Application of generalized function to dynamic analysis of thick plates

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Introduction

The structures with thick plates have been used extensively in national defence, mechanical engineering, chemical engineering, nuclear engineering, civil engineering, etc. Various theories have been established to deal with the problems of elastic plates, which include the classical theory of thin plates, the improved theory of thick plates, three-dimensional elastical theory. The classical theory cannot be expected to hold for plates, whose thicknesses are large with respect to the width. It also cannot be expected to hold when the wavenumber is large and the distribution of loads is nonuniform. However, it is difficult to obtain the analytical solutions for plates with various boundary conditions on the basis of three-dimensional elastical theory. Recently, more attentions have been paid to the improved theory, in which some of the suppositions in the classical theory are neglected and the effects of rotatory inertia and shear deformation are retained. It is applied more extensively than the classical one. Several type of equations of thick plates is listed in table 1.

In oder to calculate $\nabla^2 q$ on the right-hand of the equations in table 1. it is necessary to apply the generalized function.

In this paper, the derivative of δ-function is handled by using the generalized function. The synamic analysis of thick plates subjected the concentrated load is presented. The improved Donnell's equation of thick plates is deduced and employed as the basic equation. The generalized coordinates are solved by using the method of MWR. The gneneral expressions for the dynamic response of elastic thick plates subjected the concentrated load are given. The numerical results for rectangular plates are given herein. The results are compared with those obtained from the improved theory and the classical theory of plates

1 The Improved Vibration Equation of Elastic Thick Plates

The essential viewpoint of Donnell's modification of the classical deflection method for plates is that the deflection w_s caused by the transverse shear strains would be added to the bending deflection w_f obtained from the classical theory for plates, making the total deflection of the middle plane of the plate, i.e.

$$w_t = w_f + w_s \tag{1.1}$$

Then the relationship between w_f and w_s is made

$$w_s = \left[1 - \frac{(8-3\nu)h^2}{40(1-\nu)}\nabla^2\right]w_f \tag{1.2}$$

From Hamilton's principle

$$\delta\int_{t_1}^{t_2}(T-U)dt = 0 \tag{1.3}$$

and by means of the equation (1.2) we can obtain the improved vibration equation of elastic thick plates:

$$\begin{aligned}&\rho h\left[1 - \frac{(8-3\nu)h^2}{40(1-\nu)}\nabla^2\right]\frac{\partial^2 w_t}{\partial t^2} + \nabla^2\left[D\nabla^2 w_f - \frac{\rho h^3}{12}\cdot\frac{\partial^2 w_f}{\partial t^2}\right]\\ &= \left[1 - \frac{(8-3\nu)h^2}{40(1-\nu)}\nabla^2\right]q(x,y,t)\end{aligned} \tag{1.4}$$

For convenience we make

$$q_t(x,y,t) = \left[1 - \frac{(8-3\nu)h^2}{40(1-\nu)}\nabla^2\right]q(x,y,t) = q_s(x,y,t) + q(x,y,t) \tag{1.5}$$

If the terms with $(8-3\nu)h^2/40(1-\nu)\nabla^2$ are neglected, equation (1.4) will be reduced to Donnell's equation.

2 The General Expressions for the Dynamic Response of Elastic Thick Plates

$$w_t(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[1 - \frac{(8-3\nu)h^2}{40(1-\nu)}\nabla^2\right]W_{mn}(x,y)\cdot T_{mn}(t) \tag{2.1}$$

$$q_t(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[1 - \frac{(8-3\nu)h^2}{40(1-\nu)}\nabla^2\right]W_{mn}(x,y)q_{mn}(t) \tag{2.2}$$

where

$$\begin{aligned}T_{mn}(t) &= T_{mn}(0)\cos\omega_{mn}t + \frac{\dot{T}_{mn}(0)}{\omega_{mn}}\sin\omega_{mn}t + \frac{\omega_{mn}}{\rho h\omega^2_{mn(c)}}\cdot\\ &\int_0^t q_{tmn}(t_1)\sin\omega_{mn}(t-t_1)dt_1\end{aligned} \tag{2.3}$$

$$q_{tmn}(t) = q_{mn}(t) + q_{smn}(t) = \frac{\int_0^a\int_0^b [q(x,y,t) + q_s(x,y,t)]W_{mn}(x,y)dxdy}{\int_0^a\int_0^b [W_{mn}(x,y)]^2 dxdy} \tag{2.4}$$

For thick rectangular plates with simply supported edges we have

$$W_{mn}(x,y) = \sqrt{\frac{2}{a}}\sin\frac{m\pi x}{a}\sqrt{\frac{2}{b}}\sin\frac{n\pi y}{b} \tag{2.5}$$

$$\begin{aligned}\omega_{mn} = \omega_{mn(c)}\Big\{1 + \frac{\pi^2 h^2}{a^2}\left(m^2 + \frac{a^2}{b^2}n^2\right)\Big[\frac{1}{12} + \frac{8-3\nu}{40(1-\nu)}\\ + \left(\frac{8-3\nu}{40(1-\nu)}\right)^2\frac{\pi^2 h^2}{a^2}\left(m^2 + \frac{a^2}{b^2}n^2\right)\Big]\Big\}^{-\frac{1}{2}}\end{aligned} \tag{2.6}$$

where

$$(2.7)\qquad \omega_{mn(c)}=\frac{\pi^2}{a^2}\sqrt{\frac{D}{\rho h}}\left(m^2+\frac{a^2}{b^2}n^2\right)$$

For thick rectangular plates with fixed edges we can introduce the general variational principle.

3 The General Expressions for the Dynamic Response of Elastic Thick Plates Subjected the Concentrated Load

Assume that a thick rectangular plate is subjected to a concentrated load at (x_R, y_R),

$$(3.1)\qquad q(x,y,t)=p(t)\delta(x-x_R,y-y_R)$$

where $(x-x_R, y-y_R)$ is 2-dimensional δ-function.

$$(3.2)\qquad \delta(x-x_R,y-y_R)=\begin{cases}\infty & (x=x_R,y=y_R)\\ 0 & (\text{others})\end{cases}$$

In order to calculate $\nabla^2 q$, it is necessary to apply the generalized function.

The definition of the sobolev space is

$$(3.3)\qquad W_p^m(\Omega)=\{f\,|\,D^j f\in L^p(\Omega),|j|\leqslant m\}$$

The definition of the generalized derivative.

For $f\in D'(\Omega)$, the generalized derivative $D^j f$ of f is the generalized function:

$$(3.4)\qquad \langle D^j f,\varphi\rangle=(-1)^{|j|}\langle f,D^j\varphi\rangle,\quad \varphi\in c_0^\infty(\Omega)$$

The derivative of δ-function is handled by using the generalized function. We obtain the expressions for the dynamic response of elastic thick plates subjected the concentrated load

$$(3.5)\qquad w_t(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[1-\frac{(8-3\nu)h^2}{40(1-\nu)}\nabla^2\right]W_{mn}(x,y)T_{mn}(t)$$

where

$$(3.6)\qquad T_{mn}(t)=T_{mn}(0)\cos\omega_{mn}t+\frac{\dot{T}_{mn}(0)}{\omega_{mn}}\sin\omega_{mn}t+\frac{\omega_{mn}}{\rho h\omega_{mn(c)}^2}\int_0^t W_{mn}(x_R,y_R)$$
$$\cdot\left[1+\frac{(8-3\nu)\pi^2h^2}{40(1-\nu)a^2}\left(m^2+\frac{a^2}{b^2}\right)\right]p(t_1)\sin\omega_{mn}(t-t_1)dt_1$$

For thick rectangular plates with simply supported edges we have

$$(3.7)\qquad w_t(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[1+\frac{(8-3\nu)\pi^2h^2}{40(1-\nu)a^2}\left(m^2+\frac{a^2}{b^2}n^2\right)\right]\sqrt{\frac{2}{a}}\sin\frac{m\pi x}{a}\times$$
$$\times\sqrt{\frac{2}{b}}\sin\frac{n\pi y}{b}\left\{T_{mn}(0)\cos\omega_{mn}t+\frac{\dot{T}_{mn}(0)}{\omega_{mn}}\sin\omega_{mn}t+\right.$$
$$+\frac{\omega_{mn}}{\rho h\omega_{mn(c)}^2}\sqrt{\frac{2}{a}}\sin\frac{m\pi x_R}{a}\times\sqrt{\frac{2}{b}}\sin\frac{n\pi y_R}{b}\times$$

$$\left[1+\frac{(8-3\nu)\pi^2h^2}{40(1-\nu)a^2}\times\left(m^2+\frac{a^2}{b^2}n^2\right)\right]\times\int_o^t p(t_1)\sin\omega_{mn}(t-t_1)dt_1\right\}$$

4 The Solution on the Generalized Corrdinates by Using the Method of MWR

In order to solve

$$(4.1)\qquad \ddot{T}_{mn}(t)+2\xi_{mn}\omega_{mn}\dot{T}_{mn}(t)+\omega^2_{mu}T_{mn}(t)=P_{mn}(t)$$

the spline function as follows is used:

$$(4.2)\qquad \Omega_3(x)=\begin{cases}(x+2)^3 & x\in[-2,-1]\\ (x+2)^3-4(x+1)^3 & x\in[-1,0]\\ (2-x)^3-4(1-x)^3 & x\in[0,1]\\ (2-x)^3 & x\in[1,2)\\ 0 & |x|>2\end{cases}$$

By using the method of MWR we can obtain

$$(4.3)\qquad \tilde{T}_{mn}(t_{i+1})=\tilde{A}_{mn}\tilde{T}_{mn}(t_i)+\tilde{B}_{mn}\tilde{P}_{mn}(t_{i+1})$$

where

$$\tilde{T}_{mn}(t_{i+1})=\begin{Bmatrix}T_{mn}(t_{i+1})\\ \dot{T}_{mn}(t_{i+1})\\ \ddot{T}_{mn}(t_{i+1})\end{Bmatrix},\qquad \tilde{T}_{mn}(t_i)=\begin{Bmatrix}T_{mn}(t_i)\\ \dot{T}_{mn}(t_i)\\ \ddot{T}_{mn}(t_i)\end{Bmatrix}\quad \tilde{B}_{mn}=\begin{Bmatrix}\beta_{mn}\Delta t^2/b\\ \beta_{mn}\Delta t/2\\ \beta_{mn}\end{Bmatrix}$$

$$\tilde{A}_{mn}=\begin{bmatrix}\frac{5}{6}-\frac{\beta_{mn}}{6}(L_{1,mn}) & \left(\frac{2}{3}-\frac{\beta_{mn}L_{2,mn}}{6}\right)\Delta t & \left[\frac{1}{36}+\frac{\beta_{mn}}{36}(L_{1,mn}-2L_{2,mn})\right]\Delta t^2\\ -1-\beta_{mn}(L_{1,mn}+L_{2,mn}) & -\frac{\beta_{mn}L_{2,mn}}{2} & [1+\beta_{mn}(L_{1,mn}-2L_{2,mn})]\frac{\Delta t}{12}\\ \frac{-1-\beta_{mn}(L_{1,mn}+L_{2,mn})}{\Delta t^2} & \frac{-2-\beta_{mn}L_{2,mn}}{\Delta t} & -\frac{5}{6}+\frac{\beta_{mn}}{6}(L_{1,mn}-2L_{2,mn})\end{bmatrix}$$

$$\beta_{mn}=\frac{1}{L_{3,mn}},\quad \begin{Bmatrix}L_{1,mn}\\ L_{2,mn}\\ L_{3,mn}\end{Bmatrix}=\begin{Bmatrix}1-\xi_{mn}\omega_{mn}\Delta t+\frac{1}{6}\omega^2_{mn}\Delta t^2\\ -2-2\omega^2_{mn}\Delta t^2/3\\ 1+\xi_{mn}\omega_{mn}\Delta t+\frac{1}{6}\omega^2_{mn}\Delta t^2\end{Bmatrix}$$

5 Numerical Results and Conclusions

The calculations are performed for the rectangular plates with simply cupported edges subjected a nuit step load concentrated in the centre of plates in Fig. 1. The modulus of elasticity $E=2.1\times10^6$ kg/cm², Possion's ratio $\nu=0.3$, height-span ratios h/a=0.1, 0.2, 0.3, the length-width ratios a/b=0.25, 0.5, 0.75, 1. The responses of the square plate are given (Fig.2). The full line represents the total deflection δ_t in the centre of the plate, the dotted line represents the bending deflection δ_f in the centre of the plate. These are done nondimensional, and equal to w_t/δ_0 , w_f/δ_0 , respectively, where δ_0 represents the static deflection in centre of the plate calculated by the classical theory. A time increment Δt is taken as 1/50 of the fundamental period of the plate. * and · denote the results by the analytical method using the improved theory and the classical theory. Fig3-Fig6 show the factors of dynamical action of load for

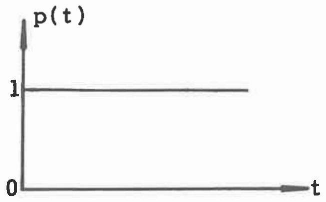

Fig. 1

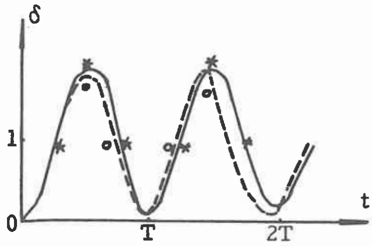

Fig. 2

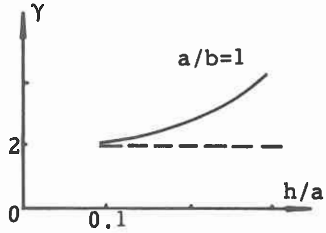

Fig. 3

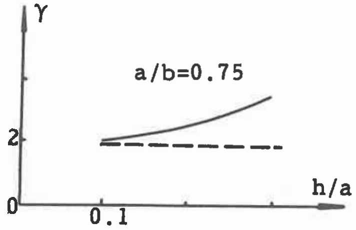

Fig. 4

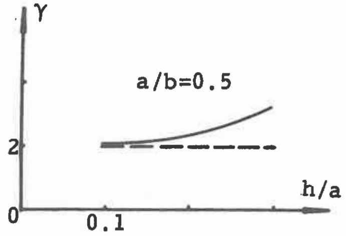

Fig. 5

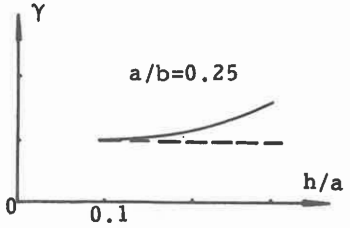

Fig. 6

the centre of plates γ which depend on h/a, a/b, and which represent ratios of the peak dynamical deflection for centre of plates of the classical, statical deflection for centre of plates. The full line and dotted line represent the results using the improved theory and the classical theory respeetively.

From thr numerical results it is evident that the analysis must be done by the improved theory when the ratios of height-spand and wavenumber are larger, and the classical theory cannot be expected to hold for the plates.

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