# A GENERAL APPROACH TO THE ESTIMATION OF VARIANCE COMPONENTS

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A general method of estimation of variance components in random-effects models of the nested and/or classification type is considered. If a given parameter is estimable with respect to some particular experimental design (i.e., an unbiased estimate of the parameter may be obtained from the experiment), then the suggested estimator may be readily computed with only the aid of a desk calculator. The estimates are always unbiased and consistent (with respect to the structure of the experimental design); in the case of balanced experiments, they coincide with those obtained from the analysis of variance.

Secondly, the problem of designing experiments to estimate variance components is briefly discussed from the point-of-view of the suggested estimation procedure. As a result, certain non-balanced designs are seen to yield more efficient estimators of particular parameters in specified situations than the corresponding balanced design using the same number of observations.

Finally, the method of estimation is shown to be applicable to models more general than the variance component one. Again it is readily computed and is unbiased and consistent.

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## INTRODUCTION

In this paper, we want to consider experiments in which the observations may be assumed to be linear functions of independent, unmeasurable random variables with zero means and unknown variances which are called variance components. The purpose of the experiment is to study the contribution of several sources of variation to the total variation among the observations by estimating the variance components. Several methods of estimating variance components have been developed. The three best known are:

(i) setting the mean squares appearing in some particular type of analysis of variance (weighted or unweighted, adjusted or unadjusted) equal to their respective expectations and solving the resulting equations for the variance components (see Anderson [2]; and Bush and Anderson [3]) ;

(ii) listing certain sums of squares and their corresponding expectations (there being more sums of squares than variance components) and performing weighted least squares (using prior estimates of the variance components or other information to determine the weights) (see Anderson [2]) ;

(iii) assuming that the random-effects obey some probability distribution function which is completely known except for a constant representing the associated variance component (for example, assuming the random-effects are normally distributed with zero means) and obtaining maximum likelihood estimates or modified maximum likelihood estimates (see Anderson and Bancroft [1]).

In balanced experiments, we may say (loosely speaking) that methods (i), (ii), and (iii) all lead to the same estimators. In addition, the estimators obtained by (i) are the best quadratic unbiased estimates (BQUE) of the corresponding parameters (see Graybill [7]; Graybill and Hultquist [8]; Hsu [10]). However, in non-balanced experiments (particularly non-balanced cross-classification-type designs), methods (i), (ii), and (iii) can become very complicated computationally; their small sample properties are, for the most part, unknown and difficult to study. Hence these methods are of limited use in studying the efficiency of different experimental designs.

With the comments made above in mind, one sees the need for another type of estimator which, in non-balanced experiments, is computationally simpler and theoretically more tractable than those estimators obtained by the methods (i), (ii), and (iii) while, in balanced experiments, coincides with them (i.e., it is BQUE). The method of estimation presented here will be seen to have the above properties.

The structure of the suggested estimate follows from basic assumptions made about the form of the covariance matrix in the assumed model. Any estimable parameter (see Appendix 1) can be estimated by forming an appropriate linear combination of certain symmetric sums of (linearly independent) random variables having the same expectation such that the resulting quadratic form is an unbiased estimate of it. In many cases, there is some choice as to which linear combination to use. As a consequence, if prior information concerning the magnitude (relative or actual) of the variance components is available, then the suggested estimator would be modified to be that linear combination which was best with respect to this information.

Because of its generality and structure, this estimation procedure may be used to evaluate the efficiency of non-balanced experimental designs. Even with it, however, results have been too difficult to obtain in a clear-cut form for the general case; but some light is shed on the problem of the proper choice of design if a few simplifying assumptions are made. Namely,

(1) the mean of the observations is assumed to be zero,

(2) the random effects are assumed to be normally distributed.

Finally, the suggested approach is valid and useful in obtaining estimators of the variances and covariances of observations from any experiment in which they are estimable with respect to the assumed model (see Appendix 1). The estimates will be of particular interest whenever the variance-covariance matrix of the observations possesses some structure; i.e., it is a function of a small number (with respect to $(N^2 + N)/2$ where N is the number of observations) of parameters which appear in it in a particular pattern. All of the above comments will be clarified by the specific applications and examples which follow.

NOTATION: <u>If a "dot" replaces a subscript, then that subscript has been summed over; if a "bar" appears over a quantity with "dots" in its subscript, then those subscripts have been averaged over.</u>

## 2. NESTED EXPERIMENTS

### 2.1 The Degenerate or One-Stage Nested Design.

Model: $Y_i = \mu + e_i$ for $i = 1, 2, \ldots, n$.

Assume $e_1, e_2, \ldots, e_n$ are independent and identically distributed as $N(0, \sigma_e^2)$.

Hence, we have that

$$E\left\{Y_i Y_j\right\} = \sigma_e^2 + \mu^2 \quad \text{if } i = j$$
$$= \mu^2 \quad \text{if } i \neq j$$

where $i, j = 1, 2, \ldots, n$.

Let us now estimate $(\sigma_e^2 + \mu^2)$ by forming the normalized symmetric sum of all of its unbiased estimators listed above; i.e., we have

$$g_t = \widehat{(\sigma_e^2 + \mu^2)} = \frac{1}{n} \sum_{i=1}^{n} Y_i^2$$

Similarly, let us estimate $\mu^2$ by the normalized symmetric sum of its unbiased estimators. Hence,

$$g_m = \hat{\mu}^2 = \frac{1}{n(n-1)} \sum_{i \neq j=1}^{n} Y_i Y_j = \frac{1}{n(n-1)} \left\{ \left(\sum_{i=1}^{n} Y_i\right)^2 - \sum_{i=1}^{n} Y_i^2 \right\} .$$

We obtain the estimator of $\sigma_e^2$ by subtracting $g_m$ from $g_t$; that is,

$$\hat{\sigma}_e^2 = g_t - g_m = \frac{1}{(n-1)n} \left\{ (n-1) \sum_{i=1}^{n} Y_i^2 - \left[\left(\sum_{i=1}^{n} Y_i\right)^2 - \sum_{i=1}^{n} Y_i^2\right] \right\}$$
$$= \frac{1}{n-1} \left\{ \sum_{i=1}^{n} Y_i^2 - \left[\left(\sum_{i=1}^{n} Y_i\right)^2 / n\right] \right\}$$
$$= \frac{1}{n-1} \left\{ \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \right\} .$$

It is well known that

$$E\,(\hat{\sigma}_e^2) = \sigma_e^2 \quad \text{and} \quad \mathrm{Var}\,(\hat{\sigma}_e^2) = \frac{2\sigma_e^4}{(n-1)}$$

and that $\hat{\sigma}_e^2$ is the minimum variance unbiased estimator of $\sigma_e^2$ for this model. As a result, we see that in this case the suggested estimation procedure leads to the usual estimator of $\sigma_e^2$.

## 2.2. The Two-Stage Nested Design.

Model: $Y_{ij} = \mu + a_i + e_{ij}$ for $i = 1, 2, \ldots, a$ and $j = 1, 2, \ldots, n_i$.

Assume

(i) $a_1, a_2, \ldots, a_a, e_{11}, \ldots, e_{1n_1}, \ldots, e_{a1}, \ldots, e_{a,n_a}$ are independent;

(ii) $a_1, a_2, \ldots, a_a$ are identically distributed as $N(0, \sigma_a^2)$;

(iii) $e_{11}, \ldots, e_{1n_1}, \ldots, e_{a1}, \ldots, e_{a,n_a}$ are identically distributed as $N(0, \sigma_a^2)$;

Hence, we have that

$$\begin{aligned} E\left\{Y_{ij}\, Y_{k\ell}\right\} &= \sigma_a^2 + \sigma_e^2 + \mu^2 && \text{if } i = k \text{ and } j = \ell \\ &= \sigma_a^2 + \mu^2 && \text{if } i = k \text{ and } j \neq \ell \\ &= \mu^2 && \text{if } i \neq k \end{aligned}$$

where $i, k = 1, 2, \ldots, a$ and $j = 1, 2, \ldots, n_i$ and $\ell = 1, 2, \ldots, n_k$.

Let us now estimate $\sigma_a^2 + \sigma_e^2 + \mu^2$, $\sigma_a^2 + \mu^2$, $\mu^2$ by forming the normalized symmetric sums of their respective unbiased estimators listed above; that is,

$$g_t = \left(\sum_{i=1}^{a} n_i\right)^{-1} \sum_{i=1}^{a} \sum_{j=1}^{n_i} Y_{ij}^2 = \frac{1}{n} \sum_{i=1}^{a} \sum_{j=1}^{n_i} Y_{ij}^2 \;,$$

$$g_a = \left\{\sum_{i=1}^{a} n_i(n_i-1)\right\}^{-1} \left\{\sum_{i=1}^{a} \sum_{j\neq\ell=1}^{n_i} Y_{ij}\, Y_{i\ell}\right\} = \frac{1}{(k_1-n)} \sum_{i=1}^{a} \left\{(\sum_{j=1}^{n_i} Y_{ij})^2 - \sum_{j=1}^{n_i} Y_{ij}^2\right\} ,$$

$$g_m = \left\{\sum_{i=1}^{a} n_i(n-n_i)\right\}^{-1} \left\{\sum_{i\neq k=1}^{a} \sum_{j=1}^{n_i} \sum_{\ell=1}^{n_k} Y_{ij}\, Y_{k\ell}\right\} = \frac{1}{(n^2-k_1)} \left\{(\sum_{i=1}^{a} \sum_{j=1}^{n_i} Y_{ij})^2 - \sum_{i=1}^{a} (\sum_{j=1}^{n_i} Y_{ij})^2\right\}$$

where $n = \sum_{i=1}^{a} n_i$ and $k_1 = \sum_{i=1}^{a} n_i^2$ .

We obtain the estimators of $\sigma_a^2$, $\sigma_e^2$, $\mu^2$ by setting $g_t$, $g_a$, $g_m$ equal to their respective expectations and solving the resulting equations. Thus, we obtain

$$\hat{\sigma}_a^2 = g_a - g_m ,$$

$$\hat{\sigma}_e^2 = g_t - g_a ,$$

$$\hat{\mu}^2 = g_m .$$

The above estimates are unbiased by construction. Their variances and covariances can be obtained by using the theory of Appendix 2; however, the resulting expressions become unwieldly and are difficult to interpret; hence, they are not considered here. On the other hand, if we assume that $\mu$ is a known constant and in particular is zero (this can be achieved by performing a transformation $Y'_{ij} = Y_{ij} - \mu$), then the estimates of the variance components are

$$\hat{\sigma}_a^2 = g_a ,$$

$$\hat{\sigma}_e^2 = g_t - g_a .$$

In this case, the variances and covariances of the estimators are not so difficult to obtain and provide some insights in the design problem.
If we let

$$g_{ti} = \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij}^2 \quad . \quad ,$$

$$g_{ai} = \frac{1}{n_i(n_i-1)} \sum_{j \neq \ell =1}^{n_i} Y_{ij} Y_{i\ell} \quad ,$$

then from the theory of Appendix 2, we obtain

$$V_{ti} = \text{Var}\,(g_{ti}) = 2\sigma_a^4 + \frac{4\sigma_a^2\,\sigma_e^2}{n_i} + \frac{2\sigma_e^4}{n_i} \quad ,$$

$$V_{ai} = \text{Var}\,(g_{ai}) = 2\sigma_a^4 + \frac{4\sigma_a^2\,\sigma_e^2}{n_i} + \frac{2\sigma_e^4}{n_i(n_i-1)} \quad ,$$

$$C_{at,i} = \text{Cov}\,(g_{ti}, g_{ai}) = 2\sigma_a^4 + \frac{4\sigma_a^2\,\sigma_e^2}{n_i} \quad ,$$

$$V_{e,i} = \text{Var}\left\{g_{ti} - g_{ai}\right\} = \frac{2\sigma_e^4}{(n_i-1)} \quad .$$

Since $(g_{ti}, g_{ai})$ is independent of $(g_{tk}, g_{ak})$ for $i \neq k = 1, 2, \ldots, a$, we have that

$$V_t = \mathrm{Var}\ (g_t) = \frac{2k_1}{n^2}\ \sigma_a^4 + \frac{4\sigma_a^2\ \sigma_e^2}{n} + \frac{2\sigma_e^4}{n} \quad ,$$

$$V_a = \mathrm{Var}\ (g_a) = \frac{2k_2\ \sigma_a^4}{(k_1-n)^2} + \frac{4k_3\ \sigma_a^2\ \sigma_e^2}{(k_1-n)^2} + \frac{2\sigma_e^4}{(k_1-n)} \quad ,$$

$$C_{at} = \mathrm{Cov}\ (g_t,g_a) = \frac{2k_4\ \sigma_a^4}{n(k_1-n)} + \frac{4\sigma_a^2\ \sigma_e^2}{n} \quad ,$$

$$V_e^{\ *} = \mathrm{Var}\ (g_t-g_a) = 2\left\{\frac{k_1}{n^2} - \frac{2k_4}{n(k_1-n)} + \frac{k_2}{(k_1-n)^2}\right\}\sigma_a^4 + 4\left\{\frac{k_3}{(k_1-n)^2} - \frac{1}{n}\right\}\sigma_a^2\ \sigma_e^2 + 2\left\{\frac{k_1}{n(k_1-n)}\right\}\sigma_e^4,$$

where $k_2 = \sum_{i=1}^{a} n_i^2\ (n_i-1)^2$ , $k_3 = \sum_{i=1}^{a} n_i\ (n_i-1)^2$ , $k_4 = \sum_{i=1}^{a} n_i^2\ (n_i-1)$.

At this time, we should note that the estimation procedure can give us a more well-known estimator of $\sigma_e^2$ than $g_t-g_a$. Thus, we observe that

$$g_{e,i} = g_{ti}-g_{ai} = \frac{1}{n_i-1}\left\{\sum_{j=1}^{n_i} (Y_{ij}-\bar{Y}_{i.})^2\right\}$$

is an unbiased estimator of $\sigma_e^2$ with variance $V_{ei} = \dfrac{2\sigma_e^4}{(n_i-1)}$ . With this in mind, an alternative estimator of $\sigma_e^2$ would be

$$g_e = \frac{1}{n-a}\left\{\sum_{i=1}^{a} (n_i-1)(g_{ti}-g_{ai})\right\} = \frac{1}{n-a}\sum_{i=1}^{a}\sum_{j=1}^{n_i} (Y_{ij}-\bar{Y}_{i.})^2 \ .$$

Its variance is, of course,

$$V_e = \mathrm{Var}\,(g_e) = \frac{2\sigma_e^4}{(n-a)} \quad .$$

Finally, one should observe that if prior estimates of $\sigma_a^2$, $\sigma_e^2$ are available, then other linear combinations of the $g_{ai}$ and/or the $g_{ti}$ may lead to better estimators of the parametric functions $\sigma_a^2$, $\sigma_a^2 + \sigma_e^2$, $\sigma_e^2$ . For example, one might consider

$$g_a^* = \left( \sum_{i=1}^{a} \frac{g_{ai}}{V_{ai}} \right)\left( \sum_{i=1}^{a} \frac{1}{V_{ai}} \right)^{-1}$$

as an alternative estimator of $\sigma_a^2$. The questions concerning what is the best way in which to use the information of the covariance matrix (if such a best way exists) will require further research and probably numerical study.

Let us now see what happens if $n_1 = n_2 = \ldots = n_a = c$. In this case, $n = ac$, $k_1 = ac^2$, $k_2 = ac^2\,(c-1)^2$, $k_3 = ac(c-1)^2$, $k_4 = ac^2(c-1)$. Hence

$$V_t = \frac{2\sigma_a^4}{a} + \frac{4\sigma_a^2\,\sigma_e^2}{ac} + \frac{2\sigma_e^4}{ac} \quad ,$$

$$V_a = \frac{2\sigma_a^4}{a} + \frac{4\sigma_a^2\,\sigma_e^2}{ac} + \frac{2\sigma_e^4}{ac\,(c-1)} \quad ,$$

$$C_{at} = \frac{2\sigma_a^4}{a} + \frac{4\sigma_a^2\,\sigma_e^2}{ac} \quad ,$$

$$V_e = V_e^* = \frac{2\sigma_e^4}{a\,(c-1)} \quad .$$

The above suggest that unless $\sigma_e^2$ is very small with respect to $\sigma_a^2$, one should have that c be of moderate size for the design to be efficient in estimating the parametric functions, $\sigma_e^2$, $\sigma_a^2 + \sigma_e^2$, and particularly $\sigma_a^2$; in addition "a" should be larger or smaller than moderate according as $\sigma_a^2$ is larger or smaller.

Finally, by computations similar to those used in the next section, one can verify that if $n_1 = n_2 = \ldots = n_a = c$, than

$$\hat{\sigma}_a^2 = g_a - g_m = \frac{1}{c}\left\{MSA - MSE\right\}$$

$$\hat{\sigma}_e^2 = g_t - g_a = g_e = MSE$$

where MSA and MSE are the mean squares due to A-classes and error respectively in the one-way analysis of variance associated with the model. Thus, for balanced experiments $\hat{\sigma}_a^2$ and $\hat{\sigma}_e^2$ coincide with the estimators obtained by the analysis of variance method of estimation and hence are BQUE (see Graybill [7] ; and Graybill and Hultquist [8]).

## 2.3. The Three-Stage Nested Design

Model: $Y_{ijk} = \mu + a_i + b_{ij} + e_{ijk}$ for $i = 1, 2, \ldots, a$;

$j = 1, 2, \ldots, m_i$; $k = 1, 2, \ldots, n_{ij}$.

Assume

(i) $a_1, a_2, \ldots, a_a;\ b_{11}, \ldots, b_{1m_1}, \ldots, b_{am_a};\ e_{111}, \ldots, e_{a,m_a,n_{am_a}}$ are independent;

(ii) the $\{a_i\}$ are identically distributed as $N(0, \sigma_a^2)$ ;

(iii) the $\{b_{ij}\}$ are identically distributed as $N(0, \sigma_b^2)$;

(iv) the $\{e_{ijk}\}$ are identically distributed as $N(0, \sigma_e^2)$ .

Hence, we have that

$$
\begin{aligned}
E\{Y_{ijk}\, Y_{tuv}\} &= \sigma_a^2 + \sigma_b^2 + \sigma_e^2 + \mu^2 && \text{if } i = t,\ j = u,\ k = v \\
&= \sigma_a^2 + \sigma_b^2 + \mu^2 && \text{if } i = t,\ j = u,\ k \neq v \\
&= \sigma_a^2 + \mu^2 && \text{if } i = t,\ j \neq u \\
&= \mu^2 && \text{if } i \neq t
\end{aligned}
$$

where $i, t = 1, 2, \ldots, a$ ; $j = 1, 2, \ldots, m_i$ ; $k = 1, 2, \ldots, n_{ij}$ .

$u = 1, 2, \ldots, m_t$ ; $v = 1, 2, \ldots, n_{tu}$

Let us estimate $\sigma_a^2 + \sigma_b^2 + \sigma_e^2 + \mu^2$, $\sigma_a^2 + \sigma_b^2 + \mu^2$, $\sigma_a^2 + \mu^2$, $\mu^2$ by forming the normalized symmetric sums of their respective unbiased estimators listed above; that is,

$$g_t = \Big(\sum_{i=1}^{a}\sum_{j=1}^{m_i} n_{ij}\Big)^{-1} \sum_{i=1}^{a}\sum_{j=1}^{m_i}\sum_{k=1}^{n_{ij}} Y_{ijk}^2 = \frac{1}{n}\sum_{i=1}^{a}\sum_{j=1}^{m_i}\sum_{k=1}^{n_{ij}} Y_{ijk}^2 \qquad ,$$

$$g_{ab} = \left(\sum_{i=1}^{a}\sum_{j=1}^{m_i} n_{ij}(n_{ij}-1)\right)^{-1}\left\{\sum_{i=1}^{a}\sum_{j=1}^{m_i}\sum_{k\neq v} Y_{ijk}Y_{ijv}\right\} = \frac{1}{(k_1-n)}\left\{\sum_{i=1}^{a}\sum_{j=1}^{m_i}\left(Y_{ij.}^2 - \sum_{k=1}^{n_{ij}} Y_{ijk}^2\right)\right\} ,$$

$$g_a = \left(\sum_{i=1}^{a}\sum_{j=1}^{m_i} n_{ij}(n_i-n_{ij})\right)^{-1}\left\{\sum_{i=1}^{a}\sum_{j\neq u} Y_{ij.}Y_{iu.}\right\} = \frac{1}{(k_2-k_1)}\left\{\sum_{i=1}^{a}\left(Y_{i..}^2 - \sum_{j=1}^{m_i} Y_{ij.}^2\right)\right\} ,$$

$$g_m = \left(\sum_{i=1}^{a} n_i(n-n_i)\right)^{-1}\left\{\sum_{i\neq t} Y_{i..}Y_{t..}\right\} = \frac{1}{(n^2-k_2)}\left\{Y_{...}^2 - \sum_{i=1}^{a} Y_{i..}^2\right\} ,$$

where $n_i = \sum_{j=1}^{m_i} n_{ij}$, $n = \sum_{i=1}^{a} n_i$, $k_1 = \sum_{i=1}^{a}\sum_{j=1}^{m_i} n_{ij}^2$, $k_2 = \sum_{i=1}^{a} n_i^2$ .

We obtain estimators of $\sigma_a^2$, $\sigma_b^2$, $\sigma_e^2$, $\mu^2$ by setting $g_t$, $g_{ab}$, $g_a$, $g_m$ equal to their respective expectations and solving the resulting equations. Thus, we obtain

$$\hat{\sigma}_a^2 = g_a - g_m ,$$

$$\hat{\sigma}_b^2 = g_{ab} - g_a ,$$

$$\hat{\sigma}_e^2 = g_t - g_{ab} ,$$

$$\hat{\mu}^2 = g_m .$$

The same remarks that were made in the previous section apply to the variances and covariances of these estimators. In addition, the approach used there can be applied to this model and hence can provide some insight to efficient designs. Finally, the procedure can be modified at various places to provide alternative estimators of the parameters. For example, in the case of $\sigma_e^2$, we consider

$$g_e = \frac{1}{\sum_{i=1}^{a}\sum_{j=1}^{m_i}(n_{ij}-1)}\left\{\sum_{i=1}^{a}\sum_{j=1}^{m_i}\sum_{k=1}^{n_{ij}}(Y_{ijk}-\bar{Y}_{ij.})^2\right\} = \frac{1}{n-m}\left\{\sum_{i=1}^{a}\sum_{j=1}^{m_i}\sum_{k=1}^{n_{ij}}(Y_{ijk}-\bar{Y}_{ij.})^2\right\}$$

where $m = \sum_{i=1}^{a} m_i$ .

If $n_{i1} = n_{i2} = \ldots = n_{im_i} = c_i$ for $i = 1, 2, \ldots, a$, then an alternative estimator of $\sigma_b^2$ is obtained by considering

$$g_{ab,i}-g_{a,i} = \frac{1}{c_i m_i (c_i-1)}\left\{\sum_{j=1}^{m_i}(Y_{ij.}^2 - \sum_{k=1}^{c_i} Y_{ijk}^2)\right\} - \frac{1}{c_i^2 m_i (m_i-1)}\left\{Y_{i..}^2 - \sum_{j=1}^{m_i} Y_{ij.}^2\right\}$$

$$= \frac{1}{c_i}\left\{[\sum_{j=1}^{m_i} c_i (\bar{Y}_{ij.} - \bar{Y}_{i..})^2/(m_i - 1)] - [\sum_{j=1}^{m_i}\sum_{k=1}^{c_i}(Y_{ijk} - \bar{Y}_{ij.})^2/m_i(c_i-1)]\right\}$$

and forming

$$\hat{\sigma}_b^2 = g_b = \sum_{i=1}^{a} c_i\,(g_{ab,i} - g_{a,i})/\sum_{i=1}^{a} c_i \quad .$$

If $c_1 = c_2 = \dots = c_a = c$, we might use

$$g_b^* = \left\{ \sum_{i=1}^{a} \sum_{j=1}^{m_i} (\bar{Y}_{ij.} - \bar{Y}_{i..})^2 \right\} / \left\{ m - a \right\} - \left\{ \sum_{i=1}^{a} \sum_{j=1}^{m_i} \sum_{k=1}^{c} (\bar{Y}_{ijk} - \bar{Y}_{ij.})^2 \right\} / \left\{ mc\ (c - 1) \right\}.$$

Again, it should be noted that it is possible to derive any number of estimators; we have only considered some of the simpler ones and some of the more traditional ones.

Suppose now that $m_1 = m_2 = \dots = m_a = b$ and $n_{11} = n_{12} = \dots = n_{1b} = \dots = n_{a1} = \dots = n_{ab} = c$; i.e., the experiment is balanced. In this case,

$$g_t = \frac{1}{abc} \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{c} Y_{ijk}^2 \quad ,$$

$$g_{ab} = \frac{1}{abc\ (c - 1)} \sum_{i=1}^{a} \sum_{j=1}^{b} \left( Y_{ij.}^2 - \sum_{k=1}^{c} Y_{ijk}^2 \right) \quad ,$$

$$g_a = \frac{1}{abc^2\ (b - 1)} \sum_{i=1}^{a} \left( Y_{i..}^2 - \sum_{j=1}^{b} Y_{ij.}^2 \right) \quad ,$$

$$g_m = \frac{1}{ab^2c^2\ (a - 1)} \left( Y_{...}^2 - \sum_{i=1}^{a} Y_{i..}^2 \right) \quad ;$$

$$\hat{\sigma}_a^2 = g_a - g_m = \frac{1}{ab^2c^2\,(b-1)(a-1)}\left\{ b(a-1)\left\{\sum_{i=1}^{a} Y_{i..}^2 - \sum_{i=1}^{a}\sum_{j=1}^{b} Y_{ij.}^2\right\} - (b-1)\left\{ Y_{...}^2 - \sum_{i=1}^{a} Y_{i..}^2\right\}\right\}$$

$$= \frac{1}{ab^2c^2\,(b-1)(a-1)}\left\{ (b-1)\left\{a \sum_{i=1}^{a} Y_{i..}^2 - Y_{...}^2\right\} - (a-1)\left\{b \sum_{i=1}^{a}\sum_{j=1}^{b} Y_{ij.}^2 - \sum_{i=1}^{a} Y_{i..}^2\right\}\right\}$$

$$= \frac{1}{bc}\left\{\left[\left(\sum_{i=1}^{a} Y_{i..}^2/bc - Y_{...}^2/abc\right)/(a-1)\right] - \left[\left(\sum_{i=1}^{a}\sum_{j=1}^{b} Y_{ij.}^2/c - \left(\sum_{i=1}^{a} Y_{i..}^2/bc\right)\right)/a(b-1)\right]\right\}$$

$$= \frac{1}{bc}\left\{\left[bc \sum_{i=1}^{a} (\bar{Y}_{i..} - \bar{Y}_{...})^2/(a-1)\right] - \left[c \sum_{i=1}^{a}\sum_{j=1}^{b} (\bar{Y}_{ij.} - \bar{Y}_{i..})^2/a(b-1)\right]\right\}$$

$$= \frac{1}{bc}\left\{MSA - MSB\right\}$$

$$\hat{\sigma}_b^2 = g_{ab} - g_a = \frac{1}{abc^2\,(b-1)(c-1)}\left\{ c(b-1)\left\{\sum_{i=1}^{a}\sum_{j=1}^{b} \left(Y_{ij.}^2 - \sum_{k=1}^{c} Y_{ijk}^2\right)\right\} - (c-1)\left\{\sum_{i=1}^{a}\left(Y_{i..}^2 - \sum_{j=1}^{b} Y_{ij.}^2\right)\right\}\right\}$$

$$= \frac{1}{abc^2\,(b-1)(c-1)}\left\{ (c-1)\left\{b \sum_{i=1}^{a}\sum_{j=1}^{b} Y_{ij.}^2 - \sum_{i=1}^{a} Y_{i..}^2\right\} - (b-1)\left\{c \sum_{k=1}^{c} Y_{ijk}^2 - \sum_{i=1}^{a}\sum_{j=1}^{b} Y_{ij.}^2\right\}\right\}$$

$$= \frac{1}{c}\left\{\left[c \sum_{i=1}^{a}\sum_{j=1}^{b} (\bar{Y}_{ij.}^2 - \bar{Y}_{i..})^2/a(b-1)\right] - \left[\sum_{k=1}^{c} (Y_{ijk} - \bar{Y}_{ij.})^2/ab(c-1)\right]\right\}$$

$$= \frac{1}{c}\left\{MSB - MSE\right\}$$

$$\hat{\sigma}_e^2 = g_t - g_{ab} = \frac{1}{abc\,(c-1)}\left\{(c-1) \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} Y_{ijk}^2 - \sum_{i=1}^{a}\sum_{j=1}^{b} Y_{ij.}^2 + \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} Y_{ijk}^2\right\}$$

$$= \left\{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} (Y_{ijk} - \bar{Y}_{ij.})^2/ab(c-1)\right\}$$

$$= MSE$$

where MSA, MSB, MSE are the mean squares due to A-classes, B-classes, and error respectively in the analysis of variance associated with the model. Thus, for balanced experiments $\hat{\sigma}_a^2$, $\hat{\sigma}_b^2$, $\hat{\sigma}_e^2$ coincide with the estimators obtained by the analysis of variance method of estimation and hence are BQUE.

2.4. An Example of the Estimation Procedure for the Three-Stage Nested Design .

The following example consists of data generated from random normal deviates with $\sigma_a^2 = 16$, $\sigma_b^2 = 1$, $\sigma_e^2 = 4$, $\mu = 0$.

| 1st stage | 2nd stage | Observations | 2nd stage Sums | 1st stage Sums | Total |
|---|---|---|---|---|---|
| | 1 | 3.230, -1.588 | 1.642 | | |
| 1 | 2 | 4.982, 1.144 | 6.126 | 7.768 | |
| | 1 | 4.701, 3.639 | 8.340 | | |
| 2 | 2 | 3.182, 3.680 | 6.862 | 15.202 | |
| | 1 | .705, -5.215 | -4.510 | | |
| 3 | 2 | .229, -1.717 | -1.488 | -5.998 | |
| | 1 | -6.529, -8.797 | -15.326 | | |
| 4 | 2 | -7.730, | -7.730 | -23.056 | |
| | 1 | 8.855, 5.041 | 13.896 | | |
| 5 | 2 | 3.373 | 3.373 | 17.269 | |
| | 1 | .757, -1.421 | -0.664 | | |
| 6 | 2 | 1.520 | 1.520 | .856 | |
| | 1 | -2.095, 0.053 | -2.042 | | |
| 7 | | | | -2.042 | |
| | 1 | 8.764, 5.882 | 14.646 | | |
| 8 | | | | 14.646 | |
| | 1 | -3.299, -4.951 | -8.250 | | |
| 9 | | | | -8.250 | 16.395 |

$\sum_{i,j,k} Y^2_{ijk} = 579.8505 \qquad n = 27 \qquad g_t = 21.48$

$\sum_{i,j} Y^2_{ij.} = 968.0249 \qquad k_1 = 51 \qquad g_{ab} = 16.17$

$\sum_{i} Y^2_{i..} = 1444.686445 \qquad k_2 = 87 \qquad g_a = 13.22$

$Y^2_{...} = 268.796025 \qquad n^2 = 729 \qquad g_m = -1.83$

$\hat{\sigma}^2_e = 5.31 \quad , \qquad \hat{\sigma}^2_b = 2.95 \quad , \qquad \hat{\sigma}^2_a = 15.05$

## 2.5. The Multi-Stage Nested Design

Model: $Y_{i_1 i_2 \ldots i_r} = \mu + e^{(1)}_{i_1} + e^{(2)}_{i_1 i_2} + \ldots + e^{(r)}_{i_1 i_2 \cdots i_r}$

where $i_1 = 1,2,\ldots, m^{(1)}$ ; $i_2 = 1,2,\ldots, m^{(2)}_{i_1}$ ; ... ; $i_r = 1,2,\ldots, m^{(r)}_{i_1 i_2 \cdots i_{r-1}}$ .

Assume

(i) The $\left\{e^{(1)}_{i_1}\right\}$, $\left\{e^{(2)}_{i_1 i_2}\right\}$, ..., $\left\{e^{(r)}_{i_1 i_2 \cdots i_r}\right\}$ are independent;

(ii) The $\left\{e^{(1)}_{i_1}\right\}$ are identically distributed as $N(0,\sigma^2_1)$ ;

(iii) The $\left\{e^{(2)}_{i_1 i_2}\right\}$ are identically distributed as $N(0,\sigma^2_2)$ ;

(iv) The $\left\{e^{(j)}_{i_1 i_2 \ldots i_j}\right\}$ are identically distributed as $N(0,\sigma^2_j)$ where $j = 3, 4, \ldots, r$ .

Hence, we have that

$$E\left\{Y_{i_1 i_2 \cdots i_r} Y_{j_1 j_2 \cdots j_r}\right\} = \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_r^2 + \mu^2 \text{ if } i_1 = j_1,\ i_2 = j_2, \ldots, i_r = j_r$$

$$= \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_{r-1}^2 + \mu^2 \text{ if } i_1 = j_1, i_2 = j_2, \ldots, i_{r-1} = j_{r-1}, i_r \neq j_r$$

$$\cdots \qquad\qquad \cdots$$

$$= \sigma_1^2 + \sigma_2^2 + \mu^2 \quad \text{if } i_1 = j_1,\ i_2 = j_2,\ i_3 \neq j_3$$

$$= \sigma_1^2 + \mu^2 \qquad \text{if } i_1 = j_1,\ i_2 \neq j_2$$

$$= \mu^2 \qquad \text{if } i_1 \neq j_1$$

where $i_1, j_1 = 1, 2, \ldots, m^{(1)};\ i_2 = 1, 2, \ldots, m^{(2)}_{i_1};\ \ldots;\ i_r = 1, 2, \ldots, m^{(r)}_{i_1 i_2 \cdots i_{r-1}}$

$j_2 = 1, 2, \ldots, m^{(2)}_{j_1};\ \ldots;\ j_r = 1, 2, \ldots, m^{(r)}_{j_1 j_2 \cdots j_{r-1}}$

We estimate $\sigma_1^2 + \ldots + \sigma_r^2 + \mu^2,\ \sigma_1^2 + \ldots + \sigma_{r-1}^2 + \mu^2, \ldots, \sigma_1^2 + \mu^2,\ \mu^2$ by forming the normalized symmetric sums of their respective unbiased estimators listed above; that is,

$$g_r = \left\{\frac{1}{n} \Sigma Y^2_{i_1 i_2 \cdots i_r}\right\},$$

$$g_{r-1} = \frac{1}{k_1 - n}\left\{\Sigma Y_{i_1 i_2 \ldots i_{r-1} i_r} Y_{i_1 i_2 \cdots i_{r-1} i_r}\right\} = \frac{1}{k_1 - n}\left\{\Sigma Y^2_{i_1 \cdots i_{r-1}.} - \Sigma Y^2_{i_1 \cdots i_{r-1} i_r}\right\},$$

$$\cdots$$

$$g_1 = \frac{1}{k_{r-1}-k_{r-2}}\left\{\Sigma Y_{i_1 i_2 \cdots i_r} Y_{i_1 j_2 \cdots j_r}\right\} = \frac{1}{k_{r-1}-k_{r-2}}\left\{\Sigma Y^2_{i_1 \cdots} - \Sigma Y^2_{i_1 i_2 \cdots}\right\} \quad ,$$

$$g_m = \frac{1}{n^2-k_{r-1}}\left\{\Sigma Y_{i_1 i_2 \cdots i_r} Y_{j_1 j_2 \cdots j_r}\right\} = \frac{1}{n^2-k_{r-1}}\left\{ Y^2_{\cdots} - \Sigma Y^2_{i_1 \cdots}\right\} \quad ,$$

where $n = \sum_{i_1} \cdots \sum_{i_{r-1}} m^{(r)}_{i_1 \cdots i_{r-1}}$ and where $n_{i_1 i_2 \cdots i_j} = \sum_{i_{j+1}} \cdots \sum_{i_{r-1}} m^{(r)}_{i_1 i_2 \cdots i_{r-1}}$ and

$k_{r-j} = \Sigma n^2_{i_1 i_2 \cdots i_j}$ . By setting $g_r$, $g_{r-1}$, ..., $g_1$, $g_m$ equal to their respective expectations and solving the resulting equations, we obtain the estimators

$$\hat{\mu}^2 = g_m \quad ,$$

$$\hat{\sigma}^2_1 = g_1 - g_m \quad ,$$

$$\hat{\sigma}^2_2 = g_2 - g_1 \quad ,$$

$$\cdots$$

$$\hat{\sigma}^2_{r-1} = g_{r-1} - g_{r-2} \, ,$$

$$\hat{\sigma}^2_r = g_r - g_{r-1} \quad .$$

The same remarks concerning the variance-covariance matrix of the estimators, the possibilities for alternative estimators, and methods of studying the efficiency of different experimental designs apply to this general model as well as to the special cases previously discussed.

Finally, we can show that if the experiment is balanced (i.e., all the $m_{i_1}^{(2)}$ are equal to $m^{(2)}$, ..., all the $m_{i_1 i_2 \ldots i_{r-1}}^{(r)}$ are equal to $m^{(r)}$), then the above estimators coincide with those obtained by the analysis of variance method of estimation and hence are BQUE. The method of proof is identical with that used in the case of the three-stage nested design and consequently exploits the special form of $k_1, k_2, \ldots, k_{r-1}$ and $k_1-n$, $k_2-k_1, \ldots$, $n^2-k_{r-1}$. As a consequence, one can observe that if the experiment is balanced from some stage and onward (i.e., all the $m_{i_1 \ldots i_{j-1}}^{(j)}$ are equal to $m^{(j)}$ for $j \geq t$), then the above estimators for the variance components from the previous stage and onward (i.e., $\sigma_{t-1}^2, \sigma_t^2, \ldots, \sigma_r^2$) are similar to those which could be obtained by an analysis of variance method of estimation. They are not BQUE, however, for reasons which will be considered in Appendix I.

# 3. CLASSIFICATION EXPERIMENTS

## 3.1 The Two-Way Classification Designs

Model: $Y_{ij} = \mu + r_i + c_j + e_{ij}$ if an observation appears in cell $(i,j)$ where $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, c$.

Assume

(i) The $\{r_i\}$, $\{c_j\}$, $\{e_{ij}\}$ are independent;

(ii) The $\{r_i\}$ are identically distributed as $N(0,\sigma_r^2)$;

(iii) The $\{c_j\}$ are identically distributed as $N(0,\sigma_c^2)$;

(iv) The $\{e_{ij}\}$ are identically distributed as $N(0,\sigma_e^2)$.

Hence, we have that

$$
\begin{aligned}
E\{Y_{ij}\, Y_{k\ell}\} &= (\sigma_r^2 + \sigma_c^2 + \sigma_e^2 + \mu^2) && \text{if } i=k,\ j=\ell \\
&= (\sigma_r^2 + \mu^2) && \text{if } i=k,\ j\neq\ell \\
&= (\sigma_c^2 + \mu^2) && \text{if } i\neq k,\ j=\ell \\
&= \mu^2 && \text{if } i\neq k,\ j\neq\ell
\end{aligned}
$$

where $i, k = 1, 2, \ldots, r$ and $j, \ell = 1, 2, \ldots, c$ provided $Y_{ij}Y_{k\ell}$ is defined.

We estimate $\sigma_r^2 + \sigma_c^2 + \sigma_e^2 + \mu^2$, $\sigma_r^2 + \mu^2$, $\sigma_c^2 + \mu^2$, $\mu^2$ by forming the normalized symmetric sums of their respective unbiased estimators listed above; that is,

$$g_t = \frac{1}{n} \sum_{i=1}^{r} \sum_{j} Y_{ij}^2 = \frac{1}{n} \Sigma \Sigma Y_{ij}^2$$

$$g_r = \frac{1}{k_1-n} \left\{ \sum_{i=1}^{r} \sum_{j \neq \ell} Y_{ij} Y_{i\ell} \right\} = \frac{1}{k_1-n} \left\{ \Sigma Y_{i.}^2 - \Sigma \Sigma Y_{ij}^2 \right\}$$

$$g_c = \frac{1}{k_2-n} \left\{ \sum_{j=1}^{c} \sum_{i \neq k} Y_{ij} Y_{kj} \right\} = \frac{1}{k_2-n} \left\{ \Sigma Y_{.j}^2 - \Sigma \Sigma Y_{ij}^2 \right\}$$

$$g_m = \frac{1}{n^2-k_1-k_2+n} \left\{ \sum_{i \neq k} \sum_{j \neq \ell} Y_{ij} Y_{k\ell} \right\} = \frac{1}{n^2-k_1-k_2+n} \left\{ Y_{..}^2 - \Sigma Y_{i.}^2 - \Sigma Y_{.j}^2 + \Sigma \Sigma Y_{ij}^2 \right\}$$

where $n_{ij} = \begin{cases} 1 & \text{if an observation appears in cell } (i,j) \\ 0 & \text{otherwise} \end{cases}$ ,

$$n_{i.} = \sum_{j=1}^{c} n_{ij},\ n_{.j} = \sum_{i=1}^{r} n_{ij},\ n = \sum_{i=1}^{r} \sum_{j=1}^{c} n_{ij} \text{ and } k_1 = \sum_{i=1}^{r} n_{i.}^2,\ k_2 = \sum_{j=1}^{c} n_{.j}^2 .$$

By setting $g_t$, $g_r$, $g_c$, $g_m$ equal to their respective expectations and solving the resulting equations, we obtain the estimators

$$\hat{\mu}^2 = g_m ,$$

$$\hat{\sigma}_r^2 = g_r - g_m ,$$

$$\hat{\sigma}_c^2 = g_c - g_m ,$$

$$\hat{\sigma}_e^2 = g_t - g_r - g_c + g_m .$$

Let us now simplify the model by assuming that $n_{1.} = n_{2.} = \ldots = n_{r.} = p$, $n_{.1} = n_{.2} = \ldots = n_{.c} = q$. Hence, $n = pr = cq$, $k_1 = rp^2 = pcq$, $k_2 = cq^2 = rpq$. Thus, we may write $g_t$, $g_r$, $g_c$, $g_m$ as

$$g_t = \frac{1}{rp} \Sigma \Sigma Y_{ij}^2 \qquad ,$$

$$g_r = \frac{1}{rp\,(p-1)} \left\{ \Sigma Y_{i.}^2 - \Sigma \Sigma Y_{ij}^2 \right\} \qquad ,$$

$$g_c = \frac{1}{cq\,(q-1)} \left\{ \Sigma Y_{.j}^2 - \Sigma \Sigma Y_{ij}^2 \right\} \qquad ,$$

$$g_m = \frac{1}{rp\left\{(rp-p-q+1)\right\}} \left\{ Y_{..}^2 - \Sigma Y_{i.}^2 - \Sigma Y_{.j}^2 + \Sigma \Sigma Y_{ij}^2 \right\} \quad ;$$

again, $\hat{\mu}^2$, $\hat{\sigma}_r^2$, $\hat{\sigma}_c^2$, $\hat{\sigma}_e^2$ are constructed from these as above.

Let us now briefly consider the variances and covariances of the estimators of the variance components. For reasons previously mentioned, we assume that $\mu$ is a known constant and, in particular, is zero. In this case, we use the estimators

$$\hat{\sigma}_r^2 = g_r \qquad ,$$

$$\hat{\sigma}_c^2 = g_c \qquad ,$$

$$\hat{\sigma}_e^2 = g_t - g_r - g_c \qquad ;$$

these may be viewed as linear combinations of the following more basic estimators

$$g_{tr,\,i} = \frac{1}{p} \sum_j Y_{ij}^2 \qquad ,$$

$$g_{r,i} = \frac{1}{p\,(p-1)} \left\{ Y_{i.}^2 - \sum_j Y_{ij}^2 \right\} \qquad ,$$

$$g_{tc,j} = \frac{1}{q} \sum_i Y_{ij}^2 \qquad ,$$

$$g_{c,j} = \frac{1}{q\,(q-1)} \left\{ Y_{.j}^2 - \sum_i Y_{ij}^2 \right\} \qquad .$$

From the theory of Appendix 2, we obtain

$$V_{tr,i} = \mathrm{Var}\,(g_{tr,i}) = 2\sigma_r^4 + \frac{4\sigma_r^2\,(\sigma_c^2 + \sigma_e^2)}{p} + \frac{2(\sigma_c^2 + \sigma_e^2)^2}{p} \qquad ,$$

$$V_{r,i} = \mathrm{Var}\,(g_{r,i}) = 2\sigma_r^4 + \frac{4\sigma_r^2\,(\sigma_c^2 + \sigma_e^2)}{p} + \frac{2(\sigma_c^2 + \sigma_e^2)^2}{p\,(p-1)} \qquad ,$$

$$C_{tr,i} = \mathrm{Cov}\,(g_{tr,i},\, g_{r,i}) = 2\sigma_r^4 + \frac{4\sigma_r^2\,(\sigma_c^2 + \sigma_e^2)}{p} \qquad ,$$

$$V_{tc,j} = \mathrm{Var}\,(g_{tc,j}) = 2\sigma_c^4 + \frac{4\sigma_c^2\,(\sigma_r^2 + \sigma_e^2)}{q} + \frac{2(\sigma_r^2 + \sigma_e^2)^2}{q}$$

$$V_{c,j} = \mathrm{Var}\,(g_{c,j}) = 2\sigma_c^4 + \frac{4\sigma_c^2\,(\sigma_r^2 + \sigma_e^2)}{q} + \frac{2(\sigma_r^2 + \sigma_e^2)^2}{q\,(q-1)} \qquad ,$$

$$C_{tc,j} = \mathrm{Cov}\,(g_{tc,j}\; g_{c,j}) = 2\sigma_c^4 + \frac{4\sigma_c^2\,(\sigma_r^2 + \sigma_e^2)}{q} \qquad ;$$

$$C_{tr,ik} = \text{Cov}\,(g_{tr,i},\ g_{tr,k}) = 2\gamma_{ik}\ \sigma_c^4/p^2 \qquad ,$$

$$C_{r,ik} = \text{Cov}\,(g_{r,i},\ g_{r,k}) = 4\lambda_{ik}\ \sigma_c^4/p^2(p-1)^2 \qquad ,$$

$$C_{tc,j\ell} = \text{Cov}\,(g_{tc,j},\ g_{tc,\ell}) = 2\delta_{j\ell}\ \sigma_r^4/q^2 \qquad ,$$

$$C_{c,j\ell} = \text{Cov}\,(g_{c,j},\ g_{c,\ell}) = 4\eta_{j\ell}\ \sigma_r^4/q^2(q-1)^2 \qquad ;$$

$$\text{Cov}\,(g_{tr,i},\ g_{c,j}) = n_{ij}\left\{2\sigma_c^4 + \frac{4\sigma_c^2\,(\sigma_r^2+\sigma_e^2)}{q}\right\}/\,p \qquad ,$$

$$\text{Cov}\,(g_{tc,j},\ g_{r,i}) = n_{ij}\left\{2\sigma_r^4 + \frac{4\sigma_r^2\,(\sigma_c^2+\sigma_e^2)}{p}\right\}/\,q \qquad ,$$

$$\text{Cov}\,(g_{r,i},\ g_{c,j}) = n_{ij}\left\{4\sigma_r^2\ \sigma_c^2/pq\right\} \qquad ,$$

$$\text{Cov}\,(g_{tr,i},\ g_{r,k}) = \text{Cov}\,(g_{tc,j},\ g_{c,\ell}) = 0$$

where $\gamma_{ik} = \sum_j n_{ij}\, n_{kj}$, $\lambda_{ik} = \binom{\gamma_{ik}}{2}$, $\delta_{j\ell} = \sum_i n_{ij}\, n_{i\ell}$, $\eta_{j\ell} = \binom{\delta_{j\ell}}{2}$ ;

i.e., $\gamma_{ik}$ is the number of units common to rows i and k while $\delta_{j\ell}$ is the number of units common to columns j and $\ell$. Note that $\sum_{i\neq k} \gamma_{ik} = \sum_j \left\{\sum_{i\neq k} n_{ij}n_{kj}\right\}$ $= cq\,(q-1)$ and $\sum_{j\neq\ell} \delta_{j\ell} = rp\,(p-1)$.

$$V_t = \mathrm{Var}\,(g_t) = \frac{2}{r^2}\left\{r\,\sigma_r^4 + \frac{2r\sigma_r^2\,(\sigma_c^2+\sigma_e^2)}{p} + \frac{r\,(\sigma_c^2+\sigma_e^2)^2}{p} + \frac{cq\,(q-1)\,\sigma_c^4}{p^2}\right\}$$

$$= \frac{2}{r^2}\left\{r\,\sigma_r^4 + \frac{r\,(2\sigma_r^2\,\sigma_c^2 + 2\sigma_r^2\,\sigma_e^2)}{p} + \frac{r\,(\sigma_c^4 + 2\sigma_c^2\,\sigma_e^2 + \sigma_e^4)}{p} + \frac{r(q-1)\,\sigma_c^4}{p}\right\}$$

$$= \frac{2}{r}\left\{\sigma_r^4 + \frac{2(\sigma_r^2\,\sigma_c^2 + \sigma_r^2\,\sigma_e^2 + \sigma_c^2 + \sigma_e^2)}{p} + \frac{\sigma_e^4}{p} + \frac{q\sigma_c^4}{p}\right\}$$

$$= \frac{2}{n}\left\{p\sigma_r^2 + 2(\sigma_r^2\,\sigma_c^2 + \sigma_r^2\,\sigma_e^2 + \sigma_c^2\,\sigma_e^2) + q\sigma_c^4 + \sigma_e^4\right\} ;$$

$$V_r = \mathrm{Var}\,(g_r) = \frac{2}{r^2}\left\{r\sigma_r^4 + \frac{2r\sigma_r^2\,(\sigma_c^2+\sigma_e^2)}{p} + \frac{r(\sigma_c^2+\sigma_e^2)^2}{p\,(p-1)} + 2\sum_{i\neq k}\lambda_{ik}\left(\frac{\sigma_c^4}{p^2(p-1)^2}\right)\right\}$$

$$= \frac{2}{n}\left\{p\sigma_r^4 + 2\sigma_r^2(\sigma_c^2+\sigma_e^2) + \frac{(\sigma_c^2+\sigma_e^2)^2}{(p-1)} + 2\Big(\sum_{i\neq k}\lambda_{ik}\Big)(\sigma_c^4/n(p-1)^2)\right\};$$

$$V_c = \mathrm{Var}\,(g_c) = \frac{2}{c^2}\left\{c\sigma_c^4 + \frac{2c\sigma_c^2(\sigma_r^2+\sigma_e^2)}{q} + \frac{c(\sigma_r^2+\sigma_e^2)^2}{q\,(q-1)} + 2\Big(\sum_{j=\ell}\eta_{j\ell}\Big)(\sigma_r^4/q^2(q-1)^2)\right\}$$

$$= \frac{2}{n}\left\{q\sigma_c^4 + 2\sigma_c^2(\sigma_r^2+\sigma_e^2) + \frac{(\sigma_r^2+\sigma_e^2)^2}{(q-1)} + 2\Big(\sum_{j\neq\ell}\eta_{j\ell}\Big)(\sigma_r^4/n(q-1)^2)\right\} ;$$

$$\mathrm{Cov}\,(g_t,\,g_r) = \frac{2}{r^2}\left\{r\left(\sigma_r^4 + \frac{2(\sigma_r^2)(\sigma_c^2+\sigma_e^2)}{p}\right)\right\}$$

$$= \frac{2}{n}\left\{p\sigma_r^4 + 2\sigma_r^2(\sigma_c^2+\sigma_e^2)\right\} ;$$

$$\mathrm{Cov}\,(g_t,\ g_c) = \frac{2}{c^2}\left\{ c(\sigma_c^4 + \frac{2\sigma_c^2(\sigma_r^2 + \sigma_e^2)}{q} \right\}$$

$$= \frac{2}{n}\left\{ q\sigma_c^4 + 2(\sigma_c^2)(\sigma_r^2 + \sigma_e^2) \right\} \qquad ;$$

$$\mathrm{Cov}\,(g_r,\ g_c) = \frac{4\sigma_r^2\ \sigma_c^2\ n}{pqrc} = \frac{4\sigma_r^2\ \sigma_c^2}{n} = \frac{2}{n}\left\{ 2\sigma_r^2\ \sigma_c^2 \right\} \quad ;$$

$$V_e = \mathrm{Var}\ (g_t - g_r - g_c) = \frac{2}{n}\left\{ 2\sigma_r^2\ \sigma_c^2 + \frac{(\sigma_c^2 + \sigma_e^2)^2}{p-1} + \frac{(\sigma_r^2 + \sigma_e^2)^2}{q-1} + \sigma_e^4 \right.$$

$$\left. + 2(\sum_{i \neq k} \lambda_{ik})(\sigma_c^4/n(p-1)^2) + 2(\sum_{j \neq \ell} \eta_{j\ell})(\sigma_r^4/n(q-1)^2) \right\} .$$

Let us now look at $V_r$, $V_c$, $V_e$ for two special cases:

a.) $\lambda_{ik} = \lambda = \delta_{j\ell}$ for all i, k and j, $\ell$.

$$V_r = \frac{2}{n}\left\{ p\sigma_r^4 + 2\sigma_r^2(\sigma_c^2 + \sigma_e^2) + \frac{(\sigma_c^2 + \sigma_e^2)^2}{(p-1)} + \frac{2\lambda r(r-1)\sigma_c^4}{n(p-1)^2} \right\} ,$$

$$V_c = \frac{2}{n}\left\{ q\sigma_c^4 + 2\sigma_c^2(\sigma_r^2 + \sigma_e^2) + \frac{(\sigma_r^2 + \sigma_e^2)^2}{(q-1)} + \frac{2\lambda c(c-1)\sigma_r^4}{n(q-1)^2} \right\} ,$$

$$V_e = \frac{2}{n}\left\{ 2\sigma_r^2\ \sigma_c^2 + \frac{(\sigma_c^2 + \sigma_e^2)^2}{p-1} + \frac{(\sigma_r^2 + \sigma_e^2)^2}{q-1} + 2\ \frac{\lambda r(r-1)\sigma_c^4}{n(p-1)^2} + \frac{2\lambda c(c-1)\sigma_r^4}{n(q-1)^2} + \sigma_e^4 \right\} .$$

Usually, we also have $r = c$, $p = q$, $\lambda(r-1) = p(q-1)$ in this case; i.e., we have a symmetric BIB design. Hence, $\frac{\lambda r(r-1)}{n(p-1)^2} = \frac{\lambda c(c-1)}{n(q-1)^2} = \frac{1}{p-1} = \frac{1}{q-1}$ in the above.

Note that in this case, $g_t$, $g_r$, $g_c$ are the simple averages of equally correlated random variables.

b.) Balanced Disjoint Rectangle Designs (see Anderson [2], Bush and Anderson [3]).

Suppose $r = vq$, $c = vp$ and $\gamma_{ik} = \begin{cases} o \text{ if rows } i, k \text{ not in same rectangle} \\ p \text{ if rows } i, k \text{ are in same rectangle} \end{cases}$

and $\delta_{j\ell} = \begin{cases} o \text{ if columns } j, \ell \text{ not in same rectangle} \\ q \text{ if columns } j, \ell \text{ are in same rectangle} \end{cases}$. The design consists of $v$ disjoint rectangles of dimensions $(q \times p)$.

$$V_r = \frac{2}{n}\left\{p\sigma_r^4 + 2\sigma_r^2(\sigma_c^2 + \sigma_e^2) + \frac{(\sigma_c^2 + \sigma_e^2)^2}{(p-1)} + \frac{2vq(q-1)p(p-1)\sigma_c^4}{2vpq(p-1)^2}\right\},$$

$$= \frac{2}{n}\left\{p\sigma_r^4 + 2\sigma_r^2(\sigma_c^2 + \sigma_e^2) + \frac{(\sigma_c^2 + \sigma_e^2)^2 + (q-1)\sigma_c^4}{(p-1)}\right\}$$

$$= 2\left\{\frac{\sigma_r^4}{vq} + \frac{2\sigma_r^2\sigma_c^2 + 2\sigma_r^2\sigma_e^2}{vqp} + \frac{2\sigma_c^2\sigma_e^2 + \sigma_e^4}{vpq(p-1)} + \frac{\sigma_c^4}{vp(p-1)}\right\}$$

$$= \frac{2}{v}\left\{\frac{\sigma_r^4}{q} + \frac{2\sigma_r^2\sigma_c^2 + 2\sigma_r^2\sigma_e^2}{pq} + \frac{2\sigma_c^2\sigma_e^2 + \sigma_e^4}{qp(p-1)} + \frac{\sigma_c^4}{p(p-1)}\right\}$$

$$V_c = \frac{2}{v}\left\{\frac{\sigma_c^4}{p} + \frac{2\sigma_r^2\sigma_c^2 + 2\sigma_c^2\sigma_e^2}{pq} + \frac{2\sigma_r^2\sigma_e^2 + \sigma_e^4}{pq(q-1)} + \frac{\sigma_r^4}{q(q-1)}\right\} ,$$

$$V_e = \frac{2}{vpq}\left\{2\sigma_r^2\sigma_c^2 + \frac{\sigma_c^4 + 2\sigma_c^2\sigma_e^2 + \sigma_e^4}{p-1} + \frac{\sigma_r^4 + 2\sigma_r^2\sigma_e^2 + \sigma_e^4}{q-1} + \frac{(q-1)\sigma_c^4}{p-1} + \frac{(p-1)\sigma_r^4}{(q-1)} + \sigma_e^4\right\}$$

$$= \frac{2}{v}\left\{\frac{2\sigma_r^2\sigma_c^2}{pq} + \frac{\sigma_c^4}{p(p-1)} + \frac{\sigma_r^4}{q(q-1)} + \frac{2\sigma_c^2\sigma_e^2}{p(p-1)q} + \frac{2\sigma_r^2\sigma_e^2}{pq(q-1)} + \frac{(pq-1)\sigma_e^4}{pq(p-1)(q-1)}\right\}.$$

The case $v = 1$ is the randomized blocks design. In the above series of designs, we reduce the correlations between the estimates from the respective rows and the respective columns by taking observations in disjoint (and hence uncorrelated) blocks. As a result, some variance terms will be decreased while others will be increased; the overall effect will determine the efficiency of the design for the situation with which an experimenter is concerned.

If we remove the assumption $n_{1.} = \dots = n_{r.} = p$ and $n_{.1} = \dots = n_{.c} = q$, there is a third general type of design which is of interest for experiments of the classification type. Here, we will call such designs "intersecting rectangles" although the term "intersecting cylinders" would be more appropriate for higher-way experiments. This class of designs includes "L-designs" (see Bush and Anderson [3]) which have the form

```
* *
* *
* *
* *
* * * * * *
* * * * * *      ,
```

"+ designs" which have the form

```
    * *
    * *
* * * * * *
* * * * * *
    * *
    * *
```

(and hence the same structure as "L-designs"), and "S-designs" (see Bush and Anderson [3]) which have the form

```
* * * * * *
* * * * * *
    * * * * * *
    * * * * * *
        * * * * * *
        * * * * * *
```

With these designs, one constructs estimators of $\sigma_r^2$, $\sigma_c^2$, $\sigma_e^2$ as follows:

(i) Partition the design into non-overlapping sets of the following type

a. isolated rectangles

b. series of balanced disjoint rectangles

c. rectangles which are the intersection of two rectangles

(ii) For each set, obtain the quadratic forms $g_t$, $g_r$, $g_c$; obtain $g_m$ for the whole experiment as indicated in the beginning of the section or in some other appropriate fashion.

(iii) Combine the estimators from the non-overlapping sets in an appropriate fashion (weighting them according to some natural criteria or according to their estimated variance-covariance matrix) to obtain $g_t$, $g_r$, $g_c$ for the whole experiment; then construct $\hat{\sigma}_r^2$, $\hat{\sigma}_c^2$, $\hat{\sigma}_e^2$ from these as before.

Such a procedure as the above is a "reasonable" though arbitrary method of estimation. It is not difficult to apply, but it does require a good deal of effort to study in the general case. However, special cases of interest to an experimenter could be readily evaluated with the aid of a computer by using the theory of Appendix 2.

Finally, if the design is balanced (in the sense that $n_{ij} = 1$ for all i,j and hence $p = c$, $r = q$), then by computations similar to those used in the next section, one can verify that

$$\hat{\sigma}_r^2 = g_r - g_m = \frac{1}{c}\left\{MSR - MSE\right\} \quad ,$$

$$\hat{\sigma}_c^2 = g_c - g_m = \frac{1}{r}\left\{MSC - MSE\right\} \quad ,$$

$$\hat{\sigma}_e^2 = g_t - g_r - g_c + g_m = MSE \quad ,$$

where MSR, MSC, MSE are the mean squares due to rows, columns, error respectively in the randomized-blocks (two-way) analysis of variance associated with the model. Thus, for this balanced experiment, the recommended estimation procedure coincides with the analysis of variance procedure and hence leads to BQUE of $\sigma_r^2$, $\sigma_c^2$, $\sigma_e^2$ (Graybill and Hultquist [8]).

## 3.2 The Two-Way Classification Design with Sub-Sampling in Cells

Model: $Y_{ijk} = \mu + r_i + c_j + (rc)_{ij} + e_{ijk}$ for $i=1,2,\ldots,r; j=1,2,\ldots,c; k=1,2,\ldots,n_{ij}$.

Assume

(i) The $\{r_i\}$, $\{c_j\}$, $\{(rc)_{ij}\}$, $\{e_{ijk}\}$ are independent,

(ii) The $\{r_i\}$ are identically distributed as $N(0, \sigma_r^2)$,

(iii) The $\{c_j\}$ are identically distributed as $N(0, \sigma_c^2)$,

(iv) The $\{(rc)_{ij}\}$ are identically distributed as $N(0, \sigma_{rc}^2)$,

(v) The $\{e_{ijk}\}$ are identically distributed as $N(0, \sigma_e^2)$.

Hence, we have that

$$
E\left\{Y_{ijk}\,Y_{uvw}\right\} = \begin{cases} (\sigma_r^2 + \sigma_c^2 + \sigma_{rc}^2 + \sigma_e^2 + \mu^2) & \text{if } i = u,\ j = v,\ k = w \\ (\sigma_r^2 + \sigma_c^2 + \sigma_{rc}^2 + \mu^2) & \text{if } i = u,\ j = v,\ k \neq w \\ (\sigma_r^2 + \mu^2) & \text{if } i = u,\ j \neq v \\ (\sigma_c^2 + \mu^2) & \text{if } i \neq u,\ j = v \\ \mu^2 & \text{if } i \neq u,\ j \neq v \end{cases}
$$

where $i, u = 1, 2, \ldots, r$; $j, v = 1, 2, \ldots, c$; and $k = 1, 2, \ldots, n_{ij}$.
$w = 1, 2, \ldots, n_{uv}$

We estimate $\sigma_r^2 + \sigma_c^2 + \sigma_{rc}^2 + \sigma_e^2 + \mu^2, \sigma_r^2 + \sigma_c^2 + \sigma_{rc}^2 + \mu^2, \sigma_r^2 + \mu^2, \sigma_c^2 + \mu^2, \mu^2$ by constructing the normalized symmetric sums of their respective unbiased estimators listed above:

$$g_t = \frac{1}{n} \sum_{i=1}^{r} \sum_{j=1}^{c} \sum_{k=1}^{n_{ij}} Y_{ijk}^2 = \frac{1}{n} \Sigma \Sigma \Sigma Y_{ijk}^2$$

$$g_{rc} = \frac{1}{k_{12}-n} \left\{ \sum_{i=1}^{r} \sum_{j=1}^{c} \sum_{k \neq w} Y_{ijk} Y_{ijw} \right\} = \frac{1}{k_{12}-n} \left\{ \Sigma \Sigma Y_{ij.}^2 - \Sigma \Sigma \Sigma Y_{ijk}^2 \right\}$$

$$g_r = \frac{1}{k_1-k_{12}} \left\{ \sum_{i=1}^{r} \sum_{j \neq v} Y_{ij.} Y_{iv.} \right\} = \frac{1}{k_1-k_{12}} \left\{ \Sigma Y_{i..}^2 - \Sigma \Sigma Y_{ij.}^2 \right\}$$

$$g_c = \frac{1}{k_2-k_{12}} \left\{ \sum_{j=1}^{c} \sum_{i \neq u} Y_{ij.} Y_{uj.} \right\} = \frac{1}{k_2-k_{12}} \left\{ \Sigma Y_{.j.}^2 - \Sigma \Sigma Y_{ij.}^2 \right\}$$

$$g_m = \frac{1}{n^2-k_1-k_2+k_{12}} \left\{ \sum_{i \neq u} \sum_{j \neq v} Y_{ij.} Y_{uv.} \right\} = \frac{1}{n^2-k_1-k_2+k_{12}} \left\{ Y_{...}^2 - \Sigma Y_{i..}^2 - \Sigma Y_{.j.}^2 + \Sigma\Sigma Y_{ij.}^2 \right\}$$

where $n_{i.} = \sum_{j=1}^{c} n_{ij}$, $n_{.j} = \sum_{i=1}^{r} n_{ij}$, $n = \sum_{i=1}^{r} \sum_{j=1}^{c} n_{ij}$, $k_{12} = \sum_{i=1}^{r} \sum_{j=1}^{c} n_{ij}^2$, $k_1 = \sum_{i=1}^{r} n_{i.}^2$,

$k_2 = \sum_{j=1}^{c} n_{.j}^2$ .

By setting $g_t$, $g_{rc}$, $g_r$, $g_c$, $g_m$ equal to their respective expectations and solving the resulting equations, we obtain the estimators

$$\hat{\mu}^2 = g_m \quad ,$$

$$\hat{\sigma}_r^2 = g_r - g_m \quad ,$$

$$\hat{\sigma}_c^2 = g_c - g_m \quad ,$$

$$\hat{\sigma}_{rc}^2 = g_{rc} - g_r - g_c + g_m \quad ,$$

$$\hat{\sigma}_e^2 = g_t - g_{rc} \quad ;$$

we could also have obtained the following alternative estimator of $\sigma_e^2$

$$g_e = \frac{1}{n-m} \left\{ \sum_{i=1}^{r} \sum_{j=1}^{c} \sum_{k=1}^{n_{ij}} (Y_{ijk} - \bar{Y}_{ij.})^2 \right\}$$ where m is the number of cells occupied

(by more than one unit) by considerations previously discussed. One can study the variance-covariance matrix of the estimators and the efficiency of different experimental designs by using the methods illustrated in the preceeding section. However, as the models become more complex, this becomes more and more difficult to do for general designs. As mentioned before, particular designs of interest to an experimenter could be evaluated in any given instance with the aid of a computer.

Suppose now that the design is balanced with p observations in each of the rc cells. In this case,

$$g_t = \frac{1}{rcp} \Sigma \Sigma \Sigma Y^2_{ijk} \quad ,$$

$$g_{rc} = \frac{1}{rcp(p-1)} \left\{ \Sigma \Sigma Y^2_{ij.} - \Sigma \Sigma \Sigma Y^2_{ijk} \right\} \quad ,$$

$$g_r = \frac{1}{rcp^2(c-1)} \left\{ \Sigma Y^2_{i..} - \Sigma \Sigma Y^2_{ij.} \right\} \quad ,$$

$$g_c = \frac{1}{rcp^2(r-1)} \left\{ \Sigma Y^2_{.j.} - \Sigma \Sigma Y^2_{ij.} \right\} \quad ,$$

$$g_m = \frac{1}{rcp^2(r-1)(c-1)} \left\{ Y^2_{...} - \Sigma Y^2_{i..} - \Sigma Y^2_{.j.} + \Sigma \Sigma Y^2_{ij.} \right\} \quad ;$$

$$\hat{\sigma}^2_r = g_r - g_m = \frac{1}{rcp^2(r-1)(c-1)} \left\{ (r-1)[\Sigma Y^2_{i..} - \Sigma\Sigma Y^2_{ij.}] - [Y^2_{...} - \Sigma Y^2_{i..} - \Sigma Y^2_{.j.} + \Sigma\Sigma Y^2_{ij.}] \right\}$$

$$= \frac{1}{cp} \left\{ \frac{1}{rp(r-1)(c-1)} [r(\Sigma Y^2_{i..} - \Sigma\Sigma Y^2_{ij.}) - (Y^2_{...} - \Sigma Y^2_{.j.})] \right\}$$

$$= \frac{1}{cp} \left\{ \frac{1}{rp(r-1)(c-1)} [r(cp\Sigma \frac{Y^2_{i..}}{cp} - p\Sigma\Sigma \frac{Y^2_{ij.}}{p}) - (rcp\frac{Y^2_{...}}{rcp} - rp\Sigma \frac{Y^2_{.j.}}{rp})] \right\}$$

$$= \frac{1}{cp} \left\{ \frac{1}{(r-1)(c-1)} [(c-1)(\Sigma \frac{Y^2_{i..}}{cp} - \frac{Y^2_{...}}{rcp}) - (\Sigma\Sigma \frac{Y^2_{ij.}}{p} - \Sigma \frac{Y^2_{i..}}{cp} - \frac{\Sigma Y^2_{.j.}}{rp} + \frac{Y^2_{...}}{rcp})] \right\}$$

$$= \frac{1}{cp} \left\{ \frac{cp\Sigma(\bar{Y}_{i..} - \bar{Y}_{...})^2}{r-1} - \frac{p\Sigma\Sigma(\bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...})^2}{(r-1)(c-1)} \right\}$$

$$= \frac{1}{cp} \left\{ MSR - MS(RC) \right\} \quad :$$

$$\hat{\sigma}^2_c = g_c - g_m = \frac{1}{rp} \left\{ MSC - MS(RC) \right\} \quad ;$$

$$\hat{\sigma}^2_{rc} = g_{rc}-g_r-g_c+g_m = \frac{1}{rcp^2(r-1)(c-1)(p-1)} \Big\{p(r-1)(c-1)(\Sigma\Sigma Y^2_{ij.}-\Sigma\Sigma\Sigma Y^2_{ijk})$$

$$-(r-1)(p-1)(\Sigma Y^2_{i..}-\Sigma\Sigma Y^2_{ij.})$$

$$-(c-1)(p-1)(\Sigma Y^2_{.j.}-\Sigma\Sigma Y^2_{ij.})$$

$$+(p-1)(Y^2_{...}-\Sigma Y^2_{i..}-\Sigma Y^2_{.j.}+\Sigma\Sigma Y^2_{ij.})\Big\}$$

$$= \frac{1}{rcp^2(r-1)(c-1)(p-1)} \Big\{(p-1)Y^2_{...}-r(p-1)\Sigma Y^2_{i..}-c(p-1)\Sigma Y^2_{.j.}+rc(p-1)\Sigma\Sigma Y^2_{ij.}$$

$$-p(r-1)(c-1)\Sigma\Sigma\Sigma Y^2_{ijk}+p(r-1)(c-1)\Sigma\Sigma\frac{Y^2_{ij.}}{p}\Big\}$$

$$= \frac{1}{p}\left\{\frac{\Sigma\Sigma(Y^2_{ij.}/p)-\Sigma(Y^2_{i..}/cp)-\Sigma(Y^2_{.j.}/rp)+(Y^2_{...}/rcp)}{(r-1)(c-1)} - \frac{\Sigma\Sigma\Sigma Y^2_{ijk}-\Sigma\Sigma(Y^2_{ij.}/p)}{rc(p-1)}\right\}$$

$$= \frac{1}{p}\Big\{MS(RC) - MSE\Big\} \quad ;$$

$$\hat{\sigma}^2_e = g_t - g_{rc} = \frac{1}{rcp(p-1)} \Big\{(p-1)(\Sigma\Sigma\Sigma Y^2_{ijk}) - (\Sigma\Sigma Y^2_{ij.}-\Sigma\Sigma\Sigma Y^2_{ijk})\Big\}$$

$$= \frac{1}{rc(p-1)} \Big\{\Sigma\Sigma\Sigma(Y_{ijk} - \bar{Y}_{ij.})^2\Big\}$$

$$= MSE$$

where MSR, MSC, MS(RC), MSE are the mean squares due to rows, columns, interaction, and error respectively in the analysis of variance associated with the model. Thus, for the balanced experiment, $\hat{\sigma}_r^2$, $\hat{\sigma}_c^2$, $\hat{\sigma}_{rc}^2$, $\hat{\sigma}_e^2$ coincide with the estimators obtained by the analysis of variance method of estimation and hence are BQUE.

## 3.3 The Multi-Way Classification Experiment

Model: $Y_{i_1 i_2 \ldots i_r u} = \mu + e^{(1)}_{i_1} + e^{(2)}_{i_2} + e^{(12)}_{i_1 i_2} \ldots + e^{(12\ldots r)}_{i_1 i_2 \ldots i_r} + e_{i_1 i_2 \ldots i_r u}$

for $i_1 = 1, 2, \ldots, m_1$; $i_2 = 1, 2, \ldots, m_2$; $\ldots$ ; $i_r = 1, 2, \ldots, m_r$;

and $u = 1, 2, \ldots, n_{i_1 i_2 \ldots i_r}$ .

Assume

(i) The $\left\{e^{(1)}_{i_1}\right\}$, $\left\{e^{(2)}_{i_2}\right\}$, $\left\{e^{(12)}_{i_1 i_2}\right\}$, ..., $\left\{e^{(12\ldots r)}_{i_1 i_2 \ldots i_r}\right\}$, $\left\{e_{i_1 i_2 \ldots i_r u}\right\}$

are independent;

(ii) The $\left\{e^{(1)}_{i_1}\right\}$ are identically distributed as $N(0, \sigma_1^2)$ ;

(iii) The $\left\{e^{(2)}_{i_2}\right\}$ are identically distributed as $N(0, \sigma_2^2)$ ;

(iv) The $\left\{e^{(12)}_{i_1 i_2}\right\}$ are identically distributed as $N(0, \sigma_{12}^2)$ ;

(v) The $\left\{e^{(j_1 j_2 \ldots j_q)}_{i_{j_1} i_{j_2} \ldots i_{j_q}}\right\}$ are identically distributed as $N(0, \sigma^2_{j_1 j_2 \ldots j_q})$

for all $q = 1, 2, \ldots, r$ and $(j_1, j_2, \ldots, j_q)$ selected from $(1, 2, \ldots, r)$

such that $j_1 < j_2 < \ldots < j_q$.

Hence, we have that

$$E\left\{Y_{i_1 i_2 \cdots i_r u} Y_{j_1 j_2 \cdots j_r w}\right\} = \mu^2+\sigma_1^2+\sigma_2^2+\sigma_{12}^2+\cdots+\sigma_{12\cdots r}^2+\sigma_e^2 \quad \text{if } i_1=j_1,\ldots,i_r=j_r, u=w$$

$$= \mu^2+\sigma_1^2+\sigma_2^2+\sigma_{12}^2+\cdots+\sigma_{12\cdots r}^2 \quad \text{if } i_1=j_1,\ldots,i_r=j_r, u\neq w$$

$$\cdots \qquad\qquad \cdots$$

$$= \mu^2+\sigma_1^2+\sigma_2^2+\sigma_{12}^2 \quad \text{if } i_1=j_1, i_2=j_2, i_3\neq j_3,\ldots,i_r\neq j_r$$

$$= \mu^2+\sigma_2^2 \quad \text{if } i_1\neq j_1, i_2=j_2, i_3\neq j_3,\ldots,i_r\neq j_r$$

$$= \mu^2+\sigma_1^2 \quad \text{if } i_1=j_1, i_2\neq j_2, i_3\neq j_3,\ldots,i_r\neq j_r$$

$$= \mu^2 \quad \text{if } i_1\neq j_1, i_2\neq j_2,\ldots,i_r\neq j_r$$

where $i_1, j_1 = 1, 2, \ldots, m_1$; $i_2, j_2 = 1, 2, \ldots, m_2$; $\ldots$; $i_r, j_r = 1, 2, \ldots, m_r$; $u = 1, 2, \ldots, n_{i_1 i_2 \cdots i_r}$; $w = 1, 2, \ldots, n_{j_1 j_2 \cdots j_r}$. We estimate $\mu^2+\sigma_1^2+\sigma_2^2+\sigma_{12}^2+\cdots+\sigma_{12\cdots r}^2+\sigma_e^2, \mu^2+\sigma_1^2+\sigma_2^2+\sigma_{12}^2+\cdots+\sigma_{12\cdots r}^2, \ldots, \mu^2+\sigma_1^2+\sigma_2^2+\sigma_{12}^2, \mu^2+\sigma_1^2, \mu^2+\sigma_2^2, \mu^2$ by forming the normalized symmetric sums of their respective unbiased estimators listed above:

$$g_t = \frac{1}{n}\left\{\Sigma Y^2_{i_1 i_2 \ldots i_r u}\right\} ,$$

$$g_{12\ldots r} = \frac{1}{k_{12\ldots r}-n}\left\{\Sigma Y_{i_1 i_2 \ldots i_r u} Y_{i_1 i_2 \ldots i_r w}\right\} = \frac{1}{k_{12\ldots r}-n}\left\{\Sigma Y^2_{i_1 i_2 \ldots i_r .} - \Sigma Y^2_{i_1 i_2 \ldots i_r u}\right\} ,$$

$$\cdots$$

$$g_1 = \frac{1}{(k_1 - k_{12} \cdots - k_{1r} + \ldots + (-1)^{r-1} k_{12\ldots r})}\left\{\Sigma Y^2_{i_1 \ldots} - \Sigma Y^2_{i_1 i_2 \ldots} + \ldots + (-1)^{r-1}\Sigma Y^2_{i_1 \ldots i_r .}\right\} ,$$

$$g_m = \frac{1}{n^2 - k_1 - \ldots - k_r + k_{12} \ldots + (-1)^r k_{12\ldots r}}\left\{Y^2_{\ldots} - \Sigma Y^2_{i_1 \ldots} - \ldots - \Sigma Y^2_{\ldots i_r} + \Sigma Y^2_{i_1 i_2 \ldots} + \ldots + (-1)^r \Sigma Y^2_{i_1 \ldots i_r .}\right\} ,$$

where $n_{\ldots i_{j_1} \ldots i_{j_q} \ldots} = \sum_C n_{i_1 i_2 \ldots i_r}$ where C is the set of all indices other than $j_1, j_2, \ldots, j_q$ and $k_{j_1 \ldots j_q} = \Sigma\, n^2_{\ldots i_{j_1} \ldots i_{j_q} \ldots}$. By setting $g_t$, $g_{12\ldots r}$, $\ldots$, $g_m$ equal to their respective expectations and solving the resulting equations, we obtain the estimators

$$\hat{\mu}^2 = g_m ,$$

$$\hat{\sigma}^2_1 = g_1 - g_m ,$$

$$\hat{\sigma}^2_2 = g_2 - g_m ,$$

$$\hat{\sigma}^2_{12} = g_{12} - g_1 - g_2 + g_m ,$$

$$\cdots$$

$$\hat{\sigma}^2_{12\ldots r} = g_{12\ldots r} - g_{12\ldots(r-1)} - \ldots + (-1)^{r-1} g_r + \ldots + (-1)^{r-1} g_1 + (-1)^r g_m ,$$

$$\hat{\sigma}^2_e = g_t - g_{12\ldots r} .$$

The same remarks concerning the variance-covariance matrix of the estimators, the possibility of alternative estimators, and methods of studying the efficiency of different experimental designs apply to this general classification model in principally the same ways as with the special cases considered earlier in the section.

In addition, we have that if the experiment is balanced (i.e. there are $n_o$ observations in every cell), then the above estimators coincide with those obtained by the analysis of variance method of estimation and hence are BQUE. The method of proof is identical with that used in the case of the two-way classification design with sub-sampling and consequently exploits the special form of $k_1, k_2, \ldots, k_{12\ldots r}$ here. Moreover, if the experiment is balanced with respect to some factors at all possible combinations of levels of other factors, then estimators of the variance components associated with these factors can be obtained by appropriately combining the estimators obtained from the analysis of variance for each possible combination of levels of the other factors. The point of this is that if the experiment has special features, then the experimenter can (if he desires) use them to construct symmetric sums which are more appropriate to his needs and which take greater advantage of his prior information and the experiment's special features. As mentioned before, the suggested estimation procedure offers a special form applicable to general problems; the principles behind it are very flexible and general and may be used to construct more appropriate estimators for specific problems.

## 4. AN EXPERIMENTS OF THE MIXED TYPE

Model: $Y_{ijk\ell} = \mu + r_i + c_j + (rc)_{ij} + u_{ik} + v_{j\ell} + (uc)_{ijk} + (rv)_{ij\ell} + (uv)_{ijk\ell}$

where $i = 1, 2, \ldots, r$; $j = 1, 2, \ldots, c$; $k = 1, 2, \ldots, n_{i.}$; $\ell = 1, 2, \ldots, n_{.j}$.

Assume

(i) The $\{r_i\}$, $\{c_j\}$, $\{(rc)_{ij}\}$, $\{u_{ik}\}$, $\{v_{j\ell}\}$, $\{(uc)_{ijk}\}$, $\{(rv)_{ij\ell}\}$, $\{(uv)_{ijk\ell}\}$ are independent

(ii) The $\{r_i\}$, $\{c_j\}$, $\{(rc)_{ij}\}$, $\{u_{ik}\}$, $\{v_{j\ell}\}$, $\{(uc)_{ijk}\}$, $\{(rv)_{ij\ell}\}$, $\{(uv)_{ijk\ell}\}$ are identically distributed as $N(0,\sigma_r^2)$, $N(0,\sigma_c^2)$, $N(0,\sigma_{rc}^2)$, $N(0,\sigma_u^2)$, $N(0,\sigma_v^2)$, $N(0,\sigma_{uc}^2)$, $N(0,\sigma_{rv}^2)$, $N(0,\sigma_{uv}^2)$ respectively .

Hence, we have that

$$
\begin{aligned}
\left\{Y_{ijk\ell}\, Y_{i'j'k'\ell'}\right\} &= \mu^2 + \sigma_r^2 + \sigma_c^2 + \sigma_{rc}^2 + \sigma_u^2 + \sigma_v^2 + \sigma_{uc}^2 + \sigma_{rv}^2 + \sigma_{uv}^2 && \text{if } i=i',\ j=j',\ k=k',\ \ell=\ell' \\
&= \mu^2 + \sigma_r^2 + \sigma_c^2 + \sigma_{rc}^2 + \sigma_u^2 + \sigma_{uc}^2 && \text{if } i=i',\ j=j',\ k=k',\ \ell\neq\ell' \\
&= \mu^2 + \sigma_r^2 + \sigma_c^2 + \sigma_{rc}^2 + \sigma_v^2 + \sigma_{rv}^2 && \text{if } i=i',\ j=j',\ k\neq k',\ \ell=\ell' \\
&= \mu^2 + \sigma_r^2 + \sigma_c^2 + \sigma_{rc}^2 && \text{if } i=i',\ j=j',\ k\neq k',\ \ell\neq\ell' \\
&= \mu^2 + \sigma_r^2 + \sigma_u^2 && \text{if } i=i',\ j\neq j',\ k=k', \\
&= \mu^2 + \sigma_c^2 + \sigma_v^2 && \text{if } i\neq i',\ j=j',\ \quad ,\ \ell=\ell' \\
&= \mu^2 + \sigma_r^2 && \text{if } i=i',\ j\neq j',\ k\neq k' \\
&= \mu^2 + \sigma_c^2 && \text{if } i\neq i',\ j=j',\ \quad ,\ \ell\neq\ell' \\
&= \mu^2 && \text{if } i\neq i',\ j\neq j'
\end{aligned}
$$

From the above, it follows that all the parameters are estimable (See Appendix 1). One obtains the estimates of the variance components by constructing the appropriate normalized symmetric sums to estimate the above parametric functions, then setting them equal to their respective expectations, and solving the resulting equations for the estimators.

# 5. SOME PROBLEMS IN MULTIVARIATE ANALYSIS

## 5.1. A Simple Two-Way Model With Sub-Sampling in the Cells.

Model: Suppose $Y_{ijk}$ where $i = 1, 2, \ldots, r$; $j = 1, 2, \ldots, c$; and $k = 1, 2, \ldots, n_{ij}$ are observations from a multivariate-normal population with the following structure:

$$E\,(Y_{ijk}) = \mu \text{ for all } i, j, k \quad ;$$

$$\mathrm{Var}\,(Y_{ijk}) = \sigma^2 \text{ for all } i, j, k \quad ;$$

$$\begin{aligned}
\mathrm{Cov}\,(Y_{ijk}, Y_{uvw}) &= \rho_{rc}\sigma^2 && \text{if } i = u,\ j = v,\ k \neq w \\
&= \rho_r\sigma^2 && \text{if } i = u,\ j \neq v \\
&= \rho_c\sigma^2 && \text{if } i \neq u,\ j = v \\
&= 0 && \text{if } i \neq u,\ j \neq v
\end{aligned}$$

wherever $n_{ijk}\, n_{uvw} \neq 0$ where $n_{ijk} = \begin{cases} 1 \text{ if } Y_{ijk} \text{ is defined.} \\ 0 \text{ otherwise} \end{cases}$ Then we estimate the parameters as follows:

$$\hat{\mu}^2 = g_m ,$$

$$\hat{\sigma}^2 = g_t - g_m ,$$

$$\hat{\rho}_{rc} = (g_{rc} - g_m)/(g_t - g_m) ,$$

$$\hat{\rho}_r = (g_r - g_m)/(g_t - g_m) ,$$

$$\hat{\rho}_c = (g_c - g_m)/(g_t - g_m) ,$$

where $g_t$, $g_{rc}$, $g_r$, $g_c$, $g_m$ are the same as in Section 3.2. Similar remarks apply to estimation in these models as were made in the case of variance components models.

One should note here that for cases where negative estimates of variance components are obtained from the model of Section 3.2, the model considered here may be more appropriate.

5.2. The One-Stage Model for Observation Vectors.

Model: $$\begin{bmatrix} Y_{ij} \\ Y_{2j} \\ \vdots \\ y_{tj} \end{bmatrix} = \underline{Y}_j = \underline{\tau} + \underline{e}_j = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_t \end{bmatrix} + \begin{bmatrix} e_{1j} \\ e_{2j} \\ \vdots \\ e_{tj} \end{bmatrix}$$

where $j = 1, 2, \ldots, n$.

Assume $\left\{\underline{e}_j\right\}$ are independently distributed as $N(\underline{0}, V)$.

Hence, we have that

$$E\left\{\underline{Y}_j\underline{Y}_k'\right\} = \underline{\tau}\,\underline{\tau}' + V \quad \text{if } j = k = 1, 2, \ldots, n$$

$$= \underline{\tau}\,\underline{\tau}' \qquad \text{if } j \neq k = 1, 2, \ldots, n$$

We estimate $\underline{\tau}\,\underline{\tau}' + V$, $\underline{\tau}\,\underline{\tau}'$ by forming the normalized symmetric sums of their respective unbiased estimators listed above; that is

$$G_t = \frac{1}{n} \sum_{j=1}^{n} \underline{Y}_j\, \underline{Y}_j' \,,$$

$$G_m = \frac{1}{n(n-1)} \sum_{j \neq k} \underline{Y}_j\, \underline{Y}_k' = \frac{1}{n(n-1)} \left\{ \underline{Y}_{\cdot}\underline{Y}_{\cdot}' - \sum_{j=1}^{n} \underline{Y}_j\underline{Y}_j' \right\} .$$

By setting $G_t$, $G_m$ equal to their respective expectations and solving the resulting equations, we obtain the estimators

$$\widehat{\underline{\tau}\,\underline{\tau}'} = G_m \;,$$

$$\hat{V} = G_t - G_m = \frac{1}{n-1} \sum_{j=1}^{n} (\underline{Y}_j - \bar{\underline{Y}})(\underline{Y}_j - \bar{\underline{Y}})' .$$

If, as in the case of the mixed model, we have $V = \sigma_c^2\, J_t + \sigma_e^2\, I_t$, then we obtain the estimators

$$\hat{\sigma}_c^2 = \frac{1}{t(t-1)} \left\{ \underset{\sim}{i}_t' \hat{V} \underset{\sim}{i}_t - \operatorname{tr}(\hat{V}) \right\} ,$$

$$\hat{\sigma}_e^2 = \frac{1}{t} \left\{ \operatorname{tr}(\hat{V}) \right\} - \hat{\sigma}_c^2 ,$$

where $\underset{\sim}{i}_t'$ is a $(1 \times t)$ vector of ones and $\operatorname{tr}(\hat{V})$ is sum of diagonal elements of $\hat{V}$.

## 5.3. A Two-Stage Nested Model for Observation Vectors

Model: $\underset{t\times 1}{\underline{Y}}{}_{jk} = \underset{t\times 1}{\underline{\tau}} + \underset{t\times 1}{\underline{c}}{}_j + \underset{t\times 1}{\underline{e}}{}_{jk}$ for $j = 1, 2, \ldots, c$; $k = 1, 2, \ldots, n_j$

Assume

(i) $\left\{\underline{c}_j\right\}, \left\{\underline{e}_{jk}\right\}$ are independent ,

(ii) $\left\{\underline{c}_j\right\}$ are identically distributed as $N(\underline{0}, V_c)$ ;

(iii) $\left\{\underline{e}_{jk}\right\}$ are identically distributed as $N(\underline{0}, V_e)$ .

Hence, we have that

$$
\begin{aligned}
E\left\{\underline{Y}_{jk}\, \underline{Y}'_{uv}\right\} &= \underline{\tau}\,\underline{\tau}' + V_c + V_e && \text{if } j = u,\ k = v \\
&= \underline{\tau}\,\underline{\tau}' + V_c && \text{if } j = u,\ k \neq v \\
&= \underline{\tau}\,\underline{\tau}' && \text{if } j \neq u
\end{aligned}
$$

We estimate $\underline{I}\,\underline{I}' + V_c + V_e$, $\underline{I}\,\underline{I}' + V_c$, $\underline{I}\,\underline{I}'$ by forming the normalized symmetric sums of their respective unbiased estimators listed above; that is

$$G_t = \frac{1}{n} \sum_{j=1}^{c} \sum_{k=1}^{n_j} \underline{Y}_{jk}\, \underline{Y}'_{jk} = \frac{1}{n} \sum_{j=1}^{c} \sum_{k=1}^{n_j} \underline{Y}_{jk}\, \underline{Y}'_{jk} \quad ,$$

$$G_c = \frac{1}{k_1 - n} \sum_{j=1}^{c} \sum_{k \neq v} \underline{Y}_{jk}\, \underline{Y}'_{jv} = \frac{1}{k_1 - n} \left\{ \sum_{j=1}^{c} \underline{Y}_{j.}\, \underline{Y}'_{j.} - \sum_{j=1}^{c} \sum_{k=1}^{n_j} \underline{Y}_{jk}\, \underline{Y}'_{jk} \right\} \quad ,$$

$$G_m = \frac{1}{n^2 - k_1} \sum_{j \neq u} \sum_{k=1}^{n_j} \sum_{v=1}^{n_v} \underline{Y}_{jk}\, \underline{Y}'_{uv} = \frac{1}{n^2 - k_1} \left\{ \underline{Y}_{..}\, \underline{Y}'_{..} - \sum_{j=1}^{c} \underline{Y}_{j.}\, \underline{Y}'_{j.} \right\} ,$$

where $n = \sum_{j=1}^{c} n_j$ and $k_1 = \sum_{j=1}^{c} n_j^2$. By setting $G_t$, $G_c$, $G_m$ equal to their respective expectations and solving the resulting equations, we obtain the estimators

$$\widehat{\underline{I}\,\underline{I}'} = G_m \; ,$$

$$\hat{V}_c = G_c - G_m \; ,$$

$$\hat{V}_e = G_t - G_c \; .$$

If $V_c = \sigma_c^2\, J_t$, $V_e = \sigma_{rc}^2\, J_t + \sigma_e^2\, I_t$, then we obtain the estimators

$$\hat{\sigma}_c^2 = \frac{1}{t^2} \underline{1}'_t\, \hat{V}_c\, \underline{1}_t , \quad \hat{\sigma}_{rc}^2 = \frac{1}{t(t-1)} \left\{ \underline{1}'_t\, \hat{V}_e\, \underline{1}_t - \mathrm{tr}\,(\hat{V}_e) \right\} ,$$

$$\hat{\sigma}_e^2 = \frac{1}{t} \left\{ \mathrm{tr}\,(\hat{V}_e) \right\} - \hat{\sigma}_{rc}^2 \quad .$$

5.4. A Two-Stage Nested Bivariate Model .

Model: $$\begin{pmatrix} Y_{ij} \\ X_{ij} \end{pmatrix} = \begin{pmatrix} \mu_y \\ \mu_x \end{pmatrix} + \begin{pmatrix} r_{yi} \\ r_{xi} \end{pmatrix} + \begin{pmatrix} e_{yij} \\ e_{xij} \end{pmatrix}$$

for $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, n_i$.

Assume

(i) $\left\{\begin{pmatrix} r_{yi} \\ r_{xi} \end{pmatrix}\right\}$, $\left\{\begin{pmatrix} e_{yij} \\ e_{xij} \end{pmatrix}\right\}$ are independent ;

(ii) $\left\{\begin{pmatrix} r_{yi} \\ r_{xi} \end{pmatrix}\right\}$ are identically distributed as $N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2_{yr}, & \rho_r \sigma_{yr} \sigma_{xr} \\ \rho_r \sigma_{yr} \sigma_{xr}, & \sigma^2_{xr} \end{pmatrix}\right)$ ;

(iii) $\left\{\begin{pmatrix} e_{yij} \\ e_{xij} \end{pmatrix}\right\}$ are identically distributed as $N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2_{ye}, & \rho_e \sigma_{ye} \sigma_{xe} \\ \rho_e \sigma_{ye} \sigma_{xe}, & \sigma^2_{xe} \end{pmatrix}\right)$ .

Hence, we have that

$$\begin{aligned} E\left\{Y_{ij}\, Y_{k\ell}\right\} &= \mu_y^2 + \sigma^2_{yr} + \sigma^2_{ye} && \text{if } i = k,\ j = \ell \quad ; \\ &= \mu_y^2 + \sigma^2_{yr} && \text{if } i = k,\ j \neq \ell \\ &= \mu_y^2 && \text{if } i \neq k \end{aligned}$$

$$\begin{aligned} E\left\{X_{ij}\, X_{k\ell}\right\} &= \mu_x^2 + \sigma^2_{xr} + \sigma^2_{xe} && \text{if } i = k,\ j = \ell \quad ; \\ &= \mu_x^2 + \sigma^2_{xr} && \text{if } i = k,\ j \neq \ell \\ &= \mu_x^2 && \text{if } i \neq k \end{aligned}$$

$$E\left\{X_{ij}\,Y_{k\ell}\right\} = \mu_x\,\mu_y + \rho_r\,\sigma_{yr}\,\sigma_{xr} + \rho_e\,\sigma_{ye}\,\sigma_{xe} \quad \text{if } i = k,\ j = \ell\,.$$

$$= \mu_x\,\mu_y + \rho_r\,\sigma_{yr}\,\sigma_{xr} \quad \text{if } i = k,\ j \neq \ell$$

$$= \mu_x\,\mu_y \quad \text{if } i \neq k$$

We estimate the above parametric functions by forming the normalized symmetric sums of their respective unbiased estimators

$$g_{ty} = \frac{1}{n}\sum_{i=1}^{r}\sum_{j=1}^{n_i} Y_{ij}^2\,,\quad g_{txy} = \frac{1}{n}\sum_{i=1}^{r}\sum_{j=1}^{n_i} X_{ij}\,Y_{ij}\,,\quad g_{tx} = \frac{1}{n}\sum_{i=1}^{r}\sum_{j=1}^{n_i} X_{ij}^2\;;$$

$$g_{ry} = \frac{1}{k_1-n}\left\{\sum_{i=1}^{r} Y_{i.}^2 - \sum_{i=1}^{r}\sum_{j=1}^{n_i} Y_{ij}^2\right\},\ g_{rxy} = \frac{1}{k_1-n}\left\{\sum_{i=1}^{r} X_{i.}Y_{i.} - \sum_{i=1}^{r}\sum_{j=1}^{n_i} X_{ij}Y_{ij}\right\}\,,$$

$$g_{rx} = \frac{1}{k_1-n}\left\{\sum_{i=1}^{r} X_{i.}^2 - \sum_{i=1}^{r}\sum_{j=1}^{n_i} X_{ij}^2\right\}\;;$$

$$g_{my} = \frac{1}{n^2-k_1}\left\{Y_{..}^2 - \sum_{i=1}^{r} Y_{i.}^2\right\},\ g_{mxy} = \frac{1}{n^2-k_1}\left\{X_{..}\,Y_{..} - \sum_{i=1}^{r} X_{i.}\,Y_{i.}\right\}\,,$$

$$g_{mx} = \frac{1}{n^2-k_1}\left\{X_{..}^2 - \sum_{i=1}^{r} X_{i.}^2\right\}\,,$$

where $n = \sum_{i=1}^{r} n_i$ and $k_1 = \sum_{i=1}^{r} n_i^2$.

Equating the above to their respective expectations and solving the resulting equations, we obtain the estimators

$$\hat{\mu}_y^2 = g_{my} , \qquad \hat{\mu}_x^2 = g_{mx} ;$$

$$\hat{\sigma}_{yr}^2 = g_{ry} - g_{my} , \qquad \hat{\sigma}_{xr}^2 = g_{rx} - g_{mx} ;$$

$$\hat{\sigma}_{ye}^2 = g_{ty} - g_{ry} , \qquad \hat{\sigma}_{xe}^2 = g_{tx} - g_{rx} ;$$

$$\hat{\rho}_r = (g_{rxy} - g_{mxy}) / \left\{ (g_{ry} - g_{my})(g_{rx} - g_{mx}) \right\}^{\frac{1}{2}} ,$$

$$\hat{\rho}_e = (g_{txy} - g_{rxy}) / \left\{ (g_{ty} - g_{ry})(g_{tx} - g_{rx}) \right\}^{\frac{1}{2}} .$$

One can observe that models of the type studied here (see Tukey [ ]) can be generalized both in the direction of the number of components per observation vector and in the direction of the number of effects. Procedures similar to those demonstrated here would be used to find the estimates of the parameters of interest in more complex situations. As always, the variances of the estimators may be obtained by appropriate application of the theory of Appendix 2. Such considerations may lead an experimenter to appropriate designs and refined estimators for the special situations of interest to him.

APPENDIX

A.1. Basic Theory of Suggested Estimation Procedure.

We assume that the observations from an experiment can be represented by a vector $\underset{n\times 1}{\underline{Y}}$. This may be a vector of univariate or multivariate observations. Let the mean vector and covariance matrix of $\underline{Y}$ be denoted as follows:

$$E(\underline{Y}) = \underline{\mu}, \quad Var(\underline{Y}) = V \quad .$$

Our purpose is to estimate the elements of V by using the information provided by $\underline{Y}$.

Definition: A parameter will be said to be estimable in the quadratic sense with respect to a particular experiment (or model representing it) if there exists a quadratic form $\underline{Y}'C\,\underline{Y}$ which is an unbiased estimator of it.

From the above, we have that

$$E\left\{\underline{Y}\,\underline{Y}'\right\} = \underline{\mu}\,\underline{\mu}' + V = S.$$

It follows that if a function of the parameters can be written as a linear function of the elements of S, then it is estimable. In the general case, not very many functions of experimental interest would be estimable in this sense. However, with the models of interest to us, $\underline{\mu}$ and V have a particular structure. Let us now look more closely at a general random effects model. We assume that the observations may be characterized by the following model

$$Y(i) = \mu + e_1(i) + e_2(i) + \ldots + e_r(i) \quad \text{for } i = 1, 2, \ldots, n$$

where $e_k(i)$, the random variable corresponding to the k th source of variation, has the following properties :

(i) $E\left\{e_k(i)\right\} = 0$ for $k = 1, 2, \ldots, r$ and $i = 1, 2, \ldots, n$;

$\operatorname{Var}\left\{e_k(i)\right\} = \sigma_k^2$

(ii) $\operatorname{Cov}\left\{e_k(i), e_j(h)\right\} = 0$ for $j \neq k = 1, 2, \ldots, r$; $i, h = 1, 2, \ldots, n$;

(iii) $\operatorname{Cov}\left\{e_k(i), e_k(h)\right\} = \sigma_k^2$ if $e_k(i) = e_k(h)$

if $e_k(i) \neq e_k(h)$

for $k = 1, 2, \ldots, s$; $i, h = 1, 2, \ldots, n$.

(iv) The $e_k(i)$ are normally distributed.

Consequently, we have that

$$E\left\{Y(i)\,Y(h)\right\} = \sum_{k=1}^{r} \sigma_k^2 + \mu^2 \text{ if } i = h = 1, 2, \ldots, n$$

$$= \sum_{k \varepsilon C_{ih}} \sigma_k^2 + \mu^2 \text{ if } i \neq h = 1, 2, \ldots, n$$

where $C_{ih} = \left\{k \mid e_k(i) = e_k(h)\right\}$. Let $\theta_{ih} = \sum_{k \varepsilon C_{ih}} \sigma_k^2 + \mu^2$.

One now sees that a function of the parameters $\mu^2, \sigma_1^2, \ldots, \sigma_r^2$ is estimable in the quadratic sense if and only if it can be written as a linear function of the $\left\{\theta_{ih}\right\}$. The $\left\{\theta_{ih}\right\}$ are estimated by constructing the normalized symmetric sums of their unbiased estimators

$$\hat{\theta}_{ih} = \frac{1}{k_{ih}} \sum_{j,\ell \varepsilon T_{ih}} Y(j)Y(\ell)$$

where $T_{ih} = \left\{(j,\ell) \,\middle|\, E\left\{Y(j)\ Y(\ell)\right\} \equiv \theta_{j\ell} = \theta_{ih}\right\}$ and

$k_{ih} = \left\{\text{number of } (j, \ell) \text{ combinations in } T_{ih}\right\}$ . If $\sigma_k^2$ is estimable, then it may be written as a linear combination of the $\left\{\theta_{ih}\right\}$; that is,

$$\sigma_k^2 = \Sigma\ d_{ih}\ \theta_{ih}$$

where it is understood that the summation is with respect to distinct values of $\theta_{ih}$. The suggested estimator of $\sigma_k^2$ is then given by

$$\hat{\sigma}_k^2 = \Sigma\ d_{ih}\ \hat{\theta}_{ih} \quad .$$

This estimator will be a good one if the $\left\{\hat{\theta}_{ih}\right\}$ are good estimators of the $\left\{\theta_{ih}\right\}$. If the distribution of $\underline{Y}$ is invariant under groups of transformations involving certain permutations of observations or groups of observations (which is the case with balanced experiments of the nested and/or classification type), then the $\left\{\hat{\theta}_{ih}\right\}$ often may be viewed as conditional expectations of unbiased estimators of the $\left\{\theta_{ih}\right\}$ given a sufficient statistic. As a result, the $\left\{\hat{\theta}_{ih}\right\}$ are efficient quadratic estimators. (See Wilks [14]). For example, in the case of a balanced two-way classification experiment, the distribution of $\underline{Y}$ is invariant under permutations of rows and . permutations of columns. As a result of these considerations, estimators which are linear combinations of the

$\left\{\hat{\theta}_{ih}\right\}$ will be efficient quadratic estimators. This result has been proven by Graybill and Hultquist [8]; it, in part, follows from the U-like structure they possess (See Hoeffding [9]).

On the other hand, if $\underline{Y}$ is not invariant under any particular group of transformations, then the suggested estimators are reasonable in the sense of being unbiased, consistent, and readily obtainable. Moreover, if they are of interest, efficient quadratic estimators of the $\theta_{ih}$ or estimable functions of them can be derived by performing weighted least squares on the elements of $\underline{Y}\,\underline{Y}'$ where it is assumed that the weight matrix is the estimated variance-covariance matrix of the elements of $\underline{Y}\,\underline{Y}'$; this procedure necessarily presupposes the existence of prior estimates of the variance components themselves.

The concepts and procedures considered here for the variance-components model naturally extend to the multivariate model introduced in the beginning of the section. Hence, if a parameter is estimable, the suggested estimator of it is an appropriate linear function of symmetric sums of elements from the matrix $\underline{Y}\,\underline{Y}'$.

## A.2. Variances of Symmetric Quadratic Forms in Normally Distributed Random Variables.

The results in this section may be found elsewhere in the literature; in particular, Lancaster [11], Anderson [2], Anderson and Bush [3].

Lemma (2.1). Let $\underset{n\times 1}{\underline{Y}}$ be distributed according to the non-singular multivariate normal distribution $N(\underset{n\times 1}{\underline{0}}, \underset{n\times n}{V})$. Let $S = \underline{Y}'\,Q\,\underline{Y}$ be a symmetric quadratic form. Then,

$$E\left\{S\right\} = \operatorname{tr}\left\{VQ\right\} ,$$

$$\operatorname{Var}\left\{S\right\} = 2\operatorname{tr}\left\{[VQ]^2\right\} .$$

Proof: Let $\underset{n\times n}{F}$ be a non-singular matrix such that $F\ V\ F' = I_n$; than $\underline{Z} = F\ \underline{Y}$ is $N(\underline{O}, I_n)$ distributed.

Let $W = (F^{-1})'\ Q\ (F^{-1})$ and let $\underset{n\times n}{P}$ be an orthogonal matrix such that $P'\ W\ P = D$ where D is a diagonal matrix with diagonal elements $(d_1, d_2, \ldots, d_n)$. Let $\underset{n\times 1}{\underline{U}}$ be defined by $\underline{Z} = P\ \underline{U}$.

Consider the transformation $\underline{U} = P'\ \underline{Z} = P'\ F\ \underline{Y}$. Then $\underline{U}$ is distributed according to $N(\underline{O}, P'\ F\ V\ F'\ P)$; i.e., $\underline{U}$ is $N(\underline{O}, I_n)$.

$$
\begin{aligned}
E\left\{S\right\} = E\left\{\underline{Y}'Q\underline{Y}\right\} &= E\left\{\underline{U}'P'(F^{-1})'\ Q\ (F^{-1})\ P\ \underline{U}\right\} \\
&= E\left\{\underline{U}'D\underline{U}\right\} \\
&= E\left\{\sum_{i=1}^{n} d_i\ U_i^2\right\} \\
&= \sum_{i=1}^{n} d_i \qquad ;
\end{aligned}
$$

$$
Var\left\{S\right\} = Var\left\{\sum_{i=1}^{n} d_i\ U_i^2\right\} = 2\sum_{i=1}^{n} d_i^2 \quad ;
$$

$$
\begin{aligned}
E\left\{S\right\} = \sum_{i=1}^{n} d_i = tr(D) = tr(P'WP) &= tr\ (W\ P\ P') = tr(W) \\
&= tr\left\{(F^{-1})'\ Q\ (F^{-1})\right\} \\
&= tr\left\{Q[(F^{-1})(F^{-1})']\right\} \\
&= tr\left\{VQ\right\}
\end{aligned}
$$

since $tr(AB) = tr(BA)$ if A,B are squares matrices of same dimensions.

Similarly, $\mathrm{Var}\left\{S\right\} = 2\sum_{i=1}^{n} d_i^2 = 2\,\mathrm{tr}(D^2)$

$$= 2\,\mathrm{tr}\,(P'WPP'WP)$$

$$= 2\,\mathrm{tr}\,(W^2)$$

$$= 2\,\mathrm{tr}\left\{F^{-1'}QF^{-1}F^{-1'}QF^{-1}\right\}$$

$$= 2\,\mathrm{tr}\left\{(VQ)^2\right\}$$

Corollary (2.2). Let $\underline{Y}$ be the same as in Lemma (2.1). If $S_1 = \underline{Y}'Q_1\underline{Y}$ and $S_2 = \underline{Y}'Q_2\underline{Y}$ are symmetric quadratic forms, then $\mathrm{Cov}\left\{S_1, S_2\right\} = 2\,\mathrm{tr}\left\{VQ_1VQ_2\right\}$.

Proof: $\mathrm{Var}\left\{S_1 + S_2\right\} = \mathrm{Var}\left\{S_1\right\} + 2\,\mathrm{Cov}\left\{S_1, S_2\right\} + \mathrm{Var}\left\{S_2\right\}$,

$$\mathrm{Var}\left\{S_1 + S_2\right\} = 2\,\mathrm{tr}\left\{[V(Q_1 + Q_2)]^2\right\}$$

$$= 2\left\{\mathrm{tr}[(VQ_1)^2] + 2\mathrm{tr}\,[VQ_1VQ_2] + \mathrm{tr}\,[(VQ_2)^2]\right..$$

since $\mathrm{tr}\,(\sum_{i=1}^{k} A_i) = \sum_{i=1}^{k} \mathrm{tr}\,A_i$.

Hence $\mathrm{Cov}\left\{S_1, S_2\right\} = 2\,\mathrm{tr}\,[VQ_1VQ_2]$ .

Lemma (2.2). Let $\underset{n\times 1}{\underline{Y}}$ be distributed according to the non-singular multivariate normal distribution $N(\underset{n\times 1}{\underline{\mu}}, \underset{n\times n}{V})$. Let $S = \underline{Y}'\,Q\,\underline{Y}$ be a symmetric quadratic form. Then,

$$E\left\{S\right\} = \mathrm{tr}\left\{(V+\underline{\mu}\,\underline{\mu}')\,Q\right\} = \mathrm{tr}\left\{VQ\right\} + \underline{\mu}'\,Q\,\underline{\mu}.$$

$$\mathrm{Var}\left\{S\right\} = 2\,\mathrm{tr}\left\{(V+2\underline{\mu}\,\underline{\mu}')\,QVQ\right\} = 2\,\mathrm{tr}\left\{(VQ)^2\right\} + 4\,\underline{\mu}'\,QVQ\,\underline{\mu}\,.$$

In addition, if $S_1 = \underline{Y}' Q_1 \underline{Y}$ and $S_2 = \underline{Y}' Q_2 \underline{Y}$ are any two symmetric quadratic forms, then

$$\mathrm{Cov}\left\{S_1, S_2\right\} = 2\ \mathrm{tr}\left\{VQ_1VQ_2\right\} + 4\,\underline{\mu}'\, Q_1\, V\, Q_2\, \underline{\mu}\ .$$

Proof: Perform the same double-diagonalizing transformation as in the proof of Lemma (2.1); i.e., $\underline{U} = P'\,F\,\underline{Y}$. In this case, $\underline{U}$ is distributed according to $N(P'F\underline{\mu}, I_n)$. as before, $S = \underline{Y}'\,Q\,\underline{Y} = \underline{U}'\,D\,\underline{U} = \sum_{i=1}^{n} d_i\, U_i^2$.

Now $U_1^2, U_2^2, \ldots, U_n^2$ are independently distributed and have non-central chi-square distributions with one degree of freedom. The respective non-centrality parameters are the squares of the elements in $P'\,F\,\underline{\mu}$; i.e., if we let $\underline{\delta}' = (\delta_1, \delta_2, \ldots \delta_n) = (P'F\,\underline{\mu})'$, then $\delta_1, \delta_2, \ldots, \delta_n$ are the respective non-centrality parameters. As a result,

$$E\left\{U_i^2\right\} = 1 + \delta_i^2 \qquad \text{for } i = 1, 2, \ldots, n;$$

$$\mathrm{Var}\left\{U_i^2\right\} = 2 + 4\,\delta_i^2 \qquad \text{for } i = 1, 2, \ldots, n.$$

Consequently, we have that

$$\begin{aligned} E(S) = E\left\{\sum_{i=1}^{n} d_i\, U_i^2\right\} &= \sum_{i=1}^{n} d_i\,(1 + \delta_i^2) \\ &= \mathrm{tr}\,(D) + \underline{\delta}'\, D\, \underline{\delta} \\ &= \mathrm{tr}\,(D) + \underline{\mu}'\, F'P\, D\, P'F\, \underline{\mu} \\ &= \mathrm{tr}\,(VQ) + \underline{\mu}'\, Q\, \underline{\mu} \quad ; \end{aligned}$$

and

$$\mathrm{Var}\ (S) = \mathrm{Var}\left\{\sum_{i=1}^{n} d_i\ U_i^2\right\} = \sum_{i=1}^{n} d_i^2\ (2 + 4\delta_i^2)$$

$$= 2\ \mathrm{tr}\ (D^2) + 4\ \underline{\delta}'\ D^2\underline{\delta}$$

$$= 2\ \mathrm{tr}\ \left\{(VQ)^2\right\} + 4\ \underline{\mu}'\ F'\ P\ D^2\ P'\ F\ \underline{\mu}$$

$$= 2\ \mathrm{tr}\ \left\{(VQ)^2\right\} + 4\ \underline{\mu}'\ F'\ W^2\ F\ \underline{\mu}$$

$$= 2\ \mathrm{tr}\ \left\{(VQ)^2\right\} + 4\ \underline{\mu}'\ [F'\ W\ F\ F^{-1}\ F'^{-1}\ F'\ W\ F]\ \underline{\mu}$$

$$= 2\ \mathrm{tr}\ \left\{(VQ)^2\right\} + 4\ \underline{\mu}'\ Q\ V\ Q\ \underline{\mu}\ .$$

If $S_1 = \underline{Y}'\ Q_1\ \underline{Y}$ and $S_2 = \underline{Y}'\ Q_2\ \underline{Y}$ where $Q_1$, $Q_2$ are symmetric, then

$$\mathrm{Cov}\ \left\{S_1,\ S_2\right\} = \tfrac{1}{2}\ [\ \mathrm{Var}\ \left\{S_1 + S_2\right\} - \mathrm{Var}\left\{S_1\right\} - \mathrm{Var}\left\{S_2\right\}\ ]$$

$$= 2\ \mathrm{tr}\left\{VQ_1\ VQ_2\right\} + 4\ \underline{\mu}'\ Q_1\ VQ_2\ \underline{\mu}\ .$$

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