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Let (X, A, P) be a given statistical model. We concern ourselves with particular families of sub-fields of A , and, for each such family we ask ourselves whether the family has maximal and minimal elements with respect to the natural partial order of inclusion relation. For example the family $\{S\}$ of sub-fields that are sufficient for A has received a great deal of attention from theoretical statisticians. Clearly, $A \in \{S\}$ and is the maximum element of $\{S\}$. It is known [4] that, in general, $\{S\}$ has no minimal element. However, if we assume that P is dominated by a σ -finite measure then it can be shown [1] that $\{S\}$ has an essentially minimum element S_0 , i.e., given any $S_1 \in \{S\}$ and any $A \in S_0$, there exists $B \in S_1$ such that

$$P(A \Delta B) \equiv 0 \text{ for all } P \in \underline{P},$$

where Δ stands for the operation of symmetric difference.

Let B be a fixed sub-field of A and let $\{C\}$ be the family of all sub-fields that are independent of B , i.e., for any such C it is true that

$$P(BC) \equiv P(B)P(C) \text{ for all } B \in \underline{B}, C \in \underline{C}, \text{ and } P \in \underline{P}.$$

The minimum element of $\{C\}$ is the trivial sub-field consisting of the null-set and the whole space. We have the following

Theorem 1: For any sub-field C that is independent of B , there exists at least one maximal subfield C^* such that $C \subset C^*$.

The proof of Theorem 1 is given in the next section.

For the next problem, let us suppose that the family \underline{P} is indexed by

two parameters θ and φ , i.e.,

$$\underline{P} = \{P_{\theta, \varphi}\}, (\theta, \varphi) \in \Theta \times \Phi.$$

Consider all sub-fields \underline{D} that are generated by statistics whose probability distributions do not involve the parameter φ , i.e., each \underline{D} is a sub-field such that, for every $D \in \underline{D}$, $P_{\theta, \varphi}(D)$ is a function of θ only.

The trivial sub-field is again the minimum element of $\{\underline{D}\}$; but, does this family have maximal elements? We have

Theorem 2: Given any sub-field \underline{D} such that the restriction of $P_{\theta, \varphi}$ to \underline{D} does not involve φ , there exists a maximal such sub-field \underline{D}^* that contains \underline{D} .

The above Theorem is an immediate generalization of Theorem 1 in [2].

In the next section we give the proof of the above two theorems. In the final section we comment on some further problems of the same kind.

2. Proofs of Theorems 1 and 2

We need the following well-known lemmas.

Lemma 1 (Zorn's Lemma): If for a partially ordered set it is true that every linearly ordered sub-set has an upper (lower) bound, then given any element x of the set there exists a maximal (minimal) element x^* in the set such that x is less (greater) than x^* .

[The terms that are underlined are defined in terms of the partial order relation].

Lemma 2 (Extension of Measures): Given a measure μ defined on a field \underline{F} of sets there exists one and only one extension μ^* of μ to the Borel-extension \underline{F}^* of \underline{F} .

Corollary: If the two measures μ and ν agree on a field \underline{F} of sets they necessarily agree on the Borel-extension \underline{F}^* of \underline{F} .

Lemma 3: If the family $\{\underline{B}_\alpha\}$ of sub-fields of \underline{A} be linearly ordered with respect to the inclusion relation then

$$\underline{B} = \bigcup \underline{B}_\alpha$$

is a field of sub-sets.

We omit the proofs of the lemmas.

Now let \underline{B} be a given sub-field of \underline{A} and let \underline{E} be the class of all sets $E \in \underline{A}$ such that E is independent of \underline{B} , i.e.

$$P(EB) = P(E)P(B) \text{ for all } B \in \underline{B}, E \in \underline{E} \text{ and } P \in \underline{P}.$$

It is easy to check that \underline{E} contains the null-set and the whole space and further that \underline{E} is closed for complementation and countable disjoint unions. In case \underline{E} is a σ -field there is nothing to prove in Theorem 1, as \underline{E} is then the maximal sub-field for which we are searching. However, \underline{E} is usually not a σ -field (see example 1).

Let $\{\underline{C}_\alpha\}$ be a family of sub-fields in \underline{E} and be linearly ordered with respect to the inclusion relation and let

$$\underline{C}_0 = \bigcup \underline{C}_\alpha.$$

Now, from Lemma 3, \underline{C}_0 is a field of sub-sets of \underline{A} and, since $\underline{C}_0 \subset \underline{E}$, every member of \underline{C}_0 is independent of \underline{B} . Choose and fix $B \in \underline{B}$ and $P \in \underline{P}$.

Consider the two measures $P(AB)$ and $P(A)P(B)$ defined for all sets $A \in \underline{A}$. These two measures agree over the field \underline{C}_0 and hence, from the corollary to Lemma 2, they agree over the Borel-extension \underline{C}_0^* of \underline{C}_0 .

Remembering that B and P were arbitrary members of \underline{B} and \underline{P} respectively, we now have

$$P(AB) \equiv P(A)P(B) \text{ for all } A \in \underline{C}^*, B \in \underline{B} \text{ and } P \in \underline{P}.$$

Thus, \underline{C}_0^* includes every \underline{C}_α and is independent of \underline{B} . The conditions of Lemma 1 are satisfied and hence the proof of Theorem 1 is complete.

We now turn our attention to Theorem 2. Let \underline{F} be the class of all sets $F \in \underline{A}$ such that

$$P_{\theta, \varphi}(F) \text{ is a function of } \theta \text{ only.}$$

Again, it is easy to check that \underline{F} contains the null-set and the whole space and is closed for complementation and countable disjoint unions. In example 2 we shall see that \underline{F} is usually not a sub-field of \underline{A} .

Let $\{\underline{D}_\alpha\}$ be a family of sub-fields in \underline{F} and be linearly ordered with respect to the inclusion relation and let

$$\underline{D}_0 = \bigcup \underline{D}_\alpha.$$

As before \underline{D}_0 is a field of sub-sets of \underline{X} . Let \underline{D}_0^* be the Borel-extension of \underline{D}_0 .

We define the measure Q_θ on \underline{D}_0 as the restriction of $P_{\theta, \varphi}$ on \underline{D}_0 . [Since $\underline{D}_0 \subset \underline{F}$, the measure Q_θ on \underline{D}_0 must be independent of φ .]

From Lemma 2, for each $\theta \in \Theta$, the extension Q_θ^* of Q_θ from \underline{D}_0 to \underline{D}_0^* is unique. From the corollary to Lemma 2, the two measures Q_θ^* and $P_{\theta, \varphi}$ must agree on \underline{D}_0^* .

In other words, the restriction of $P_{\theta, \varphi}$ to the sub-field \underline{D}_0^* is independent of φ . Also, \underline{D}_0^* includes every \underline{D}_α . The conditions of Lemma 1 are satisfied and hence the proof of Theorem 2 is complete.

3. Examples and Comments.

The following two examples demonstrate that the maximal element is usually not unique.

Example 1: Let X consist of the four points a, b, c and d and let P consist of just one probability distribution namely the uniform distribution over the four points. Consider the three sub-fields B , C_1 and C_2 each consisting of four sub-sets of X :

B consists of X , $\{a, b\}$ and their complements,

C_1 X , $\{a, c\}$

and C_2 X , $\{a, d\}$

Here, each of C_1 and C_2 is independent of B and each of them is a maximal such sub-field. Incidentally, C_1 and C_2 are also independent of each other.

Example 2: Let X consist of the five points a, b, c, d , and e and let the probability distribution over the five points be

Points:	a	b	c	d	e
Probs:	$1-\theta$	$\theta \varphi$	$\theta \varphi$	$\theta(\frac{1}{2}-\varphi)$	$\theta(\frac{1}{2}-\varphi)$

where $0 < \theta < 1$ and $0 < \varphi < \frac{1}{2}$.

The family F of all sets whose probability does not involve φ consists of 12 sets, that is

X , $\{a\}$, $\{b, d\}$, $\{b, e\}$, $\{c, d\}$, $\{c, e\}$

and their complements.

Note that F does not constitute a sub-field. There are two different maximal sub-fields in F , namely,

\mathcal{A}_1 : consisting of X , $\{a\}$, $\{b, d\}$, $\{c, e\}$
and their complements.

and \mathcal{A}_2 : consisting of \underline{X} , (a) , (b,e) , (c,d)
and their complements.

Theorems 1 and 2 only establish the existence of maximal sub-fields in \underline{E} and \underline{F} respectively. It would be of some interest to develop general methods for proving the maximality of certain given sub-fields of \underline{E} and \underline{F} . One such method, with very limited application, is given in Theorem 7 of [2].

Consider the problem where we have n independent observations x_1, x_2, \dots, x_n on a real random variable x with cumulative distribution function of the form

$$F\left(\frac{x - \varphi}{\theta}\right), \quad -\infty < \varphi < \infty, \quad 0 < \theta < \infty.$$

where the function F is known and θ and φ are the so-called scale and location parameters.

If y stand for the vector-valued statistic

$$(x_1 - x_n, x_2 - x_n, \dots, x_{n-1} - x_n)$$

then the distribution of y does not involve the location parameter φ . Is y a maximal such statistic? In the language of sub-fields, if \underline{C}_y be the sub-field generated by y then is it true that \underline{C}_y is maximal in the sense of Theorem 2? The author does not expect the answer to be 'yes' for all F .

4. Some further problems

The sub-field $\underline{B} \subset \underline{A}$ is said to be sufficient for the sub-field $\underline{C} \subset \underline{A}$ if for every $C \in \underline{C}$ there exists a \underline{B} -measurable function $f(x; C)$ mapping \underline{X} into the real line such that

$$P(BC) \equiv \int_B f(x; C) dP(x) \quad \text{for all } P \in \underline{P} \\ \text{and } B \in \underline{B}.$$

In other words, \underline{B} is sufficient for \underline{C} if, for every $C \in \underline{C}$, there exists a choice for the conditional probability (function) of C given \underline{B} that serves for all $P \in \underline{P}$.

Now, for a fixed \underline{B} , let us enquire about the family $\{\underline{C}\}$ of all sub-fields \underline{C} such that \underline{B} is sufficient for \underline{C} . Clearly, the minimum element of $\{\underline{C}\}$ is the trivial sub-field consisting of only the null-set and the whole space. Do there exist maximal elements in $\{\underline{C}\}$?

Let \underline{G} be the class of all sets $G \in \underline{A}$ such that \underline{B} is sufficient for G in the sense mentioned above, namely, for every $P \in \underline{P}$ there exists a \underline{B} -measurable $f(x;G)$ such that

$$P(GB) = \int_B f(x;G) dP(x) \quad \text{for all } P \in \underline{P} \\ \text{and } B \in \underline{B}.$$

The class \underline{G} is similar to the classes \underline{E} and \underline{F} considered before in that \underline{G} contains the null-set and the whole space and is closed for complementation and countable disjoint unions. As before, \underline{G} is usually not a sub-field. The rest of the arguments in Theorems 1 and 2 will apply if we could prove a result of the following type:

"If \underline{B} is sufficient for each member of a field \underline{C} of sets in \underline{A} , then \underline{B} is sufficient for the Borel extension \underline{C}^* of \underline{C} ."

The above statement does not seem to be true in the generality stated above.

In the particular case where \underline{B} is the trivial sub-field, the question posed above has a definite answer. For, in this case, \underline{B} can be sufficient for G if and only if

$P(G)$ is the same for all $P \in \underline{P}$,
and therefore, Theorem 2, or rather a particular case of it, namely, Theorem 1

in [2] applies.

Fraser in [3] introduced the notation of partial sufficiency in the following manner:

If $\underline{P} = \{P_{\theta, \varphi}\}$, $\theta \in \Theta$, $\varphi \in \Phi$, be a family of probability measures indexed by the two independent parameters θ and φ , then a sub-field $\underline{B} \subset \underline{A}$ will be called θ -sufficient for \underline{A} (or simply θ -sufficient) if

i) the restriction of $P_{\theta, \varphi}$ to \underline{B} does not depend on φ [i.e., \underline{B} is a sub-field of the type considered in Theorem 2.],

and ii) given any $A \in \underline{A}$, there exists a choice (of the conditional probability (function) of A given \underline{B} that does not depend on θ , i.e., for each $\theta \in \Theta$ there exists a \underline{B} -measurable function $f_{\theta}(x; A)$ such that

$$P_{\theta, \varphi}(AB) \equiv \int_{\underline{B}} f_{\theta}(x; A) dP_{\theta, \varphi} \quad \text{for all } B \in \underline{B} \\ \text{and all } (\theta, \varphi).$$

Under what conditions does a θ -sufficient sub-field exist? Does there exist an essentially minimum such sub-field?

As a final problem on the existence of minimal sub-fields consider the following:

Given two sub-fields \underline{B} and \underline{C} , let $\underline{B} \vee \underline{C}$ stand for the smallest sub-field that contains both \underline{B} and \underline{C} .

Now, for a fixed $\underline{B} \subset \underline{A}$, let us consider the family $\{\underline{C}\}$ of all sub-fields $\underline{C} \subset \underline{A}$ such that

$$\underline{B} \vee \underline{C} = \underline{A}.$$

Every $\underline{C} \in \{\underline{C}\}$ may be called a complement of \underline{B} . The family $\{\underline{C}\}$ has \underline{A} as its maximum element. Does $\{\underline{C}\}$ have minimal elements? The author expects the answer to be yes. It is easy to construct examples where there are several minimal complements to \underline{B} .

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