1), [.1, .2), ..., [.9, 1.0].

Step. 7. Graph the distribution of Z.

Results of the Simulations

The parameters α and β show values of $\alpha = 1.99$ and $\beta = 0.08$, the computed values for U^2, P_4^2 and χ^2 show $U^2 = .075$, $P_4^2 = .1533$ and $\chi^2 = 13.2$.

The critical values for U^2, P_4^2 and χ^2 are $U^2_{\infty,.05} = 0.187$ and $P^2_{4,\infty,.05} = 9.49$ and $\chi^2_{0.05} = 16.9$.

Therefore in each case the null hypothesis was not rejected. It is concluded that $Z = f(\hat{\sigma}_1^2,\hat{\sigma}_2^2)$ belongs to the class of gamma distributions.

A second simulation was performed to know how many times the value

$$t = \frac{\lambda'\tilde{\beta}}{\sqrt{\widehat{\text{var}}(\lambda'\tilde{\beta})}}$$

is rejected under the hypothesis H_o: $\lambda'\beta = 0$

against H_a: $\neq$

It was found that in 3 of 100 cases the null hypothesis was rejected. This means less than 5% of the time.

A third simulation was performed as follows:

Step 1. Generate N(0,1).

Step 2. Select an unbalanced design with two blocks, four whole plots and two split plots and assume there are missing values, say y_{221}, y_{222}.

Step 3. Assume prior values and compute the MINQUE for the variance components σ_1^2 and σ_2^2.

Step 4. Compute the approximate parameters of the gamma distribution by formulas (5.4.1.1) and (5.4.1.2).

Step 5. Generate the estimates of the variance components 100 times, and with these evaluate the function $Z = f(\hat{\sigma}_1^2,\hat{\sigma}_2^2)$. Use α, β and Z to generate a gamma distribution by using the function u = CDGAMA(Z,α,β).

Step 6. Use two criteria to test the hypothesis

H_o: $u \in \tau$

against H_a: $u \notin \tau$, where τ is the set of gamma distributions.

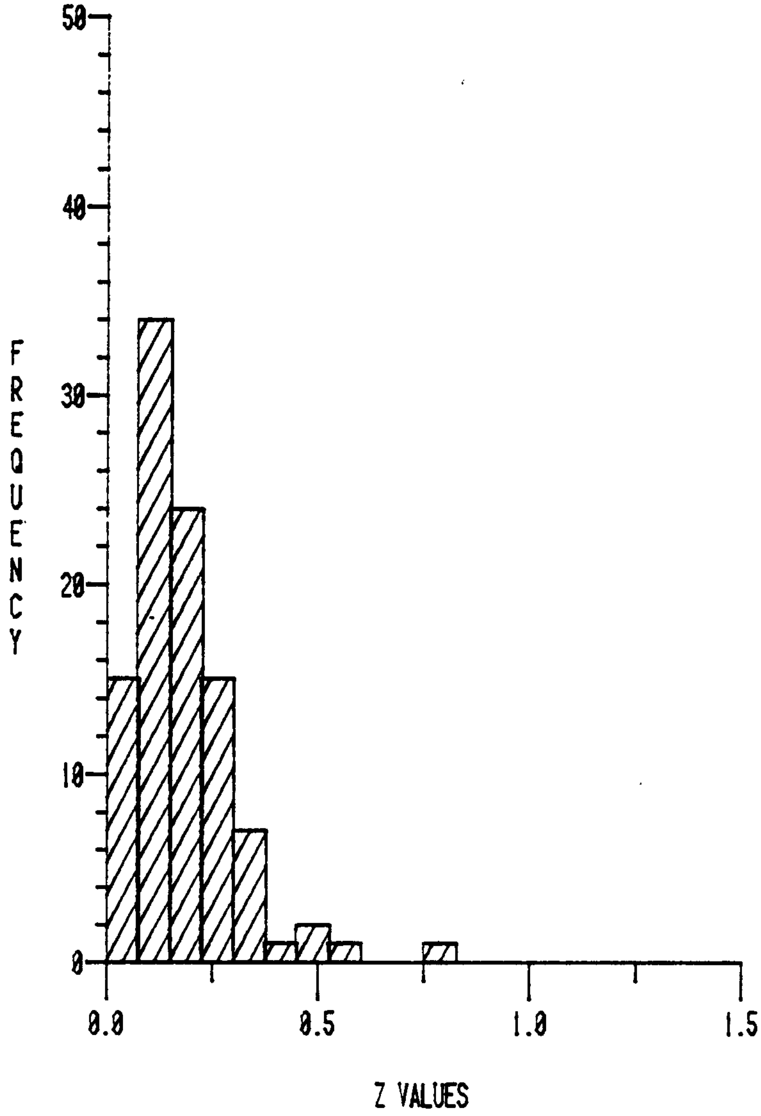

FIGURE 5.4.1 DISTRIBUTION OF Z
FOR TESTING A CONTRAST BETWEEN SPLIT PLOT EFFECTS
FOR A SPLIT PLOT MODEL

The criteria used were based on Watson's U^2 and Neyman's P_4.

Results

The parameters α and β show values of $\alpha = 1.4999$, $\beta = 0.1333$. The computed values for U^2 and P_4^2 show values of $U^2 = 0.097$ and $P_4^2 = 1.99323$. The critical points for U^2 and P_4^2 are $U^2_{\infty,0.05} = 0.187$ and $P^2_{4,\infty,.05} = 9.49$. Hence, the null hypothesis was not rejected in both cases.

Finally, a simulation was performed to know how many times the value

$$t = \frac{\lambda'\tilde{\beta}}{\sqrt{\widehat{\text{var}}(\lambda'\tilde{\beta})}}$$

was rejected under the hypothesis H_o: $\lambda'\beta = 0$

against H_a: $\neq$

It was found that in 2 of 100 cases the hypothesis was rejected. This means less than 5% of the time. We conclude from this that the distributions belong to the class of gamma distributions.

5.4.2 Simulation of the Distribution of $t = \lambda'\tilde{\beta}/\sqrt{\widehat{\text{var}}(\lambda'\tilde{\beta})}$ Under the Null Hypothesis $\lambda'\beta = 0$ By Using the MINQUE Iterated Estimators in the Computation of $\lambda'\tilde{\beta}$

The simulation that is proposed is given as follows:

Step 1. Generate $N(0,1)$.

Step 2. Select a balanced design with two blocks, four whole plots, two split plots and a linear parametric function for testing split-plot effects.

Step 3. Assume prior values.

Step 4. Generate the MINQUE estimates of the variance components σ_1^2 and σ_2^2, 100 times, by iterating the MINQUE procedure several times.

Step 5. Compute $t = \lambda'\tilde{\beta}/\sqrt{\hat{var}(\lambda'\tilde{\beta})}$ by using the MINQUE iterated estimators. Use the approximated degrees of freedom given by formula (5.1.3) to generate the distribution function of the t-distribution.

Step 6. Use three criteria to test the hypothesis

$$H_o: \ u \in \tau$$

against $H_a: \ u \notin \tau$, where τ is the set of t-distribution.

The criteria were based on Watson's U^2, Neyman's P_4 and a χ^2 test using 10 intervals.

Results of the Simulation

The computed values for U^2, P_4^2 and χ^2 shows values of U^2 = 0.0957, P_4^2 = 0.0178 and χ^2 = 10.6, with critical points $U^2_{\infty,.05}$ = 0.187, $P^2_{4,\infty,.05}$ = 9.49 and $\chi^2_{9,.05}$ = 16.9. Hence, the null hypothesis was accepted.

Also, the number of times that the null hypothesis $\lambda'\beta = 0$ was rejected at the 5% level of significance was 2 of 100. This means less than 5%.

A second simulation was performed as before but instead of step 2, now select an unbalanced design with missing values, say y_{221}, y_{222}.

Results of the Simulation

The computed values for U^2, P_4^2 and χ^2 show values of U^2 = 0.09224, P_4^2 = 2.333 and χ^2 = 3.8 with critical points $U^2_{\infty,.05}$ = 0.187, $P^2_{4,\infty,.05}$ = 9.49 and $\chi^2_{9,.05}$ = 16.9. Hence, the null hypothesis was accepted.

The number of times that the null hypothesis $\lambda'\tilde{\beta} = 0$ was rejected at the 5% level of significance was 6 of 100. This means over 5%.

As a conclusion of the above simulations, it is legitimate to assume that $t = \lambda'\tilde{\beta}/\sqrt{\widehat{var}(\lambda'\tilde{\beta})}$ be distributed approximately as t-distributions with degrees of freedom given by formula (5.1.3).

6. THE ANALYSIS OF COVARIANCE

The methodology developed in Chapters 4 and 5 will now be applied to the analysis of covariance for the unbalanced split-plot experiment. For the complete balanced experiment the model can be written as

$$y_{ijk} = \mu + a_i + \beta_1(x_{ij} - \bar{x}) + \delta_{ij} + b_k + ab_{ik} + \beta_2(x_{ijk} - x_{ij}) + e_{ijk}$$

$$i = 1, \ldots, a$$

$$j = 1, \ldots, n$$

$$k = 1, \ldots, b$$

Note that this model allows for different slopes at the whole-plot and the split-plot level.

In matrix notation this model becomes

$$Y = 1\mu + X_a\underline{a} + \beta_1 Z_1 + U_1\underline{\varepsilon}_1 + X_b\underline{b} + X_{ab}\underline{ab} + \beta_2 Z_2 + U_2\underline{\varepsilon}_2 \tag{6.1}$$

Note that $U_2 = I$, $Z_1'1 = 0$, $Z_2'U_2 = 0$, $Z_2'1 = 0$, $Z_1'Z_2 = 0$ and $Z_2'X_a = 0$. Also, vector 1, columns of X_a and Z_1 lie in the space spanned by the columns of U_1.

Without further mention, it will be assumed in the remainder of this chapter that the model (6.1) has been appropriately modified to allow for unbalanced experiments. Also, let α_1^2 and α_2^2 be prior values or gueses for σ_1^2 and σ_2^2, and use this to write $V = V_1\alpha_1^2 + V_2\alpha_2^2$.

Transforming the model (6.1) by $V^{-1/2}$ yields

$$V^{-1/2}Y = V^{-1/2}1\mu + V^{-1/2}X_a\underline{a} + V^{-1/2}\beta_1 Z_1 + V^{-1/2}U_1\underline{\varepsilon}_1 + V^{-1/2}X_b\underline{b}$$
$$+ V^{-1/2}X_{ab}\underline{ab} + V^{-1/2}Z_2P_2 + V^{-1/2}U_2\underline{\varepsilon}_2 \tag{6.2}$$

Applying generalized least squares and reordering leads to the normal equations:

$$\begin{bmatrix} \underline{1}'V^{-1}\underline{1} & \underline{1}'V^{-1}X_a & \underline{1}'V^{-1}X_b & \underline{1}'V^{-1}X_{ab} & \underline{1}'V^{-1}Z_1 & \underline{1}'V\text{-}1Z_2 \\ X_a'V^{-1}\underline{1} & X_a'V^{-1}X_a & X_a'V^{-1}X_b & X'V^{-1}X_{ab} & X_a'V^{-1}Z_1 & X_a'V^{-1}Z_2 \\ X_b'V^{-1}\underline{1} & X_b'V^{-1}X_a & X_b'V^{-1}X_b & X_b'V^{-1}X_{ab} & X_b'V^{-1}Z_1 & X_b'V^{-1}Z_2 \\ X_{ab}'V^{-1}\underline{1} & X_{ab}'V^{-1}X_a & X_{ab}'V^{-1}X_b & X_{ab}'V^{-1}X_{ab} & X_{ab}'V^{-1}Z_1 & X_{ab}'V^{-1}Z_2 \\ Z_1'V^{-1}\underline{1} & Z_2'V^{-1}X_a & Z_1'V^{-1}X_b & Z_1'V^{-1}X_{ab} & Z_1'V^{-1}Z_1 & Z_1'V^{-1}Z_2 \\ Z_2'V^{-1}\underline{1} & Z_2'V^{-1}X_a & Z_2'V^{-1}X_b & Z_2'V^{-1}X_{ab} & Z_2'V^{-1}Z_1 & Z_2'V^{-1}Z_2 \end{bmatrix} \begin{bmatrix} \mu \\ \underline{a} \\ \underline{b} \\ \underline{ab} \\ \beta_1 \\ \beta_2 \end{bmatrix} \begin{bmatrix} \underline{1}'V^{-1}Y \\ X_a'V^{-1}Y \\ X_b'V^{-1}Y \\ X_{ab}'V^{-1}Y \\ Z_1'V^{-1}Y \\ Z_2'V^{-1}Y \end{bmatrix} \qquad (6.3)$$

Let $V^{-1/2}X = V^{-1/2}[\underline{1}:X_a:X_b:X_{ab}]$ of order n x ($\underline{1}$ + a + b + ab)

and $P_V = X(X'V^{-1}X)^{-}X'V^{-1}$ of order n x n.

Also, let $V^{-1/2}Z = V^{-1/2}[Z_1:Z_2]$ of order n x 2.

Define $$T_1 = \begin{bmatrix} I & 0 \\ -\ Z'V^{-1}X(X'V^{-1}X)^{-} & I \end{bmatrix}$$

Transform (6.3) by T_1

$$\begin{bmatrix} X'V^{-1}X & X'V^{-1}Z \\ 0 & Z'V^{-1}Z\text{-}Z'V^{-1}X(X'V^{-1}X)^{-}X'V^{-1}Z \end{bmatrix} \begin{bmatrix} \mu \\ \underline{a} \\ \underline{b} \\ \underline{ab} \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} X'V^{-1}Y \\ Z'V^{-1}Y\text{-}Z'V^{-1}X(X'V^{-1}X)^{-}X'V^{-1}Y \end{bmatrix} . \qquad (6.4)$$

This leads to a solution for β_1 and β_2, say

$$\begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = (Z'(V^{-1} - V^{-1}P_V)Z)^{-}Z'(V^{-1} - V^{-1}P_V)Y. \tag{6.5}$$

In order to test hypotheses about the ab interaction, define $X_1 = [\underline{1}:X_a:X_b]$, $P_{X_1} = X_1(X_1'V^{-1}X_1)^{-}X_1'V^{-1}$

$$\text{and} \quad T_2 = \begin{bmatrix} I & 0 & 0 \\ -X_{ab}V^{-1}X_1(X_1'V^{-1}X_1)^{-} & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Transform (6.4) by T_2 to get

$$\begin{bmatrix} X_1'V^{-1}X_1 & X_1'V^{-1}X_{ab} & X_1'V^{-1}Z \\ & X_{ab}'(V^{-1}-V^{-1}P_{X_1})X_{ab} & X_{ab}'(V^{-1}-V^{-1}P_{X_1})Z \\ & & Z'(V^{-1}-V^{-1}P_X)Z \end{bmatrix} \begin{bmatrix} \mu \\ \underline{a} \\ \underline{b} \\ \underline{ab} \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} X_1'V^{-1}Y \\ X_{ab}'(V^{-1}-V^{-1}P_{X_1})Y \\ Z'(V^{-1}-V^{-1}P_X)Y \end{bmatrix}. \tag{6.6}$$

This leads to a solution for $\underline{ab}$, say

$$\underline{ab} = (X_{ab}'(V^{-1}-V^{-1}P_{X_1})X_{ab})^{-}(X_{ab}'(V^{-1}-V^{-1}P_{X_1})Y - X_{ab}'(V^{-1}-V^{-1}P_{X_1})Z\begin{pmatrix}\hat{\beta}_1 \\ \hat{\beta}_2\end{pmatrix}). \tag{6.7}$$

Hypotheses about linear functions among the interaction parameters, i.e., $\lambda'\underline{ab}$ can be tested by replacing V by $\hat{V}$, and then using the approximated t-test developed in section 5.1 in the form

$$t = \frac{\lambda'\hat{\underline{ab}} - \lambda'\underline{ab}}{\sqrt{\hat{var}(\hat{\underline{ab}})}}$$

and using formula (5.1.3) to approximate the degrees of freedom associated with the t-test.

If interest is in split-plot treatment effects, then the analysis must go one step further.

Let $X = [1:X_a]$ and $P_{X_a} = X_a(X'_a V^{-1} X_a)^- X'_a V^{-1}$.

Observe that the vector 1 lies in the space generated by the columns of X_a. Therefore, there exists a vector $\underline{q}$ such that $1 = X_a\underline{q}$. This implies that

$$X_a(X_a'V^{-1}X_a)^- X_a'V^{-1}\underline{1} = 1.$$

Define

$$T_3 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & -X_b'V^{-1}X_a(X_a'V^{-1}X_a)^- & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

Observe that $-X_b'V^{-1}X_a(X_a'V^{-1}X_a)^- X_a'V^{-1}\underline{1} - X_b'V^{-1}\underline{1} = 0$

and $-X_b'V^{-1}X_a(X_a'V^{-1}X_a)^- X_a'V^{-1}X_a - X_b'V^{-1}X_a = 0.$

Transform (6.6) by T_3 to get

$$\begin{bmatrix} 1'V^{-1}1 & 1'V^{-1}X_a & 1'V^{-1}X_b & 1'V^{-1}X_{ab} & 1'V^{-1}Z \\ 1_a'V^{-1}1 & X_a'V^{-1}X_a & X_a'V^{-1}X_b & X_a'V^{-1}X_{ab} & X_a'V^{-1}Z \\ & & X_b'(V^{-1}-V^{-1}P_{X_a})X_b & X_b'(V^{-1}-V^{-1}P_{X_a})X_{ab} & X_b'(V^{-1}-V^{-1}P_{X_a})Z \\ & & & X_{ab}'(V^{-1}-V^{-1}P_{X_a})X_{ab} & X_{ab}'(V^{-1}-V^{-1}P_{X_1})Z \\ & & & & Z'(V^{-1}-V^{-1}P_X)Z \end{bmatrix}$$

$$\cdot \begin{bmatrix} \mu \\ \underline{a} \\ \underline{b} \\ \underline{ab} \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1'V^{-1}Y \\ X_a'V^{-1}Y \\ X_b'(V^{-1}-V^{-1}P_{X_a})Y \\ X_{ab}'(V^{-1}-V^{-1}P_{X_1})Y \\ Z'(V^{-1}-V^{-1}P_X)Y \end{bmatrix} . \tag{6.8}$$

A solution for $\underline{b}$ is given by

$$\hat{\underline{b}} = (X_b'(V^{-1}-V^{-1}P_{X_a})X_b)^-(X_b'(V^{-1}-V^{-1}P_{X_a})Y - X_b'(V^{-1}-V^{-1}P_{X_a})X_{ab}\hat{\underline{ab}}$$

$$- X_b'(V^{-1}-V^{-1}P_{X_a})Z\begin{pmatrix}\hat{\beta}_1 \\ \hat{\beta}_2\end{pmatrix}) . \tag{6.9}$$

A linear combination of the parameters $\underline{b}$ can be tested by using the approximated t-test method of section 5.1 by computing

$$t = \frac{\lambda'\hat{\underline{b}} - \lambda'\underline{b}}{\sqrt{\hat{\text{var}}(\lambda'\hat{\underline{b}})}}$$

and using the formula (5.1.3) to approximate the degrees of freedom associated with the t-test.

Finally, to derive a test on whole plots treatments, let $P_1 = \underline{1}(\underline{1}'V^{-1}\underline{1})^{-}\underline{1}'V^{-1}$.

Define

$$T_4 = \begin{bmatrix} I & 0 & 0 \\ -X_a V^{-1} 1(1'V^{-1}1)^{-} & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

Transform (6.8) by T_4 to get

$$\begin{bmatrix} 1'V^{-1}1 & 1'V^{-1}X_a & 1'V^{-1}X_b & 1'V^{-1}X_{ab} & 1'V^{-1}Z \\ & X_a'(V^{-1}-V^{-1}P_1)X_a & X_a'(V^{-1}-V^{-1}P_1)X_b & X_a'(V^{-1}-V^{-1}P_1)X_{ab} & X_a'(V^{-1}-V^{-1}P_1)Z \\ & & X_b'(V^{-1}-V^{-1}P_{X_a})X_b & X_b'(V^{-1}-V^{-1}P_{X_a})X_b & X_a'(V^{-1}-V^{-1}P_{X_a})Z \\ & & & X_{ab}'(V^{-1}-V^{-1}P_{X_1})X_{ab} & X_{ab}'(V^{-1}-V^{-1}P_{X_1})Z \\ & & & & Z'(V^{-1}-V^{-1}P_X)Z \end{bmatrix} \begin{bmatrix} \mu \\ \underline{a} \\ \underline{b} \\ \underline{ab} \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1'V^{-1}Y \\ X_a'(V^{-1}-V^{-1}P_1)Y \\ X_b'(V^{-1}-V^{-1}P_{X_a})Y \\ X_{ab}'(V^{-1}-V^{-1}P_{X_1})Y \\ Z'(V^{-1}-V^{-1}P_X)Y \end{bmatrix}$$

A solution for $\underline{a}$ is given by

$$\hat{\underline{a}} = (X_a'(V^{-1}-V^{-1}P_1)X_a)^{-}X_a'(V^{-1}-P_1)(Y-X_b\hat{\underline{b}}-X_{ab}\underline{ab}-Z\binom{\hat{\beta}_1}{\hat{\beta}_2}). \qquad (6.11)$$

Now it is possible to test a parametric linear combination of the parameters $\underline{a}$ by using the approximated t-test method of section 5.1 of the form

$$t = \frac{\lambda'\hat{\underline{a}} - \lambda'\underline{a}}{\sqrt{\widehat{\text{var}}(\lambda'\hat{\underline{a}})}}.$$

Formula (5.1.3) is used to approximate the degrees of freedom associated with the t-test.

7. SIMULATION STUDIES

Previous chapters have presented asymptotic and large sample properties of the estimators obtained via generalized least squares based on an estimated variance-covariance matrix. In this chapter some insight into the small sample properties of these estimates will be obtained from a series of small simulations experiments.

These simulations all follow the same general pattern. A set of observations is generated according to a specified split-plot model. The first step in the estimation procedure is to compute values for $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$. In the second step generalized-least-squares estimates of the fixed effects in the model are obtained by using the estimated variance-covariance matrix obtained from the first step. A listing of the Fortran program used for these simulations is in Appendix A. A flow chart is given in Figure 7.1.

7.1 Standard Conditions

The first simulation was performed under standard or ideal conditions as a check on the computer program. Observations for a split-plot experiment with two blocks, four whole plots per block and two split plots per whole plot were generated. Both error components were drawn from N(0,1). Estimates for both σ_1^2 and σ_2^2 were obtained using the MINQUE equations with prior values $\alpha_1 = \alpha_2 = 1$. Note that in this case the estimates are identical with those obtained from the usual analysis of variance. Also in this case the weighted least squares estimates of the fixed effects, i.e., the treatment effects, are

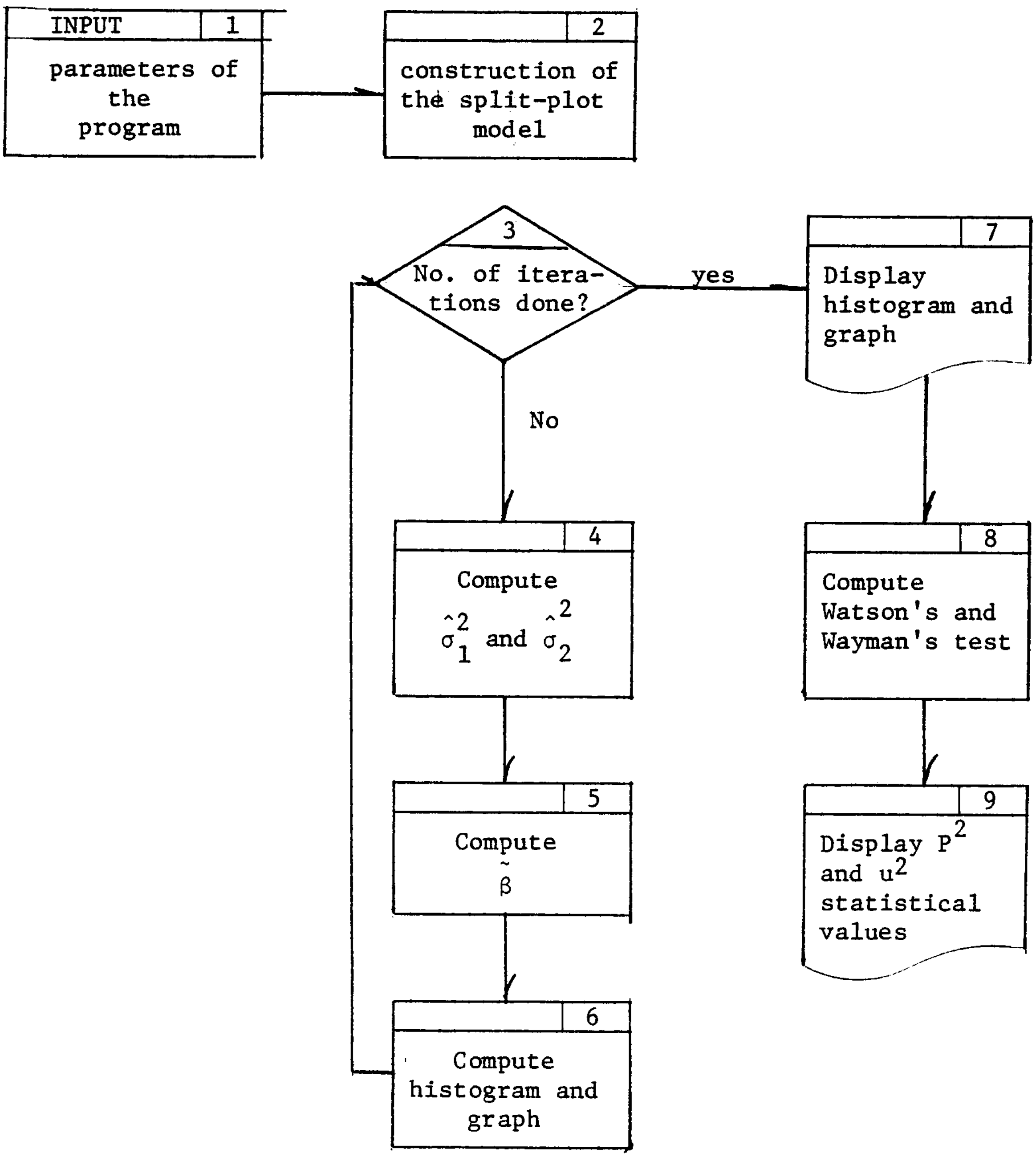

Figure 7.1. Flow chart of the simulation of the behavior of the estimate vector of parameter $\tilde{\beta}$

identical with the usual treatment means. This simulation was repeated 100 times.

The resulting "observations, i.e., estimate of the general mean, are given in Figure 7.2. Clearly, this distribution is close to normal. The Watson's U^2 statistic for testing normality is 0.03563, well beyond the critical value $U^2_{.05} = 0.187$. Neyman's P^2_4 statistic was 1.32074, again well below the critical value $P^2_{4,.05} = 9.49$.

7.2 Non-standard Conditions

The second simulation was performed as follows:
Observations for a split-plot experiment with two blocks, four whole plots per block and two split plots per whole plot with two missing observations, say y_{111} and y_{242}, were generated. Both error components were drawn from N(0,1). Estimates for both σ^2_1 and σ^2_2 were obtained using the MINQUE equations with prior values $\alpha_1 = \alpha_2 = 1$. This simulation was repeated 100 times.

The resulting "observations," i.e., estimates of the general mean, are given in Figure 7.3. The Watson's U^2 statistic for testing normality is 0.02422, well below the critical value $U^2_{.05} = 0.187$. Neyman's P^2_4 statistic was 0.74839, again well below the critical value $P^2_{4,0.05} = 9.49$.

The third simulation was performed as follows:
Observations for a split-plot experiment with two blocks, four whole plots per block and two split plots per whole plot with two missing observations, say y_{111} and y_{242}, were generated. Both error components were drawn for U(-1,1). Estimates for both σ^2_1 and σ^2_2 were obtained using the MINQUE equations with prior values $\alpha_1 = \alpha_2 = 1$. This simulation was repeated 100 times.

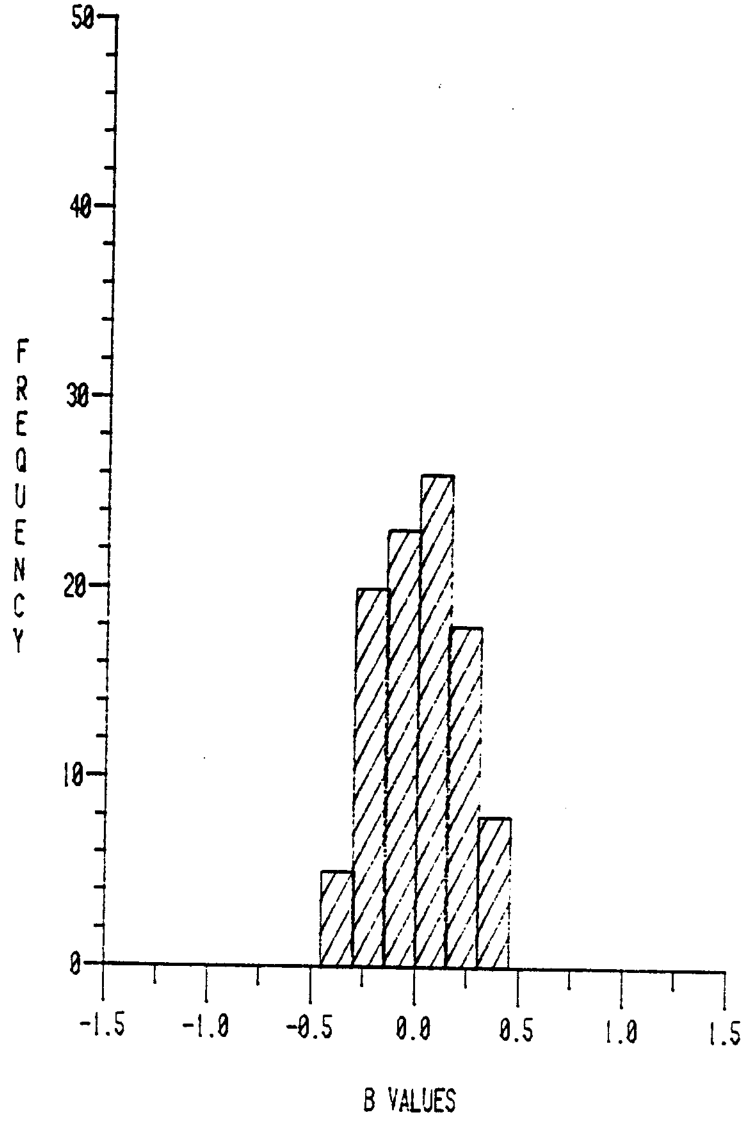

FIGURE 7.2 DISTRIBUTION OF THE ESTIMATED GENERAL MEAN
FOR THE SPLIT PLOT MODEL
UNDER NORMAL DISTRIBUTION, WHEN THE VARIANCE COMPONENTS
WERE REPLACED BY THE MINQUE ESTIMATORS

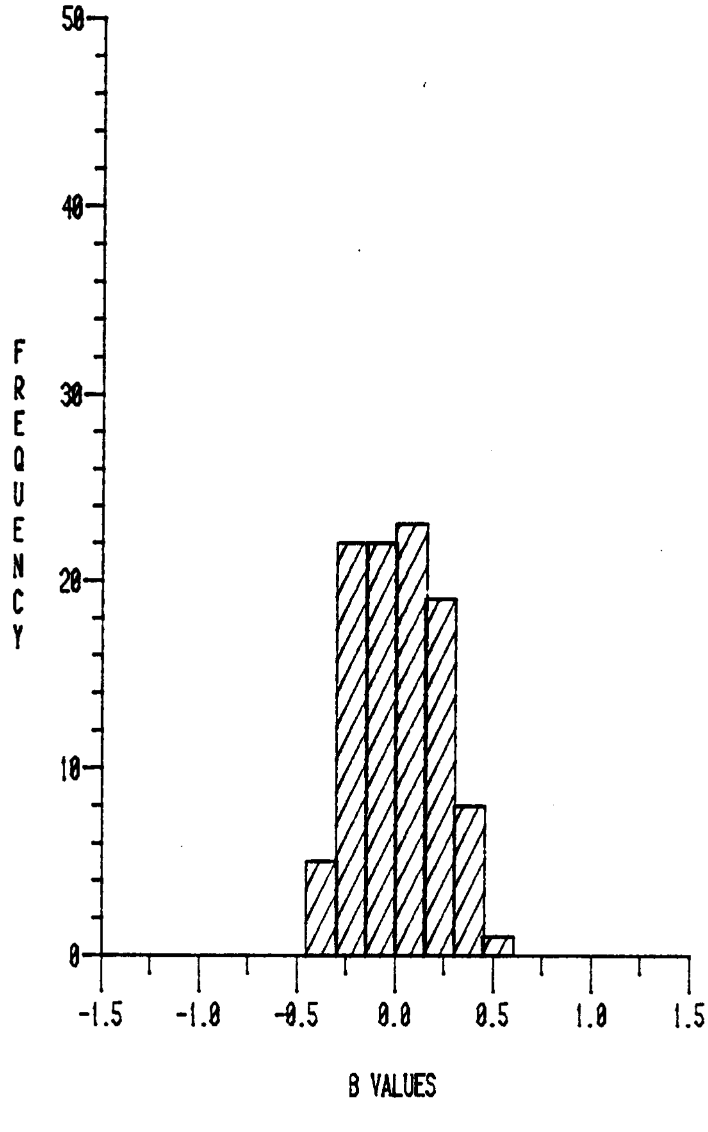

FIGURE 7.3 DISTRIBUTION OF THE ESTIMATED GENERAL MEAN FOR THE SPLIT PLOT MODEL, WITH MISSING OBSERVATIONS UNDER NORMAL DISTRIBUTION, WHEN THE VARIANCE COMPONENTS WERE REPLACED BY THE MINQUE ESTIMATORS

The resulting "observations," i.e., estimates of the general mean, are given in Figure 7.4. The Watson's U^2 statistic for testing normality is 0.04703, well below the critical value $U^2_{0.05} = 0.187$. Neyman's P^2_4 statistic was 1.15893, again well below the critical value $P^2_{4,0.05} = 9.49$.

The fourth simulation was performed as follows:
Observations for a split-plot experiment with two blocks, four whole plots per block and two split plots per whole plot, and two missing observations, y_{111} and y_{242}, were generated. Both error components were drawn from N(0,1). Estimates for both σ^2_1 and σ^2_2 were obtained using the restricted MINQUE estimators defined in section 4.2.2 with prior values $\alpha_1 = \alpha_2 = 1$. This simulation was repeated 100 times.

The resulting "observations," estimates of the general mean, are given in Figure 7.5. The Watson's U^2 statistic for testing normality is 0.02350, well beyond the critical value $U^2_{0.05} = 0.187$. Neyman's P^2_4 statistic was 0.67622, again well below the critical value $P^2_{4,0.05} = 9.49$.

The fifth simulation was performed as follows:
Observations for a split-plot experiment with two blocks, four whole plots per block, two split plots per whole plot and two missing observations, y_{111} and y_{242}, were generated. Both error components were drawn from U(-1,1). Estimates for both σ^2_1 and σ^2_2 were obtained using the restricted MINQUE estimators defined in section 4.2.2 with prior values $\alpha_1 = \alpha_2 = 1$. This simulation was repeated 100 times.

The resulting "observations," estimates of the general mean, are given in Figure 7.6. The Watson's U^2 statistic for testing normality is 0.04925, well beyond the critical value $U^2_{0.05} = 0.187$. Neyman's P^2_4

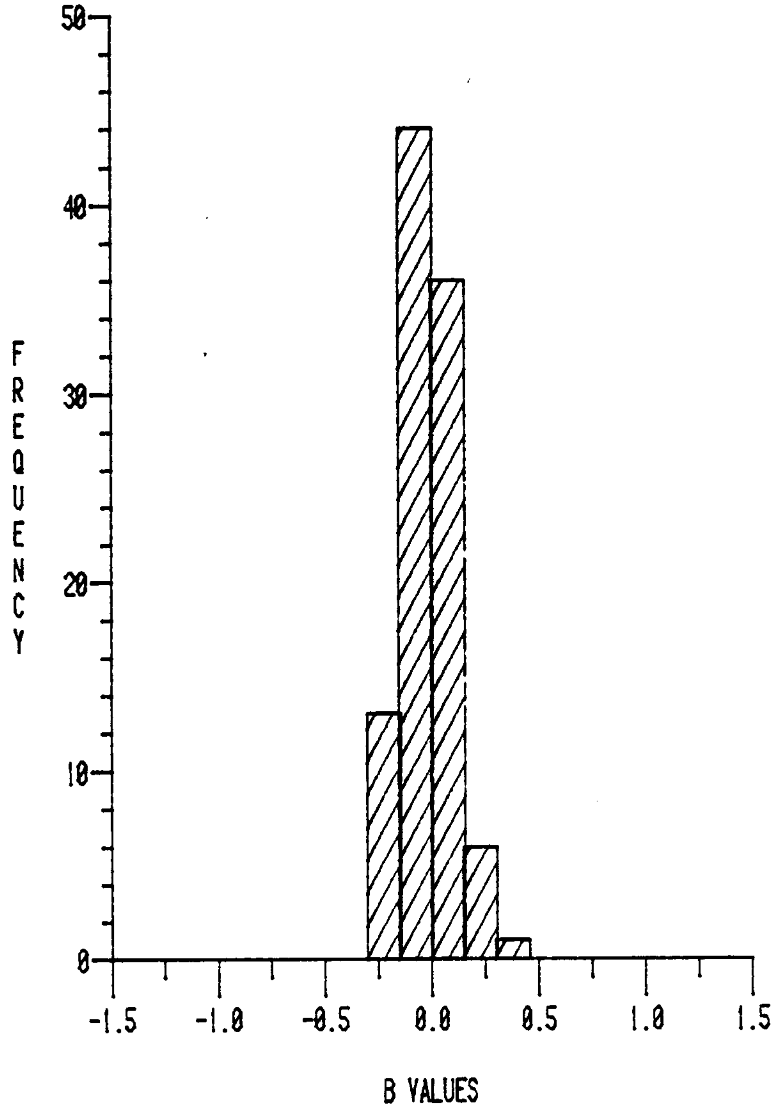

FIGURE 7.4 DISTRIBUTION OF THE ESTIMATED GENERAL MEAN FOR THE SPLIT PLOT MODEL, WITH MISSING OBSERVATIONS UNDER UNIFORM DISTRIBUTION, WHEN THE VARIANCE COMPONENTS WERE REPLACED BY THE MINQUE ESTIMATORS

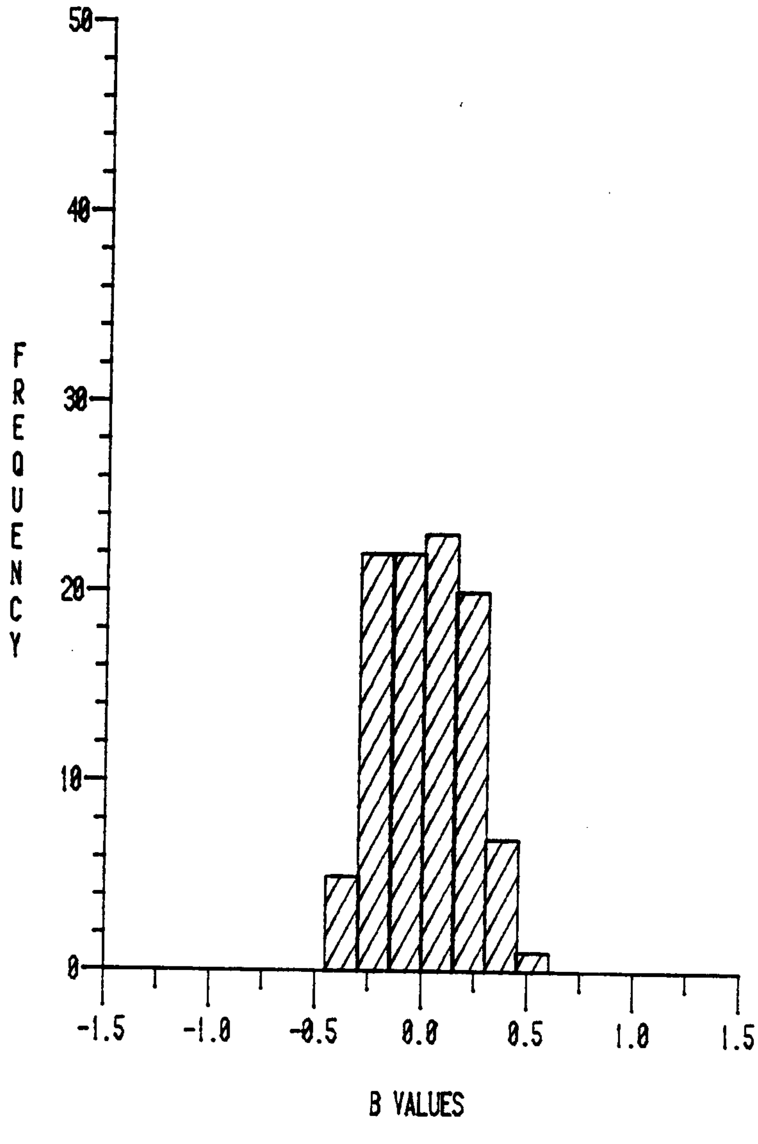

FIGURE 7.5 DISTRIBUTION OF THE ESTIMATED GENERAL MEAN FOR THE SPLIT PLOT MODEL, WITH MISSING OBSERVATIONS UNDER NORMAL DISTRIBUTION, WHEN THE VARIANCE COMPONENTS WERE REPLACED BY THE RESTRICTED MINQUE ESTIMATORS

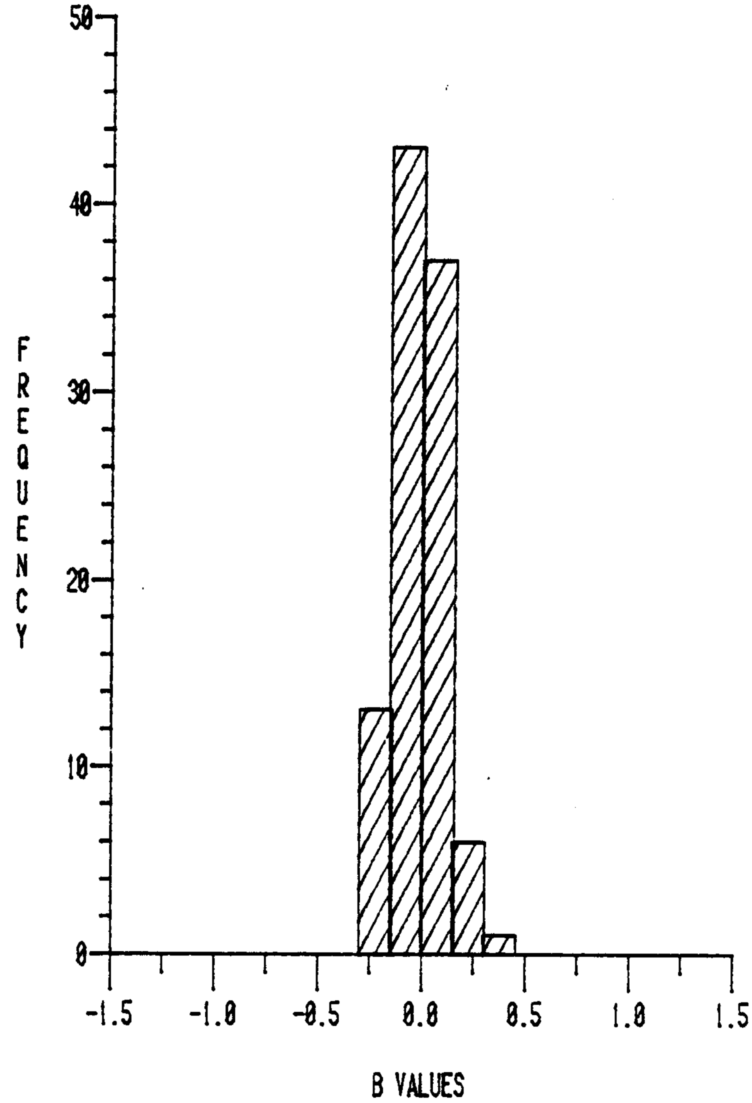

FIGURE 7.6 DISTRIBUTION OF THE ESTIMATED GENERAL MEAN FOR THE SPLIT PLOT MODEL, WITH MISSING OBSERVATIONS UNDER UNIFORM DISTRIBUTION, WHEN THE VARIANCE COMPONENTS WERE REPLACED BY THE RESTRICTED MINQUE ESTIMATORS

statistic was 1.14965. This was again well below the critical value $P^2_{4,0.05} = 9.49$.

The sixth simulation was performed as follows:
Observations for a split-plot experiment with two blocks, four whole plots per block, two split plots per whole plot, and two missing observations, say y_{111} and y_{242}, were generated. Both error components were drawn from N(0,1). Estimates for both σ_1^2 and σ_2^2 were obtained using the positive semi-definite MINQUE estimators defined in section 4.2.3 with prior values $\alpha_1 = \alpha_2 = 1$. This simulation was repeated 100 times.

The resulting "observations," estimates of the general mean, are given in Figure 7.7. The Watson's U^2 statistic for testing normality is 0.03291, well beyond the critical value $U^2_{0.05} = 0.187$. Neyman's P^2_4 statistic was 0.7935. This is again well below the critical value $P^2_{4,0.05} = 9.49$.

The seventh simulation was performed as follows:
Observations for a split-plot experiment with two blocks, four whole plots per block, two split plots per whole plot, and two missing observations, say y_{111} and y_{242}, were generated. Both error components were drawn from U(-1,1). Estimates for both σ_1^2 and σ_2^2 were obtained using the positive semi-definite MINQUE estimators defined in section 4.2.3 with prior values $\alpha_1 = \alpha_2 = 1$. This simulation was repeated 100 times.

The resulting "observations," estimates of the general mean, are given in Figure 7.8. The Watson's U^2 statistic for testing normality is 0.04645, well beyond the critical value $U^2_{0.05} = 0.187$. Neyman's P^2_4 statistic was 1.26313. This is again well below the critical value $P^2_{4,0.05} = 9.49$.

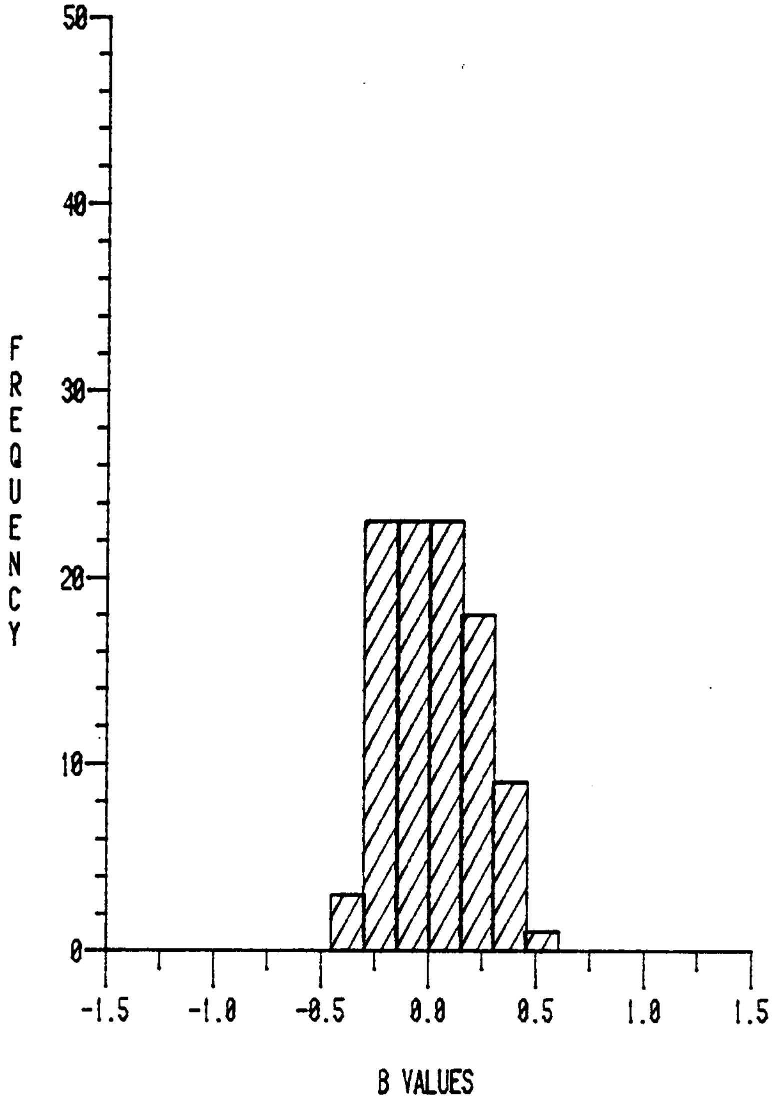

FIGURE 7.7 DISTRIBUTION OF THE ESTIMATED GENERAL MEAN FOR THE SPLIT PLOT MODEL, WITH MISSING OBSERVATIONS UNDER NORMAL DISTRIBUTION, WHEN THE VARIANCE COMPONENTS WERE REPLACED BY THE RESTRICTED MINQUE ESTIMATORS

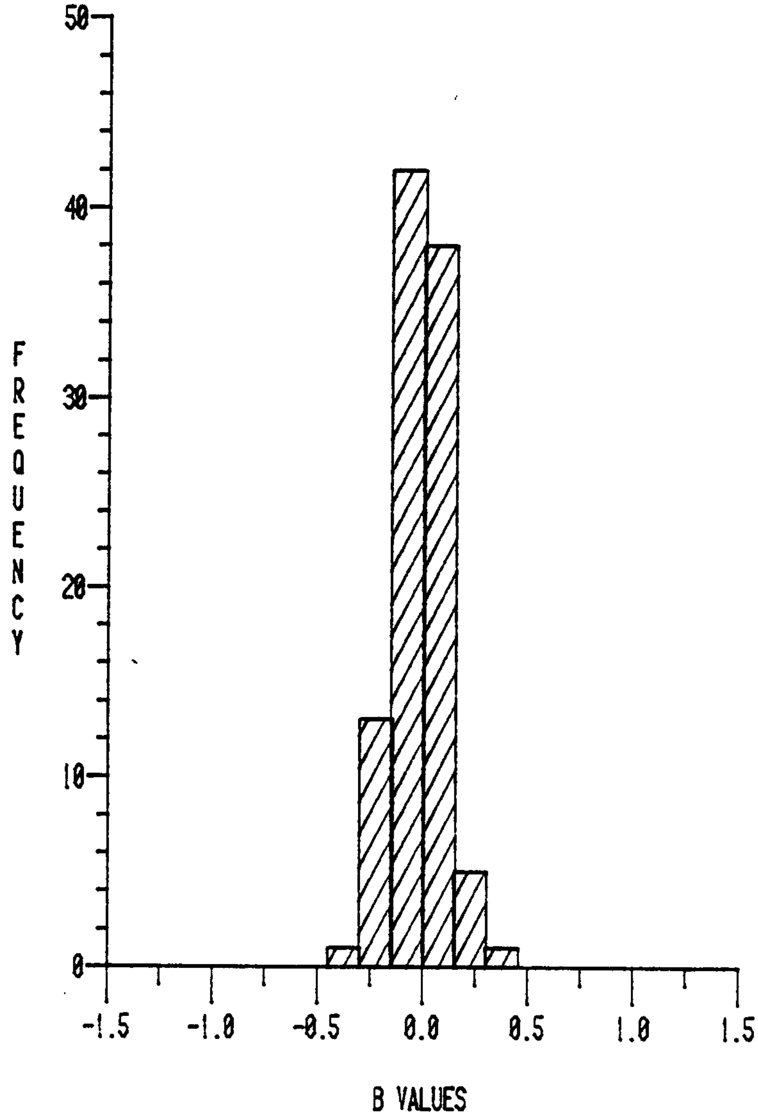

FIGURE 7.8 DISTRIBUTION OF THE ESTIMATED GENERAL MEAN FOR THE SPLIT PLOT MODEL, WITH MISSING OBSERVATIONS UNDER UNIFORM DISTRIBUTION, WHEN THE VARIANCE COMPONENTS WERE REPLACED BY THE P.S.D MINQUE ESTIMATORS

Finally, the last simulation was performed as follows: Observations for a split-plot experiment with two blocks, four whole plots per block and two split plots per whole block were generated. Both error components were drawn from the slash distribution. Estimates for both σ_1^2 and σ_2^2 were obtained using the MINQUE equations with prior values $\alpha_1 = \alpha_2 = 1$. This simulation was repeated 100 times.

The resulting "observations," estimates of the general mean, are given in Figure 7.9. This distribution is symmetrical with tails heavier than the normal. Watson's U^2 statistic for testing normality was 3.9252 which is greater than the critical value $U^2_{.05,\infty} = 0.187$. Neyman's P_4^2 statistic was 65.9737, again greater than the critical value $P^2_{4,0.05} = 9.49$.

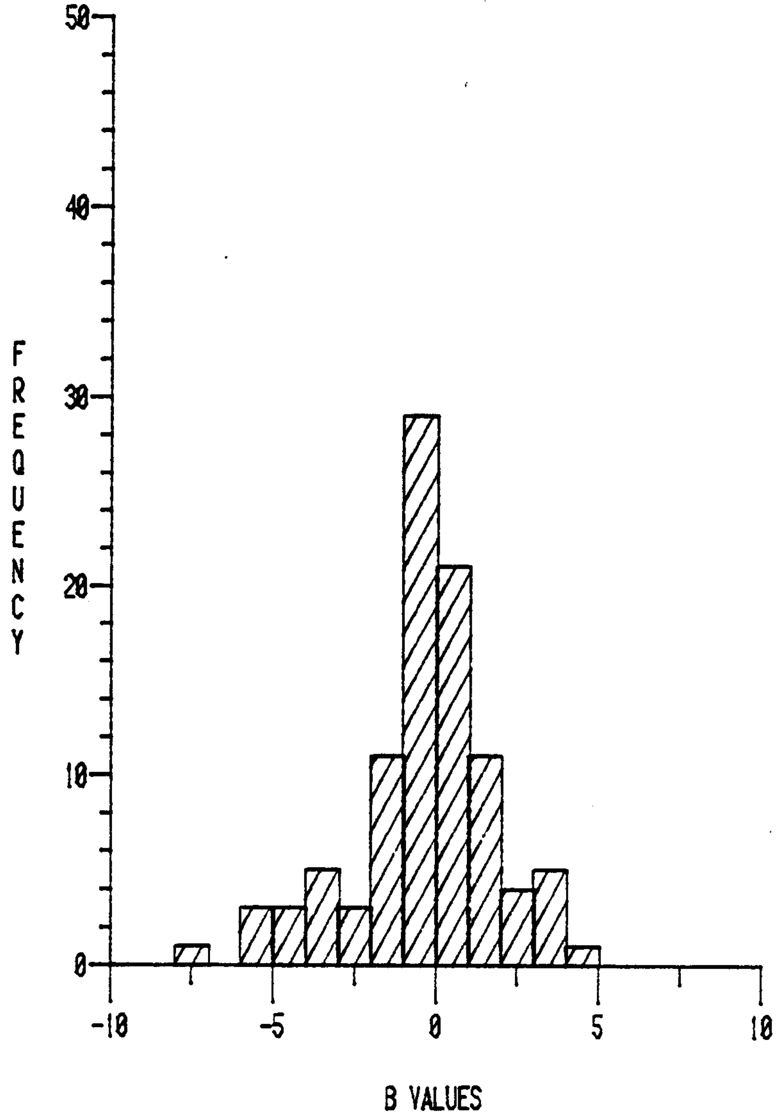

FIGURE 7.9 DISTRIBUTION OF THE ESTIMATED GENERAL MEAN FOR THE SPLIT PLOT MODEL, UNDER SLASH DISTRIBUTION WHEN THE VARIANCE COMPONENTS WERE REPLACED BY THE MINQUE ESTIMATORS

8. SUMMARY AND CONCLUSIONS

The generalized least squares analysis for the split-plot model with missing observations and estimated variance-covariance matrix was discussed in this thesis.

Many methods have been suggested to estimate the variance-covariance matrix, a few of them are: Henderson's method III, Rao's method and Seely's method. The behavior of the generalized least squares estimators of the split-plot model when using an estimated variance-covariance matrix depends on the distribution of the random errors and the method used to estimate the variance-covariance matrix.

In this thesis it was found that the estimated variance-covariance matrix $\hat{V}$, converges in probability to the variance-covariance matrix V, i.e., $\hat{V} \xrightarrow[p]{} V$, when using Seely's method under the invariance condition with respect to the fixed effects. Under the invariance condition Seely's method is equivalent to the MINQUE method.

Also, three procedures were proposed to observe the behavior of the generalized least squares estimator of the parameters β, when the variance-covariance matrix was replaced by its estimate. Under simulation it was found that each one behaves normally under the assumption that the random errors are both normal (0,1) and uniform (-1,1).

In many practical problems, the researcher wants to know how many degrees of freedom he needs to consider in a t-test for testing a linear combination of the parameters when he uses an estimate of the variance-covariance matrix. In this thesis the answer is given in terms of a formula (formula 5.1.3), when the variance-covariance matrix is estimated by the MINQUE method. Also, it is suggested that either prior values or iterated MINQ estimators be used to evaluate the formula. Simulations indicate that this formula provides a good approximation for the degrees of freedom associated with the t-test.

Finally, Appendix A shows explicit formulas for the MINQUE estimators of the variance components for a complete randomized design in both the balanced and the unbalanced case. But the latter was restricted to the condition that if a split plot in the ith whole plot is missing, the same split plot must be missing for all the repetitions of the ith whole plots.

9. RECOMMENDATIONS

It is recommended that in the future an efficient simulation study be performed by using some variance reduction techniques for the different methods used in the simulations of this thesis.

There are several methods used to reduce the variability of an estimator; one of them is the control variates. The goal of this method is to obtain an efficient estimator t of the parameters $\lambda'\beta$. This method is defined as follows:

Let $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ denote the MINQUE estimators of the variance components σ_1^2 and σ_2^2. Let α_1^2 and α_2^2 be prior values of the variance components.

Step 1. Define $t_1 = \frac{1}{n}\sum_{i=1}^{n}\lambda'\tilde{\beta}_i$ where $\tilde{\beta}_i = (X'\hat{V}^{-1}X)^{-}X'\hat{V}^{-1}Y_i$ where $\hat{V} = \hat{V}_1\hat{\sigma}_1^2 + V_2\hat{\sigma}_2^2$ and $Y_i = U_1\varepsilon_1 + U_2\varepsilon_2$ where ε_1 and $\varepsilon_2 \sim N(0,1)$ and U_1 and U_2 are known matrices.

Step 2. Define $t_2 = \frac{1}{n}\sum_{i=1}^{n}\lambda'\hat{\beta}$ where $\hat{\beta}_i = (X'V^{-1}X)^{-}X'V^{-1}Y_i$ where $V = V_1\alpha_1^2 + V_2\alpha_2^2$.

Since $\tilde{\beta}_i$ and $\hat{\beta}_i$ come from the same generalized random numbers, they are likely to be highly positively correlated, and therefore t_1 and t_2 should be positively correlated.

Step 3. Define $t = \lambda'\beta + t_1 - t_2$ as the new estimator of $\lambda'\beta$.

Step 4. Define $var(t) = var(t_1) + var(t_2) - 2\ cov(t_1,t_2)$ and observe that $var(t) < var(t_1)$ if $var(t_2) < 2\ cov(t_1,t_2)$. Since this is true because of the higher positive correlated coefficient, then the estimator t is an efficient estimator of $\lambda'\beta$ with small variance.

Another way to look at the same problem is when there is an especial interest in observing the number of times that the value

$$t = \frac{\lambda'\tilde{\beta}}{\sqrt{\hat{var}(\lambda'\tilde{\beta})}}$$

is rejected under the null hypothesis $\lambda'\beta = 0$.

$$\text{Define } \tilde{t}_i = \begin{cases} 1 & \text{if } \left|\frac{\lambda'\tilde{\beta}}{\sqrt{\hat{var}(\lambda'\tilde{\beta})}}\right| > 1.96 \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{t}_i = \begin{cases} 1 & \text{if } \left|\frac{\lambda\ \hat{\beta}}{\sqrt{\hat{var}(\lambda'\hat{\beta})}}\right| > 1.96 \\ 0 & \text{otherwise} \end{cases}$$

Define the estimated power $t = 0.05 + \sum_{i=1}^{n} \tilde{t}_i - \sum_{i=1}^{n} \hat{t}_i$ and observe that

$$var(\text{estimated power } t) = var(\sum_{i=1}^{n} \tilde{t}_i) + var(\sum_{i=1}^{n} \hat{t}_1) - 2\ cov(\sum_{i=1}^{n} \tilde{t}_i, \sum_{i=1}^{n} \hat{t}_i)$$

and again if the var(estimated power t) $\leq var(\sum_{i=1}^{n} \tilde{t}_i)$, then it is an efficient estimator.

Another method to reduce the variability of an estimator is the antithetic variates. This method is defined as follows:

Step 1. Define $\xi_1, \xi_2, \ldots, \xi_s$ as uniform random numbers and define $1 - \xi_1, 1 - \xi_2, \ldots, 1 - \xi_s$ as also uniform random numbers.

Step 2. With $\xi_1, \ldots, \xi_s$ construct normal desviates such that $U = U_1\varepsilon_1 + U_2\varepsilon_2$ where ε_1 and ε_2 are N(0,1) and U_1 and u_2 are known matrices.

Step 3. With $1 - \xi_1, \ldots, 1 - \xi_s$ construct normal desviates, such that $U_* = U_1\varepsilon_{*1} + U_2\varepsilon_{*2}$ where ε_{*1} and ε_{*2} are N(0,1) and U_1 and u_2 are known matrices.

Step 4. Let $t_1 = \frac{1}{n}\sum_{i=1}^{n} \lambda'\tilde{\beta}_i$ where $\tilde{\beta}_i = (X'\hat{V}^{-1}X)^{-}X'\hat{V}^{-1}Y_i$ where $Y_i = U_i$.

Step 5. Let $t_2 = \frac{1}{n}\sum_{i=1}^{n} \lambda'\tilde{\beta}_{*i}$ where $\tilde{\beta}_{*i} = (X'\hat{V}_*^{-1}X)^{-}X'\hat{V}_*^{-1}Y_{*i}$ where

$Y_{*i} = U_{*i}$.

Step 6. Let $t = \frac{1}{2}(t_1 + t_2)$; this will produce a new estimator such that t_1 and t_2 are negatively correlated.

Step 7. $\mathrm{Var}(t) = \frac{1}{4}\,\mathrm{var}(t_1) + \frac{1}{4}\,\mathrm{var}(t_2) + 2\,\mathrm{cov}(t_1,t_2)$ and since t_1 and t_2 are negatively correlated $\mathrm{var}(t) \leq \mathrm{var}(t_1)$, then t is an efficient estimator.

Another way to look at this is when it is desired to count the number of times that the value

$$t = \frac{\lambda'\tilde{\beta}}{\sqrt{\widehat{\mathrm{var}}(\lambda'\tilde{\beta})}}$$

is rejected under the null hypothesis $\lambda'\beta = 0$.

$$\text{Define } \tilde{t}_{1i} = \begin{cases} 1 & \text{if } \left|\dfrac{\lambda'\tilde{\beta}}{\sqrt{\widehat{\mathrm{var}}(\lambda'\tilde{\beta})}}\right| > t_{f,0.05} \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{t}_{2i} = \begin{cases} 1 & \text{if } \left| \dfrac{\lambda'\tilde{\beta}_*}{\widehat{\text{var}}(\lambda'\tilde{\beta}_*)} \right| > t_{f,0.05} \\ 0 & \text{otherwise} \end{cases}$$

where f is computed with formula (5.1.3).

Define estimated power $\tilde{t} = \frac{1}{2}\left(\sum_{i=1}^{n} \tilde{t}_{1i} + \sum_{i=1}^{n} \tilde{t}_{2i}\right)$ and var(estimated power $\tilde{t}$) $= \frac{1}{4}\,\text{var}(\sum_{i=1}^{n} \tilde{t}_{1i}) + \frac{1}{4}\,\text{var}(\sum_{i=1}^{n} \tilde{t}_{2i}) + 2\,\text{cov}(\sum_{i=1}^{n} \tilde{t}_{1i}, \sum_{i=1}^{n} \tilde{t}_{2i})$ and again var(estimated power $\tilde{t}$) $\leq \text{var}(\sum_{i=1}^{n} \tilde{t}_{1i})$.

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APPENDIX A

MINQUE ESTIMATORS FOR THE SPLIT-PLOT ERROR VARIANCES

Consider a split-plot design arranged in randomized complete blocks.

Let $y_{ijh} = \mu + \gamma_i + \delta_{ih} + n_j + (\gamma n)_{ij} + e_{ijh}$, where h = 1, ..., r;

i = 1, ..., n_h; j = 1, ..., n_{hi}.

Let $n = \sum_{hi}\sum n_{hi}$, a = max $\{n_1, n_2, \ldots, n_r\}$ and b = max $\{n_{11}, \ldots, n_{ra}\}$.

In matrix notation this becomes $Y = X\beta + U_1\delta + U_2e$ where Y is a n x 1 vector of observations, X is a n x (1 + a + b + 2b) matrix, β is a (1 + a + b + ab) x 1 vector of parameters, U_1 is an n x ra matrix, δ is an ra x 1 vector of random errors, U_2 is an n x n identity matrix and e is an n x 1 vector of random errors.

Let α_1^2 and α_2^2 be prior values or guesses for the variance components σ_1^2 and σ_2^2 and use this to write $V = \text{BDMATX}(J_{hi}\alpha_1^2 + I_{hi}\alpha_2^2)$ where J_{hi} and I_{hi} are of order n_{hi} x n_{hi}, and $V^{-1} = \text{BDMATX}\ (I_{hi}a_o + a_{hi}J_{hi})$ where $a_o = \alpha_2^{-2}$ and $a_{hi} = -\ \alpha_1^2(\alpha_2^2(n_{hi}\alpha_1^2 + \alpha_2^2))^{-1}$.

Note that $X = \text{CMATX}(X_{hi})$ where X_{hi} is an n_{hi} x (1 + a + b + ab) block matrix that contains the rows of the ith whole plot of the h'th repetition.

$$\text{Rewrite } X_{h_i} = [X_{hi1}:X_{hi2}: \ldots :X_{hit}]$$

$$= \text{RMATX}(X_{hi_\ell};\ \ell = 1, \ldots, t)$$

where t is the number of different terms which are included in the fixed effects part of the split-plot model. In the present example $t = 4$. The first submatrix

$X_{hi1} = \underline{1}_{hi}$ is an $n_{hi} \times 1$ vector of ones due to the mean in the model.

X_{hi2} is an $n_{hi} \times a$ matrix of the form $[\underline{0}: \ldots :\underline{0}:\underline{1}:0: \ldots :0]$ where the column 1 is in the ith column.

X_{hi3} is an $n_{hi} \times b$ matrix which has exactly one non-zero element equal to one in each row. This matrix can be formed from an identity matrix of order $b \times b$ by deleting the jth row of that matrix if the split plot of the ith whole plot is missing. It is also possible to write X_{hi3} = CMATX($\underline{e}_k'$) where $\underline{e}_k'$, $k = 1, \ldots, n_{ka}$ is a unitary vector.

X_{hi4} = RMATX(D_k^i; $k = 1,2, \ldots, a$) is an $n_{hi} \times ab$ row matrix where the block matrices are of order $n_{hi} \times a$ and of the form

$$D_k^i = \begin{cases} X_{hi3} & \text{if } i = k \\ 0 & \text{otherwise.} \end{cases}$$

Alternatively $X_{hi4} = [0: \ldots :0:X_{hi3}:0: \ldots :0]$, X_{hi3} is the ith position.

These definitions are now used to compute $X'V^{-1}X$.

$$X'V^{-1}X = \text{RMATX}(X_{hi}') \cdot \text{BDMATX}(a_o I_{hi} + a_{hi} J_{hi}) \cdot \text{CMATX}(X_{hi})$$

$$= \text{RMATX}\ (X_{hi}') \cdot \text{CMATX}(a_o X_{hi} + a_{hi} J_{hi} X_{hi})$$

$$= \sum_{hi}\sum a_o X_{hi}' X_{hi} + \sum_{hi}\sum a_{hi} X_{hi}' J_{hi} X_{hi}.$$

Let $\delta_{hi}^j = \begin{cases} 0 & \text{if the jth split plot treatment is missing in the ith whole plot of the hth repetition} \\ 1 & \text{otherwise.} \end{cases}$

Observe

$$
X_{hi}'X_{hi} = \begin{bmatrix}
n_{hi} & 0,\ldots,0, & n_{hi}, & 0,\ldots,0 & \delta_{hi}^{1},\ldots,\delta_{hi}^{b} & 0,\ldots,0, & \delta_{hi}^{1},\ldots,\delta_{hi}^{b}, & 0,\ldots,0 \\
 & 0 & & & 0 & 0,\ldots,0, & 0 & 0,\ldots,0 \\
 & \ddots & & & \vdots & \vdots & & \vdots \\
 & 0 & & & 0 & 0,\ldots,0, & 0 & 0,\ldots,0 \\
 & & n_{hi} & & \delta_{hi}^{1},\ldots,\delta_{hi}^{b} & 0,\ldots,0, & \delta_{hi}^{1},\ldots,\delta_{hi}^{b}, & 0,\ldots,0 \\
 & & & 0 & 0 & 0,\ldots,0, & 0 & 0,\ldots,0 \\
 & & & \ddots & \vdots & \vdots & & \\
 & & & 0 & 0 & 0,\ldots,0, & 0 & 0,\ldots,0 \\
 & & & & \delta_{hi}^{1} & & \delta_{hi}^{1} & 0 \\
 & & & & \ddots & & \ddots & \\
 & & & & \delta_{hi}^{b} & & \delta_{hi}^{j} & \\
 & & & & & 0 & & \\
 & & & & & \ddots & & \\
 & & & & & 0 & 0 & \\
 & & & & & & \delta_{hi}^{1} & 0 \\
 & & & & & & \ddots & \\
 & & & & & & \delta_{hi}^{b} & \\
 & & & & & & & 0 \\
 & & & & & & & \ddots \\
 & & & & & & & 0
\end{bmatrix}
$$

Then

$$
a_o \sum_{h}\sum_{i} X_{hi}'X_{hi} = a_o \begin{bmatrix}
n & n_1 & \cdots & n_a & \delta_{++}^{1} \cdots \delta_{++}^{b} & \delta_{+1}^{1} \cdots \delta_{+1}^{b}, & \ldots, & \delta_{+a}^{1}, \cdots \delta_{+2}^{b} \\
 & n_1 & & & \delta_{+1}^{1} \cdots \delta_{+1}^{b} & \delta_{+1}^{1} \cdots \delta_{+1}^{b} & & \\
 & & \ddots & & & & & \\
 & & & n_a & \delta_{+a}^{1} \cdots \delta_{+a}^{b} & & & \delta_{+a}^{1}, \ldots, {}^{b}_{+a} \\
 & & & & \delta_{++}^{1} & \delta_{+1}^{1} & & \delta_{+a}^{1} \\
 & & & & \ddots & \ddots & & \ddots \\
 & & & & \delta_{++}^{b} & \delta_{+1}^{b} & & \delta_{+a}^{b} \\
 & & & & & \delta_{+1}^{1} & & \\
 & & & & & \ddots & & \\
 & & & & & \delta_{+1}^{b} & & \\
 & & & & & & \ddots & \\
 & & & & & & & \delta_{+a}^{1} \\
 & & & & & & & \ddots \\
 & & & & & & & \delta_{+a}^{b}
\end{bmatrix}
$$

where $n = \sum_{h}\sum_{i} n_{hi}$, $n_i = \sum_{h} n_{hi}$, $\delta_{++}^{j} = \sum_{h}\sum_{i} \delta_{hi}^{j}$, $\delta_{+i}^{j} = \sum_{h} \delta_{hi}^{j}$.

Observe

$$X'_{hi}J_{hi}X_{hi} = \begin{bmatrix}
n^2_{hi} & 0,\ldots,0, & n^2_{hi}, & 0,\ldots,0 & n_{hi}\,\delta^1_{hi},\ldots,n_{hi}\delta^b_{hi} & 0\,,\ldots,0\,, & n_{hi}\,\delta^1_{hi},\ldots,n_{hi}\delta^b_{hi}, & 0\,,\ldots,0 \\
 & \begin{matrix}0 & & \\ & \ddots & \\ & & 0\end{matrix} & & & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \\
 & & n^2_{hi} & & n_{hi}\,\delta^1_{hi},\ldots,n_{hi}\delta^b_{hi} & 0\,,\ldots,0\,, & n_{hi}\,\delta^1_{hi},\ldots,n_{hi}\delta^b_{hi}, & 0\,,\ldots,0 \\
 & & & \begin{matrix}0 & & \\ & \ddots & \\ & & 0\end{matrix} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \\
 & & & & \begin{matrix}\delta^1_{hi}\,\delta^1_{hi},\ldots,\delta^1_{hi}\delta^b_{hi}\\ \vdots \\ \delta^b_{hi}\,\delta^1_{hi},\ldots,\delta^b_{hi}\delta^b_{hi}\end{matrix} & \begin{matrix}0\,,\ldots,0\\ \vdots\\ 0\,,\ldots,0\end{matrix} & \begin{matrix}\delta^1_{hi}\,\delta^1_{hi},\ldots,\delta^1_{hi}\delta^b_{hi},\\ \vdots \\ \delta^b_{hi}\,\delta^1_{hi},\ldots,\delta^b_{hi}\delta^b_{hi},\end{matrix} & \begin{matrix}0\,,\ldots,0\\ \\ 0\,,\ldots,0\end{matrix} \\
 & & & & & \begin{matrix}0\,, & & \\ & \ddots & \\ & & 0\end{matrix} & & \\
 & & & & & & \begin{matrix}\delta^1_{hi}\,\delta^1_{hi},\ldots,\delta^b_{hi}\delta^b_{hi}\\ \vdots \\ \delta^b_{hi}\ \ {}_{hi},\ldots,\delta^b_{hi}\delta^b_{hi}\end{matrix} & \\
 & & & & & & & \begin{matrix}0\,, & & \\ & \ddots & \\ & & 0\end{matrix}
\end{bmatrix}$$

Thus,

$$\sum\sum_{hi} a_{hi} X'_{hi1} J_{hi} X_{hi1} = \sum\sum_{hi} a_{hi} n_{hi}^2,$$

$$\sum\sum_{hi} a_{hi} X'_{hi1} J_{hi} X_{hi2} = [\sum_h a_{hi} n_{hi}^2, \ \ldots, \ {}_{h} a_{ha} n_{ha}^2],$$

$$\sum\sum_{hi} a_{hi} X'_{hi1} J_{hi} X_{hi3} = [\sum\sum_{hi} a_{hi} n_{hi} \delta_{hi}^1, \ \ldots, \ \sum\sum_{hi} a_{hi} n_{hi} \delta_{hi}^b],$$

$$\sum\sum_{hi} a_{hi} X'_{hi1} J_{hi} X_{hi4} = [\sum_h a_{hi} n_{hi} \delta_{hi}^1, \ \ldots, \ \sum_h a_{ha} n_{ha} \delta_{ha}^b],$$

$$\sum\sum_{hi} a_{hi} X'_{hi2} J_{hi} X_{hi2} = \mathrm{BDMATX}(\sum_h a_{hi} n_{hi}^2; \ i = 1,2, \ \ldots, \ a),$$

$$\sum\sum_{hi} a_{hi} X'_{hi2} J_{hi} X_{hi3} = \begin{bmatrix} \sum_h a_{hi} n_{hi} \delta_{hi}^1, & \ldots, & \sum_h a_{hi} n_{hi} \delta_{hi}^b \\ \\ \sum_h a_{ha} n_{ha} \delta_{ha}^1, & \ldots, & \sum_h a_{ha} n_{ha} \delta_{ha}^b \end{bmatrix},$$

$$\sum\sum_{hi} a_{hi} X'_{hi2} J_{hi} X_{hi4} = \begin{bmatrix} \sum_h a_{hi} n_{hi} \delta_{hi}^1, \ \ldots, \ \sum_h a_{ha} n_{hi} \delta_{hi}^b & & \\ & \ddots & \\ & & \sum_h a_{ha} n_{ha} \delta_{ha}^1, \ \ldots, \ \sum_h a_{ha} n_{ha} \delta_{ha}^b \end{bmatrix},$$

$$\sum\sum_{hi} a_{hi} X'_{hi3} J_{hi} X_{hi3} = \begin{bmatrix} \sum\sum_{hi} a_{hi} \delta_{hi}^1 \delta_{hi}^1, & \ldots, & \sum\sum_{hi} a_{hi} \delta_{hi}^1 \delta_{hi}^b \\ \\ \sum\sum_{hi} a_{hi} \delta_{hi}^b \delta_{hi}^1, & \ldots, & \sum\sum_{hi} a_{hi} \delta_{hi}^b \delta_{hi}^b \end{bmatrix},$$

$$\sum\sum_{hi} a_{hi} X'_{hi3} J_{hi} X_{hi}{}^4 = \begin{bmatrix} \sum_h a_{hi} \delta_{hi}^1 \delta_{hi}^1, \ldots, {}_{h} a_{hi} \delta_{hi}^1 \delta_{hi}^1 & & \sum_h a_{ha} \delta_{ha}^1 \delta_{ha}^1, \ldots, \sum_h a_{ha} \delta_{ha}^1 \delta_{ha}^b \\ \vdots & \cdots & \vdots \\ \sum_h a_{hi} \delta_{hi}^b \delta_{hi}^1, \ldots, \sum_h a_{hi} \delta_{hi}^b \delta_{hi}^b & & \sum_h a_{ha} \delta_{ha}^b \delta_{ha}^1, \ldots, \sum_h a_{ha} \delta_{ha}^b \delta_{ha}^b \end{bmatrix},$$

and

$$\sum_{h}\sum_{i} a_{hi} X'_{hi4} J_{hi} X_{hi4} = \begin{bmatrix} \begin{matrix} \sum_h a_{hi}\delta^1_{hi}\delta^1_{hi}, \ldots, \sum_h a_{hi}\delta^1_{hi}\delta^b_{hi} \\ \vdots \\ \sum_h a_{hi}\delta^b_{hi}\delta^1_{hi}, \ldots, \sum_h a_{hi}\delta^b_{hi}\delta^b_{hi} \end{matrix} & \\ & \ddots \\ & & \begin{matrix} \sum_h a_{ha}\delta^1_{ha}\delta^1_{ha}, \ldots, \sum_h a_{ha}\delta^1_{ha}\delta^b_{ha} \\ \vdots \\ \sum_h a_{ha}\delta^b_{ha}\delta_{ha}, \ldots, \sum_h a_{ha}\delta^b_{ha}\delta^b_{ha} \end{matrix} \end{bmatrix}$$

Let each whole plot treatment have the same number of repetitions, that is $n_i = a$ for all i.

Now consider the MINQUE estimators for the variance components in these two cases: first, when there is no missing split-plot treatment and, second, when there are missing split-plot treatments.

If there are no missing observations, then $n_{hi} = b$ for all h,i and

$$\sum_{h}\sum_{i} X'_{hi} X_{hi} = \begin{bmatrix} rab & rb\underline{1}' & ra1' & r\underline{1}' \\ rb\underline{1} & rbI & rJ & rIx\underline{1}' \\ ra\underline{1} & rJ & raI & r\underline{1}xI \\ r\underline{1} & rIx\underline{1} & r\underline{1}xI & rI \end{bmatrix}$$

Observe that $a_{hi} = -\alpha_1^2(\alpha_2^2(b\alpha_1^2 + \alpha_2^2))^{-1}$ for all h and i, so let $a_1 = a_{hi}$ for all h and i.

Thus,

$$\sum_{h}\sum_{i} X'_{hi} J_{hi} X_{hi} = \begin{bmatrix} rab^2 & rb^2\underline{1}' & rab\underline{1}' & rb\underline{1}' \\ rb^2\underline{1} & rb^2I & rbJ & rbIx\underline{1}' \\ rab\underline{1} & rbJ & raJ & rJ \\ rb\underline{1} & rbIx\underline{1} & rJ & r\cdot BDMATX(J) \end{bmatrix}$$

and hence

$$X'V^{-1}X = \begin{bmatrix} rab(a_o+a_1b) & rb(a_o+a_1b)\underline{1}' & ra(a_o+a_1b)\underline{1}' & r(a_o+a_1b)\underline{1}' \\ rb(a_o+a_1b)\underline{1} & rb(a_o+a_1b)I & r(a_o+a_1b)J & r(a_o+a_1b)I\otimes\underline{1}' \\ ra(a_o+a_1b)\underline{1} & r(a_o+a_1b)J & ra(a_oI+a_1J) & r(a_o\underline{1}\otimes I+a_1J) \\ r(a_o+a_1b)\underline{1} & r(a_o+a_1b)I\otimes\underline{1} & r(a_o\underline{1}\otimes I+a_1J) & r\cdot BDMATX(a_oI+a_1J) \end{bmatrix}$$

Notice that the columns in the design matrix X that correspond to the mean, whole plots and split plots are linear combinations of the columns of the interaction between whole and split plots. Given the inverse of the matrix $r\cdot BDMATX(a_oI+a_1J)$, the generalized inverse of the matrix $X'V^{-1}X$ can be computed. Therefore,

$$C = (C'V^{-1}X)^{-} = \begin{bmatrix} 0 & 0 \\ & \\ 0 & r^{-1}\ BDMATX(b_oI_{n_i}+b_1J_{n_i}) \end{bmatrix}$$

where $b_o = a_o^{-1}$ and $b_1 = -\ a_1\ a_o^{-1}(ba_1+a_o)^{-1}$.

Recall that $a_o = \alpha_2^{-2}$ and $a_1 = -\alpha_1^2\alpha_2^{-2}(b\alpha_1^2 + \alpha_2^2)^{-1}$, then $b_o = \alpha_2^2$ and $b_1 = \alpha_1^2$.

The quadratic form $Y'Q_VV_1Q_VY$ can now be developed.

Recall that $V_1 = V_1^bV_1^b$, then

$$V_1^bQ_VY = V_1^bV^{-1}(I-X(X'V^{-1}X)^{-}X'V^{-1})Y$$

$$= V_1^bV^{-1}Y - V_1^bV^{-1}X(X'V^{-1}X)^{-}X'V^{-1}Y\,. \qquad (A.1)$$

Observe that $V_1^bV^{-1}Y = V_1^b\cdot CMATX((a_oI_{hi} + a_iJ_{hi})Y_{hi})$

$$= CMATX(n_{hi}^{1/2}\ J_{hi}(a_oI_{hi} + a_iJ_{hi}\ )Y_{hi})$$

$$= \text{CMATX}(n_{hi}^{1/2}(a_o + a_1 n_{hi})J_{hi}Y_{hi})$$

$$= \text{CMATX}(b^{1/2}(a_o + a_1 b)y_{hit}1_{hi}) . \qquad (A.2)$$

Also observe that

$$V_1^b V^{-1} X(X'V^{-1}X)^- X'V^{-1}Y = V_1^b V^{-1} X(X'V^{-1}X)^- \text{RMATX}(X'_{hi}) \cdot \text{CMATX}((a_o I_{hi} + a_1 J_{hi})Y_{hi})$$

$$= \sum_{hi}\sum V_1^b V^{-1} \cdot \text{CMATX}(X_{h'i'})(CX'_{hi}(a_o I_{hi} + a_1 J_{hi})Y_{hi})$$

$$= \sum_{hi}\sum V_1^b V^{-1} \cdot \text{CMATX}(X_{h'i'} CX'_{hi}(a_o I_{hi} + a_i J_{hi})Y_{hi}) . \qquad (A.3)$$

Note that $X_{h'i'} CX'_{hi} = [X_{h'i'1} : X_{h'i'2} : X_{h'i'3} : X_{h'i'4}]$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^{-1} \cdot \text{BDMATX}(b_o I_b + b_1 J_b) \end{bmatrix} \begin{bmatrix} X_{hi1} \\ X_{hi2} \\ X_{hi3} \\ X_{hi4} \end{bmatrix}$$

$$= X_{h'i'4}(r^{-1} \cdot \text{BDMATX}(b_o I_b + b_r J_b))X_{hi4}$$

$$= r^{-1} X_{hi4} \cdot \text{BDMATX}(b_o I_i + b_1 J_b)X_{hi4}$$

$$= r^{-1} X_{hi3}(b_o I_b + b_1 J_o)X_{hi3}$$

$$= r^{-1}(B_o I_b + b_1 J_o) .$$

Then A.3 becomes

$$= \sum_h V_1^b V^{-1} \cdot \text{CMATX}(r^{-1}(b_o I_b + b_1 J_b)(a_o I_{h'i'} + a_i J_{h'i'})Y_{h'i'})$$

$$= {}_h V_1^b V^{-1} \cdot \text{CMATX}(r^{-1} I_{h'i'} Y_{h'i'})$$

$$= V_1^b V^{-1} \cdot \text{CMATX}(r^{-1} Y_{+i})$$

$$= \mathrm{CMATX}(r^{-1}b^{1/2}(a_o + a_1 b)J_{ki}Y_{+i})$$

$$= \mathrm{CMATX}(r^{-1}b^{1/2}(a_o + a_1 b)y_{+i+}\underline{1}_{hi}) . \qquad \text{(A.4)}$$

From (A.2) and (A.4)

$$V_1^b Q_V Y = \mathrm{CMATX}(b^{1/2}(a_o + a_1 b)(y_{hi+} - r^{-1}y_{+i+})\underline{1}_{hi}) .$$

Therefore $Y'Q_V V_1 Q_V Y = \sum_{hi}\sum (a_o + a_1 b)^2 (y_{hi+} - r^{-1}y_{+i+})^2$

Similarly $Y'Q_V Q_V Y = \sum\sum_{hij}\sum (a_o y_{hij} + a_1 y_{hi+} - r^{-1}a_o y_{+ij} - r^{-1}a_1 y_{+i+})^2$

$$= \sum\sum_{hij}\sum a_o^2 (y_{hij} - r^{-1}y_{+ij})^2 + \sum_{hi}\sum a_1(2a_o + a_1 b) \cdot (y_{hi+} - r^{-1}y_{+i+})^2$$

The $\mathrm{tr}(Q_V V_1 Q_V V_1)$ now is developed.

From (3.4.2.21)

$$\begin{aligned} \mathrm{tr}(Q_V V_1 Q_V V_1) = {} & \alpha_1^4(rab-q) - 2\alpha_1^{-4}\alpha_2^2(\sum_{hi}\sum(a_o+a_1)b - a_o q - \sum_{hi}\sum(a_o+a_1 b)t_{hi}) \\ & + \alpha_1^{-4}\alpha_2^4(\sum_{hi}\sum(a_o^2 + 2a_o a_1 + a_1^2 b)b - 2(a_o^2 q + \sum_{hi}\sum(2a_o^2 a_1 + 3a_o a_1^2 b + a_1^3 b^2)t_{hi}) \\ & + \sum_{hh'}\sum_{ii'}\sum\sum (a_o^2 t_{1_{h'i'hi}} + 2a_o(2a_o a_1 + a_1^2 b)t_{2h'i'hi} \\ & + (2a_o a_1 + a_1^2 b)(2a_o a_1 + a_1^2 b)t_{3_{h'i'hi}})) \end{aligned}$$

where $q = R(X)$,

$$t_{hi} = \mathrm{tr}(CX'_{hi}J_{hi}X_{hi})$$

$$= \mathrm{tr}(X_{hi}CX'_{hi}J_{hi})$$

$$= \mathrm{tr}(X_{hi4}r^{-1}\cdot \mathrm{BDMATX}(b_o I_{n_i} + b_1 J_{n_i})\cdot X'_{hi4}J_{hi})$$

Finally, observe that $t_{3h'i'hi} = 0$ if $i \neq i'$, then for $i = i'$

$$t_{3h'ihi} = r^{-2}(b_o+b_1b)^2b^2. \tag{A.8}$$

Therefore, from (A.5), (A.6), and (A.7),

$$\begin{aligned} tr(Q_V V_1 Q_V V_1) = \ & \alpha_1^4(rab-q) - 2\alpha_1^{-4}\alpha_2^2(ra(a_o+a_1)b - aq \\ & - ar(r^{-1}(b_o-b_1b)b)(a_o+a_1b)) \\ & + \alpha_1^{-4}\alpha_2^4(ra(a_o^2+2a_oa_1+a_1^2b)b \\ & - 2(a_o^2q+ra(2a_o^2a_1+3a_oa_1^2b+a_1^3b^2)(r^{-1}(b_o+b_1b)b) \\ & + r^2a(a_o^2(r^{-1}b(b_o^2+2b_ob_1+b_1^2b)) \\ & + 2a_o(2a_o+a_1b)r^{-2}(b_o+b_1b)^2b \\ & + a_1^2(2a_o+a_1b)(2a_o+a_1b)r^{-2}(b_o+b_1b)^2b^2). \end{aligned}$$

Also, from (3.4.2.22) and (A.5), (A.6) and (A.7)

$$\begin{aligned} tr(Q_V V_1 Q_V) = \ & \alpha_1^{-2}(\sum_{hi}\sum(a_o+a_1)b - a_oq\sum_{hi}\sum a_1(a_o+a_1b)r^{-1}(b_o+b_1b)b) \\ & - \alpha_1^2\alpha_2^2(\sum_{hi}\sum(a_o^2+2a_oa_1+a_1^2b)b \\ & - 2(a_oq + \sum_{hi}\sum(2a_o^2a_1+3a_oa_1^2b+a_1^3b^2)r^{-1}(b_o+b_1b)b \\ & + \sum_{hh'i}\sum\sum(a_o^2r^{-2}b)(b_o^2 +2b_ob_1+b_1^2b) \\ & + 2a_o(a_o+a_1b)r^{-2}(b_o+b_1b)^2b \\ & + a_1^2(2a_o+a_1b)(2a_o+a_1b)r^{-2}(b_o+b_1b)^2b^2). \end{aligned}$$

Finally, from (3.4.2.20) and (A.5), (A.6) and (A.7)

$$tr(Q_V Q_V) = \sum_{hi}\sum(a_o^2+2a_o a_1+a_1^2 b)b - 2(a_o^2 q+\sum_{hi}\sum(2a_o^2 a_1+3a_o a_1^2 b+a_1^3 b^2)r^{-1}(b_o+b_1 b)b$$

$$+ a_o^2(\sum\sum_{hh'i}\sum(r^{-2}(b(b_o^2+2b_o b_1+b_1^2 b)$$

$$+ 2a_o \sum\sum_{hh'i}\sum(2a_o a_1+a_1^2 b)r^{-2} b(b_o+b_1 b)^2$$

$$+ \sum\sum_{hh'i}\sum(2a_o a_1+a_1^2 b)(2a_o a_1+a_1^2 b)r^{-2} b^2(b_o+b_o b)^2.$$

If there are missing split plots, then in order to compute a generalized inverse for $X'V^{-1}X$, it is first necessary to compute a generalized inverse of the matrix

$$\sum_{hi}\sum(a_o X'_{hi4} X_{hi4} + a_i X'_{hi4} J_{hi} X_{hi4}) .$$

This matrix is a block diagonal matrix with blocks

$$B_i = r(a_o I_{*i} + a_i J_{*i})$$

where $I_{*_i} = \begin{bmatrix} \delta'_{hi} & & & 0 \\ & \delta^2_{hi} & & \\ & & \ddots & \\ 0 & & & \delta^b_{hi} \end{bmatrix}$ is a b x b matrix with zeros or ones on the diagonal,

and $J_{*_i} = \begin{bmatrix} \delta'_{hi} \\ \delta^2_{hi} \\ \vdots \\ \delta^b_{hi} \end{bmatrix} [\delta'_{hi}\ \delta^2_{hi}\ \cdots\ \delta^b_{hi}]$ is a b x b matrix with zeros or ones.

Observe that I_* has n_{hi} ones on the diagonal.

A generalized inverse of B_i is $B_i^- = r^{-1}(b_o I_* + b_i J_*)$, where $b_o = a_o^{-1}$ and $b_i = -a_i a_o^{-1}(\delta_i a_i + a_o)^{-1}$, where δ_i is the number of split-plot treatments in the ith whole plot.

Therefore, a generalized inverse of $X'V^{-1}X$ is

$$C = (X'V^{-1}X)^- = \begin{bmatrix} 0 & 0 \\ & \\ 0 & r^{-1}\ \mathrm{BDMATX}(b_o I_{*i} + b_i J_{*i}) \end{bmatrix}$$

where $b_o = \alpha_2^2$ and $b_i = \alpha_1^2$.

The quadratic form $Y'Q_V V_1 Q_V Y$ can now be developed.

Recall that $V_1 = V_1^b V_1^b$, then

$$V_1^b Q_V Y = V_1^b V^{-1} Y - V_1^b V^{-1} X(X'V^{-1}X)^- X'V^{-1} Y \ . \qquad \text{(A.9)}$$

Observe that

$$\begin{aligned} V_1^b V^{-1} Y &= V_1^b\ \mathrm{CMATX}(a_o I_{hi} + a_i J_{hi}) Y_{hi}) \\ &= \mathrm{CMATX}(n_{hi}^{-1/2} J_{hi}(a_o I_{hi} + a_i J_{hi}) Y_{hi}) \\ &= \mathrm{CMATX}(n_{hi}^{-1/2}\ (a_o + a_i n_{hi}) J_{hi} Y_{hi}) \\ &= \mathrm{CMATX}(n_{hi}^{-1/2}\ (a_o + a_i n_{hi}) y_{hi+} 1_{hi}) \ . \qquad \text{(A.10)} \end{aligned}$$

Also observe that

$$\begin{aligned} V_1^b V^{-1} X(X'V^{-1}X)^- X'V^{-1} Y &= \sum_{hi}\sum V_1^b V^{-1} XC(X'_{hi}(a_o I_{hi} + a_i J_{hi}) Y_{hi}) \\ &= \sum_{hi}\sum V_1^b V^{-1}\ \mathrm{CMATX}(X_{h'i'}) CX'_{hi}(a_o I_{hi} + a_i J_{hi}) Y_{hi}) \\ &= \sum_{hi}\sum V_1^b V^{-1}\ \mathrm{CMATX}(X_{h'i'} CX'_{hi}(a_o I_{hi} + a_i J_{hi}) Y_{hi}) \ . \end{aligned}$$

(A.11)

Note that

$$X_{h'i'}CX'_{hi} = [X_{h'i'1}:X_{h'i'2}:\ldots:X_{h'i'3}]$$

$$\cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^{-1}\cdot \mathrm{BDMATX}(b_o I_{*_i} + b_1 J_{*_i}) \end{bmatrix} \begin{bmatrix} X_{hi1} \\ X_{hi2} \\ X_{hi3} \\ X_{hi4} \end{bmatrix}$$

$$= X_{h'i'4}(r^{-1}\cdot \mathrm{BDMATX}(b_o I_{*_i} + b_1 J_{*_i}))X_{hi4}$$

$$= r^{-1} X_{h'i4}\cdot \mathrm{BDMATX}(b_o I_{*_i} + b_1 J_{*_i})X_{hi4}$$

$$= r^{-1} X_{h'i3}(b_o I_{*_i} + b_i J_{*_i})X_{hi3}$$

$$= r^{-1}(b_o I_{\delta_i} + b_i J_{\delta_i}).$$

Then (A.11) becomes

$$= \sum_h V_1^b V^{-1}\cdot \mathrm{CMATX}(r^{-1}(b_o I_{\ \ i} + b_i J_{\ \ i})a_o I_{hi} + a_i J_{hi})Y_{hi})$$

$$= \sum_h V_1^b V^{-1}\cdot \mathrm{CMATX}(r^{-1}(b_o a_o I_{hi} + (b_o a_i + b_i a_o + a_i b_i\ {}_i)J_{hi})Y_{hi})$$

$$= \sum_h V_1^b V^{-1}\cdot \mathrm{CMATX}(r^{-1} I_{h'i'} Y_{h'i'})$$

$$= V_1^b V^{-1}\cdot \mathrm{CMATX}(r^{-1} Y_{+i})$$

$$= \mathrm{CMATX}(r^{-1} n_{h'i'}^{-1/2}(a_o + a_{i'} n_{h'i'})J_{h'i'})J_{h'i'} Y_{+i})$$

$$= \mathrm{CMATX}(r^{-1} n_{h'i'}^{-1/2}(a_o + a_{i'} n_{h'i'})y_{+i+} \underline{1}_{h'i'}). \qquad \text{(A.12)}$$

From (A.10) and (A.12)

$$V_1^b Q_V Y = \mathrm{CMATX}(n_{hi}^{-1/2}(a_o + a_i n_{hi})(y_{hi+} - r^{-1} y_{+i+})\underline{1}_{hi}).$$

Therefore, $Y'Q_V V_1 Q_V Y = \sum_{hi}\sum (a_o+a_i n_{hi})^2 (y_{hi+}-r^{-1}y_{+i+})^2$

Similarly,

$$Y'Q_V Q_V Y = \sum_{hij}\sum\sum a_o^2 (y_{hij}-r^{-1}y_{+ij})^2 + \sum_{hi}\sum a_i(2a_o+a_i n_{hi})(y_{hi+}-r^{-1}y_{+i+})^2$$

The $tr(Q_V V_1 Q_V V_1)$ now is developed.

From

$$tr(Q_V V_1 Q_V V_1) = \alpha_1^4(\sum_{hi}\sum n_{hi}-q) - 2\alpha_1^{-4}\alpha_2^2(\sum_{hi}\sum(a_o+a_i)n_{hi} - a_o q$$

$$- \sum_{hi}\sum(a_o+a_i n_{hi})t_{hi}) + \alpha_1^{-4}\alpha_2^4(\sum_{hi}\sum(a_o^2+2a_o a_i+a_i^2 n_{hi})n_{hi}$$

$$- 2(a_o^2 q+\sum_{hi}\sum(2a_o^2 a_i+3a_o a_i^2 n_{hi}+a_i^3 n_{hi}^2)t_{hi})$$

$$+ \sum_{hh'}\sum\sum_{ii'}\sum(a_o^2 t_{1h'i'hi}+2a_o(2a_o+a_i n_{hi})t_{2h'i'hi}$$

$$+ a_i a_{i'}(2a_o+a_i n_{h'i'})(2a_o+a_i n_{hi})t_{3h'i'hi}))$$

where $q = R(X)$,

and $t_{hi} = tr(CX'_{hi}J_{hi}X_{hi})$

$= tr(X_{hi}CX'_{hi}J_{hi})$

$= tr(X_{hi4}r^{-1}\ \text{BDMATX}(b_o I_{*_i}+b_i J_{*_i})X'_{hi4}J_{hi})$

$= tr(X_{hi3}r^{-1}(b_o I_{*_i}+b_i J_{*_i})X_{hi3}J_{hi})$

$= tr(r^{-1}(b_o I_{\delta_i}+b_i J_{\delta_i})J_{hi})$

$= r^{-1}(b_o+b_i n_{hi})tr(J_{hi})$

$= r^{-1}(b_o+b_i n_{hi})n_{hi}$ (A.13)

Observe that $t_{1h'i'hi} = 0$ if $i \neq i'$, then for $i = i'$

$$t_{1h'ihi} = tr(CX'_{h'i}X_{hi}CX'_{hi}X_{h'i})$$

$$= tr(X_{h'i}CX'_{hi}X_{hi}CX'_{hi})$$

$$= tr(X_{h'i3}(r^{-1}(b_oI_{*_i}+b_iJ_{*_i}))X_{hi3}X_{hi3}(r^{-1}(b_oI_{*_i}+b_iJ_{*_i}))X_{hi3}$$

$$= r^{-2}(tr((b_o+I_{h'i}+b_iJ_{h'i})(b_oI_{hi}+b_iJ_{hi}))$$

$$= r^{-2}(tr(b_o^2I_{h'i}+(2b_ob_i+b_i^2n_{hi})J_{hi})$$

$$= r^{-1}(b_o^2+2b_ob_i+b_i^2n_{hi})n_{hi} \qquad \text{(A.14)}$$

Also observe that $t_{2h'i'hi} = 0$ if $i \neq i'$, then for $i = i'$

$$t_{2h'ihi} = r^{-1}(b_o+b_in_{hi})^2n_{hi}.$$

Finally, observe that $t_{3h'i'hi} = 0$ if $i \neq i'$, then for $i = i'$

$$t_{3h'ihi} = r^{-2}(b_o+b_in_{hi})^2n_{hi}^2 \qquad \text{(A.15)}$$

Therefore,

$$tr(Q_VV_1Q_VV_1) = \alpha_1^4(\sum_{hi}\sum n_{hi}-q) - 2\alpha_1^{-4}\alpha_2^2(\sum_{hi}\sum(b_o+a_i)n_{hi}$$

$$- \sum_{hi}\sum(r^{-1}(b_o+b_in_{hi})n_{hi})(a_o+a_in_{hi}))$$

$$+ \alpha_1^{-4}\alpha_2^4(\sum_{hi}\sum(a_o^2+2a_oa_i+a_i^2n_{hi})n_{hi}$$

$$- 2(a_o^2q+\sum_{hi}\sum(2a_o^2a_i+3a_oa_i^2n_{hi}+a_i^3n_{hi}^2)(r^{-1}(b_o+b_in_{hi})y_{hi}))$$

$$+ \sum_{hh'}\sum_{i}\sum(a_o^2(r^{-1}(b_o^2+2b_ob_i+b_i^2n_{hi})n_{hi}$$

$$+\ 2a_o(2a_o+a_i n_{hi})r^{-2}(b_o+b_i n_{hi})^2 n_{hi}$$

$$+\ a^2(2a_o+a_i n_{hi})(2a_o+a_i n_{hi})r^{-2}(b_o+b_i n_{hi})^2 n_{hi}^2.$$

Also,

$$\begin{aligned}
tr(Q_V V_1 Q_V) = {} & \alpha_1^{-2}(\sum\sum_{hi}(a_o+a_i)n_{hi} - a_o q \sum\sum_{hi} a_i(a_o+a_i n_{hi})r^{-1}(b_o+b_i n_{hi})n_{hi}) \\
& - \alpha_1^2\alpha_2^2(\sum\sum_{hi}(a_o^2+2a_o a_i+a_i^2 n_{hi})n_{hi} \\
& - 2(a_o q+\sum\sum_{hi}(2a_o^2 a_i+3a_o a_i^2 n_{hi}+a_i^3 n_{hi}^2)r^{-1}(b_o+b_i n_{hi})n_{hi} \\
& + \sum\sum_{hh'}\sum_{i}(a_o^2 r^{-2}(b_o^2+2b_o b_i+b_i^2 n_{hi})n_{hi} \\
& + 2a_o(a_o+a_i n_{hi})r^{-2}(b_o+b_i n_{hi})^2 n_{hi} \\
& + a_i^2(2a_o+a_i n_{hi})(2a_o+a_i n_{hi})r^{-2}(b_o+b_i n_{hi})^2 n_{hi}^2).
\end{aligned}$$

Finally,

$$\begin{aligned}
tr(Q_V Q_V) = {} & \sum\sum_{hi}(a_o^2+2a_o a_i+a_i^2 n_{hi})n_{hi} \\
& - 2(a_o^2 q + \sum\sum_{hi}(2a_o^2 a_i+3a_o a_i^2 n_{hi}+a_i^3 n_{hi}^2)r^{-1}(b_o+b_i n_{hi})n_{hi} \\
& + a_o^2 \sum\sum_{hh'}\sum_{i}(r^{-2}(b_o^2+2b_o b_i+b_i^2 n_{hi})n_{hi} \\
& + 2a_o \sum\sum_{hh'}\sum_{i}(2a_o a_i+a_i^2 n_{hi})r^{-1}(b_o+b_i n_{hi})^2 n_{hi} \\
& + \sum\sum_{hh'}\sum_{i}(2a_o a_i+a_i^2 n_{hi})(2a_o a_i+a_i^2 n_{hi})r^{-2}(b_o+b_i n_{hi})^2 n_{hi}^2.
\end{aligned}$$

APPENDIX B

PROGRAMS FOR THE SIMULATION OF THE BEHAVIOR OF THE ESTIMATE VECTOR OF PARAMETER $\tilde{\beta}$

```
      DIMENSION TT(4),T(4)
      REAL*8 TIT/' X'/
      REAL LA1(30),LA2(30),LA3(30),YA1(30),LA(44)
      DIMENSION X(30,30),VI(30,30),V1(30,30),QV(30,30),W(30,30)
      DIMENSION G(30,30),OP(30,30),A(30,30),POX(30,30)
      DIMENSION U1(30,30)
      DIMENSION Y(72),Z(144),RW(44),S(2,2),DELTA(2) ,ZA(72),Q1(72)
      DIMENSION SI(2,2)
      DIMENSION AV1(30,30),AQV(30,30),AS(2,2),ASI(2,2)
      DIMENSION U(1000)
      DIMENSION UU(200)
      INTEGER R,C,CA,NB(20),BL,PR(10)
      INTEGER B(30),WP(30),SP(30)
      INTEGER NR/30/,NC/30/,N3/30/
      CALL SCLPAR(PR,IT,ITM,ITN,X1,X2,CONST,R,X,NR,NC,BL,NB,C,CA)
      NU1C=CA-C
      DO 70 I=1,R
      DO 70 J=1,NU1C
70    U1(J,I)=X(I,C+J)
      X11=X1
      X12=X2
      CALL INTIAL(1)
      IDIST=PR(10)
C*** END GENERATE RANDOM ERRORS
      CALL SCRNY(Y,Z,R,IDIST)
      CALL SCRNY(YA1,Z,R,IDIST)
      CALL SCAB(YA1,U1,1,NU1C,R,YA1,Z,1,NR,NR,NR,1,NR,NR,1)
      DO 83 JJ=1,R
83    Y(JJ)=Y(JJ)+YA1(JJ)
      CALL SCSMI(X,VI,V1,QV,W,G,R,C,NR,NC,BL,NB,Z,ZA,S,SI,X1,X2,CONST)
      IF(PR(9).EQ.1) GO TO 66
      DO 99 KI =1,ITN
C*** MINQUE  ESTIMATORS
C*** COMPUTE MINQUE
      YY1=FQF(Y,V1,R,NR,NR)
      YY2=FQF(Y,QV,R,NR,NR)
      X11=YY1*SI(1,1)+YY2*SI(1,2)
      X12=YY1*SI(1,2)+YY2*SI(2,2)
      WRITE(3,602)X11,X12
602   FORMAT(2F10.6)
      CALL SCSMI(X,VI,V1,QV,W,G,R,C,NR,NC,BL,NB,Z,ZA,S,SI,
     *        X11,X12,CONST)
C  V1 ::=QV*V1*QV   , QV::=QV*QV
99    CONTINUE
      IF(PR(9).EQ.2)X1=X11
      IF(PR(9).EQ.2)X2=X12
66    CONTINUE
      CALL RSTART(563,542)
105   READ(6,2,END=106)(LA(I),I=1,C)
      ITA=0
      NDF=0
2     FORMAT(40F2.0)
      CALL SCVI(VI,R,X1,X2,BL,NB,NR,NR)
      CALL SCATB(X,VI,C,R,R,W,WA,NR,NC,NR,NR,NR,NR,1,1)
      CALL SCAB(W,X,C,R,C,G,WA,NR,NR,NR,NC,NC,NC,1,1)
      CALL SCGI(G,C,W,TR1,ZA,Z,CONST,NC,NC,NR,NR)
```

```
      CALL SCAB(VI,X,R,R,C, W,WA,NR,NR,NR,NC,NR,NR,1,1)
      CALL SCAB( W,G,R,C,C, W,Z,NR,NR,NC,NC,NR,NR,NR,1)
      CALL SCAB(W,LA,R,C,1,Q1,Z,NR,NR,NR,1,NR,1,1,1)
      CALL SCV1( A,R,BL,NB,NR,NR)
      QV1Q=FQF(Q1,A,R,NR,NR)
      CALL SCI( A,R,NR,NR)
      QQ=FQF(Q1,A,R,NR,NR)
      AA=QV1Q*X1+QQ*X2
      BB=2*(QV1Q*QV1Q*SI(1,1)+QQ*QQ*SI(2,2)+2*QV1Q*QQ*SI(1,2))
      AL=AA**2/BB
      BE=BB/AA
      DFF=2*AL
      WRITE(3,43)DFF,AL,BE,(LA(KI),KI=1,C)
43    FORMAT(//' DEGREES OF FREEDOM ',F15.1//'  AL=',F15.6//
     *'  BE=',F15.6//
     *         ' LAM'/2(30F4.0))
      IDF=DFF + .05
      DO 100 I=1,IT
C*** END GENERATE RANDOM ERRORS
      CALL SCRNY(Y,Z,R,IDIST)
      CALL SCRNY(YA1,Z,NU1C,IDIST)
      CALL SCAB(YA1,U1,1,NU1C,R,YA1,Z,1,NR,NR,NR,1,NR,NR,1)
      DO 84 JJ=1,R
84    Y(JJ)=Y(JJ)+YA1(JJ)
C*** MINQUE  ESTIMATORS
      YY1=FQF(Y,V1,R,NR,NR)
      YY2=FQF(Y,QV,R,NR,NR)
      Y1=YY1*SI(1,1)+YY2*SI(1,2)
      Y2=YY1*SI(1,2)+YY2*SI(2,2)
      CALL SCVI(VI,R,Y1,Y2,BL,NB,NR,NR)
      CALL SCATB(X,VI,C,R,R,W,WA,NR,NC,NR,NR,NR,NR,1,1)
      CALL SCAB(W,X,C,R,C,G,WA,NR,NR,NR,NC,NC,NC,1,1)
      CALL SCGI(G,C,W,TR1,ZA,Z,CONST,NC,NC,NR,NR)
      ALPL=FQF(LA,G,C,NR,NR)
      WRITE(3,609)ALPL
609   FORMAT(' Z-VALUE',F10.6)
      CALL SCAB(VI,X,R,R,C,W ,WA,NR,NR,NR,NC,NR,NR,1,1)
      CALL SCAB(W,G,R,C,C,W,Z,NR,NR,NC,NC,NR,NR,NR,1)
      CALL SCAB(Y,W,1,R,C,Q1,Z,1,NR,NR,NR,1,NR,1,1)
      CALL SCAB(Q1,LA,1,C,1,BLA,Z,1,NR,NR,1,1,1,1,1)
      TV=BLA/SQRT(ALPL)
      PROB=PRGT(TV,IDF)
      WRITE(3,610)TV,PROB
610   FORMAT(' T-VALUE',F10.6,' PROBABILITY',F10.6)
      IF(PROB.LT. .025 .OR.PROB .GT. .975 )NDF=NDF+1
      IF(CDGAMA(ALPDF,AL,BE).LE. 0.)GO TO 100
      ITA=ITA+1
      U(ITA)=CDGAMA(ALPL,AL,BE)
100   CONTINUE
      GO TO 105
106          CONTINUE
      WRITE(3,600)NDF
600   FORMAT(' N R=',I5)
C*** GOODNESS OF FIT TEST
      CALL SCCDF(U,1,6,ITA,2)
      END
```

APPENDIX C

PROGRAMS FOR FORMULA 5.1.3 FOR APPROXIMATING THE DEGREES OF FREEDOM

```
      DIMENSION TT(4),T(4)
      REAL*8 TIT/' X'/
      REAL LA1(30),LA2(30),LA3(30),YA1(30),LA(44)
      DIMENSION X(30,30),VI(30,30),V1(30,30),QV(30,30),W(30,30)
      DIMENSION G(30,30),OP(30,30),A(30,30),POX(30,30)
      DIMENSION U1(30,30)
      DIMENSION Y(72),Z(144),RW(44),S(2,2),DELTA(2) ,ZA(72),Q1(72)
      DIMENSION SI(2,2)
      DIMENSION AV1(30,30),AQV(30,30),AS(2,2),ASI(2,2)
      DIMENSION U(1000)
      DIMENSION UU(200)
      INTEGER R,C,CA,NB(20),BL,PR(10)
      INTEGER B(30),WP(30),SP(30)
      INTEGER NR/30/,NC/30/,N3/30/
      CALL SCLPAR(PR,IT,ITM,ITN,X1,X2,CONST,R,X,NR,NC,BL,NB,C,CA)
      NU1C=CA-C
C** COMPUTE U1
      DO 70 I=1,R
      DO 70 J=1,NU1C
70    U1(J,I)=X(I,C+J)
      CALL INTIAL(1)
      CALL RSTART(563,542)
C POX
      CALL SCATA(X,R,CA,G,W,NR,NC,NC,NC,NR,NR)
      CALL SCGI(G,CA,W,TR,Y,Z,CONST,NC,NC,NR,NR)
      CALL SCAB(X,G,R,CA,CA,W,WA,NR,NC,NC,NC,NR,NR,1,1)
      CALL SCABT(W,X,R,CA,R,W,Y,NR,NR,NR,NC,NR,NR,NR,1)
      CALL SCI(QV,R,NR,NR)
      CALL SCAMB(QV,W,R,R,POX,NR,NR,NR,NR,NR,NR)
C END POX
      X11=X1
      X12=X2
      IDIST=PR(10)
C*** GENERATE RANDOM ERRORS
      CALL SCRNY(Y,Z,R,IDIST)
      CALL SCRNY(YA1,Z,NU1C,IDIST)
      CALL SCAB(YA1,U1,1,NU1C,R,YA1,Z,1,NR,NR,NR,1,NR,NR,1)
      DO 83 JJ=1,R
83    Y(JJ)=Y(JJ)+YA1(JJ)
C*** END GENERATE RANDOM ERRORS
      IF(PR(9).EQ.1) GO TO 66
      DO 99 I=1,ITN
C*** COMPUTE MINQUE
      YY1=FQF(Y,V1,R,NR,NR)
      YY2=FQF(Y,QV,R,NR,NR)
      X11=YY1*SI(1,1)+YY2*SI(1,2)
      X12=YY1*SI(1,2)+YY2*SI(2,2)
      CALL SCSMI(X,VI,V1,QV,W,G,R,C,NR,NC,BL,NB,Z,ZA,S,SI,
     *       X11,X12,CONST)
C  V1 ::=QV*V1*QV  , QV::=QV*QV
99    CONTINUE
      IF(PR(9).EQ.2)X1=X11
      IF(PR(9).EQ.2)X2=X12
66    CONTINUE
      IOPT=PR(7)
      NY1=0
```

```
      NY2=0
      DO 100 I=1,IT
      G1=0.
      G2=0.
      DO 85 J=1,ITM
C*** GENERATE RANDOM ERRORS
      CALL SCRNY(Y,Z,R,IDIST)
      CALL SCRNY(YA1,Z,NU1C,IDIST)
      CALL SCAB(YA1,U1,1,NU1C,R,YA1,Z,1,NR,NR,NR,1,NR,NR,1)
      DO 84 JJ=1,R
84    Y(JJ)=Y(JJ)+YA1(JJ)
C*** END GENERATE RANDOM ERRORS
C*** MINQUE  ESTIMATORS
      Y1=FQF(Y,V1,R,NR,NR)*SI(1,1)+FQF(Y,QV,R,NR,NR)*SI(1,2)
      Y2=FQF(Y,V1,R,NR,NR)*SI(1,2)+FQF(Y,QV,R,NR,NR)*SI(2,2)
      GO TO(150,151,152),IOPT
C*** RESTRICTED MINQUE ESTIMATORS
151   CONTINUE
      IF(Y1.LT.0.)NY1=NY1+1
      IF(Y1.LT.0.)Y1=0.
      IF(Y2.LT.0.)NY2=NY2+1
      IF(Y2.LE.0.)Y2=0.01
      GO TO 150
152   CONTINUE
C*** P.S.D MINQUE ESTIMATORS
      Y2=FQF(Y,POX,R,NR,NR)/TR
      Y1=(FQF(Y,V1,R,NR,NR)-Y2*S(1,2))/S(1,1)
      IF(Y1.LT.0.)NY1=NY1+1
      IF(Y1.LT.0.)Y1=0.
150   CONTINUE
      CALL SCVI(VI,R,Y1,Y2,BL,NB,NR,NR)
      CALL SCATB(X,VI,C,R,R,W,WA,NR,NC,NR,NR,NR,NR,1,1)
      CALL SCAB(W,X,C,R,C,G,WA,NR,NR,NR,NC,NC,NC,1,1)
      CALL SCGI(G,C,W,TR1,ZA,Z,CONST,NC,NC,NR,NR)
      CALL SCAB(VI,X,R,R,C, W,WA,NR,NR,NR,NC,NR,NR,1,1)
      CALL SCAB( W,G,R,C,C, W,Z,NR,NR,NC,NC,NR,NR,NR,1)
      CALL SCAB(Y,W,1,R,C,Q1,Z,1,NR,NR,NR,1,NR,1,1)
      G1=G1+Q1(1)
85    G2=G2+Q1(2)
      G1=G1/FLOAT(ITM)
      G2=G2/FLOAT(ITM)
      U(I)=G1
C*** HISTOGRAM
      CALL HSTGM(G1,G2,G3,G4,1,2)
100   CONTINUE
C*** GRAPHS
      CALL GETPCT(IT,9)
      WRITE(3,20)
20    FORMAT(1H1,16X,' B0',24X,' B1',24X,' B2',24X,' B3')
      CALL PRNT(1,20)
      WRITE(3,21)
21    FORMAT(1H1,20X,'GRAPH OF B0'//)
      CALL GRPH(1,1)
      WRITE(3,22)
22    FORMAT(1H1,20X,'GRAPH OF B1'//)
      CALL GRPH(1,2)
```

```
      WRITE(3,24)
24    FORMAT(1H1,' MOMENTS')
      CALL RAWMOM(1,IT)
      WRITE(3,1)NY1,NY2
1     FORMAT('1NY1=',I6,' NY2=',I6)
C*** GOODNESS OF FIT TEST
      CALL SCCDF(U,1,4,IT,2)
      END
```

```
      DIMENSION TT(4),T(4)
      REAL*8 TIT/' X'/
      REAL LA1(30),LA2(30),LA3(30),YA1(30),LA(44)
      DIMENSION X(30,30),VI(30,30),V1(30,30),QV(30,30),W(30,30)
      DIMENSION G(30,30),OP(30,30),A(30,30),POX(30,30)
      DIMENSION U1(30,30)
      DIMENSION Y(72),Z(144),RW(44),S(2,2),DELTA(2) ,ZA(72),Q1(72)
      DIMENSION SI(2,2)
      DIMENSION AV1(30,30),AQV(30,30),AS(2,2),ASI(2,2)
      DIMENSION U(1000)
      DIMENSION UU(200)
      INTEGER R,C,CA,NB(20),BL,PR(10)
      INTEGER B(30),WP(30),SP(30)
      INTEGER NR/30/,NC/30/,N3/30/
      CALL SCLPAR(PR,IT,ITM,ITN,X1,X2,CONST,R,X,NR,NC,BL,NB,C,CA)
      NU1C=CA-C
      DO 70 I=1,R
      DO 70 J=1,NU1C
70    U1(J,I)=X(I,C+J)
      XX=X1
      XY=X2
      X11=X1
      X12=X2
      CALL INTIAL(1)
      IDIST=PR(10)
      CALL RSTART(563,542)
105   READ(6,2,END=106)(LA(I),I=1,C)
      ITA=0
      NDF=0
      DO 100 I=1,IT
      X1=XX
      X2=XY
C*** END GENERATE RANDOM ERRORS
      CALL SCRNY(Y,Z,R,IDIST)
      CALL SCRNY(YA1,Z,R,IDIST)
      CALL SCAB(YA1,U1,1,NU1C,R,YA1,Z,1,NR,NR,NR,1,NR,NR,1)
      DO 83 JJ=1,R
83    Y(JJ)=Y(JJ)+YA1(JJ)
      CALL SCSMI(X,VI,V1,QV,W,G,R,C,NR,NC,BL,NB,Z,ZA,S,SI,X1,X2,CONST)
      IF(PR(9).EQ.1) GO TO 66
      DO 99 KI =1,ITN
C*** MINQUE  ESTIMATORS
C*** COMPUTE MINQUE
      YY1=FQF(Y,V1,R,NR,NR)
      YY2=FQF(Y,QV,R,NR,NR)
      X11=YY1*SI(1,1)+YY2*SI(1,2)
      X12=YY1*SI(1,2)+YY2*SI(2,2)
      WRITE(3,602)X11,X12
602   FORMAT(2F10.6)
      CALL SCSMI(X,VI,V1,QV,W,G,R,C,NR,NC,BL,NB,Z,ZA,S,SI,
     *        X11,X12,CONST)
C  V1 ::=QV*V1*QV   , QV::=QV*QV
99    CONTINUE
      IF(PR(9).EQ.2)X1=X11
      IF(PR(9).EQ.2)X2=X12
66    CONTINUE
```

```
2     FORMAT(40F2.0)
      CALL SCVI(VI,R,X1,X2,BL,NB,NR,NR)
      CALL SCATB(X,VI,C,R,R,W,WA,NR,NC,NR,NR,NR,NR,1,1)
      CALL SCAB(W,X,C,R,C,G,WA,NR,NR,NR,NC,NC,NC,1,1)
      CALL SCGI(G,C,W,TR1,ZA,Z,CONST,NC,NC,NR,NR)
      CALL SCAB(VI,X,R,R,C, W,WA,NR,NR,NR,NC,NR,NR,1,1)
      CALL SCAB( W,G,R,C,C, W,Z,NR,NR,NC,NC,NR,NR,NR,1)
      CALL SCAB(W,LA,R,C,1,Q1,Z,NR,NR,NR,1,NR,1,1,1)
      CALL SCVI( A,R,BL,NB,NR,NR)
      QV1Q=FQF(Q1,A,R,NR,NR)
      CALL SCI( A,R,NR,NR)
      QQ=FQF(Q1,A,R,NR,NR)
      AA=QV1Q*X1+QQ*X2
      BB=2*(QV1Q*QV1Q*SI(1,1)+QQ*QQ*SI(2,2)+2*QV1Q*QQ*SI(1,2))
      AL=AA**2/BB
      BE=BB/AA
      DFF=2*AL
      WRITE(3,611)DFF ,AL,BE
611   FORMAT(' DF',F10.6,' A',F10.6,' B',F10.6)
      IDF=DFF + .05
      YY1=FQF(Y,VI,R,NR,NR)
      YY2=FQF(Y,QV,R,NR,NR)
      Y1=YY1*SI(1,1)+YY2*SI(1,2)
      Y2=YY1*SI(1,2)+YY2*SI(2,2)
      CALL SCVI(VI,R,Y1,Y2,BL,NB,NR,NR)
      CALL SCATB(X,VI,C,R,R,W,WA,NR,NC,NR,NR,NR,NR,1,1)
      CALL SCAB(W,X,C,R,C,G,WA,NR,NR,NR,NC,NC,NC,1,1)
      CALL SCGI(G,C,W,TR1,ZA,Z,CONST,NC,NC,NR,NR)
      ALPL=FQF(LA,G,C,NR,NR)
      WRITE(3,609)ALPL
609   FORMAT(' Z-VALUE',F10.6)
      CALL SCAB(VI,X,R,R,C,W ,WA,NR,NR,NR,NC,NR,NR,1,1)
      CALL SCAB(W,G,R,C,C,W,Z,NR,NR,NC,NC,NR,NR,NR,1)
      CALL SCAB(Y,W,1,R,C,Q1,Z,1,NR,NR,NR,1,NR,1,1)
      CALL SCAB(Q1,LA,1,C,1,BLA,Z,1,NR,NR,1,1,1,1,1)
      TV=BLA/SQRT(ALPL)
      PROB=PRGT(TV,IDF)
      WRITE(3,610)TV,PROB
610   FORMAT(' T-VALUE',F10.6,' PROBABILITY',F10.6)
      IF(PROB.LT. .025 .OR.PROB .GT. .975 )NDF=NDF+1
      ITA=ITA+1
      U(ITA)=PROB
100   CONTINUE
      GO TO 105
106          CONTINUE
      WRITE(3,600)NDF
600   FORMAT(' N R=',I5)
C*** GOODNESS OF FIT TEST
      CALL SCCDF(U,1,6,ITA,2)
      END
```

```
      SUBROUTINE POOL (U,N2,NL,IL,NSAM,W)
C
C
C     POOL AND RANK U-VALUES
C
      DIMENSION U(1000), W(1000)
      M3 = NL
      NL = NL + N2
      DO 2 I = 1,N2
 2    W(M3+I) = U(I)
      IF (NSAM.GT.IL) GO TO 6
      CALL RANK(W,NL)
      WRITE (3,1)
 1    FORMAT (//,10X,'POOLED AND RANKED U''S')
      DO 4 I = 1,NL
 4    WRITE (3,5) I,W(I)
 5    FORMAT (T2,I4,'-',F12.4)
 6    RETURN
      END

      SUBROUTINE RANK(U,N2)
C
C     SUBROUTINE RANK(U,N2)
C
      DIMENSION U(1000)
      M1=N2-1
      DO 202 I=1,M1
      K1=I+1
      DO 203 J=K1,N2
  209 IF (U(I)-U(J)) 210,210,220
  210 GO TO 203
  220 Z1=U(I)
      U(I)=U(J)
      U(J)=Z1
  203 CONTINUE
  202 CONTINUE
      RETURN
      END

      SUBROUTINE SCGI(G,M,O,TR,D,WK,CONST,N1,N2,N3,N4)
      DIMENSION G(N1,N2),O(N3,N4),D(1),WK(1)
      CALL SCI(O,M,N3,N4)
      CALL LSVDF(G,N1,M,M,O,N3,M,D,WK,IER)
      CALL SCSVIN(D,M,WK,CONST)
      CALL SCODOP(G,D,O,M,WK,M,M,N1,N2,N3,N4)
      TR=0.
      DO 33 I=1,M
      IF(D(I).NE.0.)TR=TR+1.
33    CONTINUE
      RETURN
      END
```

```
      SUBROUTINE TESTS(U,N2)
C
C     SUBROUTINE TO COMPUTE TEST STATISTICS FOR UNIFORMITY, U2,NS2, NS3,NS4
C
C     COMPUTE NEYMAN SMOOTH TESTS: NS2, NS3,NS4
      IMPLICIT REAL*8(A-H,O-Z)
      REAL*4 U(1000)
      PI1=0
      PI2=0
      PI3=0
      PI4=0
      DO 1 I = 1,N2
      PI1 = PI1 + (DSQRT(12.D0))*(U(I) -.5)
      PI2 = PI2 + (DSQRT(5.D0))*( 6*(U(I) - .5)**2 - .5)
      PI3 = PI3+ (DSQRT(7.D0))*(20*(U(I) - .5)**3 -3*(U(I) -.5))
  1   PI4 = 210*(U(I) - .5)**4 - 45*(U(I) -.5)**2 + 9./8 +PI4
      AS2 = (PI1**2 + PI2**2)/N2
      AS3 = AS2 + PI3**2/N2
      AS4 = AS3 + PI4**2/N2
C
C     COMPUTE WATSON STATISTIC: U2
C
      WS = 1./12./N2
      DO 7 I=1,N2
  7   WS = WS + (( 2.*I - 1 )/2./N2 - U(I))**2
      USQ = 0.0
      DO 791  I = 1,N2
 791  USQ = USQ + U(I)
      USQ=WS  - N2*((USQ/N2 - 0.5 )**2)
      U2   = (USQ-0.1/N2+0.1/N2/N2)*(1.0+0.8/N2)
      WRITE (3,5)
  5   FORMAT(//,7X,'WATSON U2',4X,'NEYMAN SMOOTH 2',8X,'NS3',7X,
     *'NS4',12X,'NUMBER OF U''S')
      WRITE (3,6) U2,AS2,AS3,AS4,N2
  6   FORMAT (//,5X,F10.5,7X,F10.5,8X,F8.4,2X,F8.4,9X,I5,/)
      RETURN
      END
```

```
      SUBROUTINE TRANS(X,N,NP,U,MOD)
      DIMENSION X(100),U(100),Z(100)
      GO TO (1,2,3,4,5,6,7) , MOD
C
C   SCALE PARAMETER EXPONENTIAL CLASS, MOD = 1
C
 1    SUM1 = 0.0
      SUM2 = X(1)
      DO 101 I = 2,N
      SUM1 = SUM1 + X(I-1)
      SUM2 = SUM2 + X(I)
 101  U(I-1) = 1 -  (SUM1/SUM2)**(I-1)
      GO TO 200
C   ONE PARAMETER (LOCATION) EXPONENTIAL CLASS , MOD = 2
 2    CALL SOR(X,N,T1,Z)
      N2 = N-1
      DO 161 I = 1,N2
 161  U(I) = 1 -  EXP(T1-Z(I))
      GO TO 200
C
C   TWO-PARAMETER EXPONENTIAL CLASS, MOD = 3
C
 3    CALL SOR(X,N,T1,Z)
      N1 = N-1
      SUM1 = 0.0
      SUM2 = Z(1) - T1
      DO 171 I = 2,N1
      SUM1 = SUM1 + Z(I-1) - T1
      SUM2 = SUM2 + Z(I) - T1
 171  U(I-1) =  1 -  (SUM1/SUM2)**(I-1)
      GO TO 200
C
C     TWO-PARAMETER NORMAL CLASS, MOD = 4
C
 4    T1 = X(1) + X(2)
      DO 201 I = 3,N
      T1 = T1 + X(I)
      T2 = 0.0
      DO 202 J = 1,I
 202  T2 = T2 + (X(J) - T1/I)**2
      A=(SQRT(I-2.))*(X(I)-T1/I)/(SQRT(ABS((I-1.)*T2/I-(X(I)-T1/I)**2)))
 201  U(I-2) = FRGT(A,I-2)
      GO TO 200
C
C     TWO-PARAMETER LOGNORMAL CLASS, MOD = 5
C
 5    DO 300 I = 1,N
 300  X(I) = ALOG(X(I))
      GO TO 4
C
C     TWO-PARAMETER UNIFORM, MOD = 6
C
 6    T1 = X(1)
      T2 = X(1)
      DO 601 I = 2,N
      IF (T1.LE.X(I)) GO TO 601
```

```
      T1 = X(I)
  601 CONTINUE
      DO 602 I = 2,N
      IF (X(I).LE.T2) GO TO 602
      T2 = X(I)
  602 CONTINUE
      DO 603 I = 1,N
      IF (X(I).EQ.T1) GO TO 604
      GO TO 603
  604 I10 = I
  603 CONTINUE
      DO 605 I = 1,N
      IF (X(I).EQ.T2) GO TO 606
      GO TO 605
  606 I20 = I
  605 CONTINUE
      IF (I20-I10) 20,20,21
   20 IMIN = I20
      IMAX = I10
      GO TO 22
   21 IMIN = I10
      IMAX = I20
   22 IF (IMIN.EQ.1) GO TO 10
      IMIN1 = IMIN - 1
      DO 11 I = 1, IMIN1
   11 Z(I) = X(I)
   10 IMIN2 = IMIN + 1
   12 IF (IMAX - IMIN) 13,13,14
   14 IMAX1 = IMAX - 1
      DO 15 I = IMIN2, IMAX1
   15 Z(I-1) = X(I)
   13 IMAX2 = IMAX + 1
      IF (N - IMAX) 16,16,17
   17 IMAX2 = IMAX + 1
      DO 18 I = IMAX2, N
   18 Z(I - 2) = X(I)
   16 N2 = N - 2
      DO 19 I = 1,N2
   19 U(I) = (Z(I) - T1)/(T2 - T1)
      GO TO 200
C
C     TWO-PARAMETER PARETO CLASS. MOD = 7
C
    7 DO 613 I = 1,N
  613 X(I) = ALOG(X(I))
      GO TO 3
  200 RETURN
      END
```

```
      REAL FUNCTION PRGT*4(TVALUE,DF)
C
C                 T-DISTRIBUTION FUNCTION SUBROUTINE
C
      IMPLICIT REAL*8(A-H,O-Z)
      REAL*4 TVALUE
      INTEGER DF
C
C     FOR DF<=20 COMPUTE EXACT PROB>|T|
C     REF: JOURNAL OF QUALITY TECHNOLOGY, VOL 4, NO.4, OCT.1972, P 196
C
      TSQ=TVALUE*TVALUE
      PRGT=1.D0
      IF(DF.LT.1)RETURN
      IF(TSQ.LT.1.D-10)GO TO 35
      V=DF
      IF(DF.GT.20)GO TO 50
      IF(TSQ.GT.1.D8)GO TO 30
C
      THETA=DATAN(DSQRT(TSQ/V))
      M=MOD(DF,2)
      T=DSIN(THETA)
      C=DCOS(THETA)
      IF(M.EQ.0)GO TO 10
      PRGT=1.D0-2.D0*THETA/3.141592653589793
      IF(DF.EQ.1)GO TO 25
      T=2.D0*T*C/3.141592653589793
C
10    PRGT=PRGT-T
      NT=(DF-M-2)/2
      IF(NT.LT.1)GO TO 25
      C2=C*C
      D=M
      DO 15 I=1,NT
      D=D+2.D0
      T=T*C2*(D-1.D0)/D
15    PRGT=PRGT-T
C
25    IF(PRGT.GT.0.0)GO TO 35
30    PRGT=0.0
35    PRGT=PRGT/2.0
      IF(TVALUE.GE.0.0)PRGT=1.0-PRGT
      RETURN
C
C     FOR DF>20 USE FISHER'S EXPANSION (FIRST 3 TERMS)
C     ABSOLUTE ERROR < .00002     REF: JOHNSON & KOTZ  P. 102
C     'CONTINUOUS UNIVARIATE DIST-2'  HOUGHTON MIFFLIN CO. 1970
C
50    IF(TSQ.GT.36.D0)GO TO 30
      T=DSQRT(TSQ)
      X=T/(2.D0*V)*(TSQ+1.D0-(3.D0+TSQ*(5.D0+TSQ*(7.D0-3.D0*TSQ)))/
     *  (24.D0*V)-(15.D0+TSQ*(3.D0-TSQ*(6.D0+TSQ*(14.D0-TSQ*(11.D0-TSQ))
     *  )))/(96.D0*V*V))
      PRGT=DERFC(T/1.41421356237)+DEXP(-.9189385332-TSQ/2.D0)*X
      GO TO 25
C
```

```
C      END T-DISTRIBUTION SUBROUTINE
C
       END

       SUBROUTINE SCCDF(X,NSAM,MOD,N,NP)
C
C      CPIT MODEL TESTING PROGRAM
       DIMENSION X(100),U(1000),W(1000),Z(100)
C
C
C      CONTROL CARDS
C
       NL = 0
       DO 833 IL = 1, NSAM
 100   FORMAT (3I5)
       WRITE (3,1) NSAM,MOD
 1     FORMAT (//,5X,'CPIT MODEL ANALYSIS PROGRAM, ON THIS RUN WE ANALYZE
      *',2X,I3,2X,'SAMPLES, THE MODEL ASSUMED IS MOD = ',I3,'(SEE MODEL C
      *ODE IN PROGRAM)')
C
C      MOD IS AN INDEX FOR THE NULL HYPOTHESIS MODEL TO BE TESTED:
C      1 - SCALE PARAMETER EXPONENTIAL
C      2 - LOCATION PARAMETER EXPONENTIAL
C      3 - TWO-PARAMETER EXPONENTIAL
C      4 - TWO-PARAMETER NORMAL
C      5 - TWO-PARAMETER LOGNORMAL
C      6 - TWO-PARAMETER UNIFORM
C      7 - TWO-PARAMETER PARETO
C      ETC. (ADD OTHER FAMILIES)
C      N = NO. OF OBSERVATIONS
C      NP = NO. OF PARAMETERS
C   NSAM = NUMBER OF SAMPLES, MUST BE SPECIFIED IN A PROGRAM CARD
C
C      DATA INPUT
C
  102 FORMAT(F10.5)
C
C      NULL CLASS TRANSFORMATION
C
       WRITE (3,821)
 821   FORMAT(1H0,//,T7,'ORIGINAL DATA')
       DO 822  I = 1,N
 822   WRITE (3,823) I,X(I)
 823   FORMAT (T2,I4,' - ',F12.4)
       N2 = N-NP
 721   CONTINUE
       CALL TRANS(X,N,NP,U,MOD)
       CALL RANK (U,N2)
       CALL TESTS (U,N2)
       CALL POOL (U,N2,NL,IL,NSAM,W)
 833   CONTINUE
       CALL TESTS (W,NL)
       CALL EXIT
       STOP
       END
```

```
      SUBROUTINE SCX(X,B,W,S,L,N1,N2,BL,NB,C,CA)
      DIMENSION X(N1,N2)
      INTEGER B(1),W(1),NB(1),BL,C,CA,S(1)
      INTEGER BMAX,WMAX,SMAX,WA,BA
      BMAX=B(1)
      WMAX=W(1)
      SMAX=S(1)
      DO 1 I=2,L
      IF(BMAX.LT.B(I))BMAX=B(I)
      IF(WMAX.LT.W(I))WMAX=W(I)
      IF(SMAX.LT.S(I))SMAX=S(I)
1     CONTINUE
      BL=0
      WA=W(1)
      BA=B(1)
      N=1
      DO 2 I=2,L
      IF(BA.EQ.B(I)  .AND. WA.EQ.W(I))GO TO 3
      BL=BL+1
      NB(BL)=N
      WA=W(I)
      BA=B(I)
      N=1
      GO TO 2
3     N=N+1
2     CONTINUE
      BL=BL+1
      NB(BL)=N
      M1=1
      M2=M1+BMAX
      M3=M2+WMAX
      M4=M3+SMAX
      M5=M4+WMAX*SMAX
      C=M5
      CA=C+WMAX*BMAX
      DO 4 I=1,L
4     X(I,1)=1.
      DO 5 I=1,L
      DO 6 J=2,CA
6     X(I,J)=0.
      X(I,M1+B( I))=1.
      X(I,M2+W(I))=1.
      X(I,M3+S(I))=1.
      X(I,M4+(W( I)-1)*SMAX+S(I))=1.
5     X( I,M5+(B(I)-1)*WMAX+W(I))=1.
      RETURN
      END
```

```
      SUBROUTINE SOR(X,N,T1,Z)
C
      DIMENSION X(100),Z(100)
      T1 = X(1)
      DO 111 I=2,N
      IF (T1.LE.X(I)) GO TC 111
      T1 = X(I)
 111  CONTINUE
      N1 = N - 1
      CO 121 I = 1,N
      IF (X(I).EC.T1) GO TC 131
      GO TO 121
 131  IMIN = I
 121  CONTINUE
      IF (IMIN - 1) 1,1,2
 1    CO 3 I=2,N
 3    Z(I-1) = X(I)
      GO TO 4
 2    IF (N - IMIN) 5,5,6
 5    DO  7 I = 1, N1
 7    Z(I) = X(I)
      GO TC 4
 6    IMIN1 = IMIN-1
      CO 8 I = 1,IMIN1
 8    Z(I) = X(I)
      DO 9  I = IMIN, N1
 9    Z(I) = X(I+1)
 4    RETURN
      END
```

```
      SUBROUTINE SCLBM(R,C,X,XA,IOP,N1,N2,N3,N4)
      DIMENSION X(N1,N2),XA(N3,N4)
      INTEGER R,C,IR(100),IC(100)
      READ(7,1,END=50)N,M
      IF(IOP.EQ.1)GO TO 49
      READ(7,1,END=50)(IR(I),I=1,N)
      READ(7,1,END=50)(IC(I),I=1,M)
49    CALL SCBMAT(X,XA,N,M,IOP,IR,IC,N1,N2,N3,N4)
      R=N
      C=M
50    RETURN
1     FORMAT(40I2)
      END
```

```
      SUBROUTINE SPRINT(T.A.L.M.PR.N1.N2)
      DIMENSION A(N1.N2)
      INTEGER PR
      REAL F1(08)/'(7X,','     ','(11H','     ','     ',
     *           'COL.','I3)','   )'/
      REAL F2(9)/'    1','    2','    3','    4','    5','    6',
     *           '    7','    8','    9'/
      IF(PR.NE.1)RETURN
      WRITE(3.1)T
1     FORMAT(1X.A8)
      K1=-8
      K2=0
3     K1=K1+9
      K2=K2+9
      IF(K2.GT.M)K2=M
      K=9
      IF(K2.EQ.M)K=K2-K1+1
      F1(2)=F2(K)
      WRITE(3.F1)(I.I=K1.K2)
      IF(K2.LT.M)GO TO 3
      DO 4 I=1.L
4     WRITE(3.10)I.(A(I.J).J=1.M)
10    FORMAT(1X.'ROW'.I3.9E14.6.100(/7X.9E14.6) )
      RETURN
      END

      SUBROUTINE SCRNY(Y.Z.R.IDIST)
      DIMENSION Y(1).Z(1)
      INTEGER R
      GO TO(103.104.105).IDIST
103   DO 102 L=1.R
102   Y(L)=RNOR(0)
      GO TO 106
104   DO 107 L=1.R
107   Y(L)=VNI(0)
      GO TO 106
105   DO 108 L=1.R
      Y(L)=RNOR(0)
      Z(L)=VNI(0)
      IF(Z(L).EQ.0.)Y(L)=0.
108   IF(Z(L).NE.0.)Y(L)=Y(L)/Z(L)
106   CONTINUE
      RETURN
      END

      FUNCTION FQF(Y.A.L.N1.N2)
      DIMENSION A(N1.N2).Y(1)
      FQF=0.
      DO 1 I=1.L
      DO 1 J=1.L
1     FQF=FQF+Y(I)*A(I.J)*Y(J)
      RETURN
      END
```

```
      SUBROUTINE SCQV(V,X,G,L,M,QV,W,N1,N2,N3,N4,N5,N6,N7,N8,N9,N10)
      DIMENSION V(N1,N2),X(N3,N4),G(N5,N6),QV(N7,N8),W(N9,N10)
      CALL SCAB(V,X,L,L,M,CV,WA,N1,N2,N3,N4,N7,N8,1,1)
      CALL SCAB(CV,G,L,M,M,QV,W,N7,N8,N5,N6,N7,N8,N9,1)
      CALL SCABT(QV,X,L,M,L,QV,W,N7,N8,N3,N4,N7,N8,N9,1)
      CALL SCAB(CV,V,L,L,L,CV,W,N7,N8,N1,N2,N7,N8,N9,1)
      CALL SCAMB(V,QV,L,L,QV,N1,N2,N7,N8,N7,N8)
      RETURN
      END


      SUBROUTINE SCV1(V1,L,N,NB,N1,N2)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION V1(N1,N2),NB(1)
      K1=1
      K2=NE(1)
      II=1
      DO 1 I=1,L
      DO 2 J=1,L
      IF(J.LT.K1.OR.J.GT.K2)GO TO 3
      V1(I,J)=1.
      GO TO 2
3     V1(I,J)=0.
2     CONTINUE
      IF(I.LT.K2.OR.I.EQ.L)GO TO 1
      K1=K1+NB(II)
      II=II+1
      K2=K2+NB(II)
1     CONTINUE
      RETURN
      END

      SUBROUTINE SVECIN(D,N)
      REAL*8 TIT/' D'/
      DIMENSION D(1)
      DO 100 I=1,N
      IF(D(I)*D(I).LT..01)D(I)=0.
100   IF(D(I).NE.0.)D(I)=1/D(I)
      RETURN
      END


      FUNCTION FCTRCE(A,L,N1,N2)
      DIMENSION A(N1,N2)
      FCTRCE=0.
      DO 1 I=1,L
1     FCTRCE=FCTRCE+A(I,I)
      RETURN
      END
```

```
      SUBROUTINE SCAAT(A,M,N,C,W,N1,N2,N3,N4,N5,N6)
      IMPLICIT REAL*8 (A-H,C-Z)
      DIMENSION A(N1,N2),C(N3,N4),W(N5,N6)
      IF(N6.EQ.1)GO TO 21
      DO 19 I=1,M
      DO 19 J=1,M
      W(I,J)=0.
      DO 19 K=1,N
19    W(I,J)=W(I,J)+A(I,K)*A(J,K)
      DO 20 I=1,M
      DO 20 J=1,M
20    C(I,J)=W(I,J)
      RETURN
21    IF(N5.EQ.1)GO TO 24
      DO 22 I=1,M
      DO 23 J=1,M
      W(J,1)=0.
      DO 23 K=1,N
23    W(J,1)=W(J,1)+A(I,K)*A(J,K)
      DO 22 J=1,M
22    C(I,J)=W(J,1)
      RETURN
24    DO 25 I=1,M
      DO 25 J=1,M
      C(I,J)=0.
      DO 25 K=1,N
25    C(I,J)=C(I,J)+A(I,K)*A(J,K)
      RETURN
      END

      SUBROUTINE SCATB(A,B,L,M,N,C,W,N1,N2,N3,N4,N5,N6,N7,N8)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(N1,N2),B(N3,N4),C(N5,N6),W(N7,N8)
      IF(N8.EQ.1)GO TO 5
      DO 3 I=1,L
      DO 3 J=1,N
      W(I,J)=0.
      DO 3 K=1,M
3     W(I,J)=W(I,J)+A(K,I)*B(K,J)
      DO 4 I=1,L
      DO 4 J=1,N
4     C(I,J)=W(I,J)
      RETURN
5     IF(N7.EQ.1) GO TO 9
      DO 7 I=1,L
      DO 8 J=1,N
      W(J,1)=0.
      DO 8 K=1,M
8     W(J,1)=W(J,1)+A(K,I)*B(K,J)
      DO 7 J=1,N
7     C(I,J)=W(J,1)
      RETURN
9     DO 10 I=1,L
      DO 10 J=1,N
      C(I,J)=0.
      DO 10 K=1,M
10    C(I,J)=C(I,J)+A(K,I)*B(K,J)
      RETURN
      END
```

```
      SUBROUTINE SCATA(A,M,N,C,W,N1,N2,N3,N4,N5,N6)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(N1,N2),C(N3,N4),W(N5,N6)
      IF(N6.EQ.1)GO TO 14
      DO 12 I=1,N
      DO 12 J=1,N
      W(I,J)=0.
      DO 12 K=1,M
12    W(I,J)=W(I,J)+A(K,I)*A(K,J)
      DO 13 I=1,N
      DO 13 J=1,N
13    C(I,J)=W(I,J)
      RETURN
14    IF(N5.EQ.1)GO TO 17
      DO 15 I=1,N
      DO 16 J=1,N
      W(J,1)=0.
      DO 16 K=1,M
16    W(4,1)=W(J,1)+A(K,I)*A(K,J)
      DO 15 J=1,N
15    C(I,J)=W(J,1)
      RETURN
17    DO 18 I=1,N
      DO 18 J=1,N
      C(I,J)=0.
      DO 18 K=1,N
18    C(I,J)=C(I,J)+A(K,I)*A(K,J)
      RETURN
      END

      SUBROUTINE SCAB(A,B,L,M,N,C,W,N1,N2,N3,N4,N5,N6,N7,N8)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(N1,N2),B(N3,N4),C(N5,N6),W(N7,N8)
      IF(N8.EQ.1) GO TO 5
      DO 3 I=1,L
      DO 3 J=1,N
      W(I,J)=0.
      DO 3 K=1,M
3     W(I,J)=W(I,J)+A(I,K)*B(K,J)
      DO 4 I=1,L
      DO 4 J=1,N
4     C(I,J)=W(I,J)
      RETURN
5     IF(N7.EQ.1) GO TO 9
      DO 7 I=1,L
      DO 8 J=1,N
      W(J,1)=0.
      DO 8 K=1,M
8     W(J,1)=W(J,1)+A(I,K)*B(K,J)
      DO 7 J=1,N
7     C(I,J)=W(J,1)
      RETURN
9     DO 10 I=1,L
      DO 10 J=1,N
      C(I,J)=0.
      DO 10 K=1,M
10    C(I,J)=C(I,J)+A(I,K)*B(K,J)
      RETURN
      END
```

```
      SUBROUTINE SCAPB(A,B,L,M,C,N1,N2,N3,N4,N5,N6)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(N1,N2),B(N3,N4),C(N5,N6)
      DO 32 I=1,L
      DO 32 J=1,M
32    C(I,J)=A(I,J)+B(I,J)
      RETURN
      END

      SUBROUTINE SCVI(VI,L,A,B,N,NB,N1,N2)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION VI(N1,N2),NB(1)
      K1=1
      K2=NB(1)
      II=1
      DO 1 I=1,L
      DO 2 J=1,L
      IF(J.LT.K1.OR.J.GT.K2)GO TO 3
      C=-A/(B*(NB(II)*A+B))
      IF(I.EQ.J)C=C+1/B
      VI(I,J)=C
      GO TO 2
3     VI(I,J)=0.
2     CONTINUE
      IF(I.LT.K2.OR.I.EQ.L)GOTO 1
      K1=K1+NB(II)
      II=II+1
      K2=K2+NB(II)
1     CONTINUE
      RETURN
      END

      SUBROUTINE SCAMB(A,B,L,M,C,N1,N2,N3,N4,N5,N6)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(N1,N2),B(N3,N4),C(N5,N6)
      DO 31 I=1,L
      DO 31 J=1,M
31    C(I,J)=A(I,J)-B(I,J)
      RETURN
      END

      SUBROUTINE SCI(A,L,N1,N2)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(N1,N2)
      DO 34 I=1,L
      DO 34 J=1,L
      A(I,J)=0.
      IF(I.EQ.J)A(I,J)=1.
34    CONTINUE
      RETURN
      END
```

```
      SUBROUTINE SCODOP(G,D,O,L,WK,M,N,N1,N2,N3,N4)
      DIMENSION G(N1,N2),O(N3,N4),D(1),WK(1)
      DO 1 I=1,L
      DO 2 J=1,N
      WK(J)=0.
      DO 2 K=1,M
2     WK(J)=WK(J)+G(I,K)*D(K)*O(K,J)
      DO 1 J=1,N
1     G(I,J)=WK(J)
      RETURN
      END

      SUBROUTINE SCODOP(O,D,OP,L,W,M,N,N1,N2,N3,N4)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION O(N1,N2),OP(N3,N4),D(1),W(1)
      DO 2 J=1,N
      DO 3 I=1,L
      W(I)=0.
      DO 3 K=1,M
3     W(I)=W(I)+O(I,K)*OP(K,J)*D(I)
      DO 2 I=1,L
2     OP(I,J)=W(I)
      RETURN
      END


      SUBROUTINE SCABT(A,B,L,M,N,C,W,N1,N2,N3,N4,N5,N6,N7,N8)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(N1,N2),B(N3,N4),C(N5,N6),W(N7,N8)
      IF(N8.EQ.1)GO TO 5
      DO 3 I=1,L
      DO 3 J=1,N
      W(I,J)=0.
      DO 3 K=1,M
3     W(I,J)=W(I,J)+A(I,K)*B(J,K)
      DO 4 I=1,L
      DO 4 J=1,N
4     C(I,J)=W(I,J)
      RETURN
5     IF(N7.EQ.1) GO TO 9
      DO 7 I=1,L
      DO 8 J=1,N
      W(J,1)=0.
      DO 8 K=1,M
8     W(J,1)=W(J,1)+A(I,K)*B(J,K)
      DO 7 J=1,N
7     C(I,J)=W(J,1)
      RETURN
9     DO 10 I=1,L
      DO 10 J=1,N
      C(I,J)=0.
      DO 10 K=1,M
10    C(I,J)=C(I,J)+A(I,K)*B(J,K)
      RETURN
      END
```

```
      SUBROUTINE SCSVIN(D,N,W,CONST)
      DIMENSION D(1),W(1)
      DO 1 I=1,N
      W(I)=D(I)
1     W(N+I)=I
      SUM=0.
      NM1=N-1
      DO 2 I=1,NM1
      IP1=I+1
      DO 3 J=IP1,N
      IF(W(I).LT.W(J))GO TO 3
      WA=W(I)
      W(I)=W(J)
      W(J)=WA
      WA=W(N+I)
      W(N+I)=W(N+J)
      W(N+J)=WA
3     CONTINUE
      SUM=SUM+W(I)*W(I)
      IF(SUM.LT.CONST)D(W(N+I))=0.
      IF(SUM.GT.CONST)GO TO 4
2     CONTINUE
4     DO 5 I=1,N
      IF(D(I).NE.0.)D(I)=1./D(I)
5     CONTINUE
      RETURN
      END
```

```
      SUBROUTINE SCTRS(QV,V1,L,S,W1,W2,N1,N2,N3,N4,N5,N6)
      DIMENSION QV(N1,N2),V1(N3,N4),S (2,2),W1(N5,N6),W2(1)
      CALL SCAB(QV,QV,L,L,L,W1,W2,N1,N2,N1,N2,N5,N6,1,1)
      S(2,2)=FCTRCE(W1,L,N5,N6)
      CALL SCAB(QV,V1,L,L,L,W1,W2,N5,N6,N1,N2,N5,N6,1,1)
      CALL SCAB(W1,QV,L,L,L,W1,W2,N5,N6,N1,N2,N5,N6,N7,1)
      S(1,2)=FCTRCE(W1,L,N5,N6)
      S(2,1)=S(1,2)
      CALL SCAB(W1,V1,L,L,L,W1,W2,N5,N6,N3,N4,N5,N6,N7,1)
      S(1,1)=FCTRCE(W1,L,N5,N6)
      RETURN
      END
```

```
      DOUBLE PRECISION FUNCTION CDGAMA(X,A,B)
      IMPLICIT REAL*8(A-H,O-Z)
      IF(A.GT.0.D0 .AND. B.GT.0.D0)GO TO 10
      CDGAMA=-1.D0
      RETURN
10    IF(X.GT.0.D0) GO TO 20
      CDGAMA=0.D0
      RETURN
20    Z=X/B
      IF(X.LT.A*B)GO TO 200
      A1=1.D0
      A2=1.D0+Z-A
      B1=1.D0
      B2=Z
      N=1
      F1=A2/B2
      AM=1.D0-A
      AN=0.D0
100   N=N+1
      IF(N.GT.51)GO TO 400
      IF(MOD(N,2).EQ.1)GO TO 110
      AN=AN+1.D0
      A3=A2+AN*A1
      B3=B2+AN*B1
      GO TO 120
110   AM=AM+1.D0
      A3=Z*A2+AM*A1
      B3=Z*B2+AM*B1
120   F2=A3/B3
      IF(DABS(F2-F1).LT.1.D-10)GO TO 300
      F1=F2
      A1=A2
      A2=A3
      B1=B2
      B2=B3
      GO TO 100
200   A1=1.D0
      A2=A+1.D0-Z
      B1=1.D0
      B2=A+1.D0
      N=1
      F1=A2/B2
      AM=-A*Z
      AN=0.D0
      BC=A+1.D0
210   N=N+1
      IF(N.GT.51)GO TO 400
      IF(MOD(N,2).EQ.1)GO TO 220
      AN=AN+Z
      BC=BC+1.D0
      A3=BC*A2+AN*A1
      B3=BC*B2+AN*B1
      GO TO 230
220   AM=AM-Z
      BC=BC+1.0D0
      A3=BC*A2+AM*A1
```

```
      B3=BC*B2+AM*B1
230   F2=A3/B3
      IF(DABS(F2-F1).LT.1.D-10) GO TO 300
      F1=F2
      A1=A2
      A2=A3
      B1=B2
      B2=B3
      GO TO 210
300   COEF=A*DLOG(Z)-Z-DLGAMA(A)
      COEF=DEXP(COEF)
      IF(X.LT.A*B)GO TO 310
      CDGAMA=1.D0-COEF/(Z*F2)
      RETURN
310   CDGAMA=COEF/(A*F2)
      RETURN
400   CDGAMA=-2.D0
      RETURN
      END
```

```
      SUBROUTINE SCLPAR(PR,IT,ITM,ITN,X1,X2,CONST,R,X,NR,NC,BL,NB,C,CA)
      DIMENSION X(NR,NC)
      INTEGER PR(10),R,BL,NB(1),C,CA
      INTEGER B(72),WP(72),SP(72)
      READ(1,1)PR,IT,ITM,ITN,X1,X2,CONST
1     FORMAT(10I1,3I5,3F10.0)
      WRITE(3,5)
5     FORMAT(/'   B WP SP')
      R=1
11    READ(5,3,END=10)B(R),WP(R),SP(R)
      WRITE(3,6)B(R),WP(R),SP(R)
6     FORMAT(1X, 3I3)
3     FORMAT(3I1)
      R=R+1
      GO TO 11
10    R=R-1
      CALL SCX(X,B,WP,SP,R,NR,NC,BL,NB,C,CA)
      WRITE(3,4)PR,R,C,CA,IT,ITM,ITN,X1,X2,CONST,BL,(NB(I),I=1,BL)
4     FORMAT(' OP........',10I1
     */' ROWS......',I3
     */' COL.......',I3
     */' COL X.....',I3
     */' ITER......',I6
     */' ITERM.....',I6
     */' ITERN.....',I6
     */' A1........',F10.3
     */' A2........',F10.3
     */' CONST.....',E13.6
     */' N BL......',I3
     */' N OB IN BL.',20I3
     *)
       RETURN
       END
```