Combination of Modal Responses

A.K. Gupta, K. Cordero
North Carolina State University, Department of Civil Engineering,
Box 5993, Raleigh, North Carolina 27650, U.S.A.

In the response spectrum method of analyzing structures under seismic loads, values of the maximum probable responses are calculated in each mode of vibration. These responses can be combined using a double sum equation if the correlation between the modal responses is known. An approximate expression for the correlation coefficient given by Rosenblueth and Elorduy[1] does not apply in the low and high frequency ranges. It is shown that the modal responses can be assumed to consist of two components, a rigid component, and a damped periodic component. Various modal rigid components are perfectly correlated. An expression for the correlation between the damped periodic components is presented which is similar to that presented by Rosenblueth and Elorduy[1]. An equation is derived for the correlation between modal responses on this basis, which is shown to be in good agreement with the numerically obtained values.

1. Introduction

In the response spectrum method of analyzing structures under seismic loads, the values of the maximum probable responses are calculated in each mode of vibration. The variation of responses with time is not known, therefore, there is no "exact" way of combining the maximum modal responses. If the maximum response in mode i is R_i, it can be shown that the combined maximum response, R, would satisfy the following inequality

$$\sum_i |R_i| \geqslant R \geqslant \sum_i R_i^2 \qquad (1)$$

the left hand side being an upper bound, and the right hand side a lower bound. In many cases, when the modal frequencies are sufficiently far apart, the right hand side, commonly known as the SRSS (square root of the sum of the squares) combination, gives reasonably accurate values. Statistically, in these cases, the modal responses can be considered to be independent, or uncorrelated. When the modal frequencies are relatively close, however, the modal responses no longer remain independent, and then the combined response can be significantly greater than the SRSS value. In fact, when two of the modes have identical frequencies, it stands reasons that their responses should be directly added in which cases the two responses are perfectly correlated.

In general, for combining responses from two modes one can write

$$R^2 = R_1^2 + R_2^2 + 2\varepsilon_{12}R_1R_2 \qquad (2)$$

in which

$$1 \geqslant \varepsilon_{12} \geqslant 0$$

which depends on the correlation between the modal responses. Theoretically, one could think of a negative correlation also, or when $\varepsilon_{12} < 0$, but as will be seen later, it is not very likely. Assuming that the earthquake motion can be represented by a finite duration segment of the white noise, Rosenblueth and Elorduy[1] have obtained an approximate equation similar to Eq(2). Based on their work, the value of ε_{12} can be written as

$$\varepsilon_{12} = \left\{ 1 + \left[\frac{f_2 - f_1}{\zeta(f_2 + f_1) + 2/(\Pi t_d)} \right]^2 \right\}^{-1} \qquad (3)$$

where

f_1, f_2 = modal frequencies, Hz

ζ = critical damping ratio

t_d = duration of the earthquake motion, secs

It can be seen from Eq(3) that $\varepsilon_{12}=1$ for $f_2 = f_1$, and $\varepsilon_{12} \to 0$ for $f_2 >> f_1$. Although, Eq(3) is approximate it works well for a range of frequencies. Some of the problems are discussed below.

One of the problems with Eq(3) is that it is not clear as to what value of t_d should be used. When the frequecies, f_1 and f_2, are sufficiently large, and for relatively large values of the critical damping ratio, the term consisting of t_d does not play a significant role. In those cases, it does not matter what value of t_d is used in the equation as long as it is reasonable. However, when the frequencies are small, the $2/(\Pi t_d)$ term increases the effective value of the damping, as intended, thus giving a larger value of the ε_{12}. It is in this case the value of ε_{12} is quite sensitive to the value of t_d. Using the complete duration of the ground motion for t_d does not appear to be rational.

There are two other potential problems pointed out by Kennedy[2]. First, when the modal frequencies are higher than the maximum ground motion frequency, Eq(3), does not hold. In fact, the response time histories will be practically scaled input time histories, and would be almost perfectly correlated, in which case $\varepsilon_{12} = 1.0$, even when f_1 and f_2 are sufficiently apart. The writers found that even at other frequencies in the range greater than 1Hz, significant correlation between modes existed. This particular aspect is discussed further in the subsequent sections. The other potential problem pointed by Kennedy[2] is the following. When the modal frequencies are sufficiently apart, beyond a certain point, the correlation between the modal responses may start increasing, rather than decrease as predicted by Eq(3). Heuristically, the reason is simply that it would be quite likely that the high frequency response, can easily be maximum about the same time when the low frequency response reaches the maximum. In the numerical work done for the present study, however, no such trend was evident. The explanation for that is perhaps that different segments of the ground motion have different frequency contents. As such, it is unlikely that the same segment of the motion would excite two modes with widely disparate frequencies. Thus, the second potential problem was not found to exist to any discernable degree.

A practical method of calculating the correlation factor ε_{12}, which has a wide range of applicability, is presented in the following section. The numerical work on which this method is based, is presented in the subsequent section.

2. Proposed Method of Calculating ε_{ij}

Based on the observation of the modal responses and their combinations, a heuristic assumption is made: Any modal response R_i consists of two parts, a damped periodic response, R_i^p, which has characteristics similar to that obtained by using a finite segment of the white noise[1], and a rigid response, R_i^r, which is perfectly correlated with the input ground motion. It is further assumed that the two parts are mutually uncorrelated, ie.

$$R_i^2 = R_i^{p\,2} + R_i^{r\,2} \qquad (4)$$

Thus we can write

$$R_i^r = \alpha_i R_i$$

and $$R_i^p = \sqrt{1-\alpha_i^2}\, R_i \qquad (5)$$

When two modal responses R_1 and R_2 with frequencies f_1 and f_2 are to be combined, then the combines response is given by

$$R^2 = R^{r\,2} + R^{p\,2} \qquad (6)$$

in which

$$R^r = \alpha_1 R_1 + \alpha_2 R_2 \quad \text{(perfectly correlated)}$$

$$R^{p\,2} = R_1^{p\,2} + R_2^{p\,2} + 2\varepsilon_{12}^p R_1^p R_2^p \qquad (7)$$

In Eq(7) ε_{12}^p is the correlation coefficient for the damped periodic part of the responses, and is defined by an equation similar to Eq(3)

$$\varepsilon_{12}^p = \left\{ 1 + \left[\frac{f_2 - f_1}{\zeta(f_2 + f_1) + c_{12}} \right]^2 \right\}^{-1} \qquad (8)$$

where

$$c_{12} = (1-3\zeta)\ (.036 - |f_2^{\ 2} - f_1^{\ 2}|) \geqslant 0 \qquad (9)$$

when Eq(9) gives a negative c_{12} value it is taken to be zero. The term c_{12} here replaces the $2/(\pi t_d)$ term in Eq(3). In effect, thus the value of t_d to be used with Eq(3) varies with the amount of critical damping and with $|f_2^{\ 2} - f_1^{\ 2}|$. Equations (2), (4) - (7) yield

$$\varepsilon_{12} = \alpha_1\alpha_2 + \sqrt{(1-\alpha_1^{\ 2})\ (1-\alpha_2^{\ 2})}\quad \varepsilon^p_{12} \qquad (10)$$

It was found that Eq(10) gives values of ε_{12} which are quite close to the numerically calculated values for a wide range of frequencies including high frequencies.

When $f_2 \to \infty$, $\alpha_2 = 1$, $\varepsilon^p_{12} = 0$, and Eq(10) gives

$$\varepsilon_{12} = \alpha_1$$

which is actually the mathematical definition of α implicit in the present study. The values of α vary with the modal frequency, and are also a function of the critical damping ratio. The following equation can be used to evaluate the values of α, which was found to be in reasonable agreement with the numerically calculated values

$$(\alpha + 0.1)\ (\alpha - m\ \ell n\ f + a) = b, \quad -0.1 \leqslant \alpha \leqslant 1.0 \qquad (12)$$

If the above equation gives a value of α greater than 1, it should be taken equal to 1.0, and similarly, if the equation gives α less than -0.1, it should be set equal to -0.1. The values of the constant m, a and b vary with the damping ratio and are given by

$$m = 0.07373\ \ell n\ \left(\frac{17.34}{\zeta}\right)$$

$$a = -0.3437\ \ell n\ (7.594\ \zeta)$$

$$b = -.03237\ \ell n\ (14.28\ \zeta) \qquad (13)$$

3. Numerican Results

Consider the response time histories $\overline{R}_1(t)$ and $\overline{R}_2(t)$ of two single degree of freedom systems, frequenceis f_1 and f_2, critical damping ratio ζ, subjected to the same earthquake ground motion. The standard deviations, σ_1 and σ_2, and the covariance σ_{12} are defined as follows:

$$\sigma_1^{\ 2} = \frac{1}{t_d} \int_0^{t_d} \overline{R}_1^{\ 2}(t)dt$$

$$\sigma_2^{\ 2} = \frac{1}{t_d} \int_0^{t_d} \overline{R}_2^{\ 2}(t)dt$$

$$\sigma_{12}^{\ 2} = \frac{1}{t_d} \int_0^{t_d} \overline{R}_1(t)\ R_2(t)\ dt \qquad (14)$$

The correlation between the two responses is given by [3]

$$\varepsilon_{12} = \frac{\sigma_{12}^{\ 2}}{\sigma_1\ \sigma_2} \qquad (15)$$

It is noted that whereas σ_1, σ_2 and σ_{12} are quite sensitive to the value of the duration of ground motion, t_d, the value of ε_{12} given by Eq(15) is practically independent of the value of t_d, as long as all significant part of the responses are covered in the assumed duration. The statement can be easily verified, therefore, is not pursued any further here.

The intent of the present study is to arrive at the values of ε_{12} which can be used in conjunction with the response spectrum method of analysis. The response spectra which are commonly used in design, such as in ref[4] are based on a number of measured earthquake ground motions. Similarly, the present study is performed using ten such motion time histories, which are listed below

Parkfield, 1966, N50E and N40W

El Centro, 1940, NS and EW

El Centro, 1934, NS and EW

Taft, 1952, N21E and S69E

Olympia, 1949, S04E and S86W

For any pair of frequencies f_1 and f_2, the mean and the standard deviation of ε_{12} were calculated using the responses from the ten time histories.

When f_2 is infinite, $\overline{R}_2(t)$ is equal to scaled input motion time history. For any frequency f_1, the factor α_1, is then given by ε_{12} as stated in Eq(11). The mean values of α were calculated for various frequencies and damping ratios, and were found to be in good agreement with Eq(12).

Fig 1 shows the comparison of ε_{12} values calculated using Eq(10), and those obtained from the average of the values calculated using the ten motion time histories, for the critical damping ratio of 1%. In Eq(10), actually calculated mean values of α's were used. As shown, the agreement between the calculated values and those given by Eq(10) is good. The agreements for other damping ratios (2, 4 and 7%) were also found to be good. This in effect verifies the assumptions made in this study, viz, a modal response have two components: rigid and damped periodic and that they are uncorrelated. When the frequency f_1 is small, it is seen in Fig 2 that for certain range of f_2 values the correlation coefficients have a small negative value. The writers feel that perhaps these negative values can be attributed to a limited data in terms of using a small number of earthquake time histories. In any case, since these values are numerically quite small, writers did not attach much significance to them, and decided not to investigate them further.

Next, the effect of any error in the value of ε_{12} is investigated. It is noted that the value of the combined response would be most sensitive to any errors in the values of the correlation coefficents, when there are only two significant modal responses and when they are also equal in magnitude. The value of the combined response is then given by

$$R^2 = 2R_1^{\,2} \times (1 + \varepsilon_{12}) \;,\; R_1 = R_2 \qquad (16)$$

If a different value of the correlation coefficient, ε_{12}', were used, a new value of the combined response, R', would be obtained. The percentage change in the value of the combined response is given by

$$\Delta R\% = \left(\sqrt{\frac{1 + \varepsilon_{12}'}{1 + \varepsilon_{12}}} - 1 \right) \times 100 \qquad (17)$$

There are two types of variations in ε_{12} of interest here. One is the dispersion in the value of ε_{12} represented by the standard deviation. The other is the error introduced because of using the idealized equations such as Eqs(10) and (12), which attempt to calculate the mean value of ε_{12}. In order to investigate the accuracy of Eqs(10) and (12), and the effect of dispersion in the value of ε_{12}, Eq(17) is used to calculate the maximum error in the response value by substituting in the denominator the value of ε_{12} from

Eqs (10) and (12), and in the numerator substituting for $\varepsilon_{12}^{\hat{}}$ the calculated mean + one standard deviation. The maximum of all ΔR values for f_2/f_1 = 1 to 10 for four f_1 frequencies are given below.

Frequency	Maximum Error at ζ			
f_1 Hz	.01	.02	.04	.07
0.1	11.0	10.5	9.8	8.8
1.0	3.4	3.9	3.5	3.4
10.0	5.0	6.8	7.1	5.6
21.5	4.6	2.0	0.5	0.0

The highest error of 11% is encountered at the lowest frequency and damping. Generally, as the frequency and the damping go up the error diminishes until it becomes zero. Since the worst cases are considered, in most practical situations the actual errors are likely to be much smaller.

4. Conclusions

A modal response can be assumed to consist of two components which are uncorrelated, a rigid component and a damped periodic component. Whereas the all the modal rigid components are perfectly correlated, the correlation between the modal damped periodic components is a function of the modal frequencies; an expression for the correlation coefficient ε_{12}^{p}, which is similar to one given in Ref[1], is given, Eq(8). On this basis then, the correlation coefficient for modal responses is evaluated which is given by Eq(10). It is shown that the theoretically calculated modal correlation coefficients are in good agreement with those calculated directly from the response time histories of single degree of freedom systems subjected to actual ground motion histories. This in effect verifies the assumptions made in the present study.

5. References

[1] Rosenblueth, E. and Elorduy, J., "Response of Linear Systems to Certain Transient Disturbances," Proc., fourth World Conference on Earthquake Engineering, Santiago, Chile, 1969, A-1, pp. 185-196.

[2] Kennedy, R. P., "Recommendations for Changes and Additions to Standard Review Plans and Regulatory Guides Dealing with Seismic Design Requirements for Structures," Report prepared for Lawrence Livermore Laboratory, Published in Nureg/CR-1161, June, 1979. (Also personal communications with Kennedy)

[3] Gungor, I, "A Study of Stochastic Models for Predicting Maximum Earthquake Structural Response," Ph.D. Thesis, University of Illinois at Urbana - Champaign, August, 1971.

[4] U. S. Nuclear Regulatory Commission, "Design Response Spectra for Seismic Design of Nuclear Power Plants," Regulatory Guide 1.60, Revision 1, December 1973.

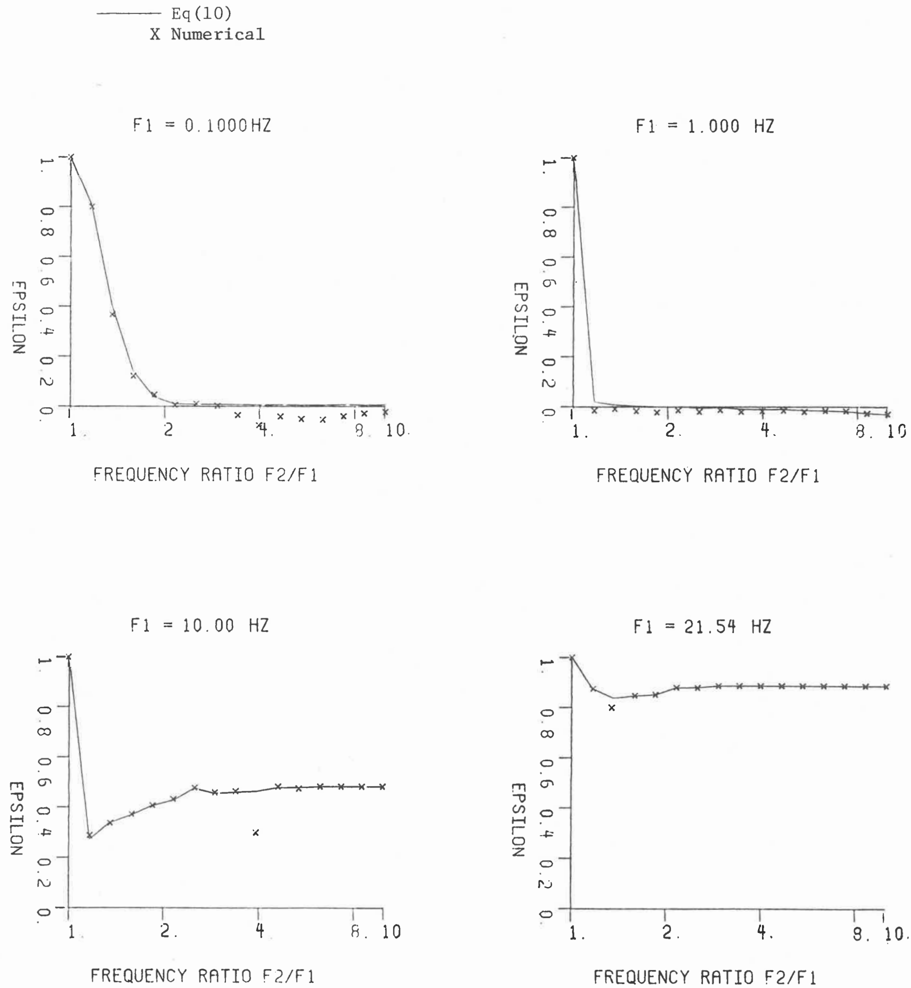

Figure 1 - Comparison of Numerically Calculated Correlation Coefficients, ε with Eq(10), ζ = 0.01