STATISTICS FOR TESTING UNIFORMITY ON THE UNIT INTERVAL

by

Forest Leonard Miller, Jr.

Institute of Statistics
Mimeograph Series No. 983
Raleigh, N. C. 1975

STATISTICS FOR TESTING UNIFORMITY ON THE UNIT INTERVAL

by

FOREST LEONARD MILLER, JR.

A thesis submitted to the Graduate Faculty of
North Carolina State University at Raleigh
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

DEPARTMENT OF STATISTICS

Raleigh

1974

APPROVED BY:

______________________ ______________________

______________________ ______________________
Chairman of Advisory Committee

BIOGRAPHY

Forest L. Miller, Jr. was born June 18, 1936 in Cincinnati, Ohio. He received his elementary and secondary education in Washington, D. C. and Ankara, Turkey, graduating from Woodrow Wilson High School in Washington, D. C. in 1954.

He received the Bachelor of Science degree with a major in statistics from Purdue University in 1958 and the Master of Science degree with a major in statistics from Purdue University in 1959.

From 1959 until 1964, he was a statistician in the Oak Ridge complex managed by Union Carbide Corporation for the Atomic Energy Commission. From 1964 until 1967 he was a student in the Department of Experimental Statistics of North Carolina State University. Since then he has been a statistician in what is now called the Mathematics and Statistics Research Department of the Computer Sciences Division of Union Carbide Corporation, Nuclear Division.

The author is married to the former Miss Mitzi Lu Gwyn, and they have three sons--Mark, Nathaniel and David.

ACKNOWLEDGEMENTS

The author wishes to express gratitude to his major advisor, Professor C. P. Quesenberry, for his advice and assistance.

Appreciation is also extended to Professor A. H. E. Grandage, R. J. Monroe and R. G. D. Steel for serving on his committee and to the author's colleagues at work for their advice and encouragement.

The author gratefully acknowledges financial support from the Department of Statistics while he was in residence.

The work reported here was performed while the author was a member of the Mathematics and Statistics Research Department, Computer Sciences Division, Union Carbide Corporation, Nuclear Division; operated by Union Carbide Corporation for the U. S. Atomic Energy Commission.

Finally, the author expresses thanks to his wife, Mitzi, and his children for their patience and forbearance during the course of this study.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

1. INTRODUCTION AND SUMMARY

Problems relating to measuring the agreement between theoretical distributions and sets of observations, goodness-of-fit problems, have been extensively investigated because of their practical importance. Many omnibus and special purpose goodness-of-fit tests have been proposed but an organized general theory is lacking. The development of the conditional probability integral transformation (CPIT) theory by O'Reilly and Quesenberry (1973) may bring more order to this area of research because it reduces many problems to the problem of testing uniformity on (0,1).

Let $X_1, \ldots, X_N$ be a random sample from a distribution F. Then if F is a member of a class $\mathcal{F}$ of continuous distributions a conditional probability integral transformation, depending on the class $\mathcal{F}$, will transform $X_1, \ldots, X_N$ into $U_1, \ldots, U_n$, $n \leq N$, where the U_i; $i = 1, \ldots, n$; are independently, identically and uniformly distributed on (0,1). Thus a test of the composite null hypothesis

$$H_0: \quad F \in \mathcal{F} \tag{1.1}$$

can be reduced to testing the simple hypothesis

$$H_0: \quad U_1, \ldots, U_n \text{ are i.i.d. } U(0,1) \tag{1.2}$$

because if $F \notin \mathcal{F}$, then, in general, the U_i; $i = 1, \ldots, n$; will not be i.i.d. U(0,1). The problem of testing uniformity and related distribution theory has received considerable attention in the literature, as will be indicated in Chapter 2. Interest here is motivated by the need to find test statistics which have good power properties against a wide range of alternative distributions, especially for small sample sizes. If such tests can be found, then they would be of immediate practical usefulness because the null hypothesis (1.2) can be used to test many continuous classes $\mathcal{F}$. The advantage of this is considerable, since only one null hypothesis distribution is involved and only one table of significance points is required.

Any goodness-of-fit test for a simple null hypothesis may be used to test (1.2). In this thesis a power study comparing nine test statistics is reported. Two new test statistics are considered as well as seven statistics which appear in the literature.

The alternative distributions used in the power study are defined in Chapter 3. The random variables produced by the CPIT tend to cluster together when the data are not from distribution specified by the null hypothesis. The four families of alternative distributions selected represent different types of clustering that have been observed in practice. The nine test statistics used in the power study and a transformation proposed by Durbin (1961) to increase the power of the test statistics are discussed in Chapter 4. The emphasis of the power study, described in Chapter 5, is on the effectiveness of these statistics in detecting non-uniformity in small samples. Textbook examples of simple linear regression have been transformed by the appropriate conditional probability integral transformation in Chapter 6 to illustrate the use of the method, and the usefulness of the power results reported here.

2. REVIEW OF THE LITERATURE

Testing the goodness of fit of models to data is a major theme of the statistical literature. Because of the development of the conditional probability integral transformation by O'Reilly and Quesenberry (1973) the concern here is with testing whether a set of random variables $U_1, \ldots, U_n$ are independently and identically distributed with a uniform distribution on the unit interval.

There are several tests of goodness-of-fit of wide applicability. The statistic that is perhaps the most widely known and used is called X^2 or chi-square, and was proposed by Karl Pearson (1900). The paper of Cochran (1952) is an excellent discussion of this test statistic. The statistic is the sum, based on arbitrary class groupings, of the ratios of the square of the difference between the numbers of data points observed in the classes and the numbers expected in the classes under the model hypothesized to the numbers expected. Kolmogorov (1933) proposed a statistic, here called the Kolmogorov-Smirnov statistic, which is a function of the largest difference between the empirical distribution function of the data and the distribution function hypothesized. Cramér (1928) and von Mises (1931) independently proposed a test statistic which is the integral of the squared difference between the empirical distribution function of the data and the hypothesized distribution function. The paper of Darling (1957) is a unified discussion of these two test statistics. Watson (1961) proposed a variant of the Cramér-von Mises statistic, W^2, which

> "has the form of a variance while W^2 has the form of a second moment about the origin, i.e. the modification corresponds to a correction for the mean."

Anderson and Darling (1952, 1954) proposed another modification of the Cramér-von Mises statistic in which the tails are given greater weight.

Consider a random sample $x_1, \ldots, x_n$ on $(0,1)$. Let $x_{(1)}, \ldots, x_{(n)}$ be the order statistics of this sample and define

$$y_i = x_{(i)} - x_{(i-1)}, \quad i = 1, \ldots, n + 1, \tag{2.1}$$

where $x_{(0)} = 0$ and $x_{(n+1)} = 1$. The y_i, $i = 1, \ldots, n + 1$ are called spacings or coverings. Although discussed earlier by Whitworth (1887) and others the development of goodness-of-fit tests based on spacings received its principal impetus from Greenwood (1946). To test the uniformity of the sample he proposed

$$G_n = \sum_{i=1}^{n+1} y_i^2, \tag{2.2}$$

as a test statistic. Moran (1947) showed that G_n becomes normally distributed as $n \to \infty$. Gardner (1952) found the exact distribution of G_3 which stimulated Moran (1953) who was able to find the lower one and five percent points of G_n for $n \leq 9$.

The paper of Greenwood and its discussion stimulated a series of papers proposing other statistics. In a series of short papers Weiss (1955, 1956, 1957a, 1957b, 1957c, 1958, 1959, 1961) carried forward the theoretical development of statistics based on spacings, mainly from the asymptotic point of view. Kimball (1947, 1950) proposed

$$K_n = \sum_{i=1}^{n+1} y_i^r, \quad r > 0. \tag{2.3}$$

Sherman (1950), following a suggestion of Kendall in the discussion of Greenwood's paper studied

$$S_n = \sum_{i=1}^{n+1} \left| y_i - \frac{1}{n+1} \right|, \tag{2.4}$$

as a test statistic for uniformity. Kendall, in that same discussion, also suggested the largest spacing minus the smallest and the ratio of the largest to the smallest as test statistics for uniformity. Darling (1953) found their limiting distributions in a survey paper and suggested two more test statistics

$$L_n = \sum_{i=1}^{n+1} \ln y_i, \tag{2.5}$$

and,

$$R_n = \sum_{i=1}^{n+1} y_i^{-1}. \tag{2.6}$$

Bartholomew (1954) computed some tables of S_n and noted a connection with exponentially distributed random variables. If t_i, $i = 1, \ldots, n$, are exponentially distributed with common parameter λ, then

$$T_n = \frac{1}{2(n+1)\bar{t}} \sum_{i=1}^{n+1} |t_i - \bar{t}|, \tag{2.7}$$

where $\bar{t} = \frac{1}{n+1} \sum_{i=1}^{n+1} t_i$, has the same distribution as S_n. Bartholomew (1956) compared T_n as a test of exponentiality with several other statistics. Pyke (1965) wrote a survey paper on spacings in which he included work by Proschan and Pyke (1962).

Mauldon (1951) studied the distribution of the k largest spacings. Fisher (1929) had solved the problem for k = 1. Barton and David (1955, 1956a, 1956b) derived the distribution of the sum of a set of consecutively ordered spacings, the difference of two ordered spacings and the ratio of two ordered spacings and proposed them as tests for uniformity.

Blumenthal (1966a, 1966b, 1967) studied the difference between the largest and smallest spacings and their ratio as test statistics for the two sample case.

David and Johnson (1948) considered the probability integral transformation when location and scale parameters appearing in the distribution function are replaced by estimates computed from the sample. They showed that the transformed sample values using such an estimated distribution function have distributions that do not depend upon the location and scale parameters. Lilliefors (1967, 1969), guided by this result, defined a Kolmogorov-Smirnov type statistic by replacing the usual null hypothesis distribution function in that statistic by the estimated distribution function of David and Johnson for the normal and exponential distributions. Quoting from the David and Johnson paper:

> "... we have noted that given n independent random variables, x, if s sample moments are calculated from them and used as estimates of the parameters of the probability law, then it appears that there will be s independent relationships between the y's. Thus in this case the point $y_1, \ldots, y_n$ is constrained to move in an n-s dimensioned space within an n dimensioned cube, and we have the exact analogue to the loss of degrees of freedom with X^2 when the parameters have to be estimated from the data. .... one may seek for some transformation of variables so that instead of the correlated y_i we obtain n-s new independent variables following some distributions which are independent of the original p(x)."

O'Reilly and Quesenberry (1973) show that certain conditional distribution functions may, under general conditions, be used to transform a sample from an arbitrary member of a parametric class into a set of N-k i.i.d. U(0,1) random variables and a k dimensional sufficient statistic. This result is called the conditional probability integral transformation and provides a method for constructing goodness-of-fit tests for many composite null hypothesis classes. Obviously, it is desirable to use as the test statistic for uniformity a statistic with good power properties for detecting various deviations from uniformity. It is this point which has motivated the present work.

The paper of Stephens (1974) contains a power study comparing seven statistics as tests of uniformity on (0,1). Four of these test statistics are included in the nine test statistics evaluated here. The shapes of his alternative distributions are the same as those considered here, and two distributions are common to both power studies.

3. ALTERNATIVE DISTRIBUTIONS

3.1 Introduction

Empirical experience with the conditional probability integral transformation has shown for a fairly large number of different null hypotheses and alternatives that under the alternatives certain patterns for the random variables produced by the transform (observations) appear quite often. One such pattern is for the observations to be pushed toward one end of the unit interval. A second pattern of interest is for them to tend to cluster near the center of the (0,1) interval. Another pattern of interest is for the observations to cluster at both ends of the (0,1) interval. These last two patterns would be expected for intersection ($\cap$) and union ($\cup$) shaped alternatives, respectively. The families of alternatives defined in the next section are of these types. Each family contains the uniform distribution, indexed as $i = 0$, and as i increases the distributions increasingly become less uniform and concentrate at the middle, or one, or both ends, depending on the family.

3.2 Families of Alternative Distributions

The first family, powers of uniform random variables, increasingly concentrates the probability mass near zero as i increases.

Let U be a $U(0,1)$ random variable and define

$$Y_{1i} = U^{1+i}, \quad i = 1, 2, 3, 4. \tag{3.1}$$

Then Y_{1i} has density function

$$h_{1i}(y) = (1/(1+i))\, y^{-i/(1+i)}\, I_{(0,1)}(y);\ i = 1, 2, 3, 4; \tag{3.2}$$

where $I_{(0,1)}(y) = 1$ if $0 < y < 1$ and 0 otherwise.

Graphs of these densities are given in Figure 3.1. This is called the Type 1 family of alternatives. The second family, averages of uniform random variables, increasingly concentrates the probability mass near .5 as i increases.

Let

$$Y_{2i} = (1/(1+i)) \sum_{k=1}^{1+i} U_k,\ i = 1, 2, 3, 4. \tag{3.3}$$

The Y_{2i} have densities shown below, as given by Wilks (1962) and due to Laplace (1814).

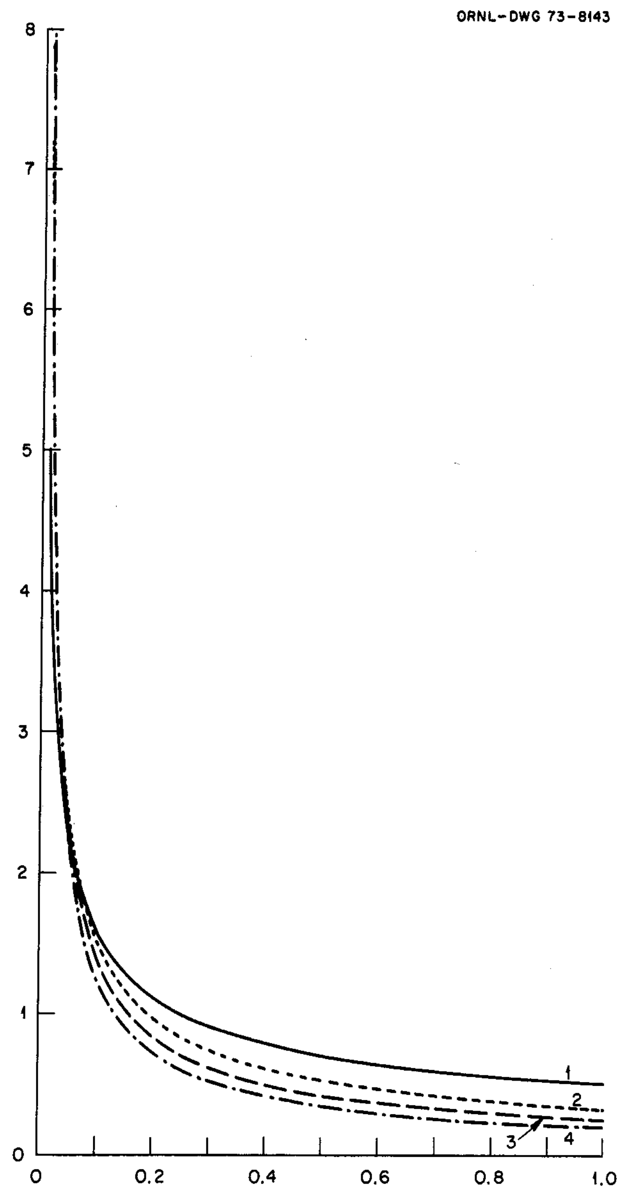

Figure 3.1 Type 1 Family of Alternative Distributions

$$h_{21}(y) = \begin{cases} 4\ y, & 0 < y \le .5, \\ 4 - 4\ y, & .5 < y \le 1. \end{cases} \tag{3.4}$$

$$h_{22}(y) = 3/2 \begin{cases} 9\ y^2, & 0 < y \le .33, \\ -18\ y^2 + 18\ y - 3, & .33 < y \le .67, \\ 9\ y^2 - 18\ y + 9, & .67 < y \le 1. \end{cases} \tag{3.5}$$

$$h_{23}(y) = 2/3 \begin{cases} 64\ y^3, & 0 < y \le .25 \\ -192\ y^3 + 192\ y^2 - 48\ y + 4, & .25 < y \le .5 \\ 192\ y^3 - 384\ y^2 + 240\ y - 44, & .5 < y \le .75 \\ -64\ y^3 + 192\ y^2 - 192\ y + 64, & .75 < y \le 1. \end{cases} \tag{3.6}$$

$$h_{24}(y) = 5/24 \begin{cases} 625\ y^4, & 0 < y \le .2, \\ -2500\ y^4 + 2500\ y^3 - 750\ y^2 + 100\ y - 5, & .2 < y \le .4, \\ 3750\ y^4 - 7500\ y^3 + 5250\ y^2 - 1500\ y + 155, & .4 < y \le .6, \\ -2500\ y^4 + 7500\ y^3 - 8250\ y^2 + 3900\ y - 655, & .6 < y \le .8, \\ 625\ y^4 - 2500\ y^3 + 3750\ y^2 - 2500\ y + 625, & .8 < y \le 1. \end{cases} \tag{3.7}$$

Graphs of these densities, called the Type 2 family of alternatives, are given in Figure 3.2. The third family, U-shaped distributions constructed by combining distributions of Type 1, increasingly concentrates the probability mass near 0 and 1 as i increases. For U a U(0,1) random variable and D an independent random variable with $p(D = 0) = P(D = 1) = \frac{1}{2}$, let

$$Y_{3i} = DU^{1+i} + (1-D)\ (1-U^{1+i}),\ i = 1, 2, 3, 4. \tag{3.8}$$

The Y_{3i} have densities

$$h_{3i}(y) = (1/2(1+i))\ (y^{-i/(1+i)} + (1-y)^{-i/(1+i)})\ I_{(0,1)}. \tag{3.9}$$

Graphs of these densities are shown in Figure 3.3.

The fourth family of distributions, a rearrangement of the distributions of the second family to produce U-shaped distributions, also increasingly concentrates the probability mass near 0 and 1. The two families differ in that distributions of Type 4 are more broadshouldered and less dense near .5 than distributions of Type 3. Let

$$Y_{4i} = (Y_{2i} + .5)\ I_{(0,.5)}\ (Y_{2i}) + (Y_{2i} - .5)\ I_{(.5,1)}\ (Y_{2i}), \tag{3.10}$$

where the densities of the Y_{2i}, i = 1, 2, 3, 4 are (3.4), (3.5), (3.6) and (3.7). The density functions h_{4i} are graphed in Figure 3.4 for i = 1, 2, 3, 4.

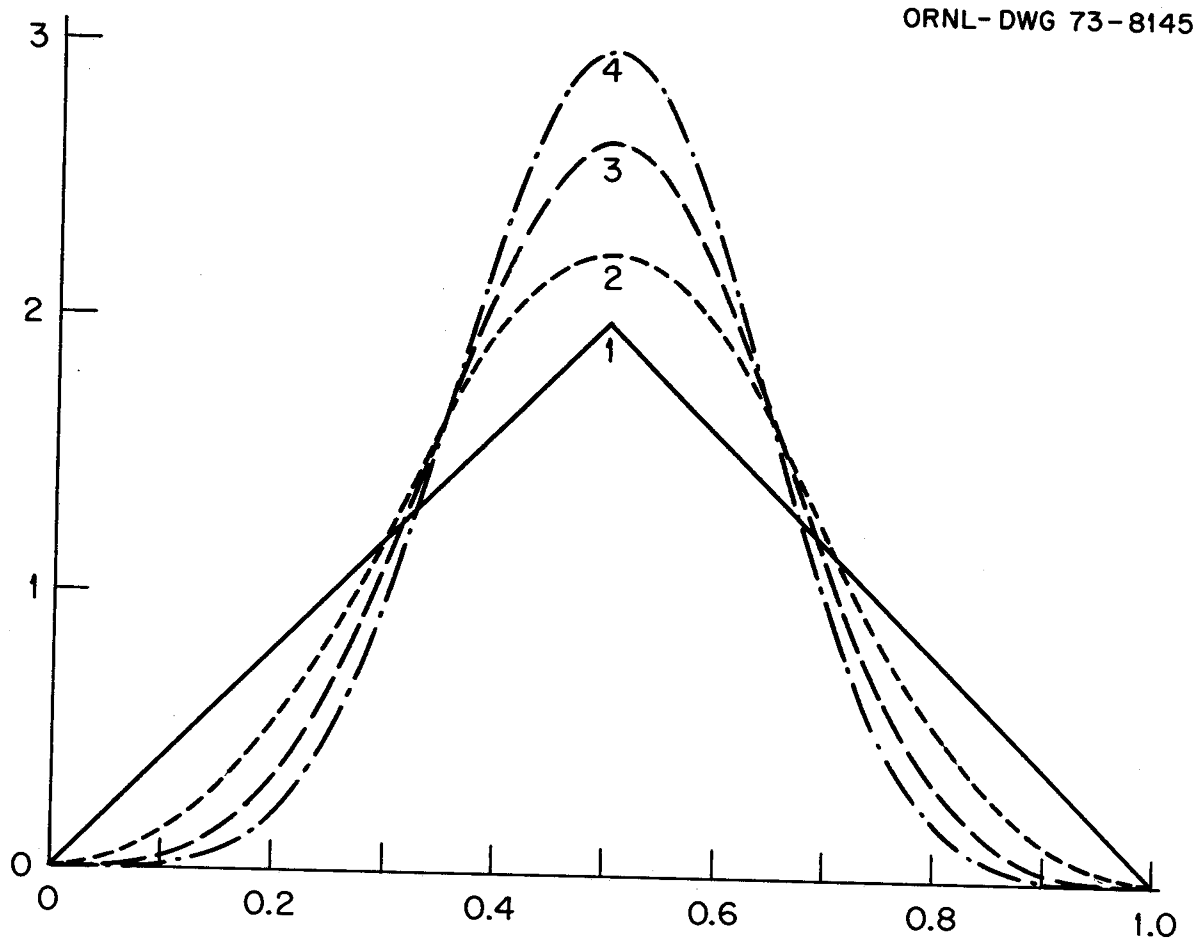

Figure 3.2 Type 2 Family of Alternative Distributions

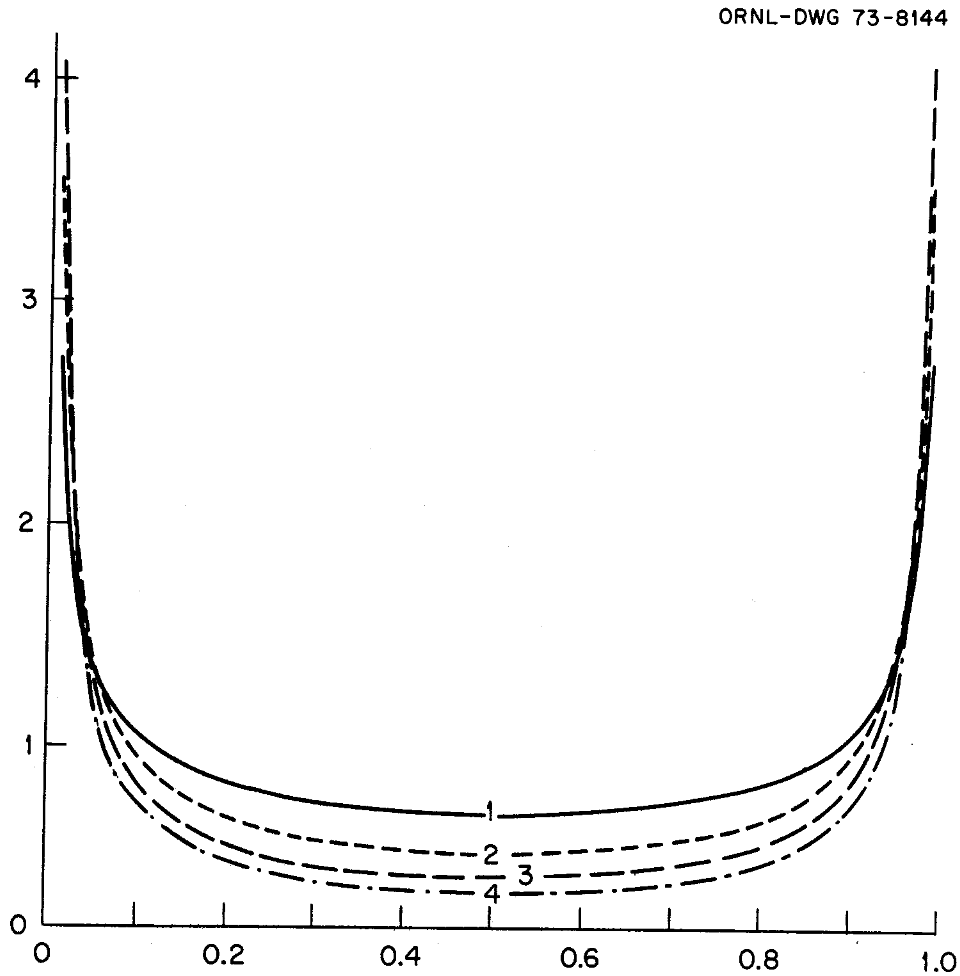

Figure 3.3   Type 3 Family of Alternative Distributions

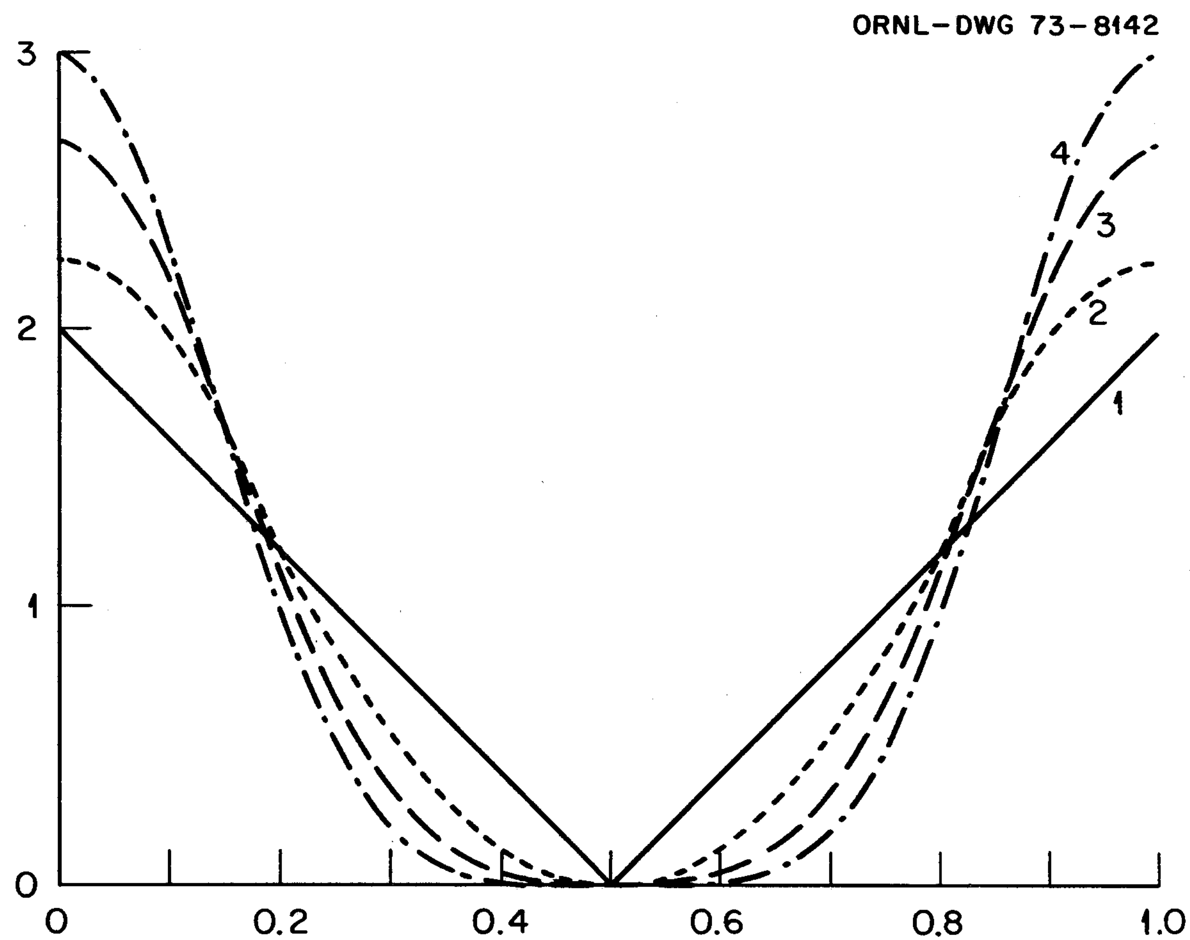

Figure 3.4 Type 4 Family of Alternative Distributions

4. TEST STATISTICS

4.1 Introduction

There are many test statistics which can be used to test the null hypothesis that $U_1, \ldots, U_n$ are i.i.d. $U(0,1)$ random variables. The performance of nine of these statistics is compared in this thesis. The discrete Pearson X_k^2 statistic is considered for $k = 10, 20$ cells. Five well-known test statistics with continuous distributions are considered: the Kolmogorov-Smirnov, the Cramér-von Mises, the Watson, the Anderson-Darling, and a statistic proposed by Greenwood. Two new test statistics with continuous distributions to be defined below are also studied. In addition, a transformation proposed by Durbin to improve the power of tests of uniformity on (0,1) has been studied. Some of the statistics considered here have previously been mentioned in Chapter 2, but are defined here for completeness.

4.2 The Pearson X_k^2 Statistic

This statistic is discrete and tables of possible values for $k = 10, 20$ were computed for the sample sizes used in the power study (2, 5, 10, 15, 20 and 50). These tables, and the routines necessary to compute them, are discussed in Appendix 8.5. Because of the discreteness of the statistic a randomization procedure was required to obtain exact α tests to compare with the other test statistics. If A is any set denote by I_A the indicator function of the set A, i.e., $I_A(x) = 1$ if $x \epsilon A$ and is zero otherwise. Let k be a positive integer greater than one and let $p_1, \ldots, p_k$ be positive numbers such that $p_1 + \ldots + p_k = 1$.

Let $B_j = \{x: \sum_{i=0}^{j-1} p_i < x \leq \sum_{i=1}^{j} p_i\}$ for $j = 1, \ldots, k$ and $p_o = 0$. Then put

$$n_j = \sum_{i=1}^{n} I_{B_j}(U_i), \; j = 1, \ldots, k. \tag{4.1}$$

If $U_1, \ldots, U_n$ are i.i.d. $U(0,1)$, then the vector $(n_1, \ldots, n_k)$ has a multinomial distribution with probability parameters $p_1, \ldots, p_k$, and $n_1 + \ldots + n_k = n$. The statistic

$$X_k^2 = \sum_{j=1}^{k} (n_j - n\, p_j)^2 / n\, p_j, \tag{4.2}$$

has a discrete Pearson X_k^2 distribution. As in O'Reilly and Quesenberry (1973), the values $p_1 = \ldots = p_k = 1/k$ will be used here. With this choice of p_j's the interval $(0,1]$ is partitioned into k intervals of length $1/k$ each: $B_j = \{x: (j-1)/k < x \leq j/k\}$; $j = 1, \ldots, k$; and

$$X_k^2 = (k/n) \sum_{j=1}^{k} n_j^2 - n. \tag{4.3}$$

4.3 The Kolmogorov-Smirnov Statistic

For $U_{(1)}, \ldots, U_{(n)}$, the order statistics of the sample $U_1, \ldots, U_n$, the statistic suggested by Kolmogorov (1933) is given by

$$D_n = \max_{i \in \{1,\ldots,n\}} [\max \{|U_{(i)} - (i-1)/n|, |U_{(i)} - i/n|\}]. \tag{4.4}$$

This is a well-known statistic and tables for its distribution when $U_1, \ldots, U_n$ are i.i.d. U(0,1) are available in Miller (1956) or Pearson and Hartley (1972). For this statistic, and others when possible, the notation of Pearson and Hartley is used.

4.4 The Cramér-von Mises Statistic

A well-known statistic suggested by Cramér (1928) and independently by von Mises (1931) is given by

$$W_n^2 = \frac{1}{12n} + \sum_{i=1}^{n} [(2i-1)/2n - U_{(i)}]^2. \qquad (4.5)$$

This statistic is also tabled by Pearson and Hartley (1972).

4.5 The Watson Statistic

A statistic suggested by Watson (1961) and tabled by Pearson and Hartley (1972) is

$$U_n^2 = W_n^2 + n\ (\overline{U} - .5)^2, \qquad (4.6)$$

where W_n^2 is defined in (4.5) and $\overline{U}$ is the mean of the sample $U_1, \ldots, U_n$. Testing revealed that the critical values of Pearson and Hartley (1972) are inadequate for small n. The percentage points of U_n^2 were tabulated by Monte Carlo methods for n = 2(1)10. For each value of n, 27,000 samples of size n were drawn from the U(0,1) distribution. The ensemble of 27,000 U_n^2 values was sorted and percentage points printed out. The sample size of 27,000 was chosen, using the results of Massey (1951), so that the probability of the empirical distribution function deviating from the true distribution functions by more than .01 in absolute value is less than .01. The critical values of U_n^2 tabled in Pearson and Hartley were developed by M. A. Stephens who found that, for the critical values tabled, $[U_n^2 - 1/(10n) - 1/(10n^2)]\ [1 + .8/n]$ is independent of n. The .1, .05 and .01 critical values of the modified Watson statistic are .152, .187 and .267, respectively. Below are critical values obtained by Monte Carlo methods and their modified values. The values of Stephens are adequate at the .1 and .05 levels for $n \geq 4$ and at the .01 level for $n \geq 9$.

TABLE 4.1

Empirical Critical Values of Watson's U^2

	U^2			U^2_{mod}		
n	α = .1	α = .05	α = .01	α = .1	α = .05	α = .01
2	.142	.154	.164	.164	.181	.195
3	.147	.173	.213	.158	.191	.242
4	.145	.175	.230	.152	.187	.254
5	.147	.176	.237	.152	.185	.256
6	.148	.179	.243	.151	.187	.260
7	.148	.180	.247	.151	.187	.262
8	.149	.182	.249	.152	.188	.262
9	.150	.183	.258	.152	.189	.270
10	.148	.181	.255	.151	.185	.265

4.6 The Anderson-Darling Statistic

Anderson and Darling (1954) suggested a statistic, tabled by Pearson and Hartley (1972) of the form

$$A_n = \left(-\{ \sum_{i=1}^{n} (2i-1) [\ln U_{(i)} + \ln(1-U_{(n+1-i)})]\}/n\right) - n. \quad (4.7)$$

4.7 The Greenwood Statistic

Greenwood (1946) proposed a test for uniformity based on differences of ordered observations (spacings). For $U_{(1)}, \ldots, U_{(n)}$ as before, $U_{(0)} = 0$ and $U_{(n+1)} = 1$, the statistic is

$$G_n = \sum_{j=1}^{n+1} [U_{(j)} - U_{(j-1)}]^2. \quad (4.8)$$

Although this statistic is not as well known as some of the other statistics considered here, there is, in fact, a rather large literature devoted to it. Moran (1947) showed that as $n \to \infty$ G_n is normally distributed with mean $2/(n+2)$ and variance $4 \{(n+6)/[(n+2)(n+3)(n+4)] -1/(n+2)\}$ although the asymptotic distribution is approached slowly. This effect is more pronounced for the lower percentage points, in which both Greenwood and Moran were interested, than in the upper percentage points, as can be seen by examination of Figures 4.1, 4.2, 4.3 and 4.4 in which the estimated distribution functions obtained by Monte Carlo methods are compared with normal distributions with mean and variance as above.

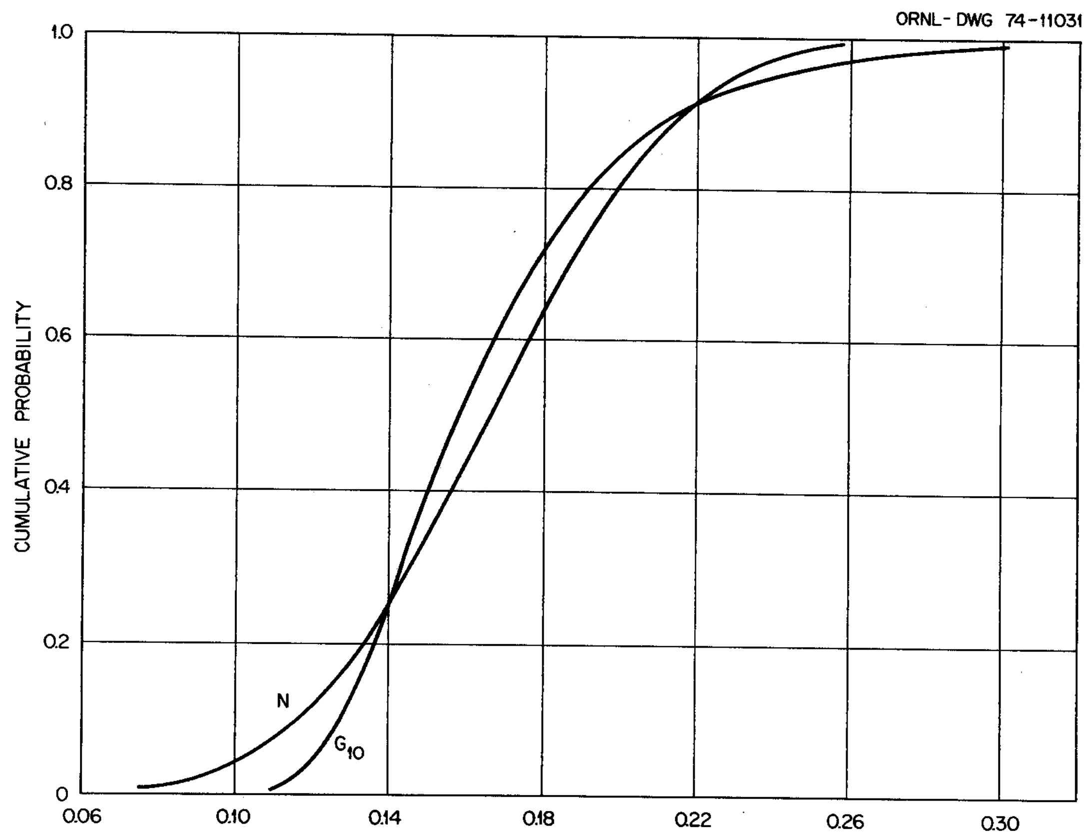

Figure 4.1   Comparison of Normal and G_{10}

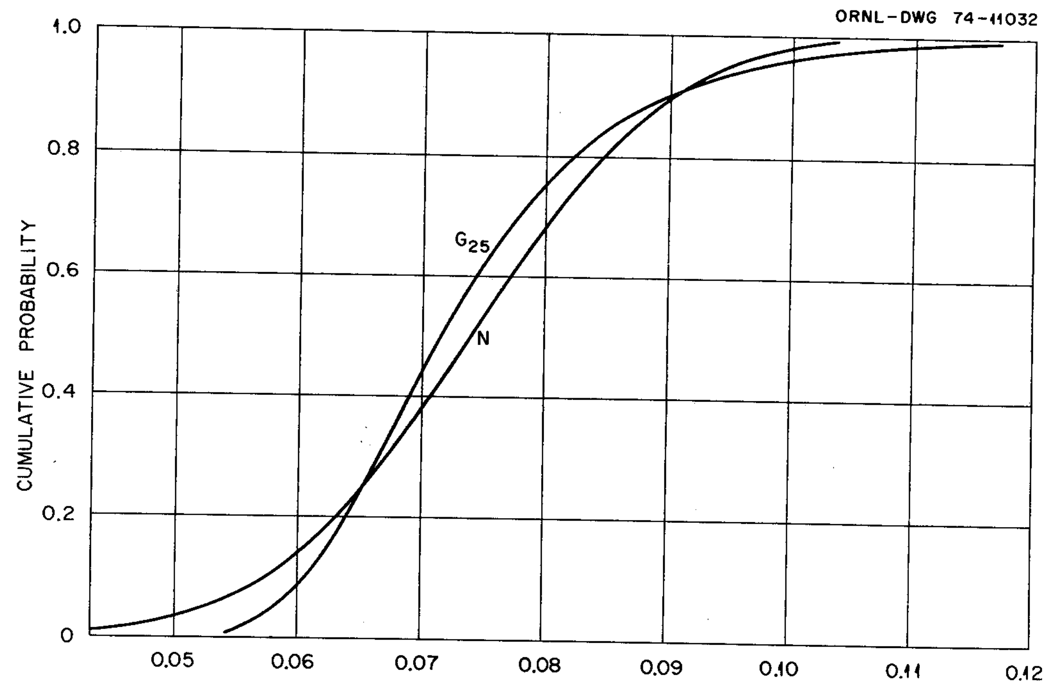

Figure 4.2 Comparison of Normal and G_{25}

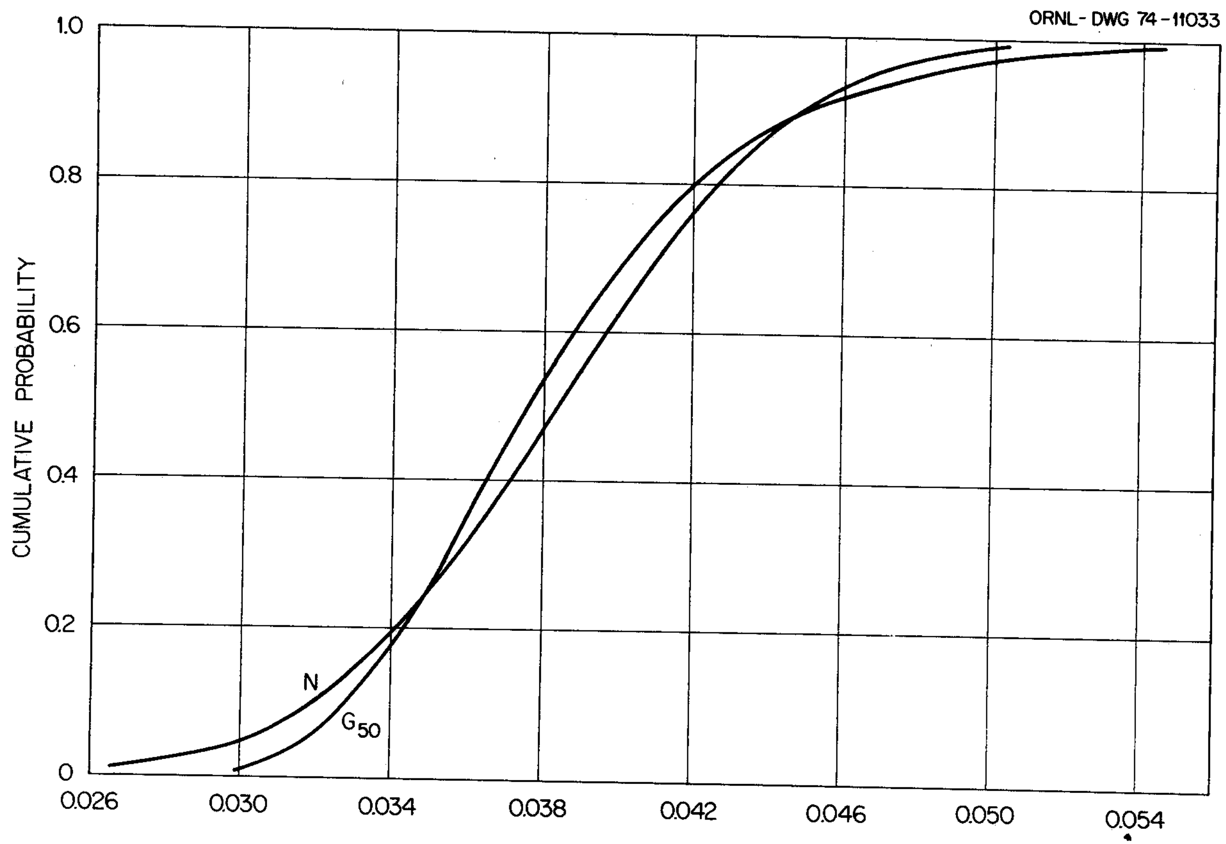

Figure 4.3 Comparison of Normal and G_{50}

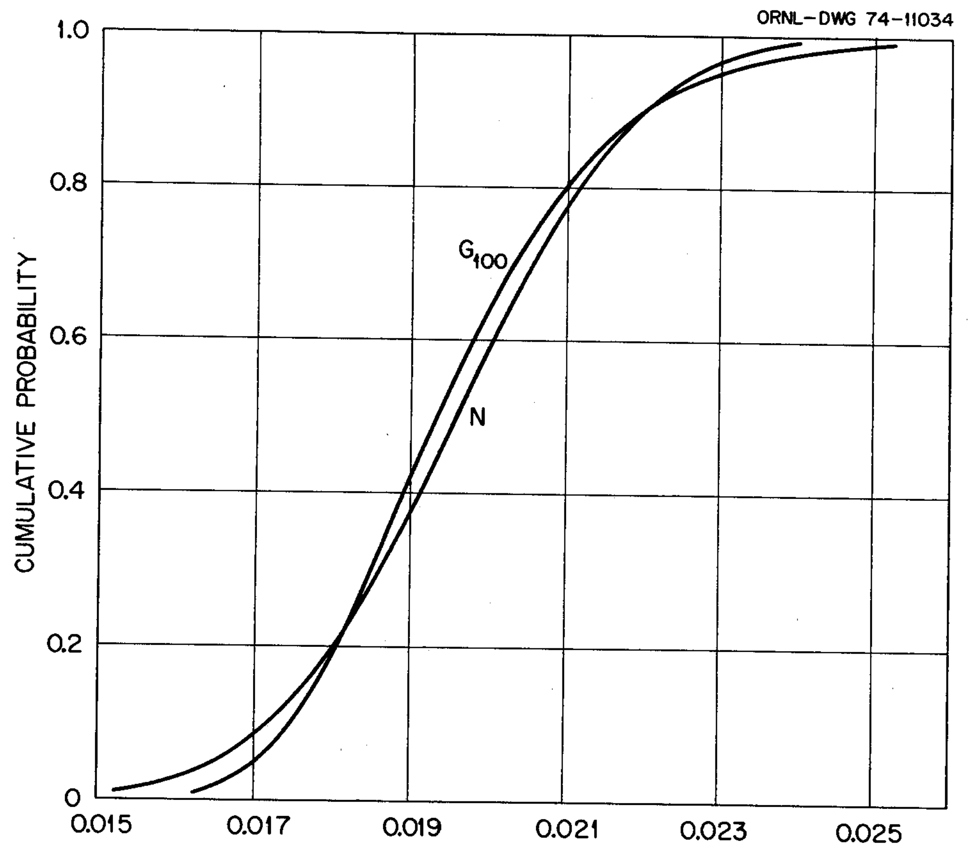

Figure 4.4 Comparison of Normal and G_{100}

The minimum value of G_n is attained when the spacings are of equal size, $1/(n+1)$, while the maximum, 1, is attained when the U_i, $i = 1, \ldots, n$ are all either 0 or 1. The percentage points of G_n have been tabulated by Monte Carlo methods for $n = 2(1)50(5)100$. For each value of n, 27,000 samples of size n were drawn from the $U(0,1)$ distribution. The ensemble of 27,000 G_n values was sorted and percentage points printed out. The sample size of 27,000 was chosen, using the results of Massey (1951), so that the probability of the empirical distribution function deviating from the true distribution function by more than .01 in absolute value is less than .01. The random number generation algorithm used in this thesis is described in Appendix 8.1, while selected percentage points of G_n are tabled in Appendix 8.2.

4.8 The Statistic Q_n

While G_n does reflect the size distribution of the sample spacings it does not contain information about uniformity provided by the order in which these spacings occur. The inclusion of information of this type should increase the power of test statistics to detect non-uniformity. A first step is to include "nearest neighbor" information and the statistic is defined as

$$Q_n = G_n + \sum_{j=1}^{n} [U_{(j)} - U_{(j-1)}] [U_{(j+1)} - U_{(j)}], \qquad (4.9)$$

where as before $U_{(0)} = 0$ and $U_{(n+1)} = 1$. The statistic Q_n does not attain its minimum value when the spacings are of equal size. The spacings which lead to a minimum value of Q_n are discussed in Appendix 8.6.

4.9 The Statistic QM

The success of Q_n relative to G_n in detecting non-uniformity on (0,1) led to consideration of a statistic which contained products of greater numbers of spacings. Let $U_{(1)}, \ldots, U_{(n)}$ be the order statistics of a random sample of size n drawn from U(0,1). Put $y_i = U_{(i)} - U_{(i-1)}$, $i = 1, \ldots, n + 1$, where $U_{(0)} = 0$ and $U_{(n+1)} = 1$. Let

$$d_i^{(j)} = (y_i \, y_{i+1} \cdots y_{i+j})^{1/(1+j)}, \quad \begin{matrix} i = 1, \ldots, n+1-j, \\ j = 1, \ldots, n. \end{matrix} \qquad (4.10)$$

Let $d_{(1)}^{(j)}, \ldots, d_{(n+1-j)}^{(j)}$ denote the ordered values of $d_1^{(j)}, \ldots, d_{n+1-j}^{(j)}$.

Put

$$d^{(j)} = \sum_{i=L_1}^{L_2} d_{(i)}^{(j)} - \sum_{i=1}^{L_3} d_{(i)}^{(j)} \qquad j = 0, \ldots, k, \qquad (4.11)$$

where $L_1 = n + 2 - j - \left[\frac{n+1-j}{2}\right]$, $L_2 = n + 1 - j$, $L_3 = \left[\frac{n+1-j}{2}\right]$, $k = \left[\left(\frac{n+1}{2}\right)^{\frac{1}{2}}\right]$ and [x] is the greatest integer in x. Then

$$QM_n = d^{(0)} + d^{(1)} + \ldots + d^{(k)}. \qquad (4.12)$$

The statistic QM_n cannot be negative and is zero for $y_i = 1/(n+1)$, $i = 1, \ldots, n + 1$.

The percentage points of QM_n have been tabulated by Monte Carlo methods for n = 2(1)50(5)100. For each value of n, 27,000 samples of size n were drawn from U(0,1) distribution. From each of these samples QM_n was computed and the ensemble of 27,000 QM_n values was sorted and percentage points printed out. The sample size of 27,000 was chosen so that the probability of the empirical distribution function deviating from the true distribution function by more than .01 in absolute value is less than .01. The 50, 75, 90, 95 and 99% points of QM_n are tabled in Appendix 8.4.

4.10 The Durbin Transformation

Durbin (1961) proposed that the random variables $U_{(0)}, U_{(1)}, \ldots, U_{(n+1)}$ be transformed to another set of values before a test of uniformity is performed. The Durbin transformation is made as follows. Put

$$y_i = U_{(i)} - U_{(i-1)}; \quad i = 1, \ldots, n + 1; \qquad (4.13)$$

and

$$g_i = (n+2-i)\ (y_{(i)} - y_{(i-1)}); \quad i = 1, \ldots, n + 1; \qquad (4.14)$$

where $y_{(0)} = 0$.

Next, let

$$W_r = \sum_{i=1}^{r} g_i; \quad r = 1, \ldots, n. \qquad (4.15)$$

Durbin (1961) showed that if $U_{(1)}, \ldots, U_{(n)}$ are order statistics of a sample from a $U(0,1)$ distribution, then $(W_1, \ldots, W_n)$ and $(U_{(1)}, \ldots, U_{(n)})$ are identically distributed. Quoting from Durbin:

> "It might be asked what has been gained by transforming to a set of values which have the same distribution as have the values we started with. The answer is that we hope to gain power."

Since interest here is primarily in finding statistics with good power for testing uniformity against a range of alternatives, Durbin's remark is highly relevant. Thus for each of the nine statistics described above, a new statistic was computed by first performing the Durbin transformation. This, in effect, doubles the number of test statistics studied here from nine to eighteen.

5. THE POWER STUDY

5.1 Introduction

Four members of each family of alternative distributions, corresponding to $i = 1, 2, 3, 4$ were used in this power study. This was an engineering decision based, not only on the shape of their density functions, but also on the degree of contrast between the power results of $i = 1$ and $i = 4$ in the four families. The sample sizes of 2, 5, 10, 15, 20 and 50 were chosen to span the "small sample" range as well as give an idea of medium sample properties ($n = 50$). For the power study 10,000 samples were drawn for each sample size for each of the 16 alternative distributions and the nine test statistics were computed both before and after application of the Durbin transformation to each sample. The tables in Appendix 8.7 display the percent rejected by the test statistics for the α levels of .1, .05, and .01.

5.2 The Durbin Transformation

The tables in Appendix 8.7 contain two entries for each statistic--sample size--alternative distribution combination. The left entry is the percent of samples rejected when the Durbin transformation was not used while the right entry is the percent of samples rejected when the Durbin transformation was used. The transformation did not improve power in over 72% of the tabled positions and in particular it did not improve the power of the best statistic for any alternative distribution nor boost a competitor into first place. It did not improve power for any statistic against the Type 1 family of alternatives. The transformation slightly improved the power of the Kolmogorov-Smirnov statistic, D_n, and the Cramér-von Mises statistic, W^2, for the other three families of alternatives and, in addition, slightly improved the power of A, G, Q and QM for type 2 alternatives. Therefore, it is not recommended and will not be further discussed in the following sections of this chapter. Durbin's hope that the transformation would generally increase power appears to be not realized.

5.3 Type 1 Family of Alternative Distributions

Tables 8.4 - 8.6 of Appendix 8.7 list the rejection rates of nine test statistics for the Type 1 family of alternative distributions for α levels of .1, .05 and .01. The Anderson-Darling statistic, A, in the notation of Pearson and Hartley (1972), uniformly dominates the other test statistics. No statistic suffers an extreme loss of power relative to A although X^2_{10}, X^2_{20}, and U^2 are the least preferred; G, dominated by Q and QM, is next while D and W^2 most closely approach A in terms of success in rejecting these alternative distributions.

5.4 Type 2 Family of Alternative Distributions

Tables 8.7 - 8.9 of Appendix 8.7 list the rejection rates of nine test statistics for the Type 2 family of alternative distributions. Only X^2_{10}, X^2_{20} and the dominating statistic U^2 fail to suffer a catastrophic loss of power for these alternative distributions. Of the discrete test statistics X^2_{10} is preferred to X^2_{20}. Durbin (1961) and Holm (1969) have mentioned that D is poor at detecting differences in the tails. Apparently the other statistics share the same defect, at least for this alternative family.

5.5 Type 3 Family of Alternative Distributions

The rejection rates of the test statistics for the Type 3 family of alternative distributions are presented in Tables 8.10 - 8.12 of Appendix 8.7 for α levels of .1, .05 and .01. The dominant test statistic is A, followed by QM, Q and G, then U^2, then D and W^2. The X^2 statistic is the least powerful for $n \leq 5$ but is superior to D and W^2 for $n > 5$.

5.6 Type 4 Family of Alternative Distributions

A test statistic based on sample spacings, Q, dominates the power comparisons of Tables 8.13 - 8.15 for the Type 4 family of alternative distributions. The X^2 test statistics are least powerful, then QM, D, W^2, A, U^2 and G in order of increasing power.

5.7 Comparisons Among Test Statistics

For purposes of discussion the test statistics can be divided into three groups: discrete, those based on sample spacings, and the remainder. The statistic X^2_{10} dominates X^2_{20} for Types 1, 2 and 4 alternative distributions and is not seriously inferior for the Type 3 distributions. The relationship between G, Q and QM is more complex. The statistic G is dominated by Q for alternative distributions of Types 1, 3 and 4 while the reverse is true for the Type 2 alternative distributions, where neither has appreciable power for small sample sizes. The test statistic QM, although most powerful of the three for alternative distributions of Type 3 is apparently not a consistent test statistic for Types 2 or 4 alternatives. As i increases the probability of rejection decreases for larger sample sizes. The Q statistic is preferred among these three. It is most powerful for Type 4 alternative distributions, a strong contender for Type 3 distributions and superior to G and QM for Type 1 distributions. Of the remaining four test statistics only Watson's U^2 is competitive against Type 2 alternative families. It also has good power against Type 4 distributions. The Anderson-Darling test statistic, A, dominates the comparisons for alternative distribution families of Types 1 and 3. The Kolmogorov-Smirnov statistic, D, is dominated by the Cramér-von Mises statistic, W^2, for Type 1 alternative distributions and along with W^2 has poor power for Types 2, 3 and 4 families of alternatives.

5.8 General Conclusions

For each family of alternative distributions there is one test statistic which dominates all the others, except, perhaps, for some α levels for samples of size 2. For Type 1 and Type 3 alternatives the most powerful test statistic is the Anderson-Darling statistic, A; for Type 2 it is Watson's U^2, while Q is the most powerful test statistic for alternative distributions of Type 4.

From these results it is apparent that none of the eighteen statistics is best for all alternatives. Indeed, it is felt that there is little hope that such a test statistic exists. However, it is felt that if one of these test statistics is to be used exclusively a good choice is Watson's U^2. The U^2 statistic is best for Type 2 alternatives, competitive for Type 4 alternatives for all except the smallest sample sizes (n = 2, 5) and does not suffer a catastrophic loss of power for alternative distributions of Types 1 and 3. The U^2 statistic also appears to be unbiased for all alternative distributions considered here.

It may also be noted that some pairs of statistics have complementary properties. For example, U^2 competes less well against those alternatives for which A performs well (Types 1, 3 and 4); and, conversely, when U^2 is best (Type 2), A performs poorly. Finally, with increasing sample size the power for most of the statistics increases rapidly and many of the tests give reasonable power for samples as small as ten. The paper of Stephens (1974) contains a similar power study using alternative distributions of the same general shape as the families of alternative distributions discussed here. Seven test statistics are included in the power study of Stephens although only four are represented for each sample size used for each of the seven alternative distributions. The power studies are directly comparable in that two alternative distributions and four test statistics are common to both. The power results of Stephens are based on "at least 1,000 samples" and are reported for $\alpha = .1$. The $i = 1$ member of the Type 2 family of alternatives is common to both power studies for samples of size 10 and 20.

TABLE 5.1

Comparison with Stephen's Results for Type 2, $i = 1$ Alternative Distribution

Percent of Samples Rejected

	Sample Size	D	W^2	U^2	A
Stephens	10	9	7	44	6
Miller	10	11	9	47	5
Stephens	20	25	25	77	28
Miller	20	27	27	78	28

The $i = 1$ member of the Type 4 family of alternative distributions is also common to both power studies for samples of size 20.

TABLE 5.2

Comparison with Stephen's Results for Type 4, $i = 1$ Alternative Distribution

Percent of Samples Rejected

	Sample Size	D	W^2	U^2	A
Stephens	20	47	44	77	54
Miller	20	47	41	78	54

The agreement between the two power studies is quite good and U^2 also emerges from Stephens' power study as the statistic of choice.

6. APPLICATIONS

6.1 Introduction

From the foregoing results it is felt that Watson's U^2 statistic composed with the conditional probability integral transformation (CPIT-U^2) will provide a reasonable goodness-of-fit test for a large class of statistical models. The simple linear regression model is important in practice and the CPIT-U^2 method of testing the model will be illustrated here using data from examples in well-known textbooks.

6.2 The Transform for Simple Linear Regression

The conditional probability integral transform for simple linear regression is of the following form. Define $X_j' = \begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_j \end{bmatrix}$, $\underline{x}_j' = (1 \; x_j)$, $\underline{y}_j' = (y_1 \ldots y_j)$, and $\underline{t}_j' = \left(\sum_{i=1}^{j} y_i \;\; \sum_{i=1}^{j} x_i \, y_i \right)$; $j = 4, \ldots, N$. Under the null hypothesis that the y_j's are normally and independently distributed with mean $\beta_o + \beta_1 x_j$ and variance σ^2; β_o, β_1 and σ^2 assumed unknown; O'Reilly (1971) showed that

$$V_j = \frac{(j-3)^{\frac{1}{2}} \left(y_j - \underline{x}_j' \, (X_j' \, X_j)^{-1} \, \underline{t}_j \right)}{\left\{ \left[1 - x_j' \, (X_j' \, X_j)^{-1} \, \underline{x}_j \right] S_j - \left[y_i - \underline{x}_j' \, (X_j' \, X_j)^{-1} \, \underline{t}_j \right]^2 \right\}^{\frac{1}{2}}}, \tag{6.1}$$

where $S_j = \underline{y}_j' \left[I - X_j \, (X_j' \, X_j)^{-1} \, X_j' \right] \underline{y}_j$; is distributed as student's t with $(j-3)$ degrees of freedom for $j = 4, \ldots, N$. Let

$$U_j = \frac{\Gamma[(j-2)/2]}{[\pi\ (j-3)]^{\frac{1}{2}}\ \Gamma[(j-3)/2]} \int_{-\infty}^{V_j} [1 + t^2/(j-2)]^{-\frac{j-2}{2}}\, dt;\ j = 4,\ \ldots,\ N. \qquad (6.2)$$

If the null hypothesis mentioned above holds, then the U_j; $j = 4, \ldots, N$; are i.i.d. $U(0,1)$. A subroutine to calculate the V_j, $j = 4, \ldots, N$ is listed in Appendix 8.8.

6.3 Examples

Seven sets of data, used as examples of observations following the simple linear hypothesis, were selected and subjected to this conditional probability integral transformation. The uniformity of the resulting numbers was tested using Watson's U^2 statistic as modified by Stephens (1974).

6.3.1 Yield of Dressed Grain as a Function of Year

Fisher (1963) used the difference between two types of fertilizer in yield per acre of dressed grain as a function of year of harvest for 30 consecutive years as an example of simple linear regression. The CPIT-U^2 statistic computed from equations (6.1) and (6.2) for this set of data is .145. The critical value for $\alpha = .1$ is .152 so the model assumed by Fisher seems reasonable.

6.3.2 Food Consumption of White Leghorn Hens

Steel and Torrie (1960) used the food consumption of 50 white leghorn hens as a function of their average body weight for ten strains as an example of simple linear regression. The CPIT-U^2 statistic for this set of data is .112. The critical value for $\alpha = .1$ is .152 so the model assumed seems reasonable.

6.3.3 Horses on Canadian Farms as a Function of Year

Steel and Torrie also used the number of horses on Canadian farms as a function of the year for a five-year period as an example of simple linear regression. The CPIT-U^2 statistic for this set of data is .082. The critical value for $\alpha = .1$ is .164 so the model assumed by Steel and Torrie seems reasonable.

6.3.4 Wormy Apples as a Function of Crop Size

Snedecor and Cochran (1967) used the percentage of wormy apples as a function of crop size for 12 trees as an example of simple linear regression. The CPIT-U^2 statistic for this set of data is .141. The critical value for $\alpha = .1$ is .152 so the model assumed seems reasonable.

6.3.5 Gain in Weight as a Function of Initial Weight in Rats

Snedecor and Cochran also used the gain in weight as a function of initial weight for 15 female rats as an example of simple linear regression. The CPIT-U^2 statistic for this set of data is .182. The critical value for $\alpha = .05$ is .187 so the model assumed seems reasonable.

6.3.6 Lobster Catch per Unit of Effort

Anderson and Bancroft (1952) proposed that a simple linear regression model be used to relate total lobster catch per unit of effort for 17 time intervals. The CPIT-U^2 statistic for this set of data is .188.

The critical value for $\alpha = .05$ is .187 so the model does not seem reasonable. It is possible that the decrease in variability with increase in catch caused the rejection.

6.3.7 Disposable Income as a Function of per Capita Income

Anderson and Bancroft also suggested a simple linear regression model for the relationship between disposable income and per capita income, both adjusted for the cost of living, for 20 years. The CPIT-U^2 statistic for this set of data is .285. The critical value for $\alpha = .01$ is .267 so the model does not seem to be appropriate. It is possible that an extreme data point caused the rejection.

7. LIST OF REFERENCES

Anderson, R. L. and T. A. Bancroft. 1952. Statistical Theory in Research. McGraw-Hill, New York.

Anderson, T. W. and D. A. Darling. 1952. Asymptotic Theory of Certain "Goodness of Fit" Criteria Based on Stochastic Processes. Ann. Math. Stat. 23:193-212.

Anderson, T. W. and D. A. Darling. 1952. A Test of Goodness of Fit. Jour. Am. Stat. Assn. 49:765-769.

Bartholomew, D. J. 1954. Note on the Use of Sherman's Statistic as a Test for Randomness. Biometrika 41:556-558.

Bartholomew, D. J. 1956. A Sequential Test for Randomness of Intervals. Jour. Roy. Stat. Soc. (B) 18:95-103.

Bartholomew, D. J. 1957. Testing for Departure from the Exponential Distribution. Biometrika 44:253-257.

Barton, D. E. and F. N. David. 1955. Sums of Ordered Intervals and Distances. Mathematika 2:150-159.

Barton, D. E. and F. N. David. 1956a. Some Notes on Ordered Random Intervals. Jour. Roy. Stat. Soc. (B) 18:79-94.

Barton, D. E. and F. N. David. 1956b. Tests for Randomness of Points on a Line. Biometrika 43:104-112.

Blumenthal, S. 1966a. Contributions to Sample Spacings Theory, I: Limit Distributions of Sums of Ratios of Spacings. Ann. Math. Stat. 37:904-924.

Blumenthal, S. 1966b. Contributions to Sample Spacings Theory, II: Tests of the Parametric Goodness of Fit and Two Sample Problems. Ann. Math. Stat. 37:925-939.

Blumenthal, S. 1967. Limit Theorems for Functions of Shortest Two-Sample Spacings and a Related Test. Ann. Math. Stat. 38:108-116.

Cochran, W. G. 1952. The X^2 Test of Goodness of Fit. Ann. Math. Stat. 23:315-345.

Cramér, H. 1928. On the Composition of Elementary Errors. Skandinavisk Aktuarietidskrift 11:13-74 and 141-180.

Darling, D. A. 1953. On a Class of Problems Related to the Random Division of an Interval. Ann. Math. Stat. 24:239-253.

Darling, D. A. 1957. The Kolmogorov-Smirnov, Cramér-von Mises Tests. Ann. Math. Stat. 28:823-838.

David, F. N. and N. L. Johnson. 1948. The Probability Integral Transformation when Parameters are Estimated from the Sample. Biometrika 35:182-192.

Durbin, J. 1961. Some Methods of Constructing Exact Tests. Biometrika 48:41-55.

Fisher, R. A. 1929. Tests of Significance in Harmonic Analysis. Proc. Roy. Soc. (A) 125:54-59.

Fisher, R. A. 1963. Statistical Methods for Research Workers. Oliver and Boyd, Edinburgh, 13th edition.

Gardner, A. 1952. Greenwood's "Problem of Intervals"; An Exact Solution for N = 3. Jour. Roy. Stat. Soc. (B) 14:135-139.

Greenwood, M. 1946. The Statistical Study of Infectious Diseases (with discussion). Jour. Roy. Stat. Soc. 109:85-110.

Holm, S. 1969. On Probability Paper Tests. Skandinavisk Aktuarietidskrift 52:1-19.

Kimball, B. F. 1947. Some Basic Theorems for Developing Tests of Fit for the Case of the Non-Parametric Probability Function, I. Ann. Math. Stat. 18:540-548.

Kimball, B. F. 1950. On the Asymptotic Distribution of the Sum of Powers of Unit Frequency Differences. Ann. Math. Stat. 21:263-271.

Kolmogorov, A. 1933. Sulla Determinazione Empirica Di Una Legge Di Distribuzione. Giornale Dell Istituto Italiano Degli Attuardi 4:83-91.

LaPlace, P. S. 1814. Theorie Analytique Des Probabilites. Paris, 2nd edition.

Lehmer, D. H. 1964. Applied Combinatorial Mathematics, edited by E. F. Beckenbach. Wiley, New York.

Lilliefors, H. W. 1967. On the Kolmogorov-Smirnov Test for Normality with Mean and Variance Unknown. Jour. Am. Stat. Assn. 62:399-402.

Lilliefors, H. W. 1969. On the Kolmogorov-Smirnov Test for the Exponential Distribution with Mean Unknown. Jour. Am. Stat. Assn. 64:387-389.

Marshall, A. W. 1958. The Small Sample Distribution of n W^2. Ann. Math. Stat. 29:307-309.

Massey, F. J. 1951. The Kolmogorov-Smirnov Test for Goodness of Fit. Jour. Am. Stat. Assn. 46:68-78.

Mauldon, J. G. 1951. Random Division of an Interval. Proc. Camb. Phil. Soc. 47:331-336.

Miller, L. H. 1956. Table of Percentage Points of Kolmogorov Statistics. Jour. Am. Stat. Assn. 51:111-121.

Moran, P. A. P. 1947. The Random Division of an Interval. Jour. Roy. Stat. Soc. (B) 9:92-98.

Moran, P. A. P. 1951. The Random Division of an Interval--Part II. Jour. Roy. Stat. Soc. (B) 13:147-150.

Moran, P. A. P. 1953. The Random Division of an Interval--Part III. Jour. Roy. Stat. Soc. (B) 15:77-80.

O'Reilly, F. J. 1971. On Goodness-of-Fit Tests Based on Rao-Blackwell Distribution Function Estimators. Ph.D. thesis, Department of Statistics, North Carolina State University at Raleigh. University Microfilms, Ann Arbor, Michigan.

O'Reilly, F. J. and C. P. Quesenberry. 1973. The Conditional Probability Integral Transformation and Applications to Obtain Composite Chi-Square Goodness-of-Fit Tests. Ann. Stat. 1:74-83.

Pearson, E. S. and H. O. Hartley. 1972. Biometrika Tables for Statisticians, Vol. 2. Cambridge University Press, Cambridge.

Pearson, K. 1900. On the Criterion that a Given System of Deviations from the Probable in the Case of a Correlated System of Variables is Such That It Can Be Reasonably Supposed to Have Arisen from Random Sampling. Philos. Mag. Series 5, 50:157-172.

Proschan, F. and R. Pyke. 1962. General Limit Theorems for Spacings. Ann. Math. Stat. 33:307.

Pyke, R. 1965. Spacings (with discussion). Jour. Roy. Stat. Soc. (B) 27:395-449.

Sherman, B. 1950. A Random Variable Related to Spacings of Sample Values. Ann. Math. Stat. 21:339-361.

Snedecor, G. W. and W. G. Cochran. 1967. Statistical Methods. Iowa State University Press, Ames, 6th edition.

Steel, R. G. D. and J. H. Torrie. 1960. Principles and Procedures of Statistics, McGraw-Hill, New York.

Stephens, M. A. 1974. EDF Statistics for Goodness of Fit and Some Comparisons. Jour. Am. Stat. Assn. 69:730-737.

Uppuluri, V. R. R. and J. A. Carpenter. 1969. The Inverse of a Matrix Occurring in First-Order Moving-Average Models. Sankhya A 31:79-82.

von Mises, R. 1931. Wahrscheinlichkeitsrechnung. Leipzig-Wien.

Watson, G. S. 1961. Goodness-of-Fit Tests on a Circle. Biometrika 48:109-114.

Weiss, L. 1955. The Stochastic Convergence of a Function of Sample Successive Differences. Ann. Math. Stat. 26:532-536.

Weiss, L. 1956. A Certain Class of Tests of Fit. Ann. Math. Stat. 27:1165-1170.

Weiss, L. 1957a. The Convergence of Certain Functions of Sample Spacings. Ann. Math. Stat. 28:778-782.

Weiss, L. 1957b. The Asymptotic Power of Certain Tests of Fit Based on Sample Spacings. Ann. Math. Stat. 28:783-786.

Weiss, L. 1957c. Tests of Fit in the Presence of Nuisance Location and Scale Parameters. Ann. Math. Stat. 38:1016-1020.

Weiss, L. 1958. Limiting Distributions of Homogeneous Functions of Sample Spacings. Ann. Math. Stat. 29:310-312.

Weiss, L. 1959. The Limiting Joint Distribution of the Largest and Smallest Sample Spacings. Ann. Math. Stat. 30:590-593.

Weiss, L. 1961. Tests of Fit Based on the Number of Observations Falling in the Shortest Sample Spacings Determined by Earlier Observations. Ann. Math. Stat. 32:838-845.

Whitworth, W. A. 1887. Choice and Chance. Cambridge University Press, Cambridge, 3rd edition.

Wilks, S. S. 1962. Mathematical Statistics. Wiley, New York.

APPENDIX 8.1

Random Number Generation

The algorithm used to produce U(0,1) random numbers for the power study and evaluation of the critical values of the test statistics is of the multiplicative congruential type. The generator is 5^{15} (mod 2^{32}). This generator has a period of 2^{46} and no cycle smaller than 2^{46}.

Let $\lambda = 5^{15}$ (mod 2^{32}), $P = 2^{48}$ and $t_o = 1$. Then

$$t_{n+1} = \lambda\ t_n \pmod{P}. \qquad (8.1)$$

Now $t_n = P\left\{\frac{\lambda^n}{P}\right\}$ where $\{x\}$ is the fractional part of x and $U_n = \left\{\frac{\lambda^n}{P}\right\}$ where U_n is the nth pseudo random number drawn from U(0,1).

About 1.5×2^{27} random numbers were used, none more than once. It took 55,323,000 random deviates to estimate the critical values of each of G_n, Q_n and QM_n; 23,085,000 were used to estimate the repeatability of some Q_n critical values and 17,340,000 were used in the power study.

The algorithm used to generate the random numbers is listed below.

Random Number Algorithm

```
FLTRN    START
*        FUNCTION FLTRNF(0)
*  PROGRAM AUTHOR  J. G. SULLIVAN
*  OAK RIDGE NATIONAL LABORATORY, OAK RIDGE, TENNESSEE
         USING *,15
         STM   0,4,SAVE
         L     4,Q
         USING RANDOM,4
         L     1,RANDOM+4
         M     0,GENERA
         LTR   0,0
         BC    10,PLUS
         A     0,GENERA
PLUS     ST    1,RANDOM+4
         ST    0,SAVE
         L     1,RANDOM
         M     0,GENERA
         AL    1,SAVE
         STH   1,RANDOM+2
         LD    0,ZERO
         AD    0,RANDOM
         LM    0,4,SAVE
         BR    14
Q        DC    A(RANDOM)
         DS    0D
ZERO     DC    X'4200000000000000'
SAVE     DS    7F
RNDOM    CSECT
RANDOM   DC    X'420000071AFD498D'
GENERA   DC    X'1AFD498D'
         END
```

APPENDIX 8.2

Critical Values of G_n

The critical values of the test statistic G_n listed in Table 8.1 are percentage points of empirical distributions obtained by Monte Carlo methods. The behavior of some critical values, as a function of n, is shown in Figure 8.1. The description of G_2 was partially tabled by Greenwood (1946). The values obtained by Monte Carlo are compared below with those given by Greenwood.

	$P(G_2 \leq g_2)$	
g_2	Greenwood	Monte Carlo
.4	.243	.246
.4667	.486	.483
.533	.692	.691
.6	.805	.805
.667	.879	.880
.733	.929	.931
.8	.964	.965

The distribution of G_3 was partially tabulated by Gardner (1952). The values obtained by Monte Carlo are compared below with those obtained by Gardner.

g_3	$P(G_3 \le g_3)$ Gardner	Monte Carlo
.2875	.091	.092
.325	.258	.26
.3333	.301	.30
.4	.600	.600
.475	.804	.804
.5	.8483	.845
.525	.881	.878
.55	.922	.905
.614	.944	.948
.7	.976	.98
.8	.984	.995
.9	.996	> .999

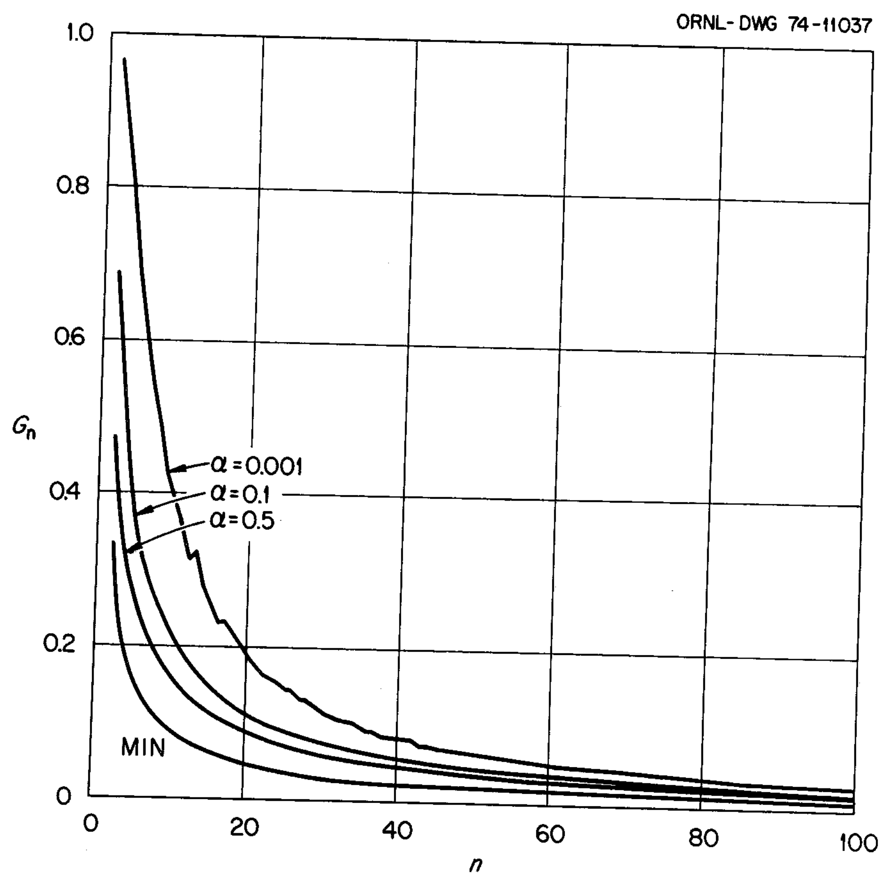

Figure 8.1 Critical Values of G

TABLE 8.1

Upper Percentage Points of G_n

n	α = .5	α = .25	α = .1	α = .05	α = .01
2	.471	.563	.688	.767	.892
3	.375	.450	.546	.617	.754
4	.313	.373	.449	.507	.639
5	.268	.318	.382	.430	.541
6	.236	.278	.331	.371	.467
7	.210	.247	.293	.329	.409
8	.189	.221	.262	.294	.369
9	.173	.201	.237	.265	.332
10	.158	.184	.215	.241	.301
11	.146	.169	.198	.219	.276
12	.136	.156	.182	.201	.247
13	.127	.146	.170	.188	.233
14	.120	.137	.158	.175	.215
15	.113	.129	.148	.164	.203
16	.107	.122	.139	.153	.185
17	.101	.115	.132	.145	.178
18	.096	.109	.125	.136	.167
19	.092	.104	.118	.129	.156
20	.088	.099	.112	.123	.147
21	.084	.095	.107	.117	.141
22	.081	.091	.103	.112	.135
23	.077	.087	.099	.107	.128

TABLE 8.1 (continued)

n	α = .5	α = .25	α = .1	α = .05	α = .01
24	.074	.083	.095	.102	.123
25	.072	.080	.090	.098	.117
26	.069	.077	.087	.094	.112
27	.067	.075	.084	.091	.108
28	.065	.072	.081	.087	.103
29	.062	.070	.078	.084	.100
30	.061	.068	.075	.082	.096
31	.059	.065	.073	.078	.092
32	.057	.063	.071	.076	.088
33	.056	.062	.069	.074	.086
34	.054	.060	.066	.071	.084
35	.053	.058	.065	.069	.081
36	.051	.057	.063	.067	.078
37	.050	.055	.061	.065	.076
38	.049	.054	.059	.064	.074
39	.048	.052	.058	.062	.071
40	.046	.051	.056	.060	.069
41	.045	.050	.055	.059	.068
42	.044	.049	.054	.057	.066
43	.044	.048	.052	.056	.064
44	.043	.047	.051	.055	.063
45	.042	.046	.050	.053	.061
46	.041	.045	.049	.052	.059

TABLE 8.1 (continued)

n	α = .5	α = .25	α = .1	α = .05	α = .01
47	.040	.044	.048	.051	.058
48	.039	.043	.047	.050	.057
49	.038	.042	.046	.049	.055
50	.038	.041	.045	.048	.055
55	.034	.037	.041	.043	.049
60	.032	.034	.037	.039	.044
65	.029	.032	.034	.036	.040
70	.027	.030	.032	.034	.038
75	.026	.028	.030	.031	.035
80	.024	.026	.028	.029	.032
85	.023	.024	.026	.027	.030
90	.021	.023	.025	.026	.028
95	.020	.022	.023	.024	.027
100	.019	.021	.022	.023	.025

APPENDIX 8.3

Critical Values of Q_n

The critical values of the test statistic Q_n in Table 8.2 are percentage points of empirical distributions of Q_n obtained by Monte Carlo methods. Nine simulations were made of the distribution of Q_n for n = 5, 20 and 70 to check the repeatability of the critical values. Estimated standard deviations of some critical values are tabled below.

Standard Deviation of a Critical Value

n	α = .5	α = .25	α = .1	α = .05	α = .01
5	.00037	.00040	.00129	.00109	.00260
20	.00015	.00017	.00041	.00064	.00064
70	.00003	.00003	.00004	.00005	.00014

The precision is worst for small n and small α. Even there it seems to be relatively good. For example, for $\alpha = 5$ the averages of the nine 98, 99 and 99.5% points are .60792, .64682 and .68094; at least 13 standard deviations of an individual 99% point from each other.

As with G_n the critical values of Q_n decrease with increasing n, as shown in Figure 8.2.

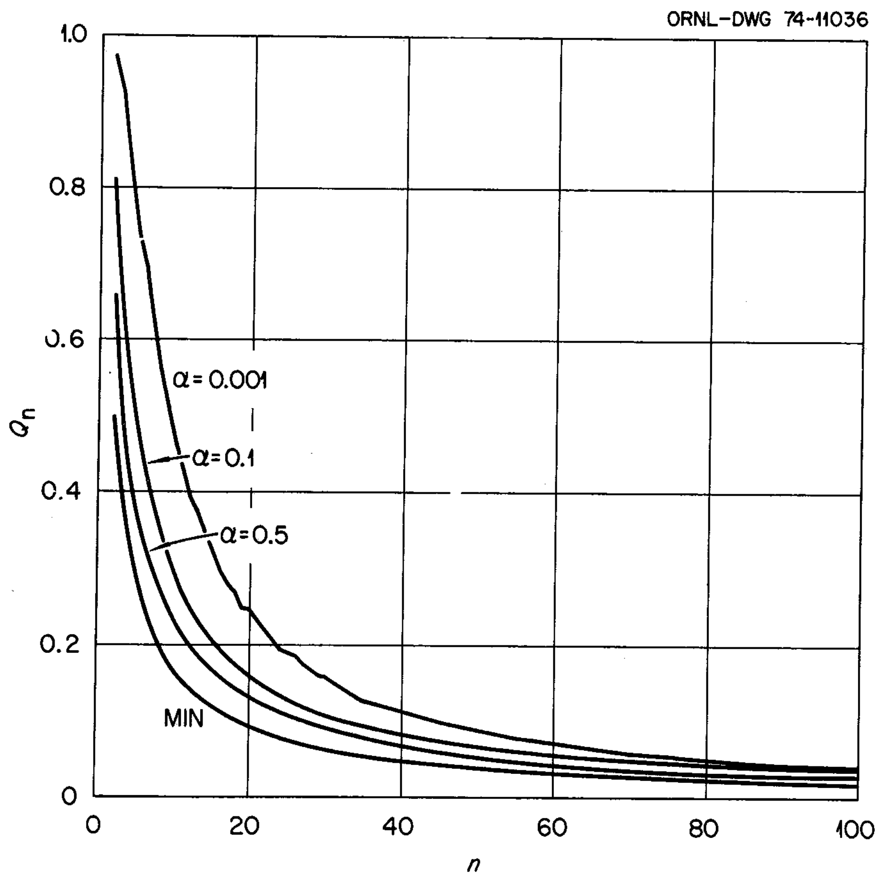

Figure 8.2 Critical Values of Q

TABLE 8.2

Upper Percentage Points of Q_n

n	α = .5	α = .25	α = .1	α = .05	α = .01
2	.658	.739	.811	.859	.932
3	.530	.608	.691	.736	.831
4	.447	.510	.586	.635	.727
5	.388	.440	.505	.551	.642
6	.343	.387	.442	.483	.573
7	.307	.346	.393	.429	.512
8	.278	.311	.355	.387	.463
9	.254	.284	.322	.350	.423
10	.234	.261	.294	.319	.378
11	.217	.242	.272	.294	.351
12	.202	.224	.251	.272	.318
13	.189	.209	.234	.253	.298
14	.177	.197	.219	.237	.279
15	.168	.185	.206	.222	.259
16	.159	.175	.195	.209	.245
17	.150	.165	.184	.197	.230
18	.143	.157	.174	.187	.218
19	.137	.150	.166	.177	.206
20	.131	.143	.158	.168	.196
21	.125	.137	.151	.162	.187
22	.120	.131	.144	.154	.178
23	.115	.126	.138	.147	.169

TABLE 8.2 (continued)

n	α = .5	α = .25	α = .1	α = .05	α = .01
24	.111	.121	.133	.141	.163
25	.107	.117	.128	.136	.156
26	.103	.112	.123	.131	.148
27	.100	.109	.118	.126	.144
28	.097	.105	.114	.121	.138
29	.094	.101	.111	.117	.134
30	.091	.098	.107	.114	.129
31	.088	.095	.104	.110	.125
32	.086	.093	.101	.106	.120
33	.083	.088	.097	.103	.116
34	.081	.087	.094	.100	.112
35	.079	.085	.092	.097	.109
36	.077	.082	.090	.095	.107
37	.075	.080	.087	.092	.103
38	.073	.079	.085	.089	.100
39	.071	.076	.083	.087	.097
40	.070	.075	.081	.085	.095
41	.068	.073	.079	.083	.092
42	.067	.071	.077	.081	.090
43	.065	.070	.075	.079	.088
44	.064	.068	.073	.077	.086
45	.062	.067	.072	.075	.084
46	.061	.065	.070	.074	.082

TABLE 8.2 (continued)

n	α = .5	α = .25	α = .1	α = .05	α = .01
47	.060	.064	.069	.072	.080
48	.059	.063	.067	.070	.078
49	.058	.061	.066	.069	.077
50	.056	.060	.064	.068	.075
55	.052	.055	.059	.061	.068
60	.047	.050	.054	.056	.062
65	.044	.047	.050	.052	.057
70	.041	.043	.046	.048	.052
75	.038	.041	.043	.045	.048
80	.036	.038	.040	.042	.045
85	.034	.036	.038	.039	.042
90	.032	.034	.036	.037	.040
95	.031	.032	.034	.035	.038
100	.029	.031	.032	.033	.036

APPENDIX 8.4

Critical Values of QM_n

The critical values of the test statistic QM_n in Table 8.3 are percentage points of empirical distributions obtained by Monte Carlo methods. As Figure 8.3 shows, the structure of QM_n induces discontinuities in the critical values over n. The critical values for n of 71, 96 and 97 in Figure 8.3 are extrapolations. Had QM_n been an attractive statistic for large n these values would also have been estimated by Monte Carlo methods.

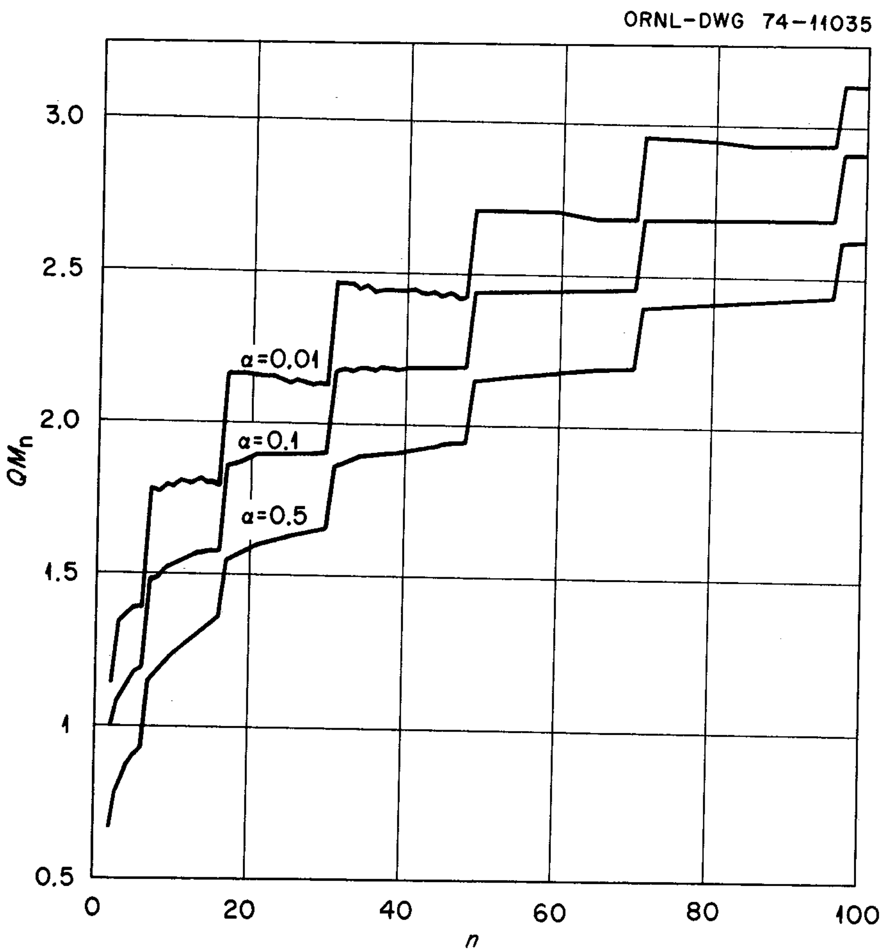

Figure 8.3 Critical Values of QM

TABLE 8.3

Upper Percentage Points of QM_n

n	α = .5	α = .25	α = .1	α = .05	α = .01
2	.658	.863	1.003	1.064	1.147
3	.790	.954	1.095	1.195	1.343
4	.857	1.006	1.137	1.226	1.370
5	.905	1.046	1.174	1.252	1.393
6	.936	1.068	1.187	1.261	1.389
7	1.151	1.314	1.480	1.581	1.780
8	1.179	1.337	1.489	1.589	1.777
9	1.217	1.369	1.522	1.616	1.798
10	1.236	1.383	1.528	1.625	1.797
11	1.258	1.403	1.544	1.634	1.808
12	1.276	1.417	1.549	1.635	1.800
13	1.295	1.433	1.566	1.651	1.810
14	1.307	1.443	1.567	1.652	1.813
15	1.321	1.449	1.575	1.653	1.799
16	1.329	1.454	1.573	1.650	1.795
17	1.546	1.703	1.861	1.965	2.164
18	1.560	1.715	1.868	1.965	2.165
19	1.572	1.724	1.873	1.968	2.165
20	1.582	1.734	1.883	1.974	2.158
21	1.596	1.741	1.883	1.975	2.157
22	1.600	1.749	1.891	1.978	2.152
23	1.611	1.755	1.891	1.977	2.155

TABLE 8.3 (continued)

n	α = .5	α = .25	α = .1	α = .05	α = .01
24	1.618	1.757	1.891	1.978	2.146
25	1.624	1.762	1.890	1.976	2.132
26	1.630	1.767	1.898	1.982	2.141
27	1.636	1.771	1.899	1.978	2.139
28	1.643	1.775	1.901	1.979	2.128
29	1.648	1.777	1.903	1.979	2.134
30	1.655	1.782	1.902	1.980	2.129
31	1.867	2.023	2.174	2.272	2.469
32	1.871	2.026	2.176	2.267	2.469
33	1.878	2.031	2.175	2.275	2.464
34	1.882	2.031	2.177	2.268	2.445
35	1.889	2.038	2.181	2.267	2.454
36	1.889	2.035	2.176	2.264	2.441
37	1.898	2.045	2.187	2.275	2.445
38	1.899	2.045	2.185	2.275	2.440
39	1.904	2.044	2.180	2.262	2.431
40	1.911	2.054	2.186	2.269	2.435
41	1.913	2.050	2.185	2.273	2.448
42	1.916	2.054	2.186	2.270	2.431
43	1.919	2.057	2.189	2.274	2.429
44	1.922	2.058	2.189	2.269	2.434
45	1.930	2.062	2.193	2.272	2.426
46	1.930	2.064	2.191	2.271	2.429

TABLE 8.3 (continued)

n	α = .5	α = .25	α = .1	α = .05	α = .01
47	1.930	2.063	2.190	2.269	2.422
48	1.935	2.064	2.188	2.267	2.418
49	2.131	2.293	2.440	2.534	2.723
50	2.143	2.295	2.444	2.536	2.717
55	2.157	2.302	2.443	2.539	2.708
60	2.171	2.314	2.448	2.532	2.708
65	2.180	2.316	2.447	2.529	2.678
70	2.190	2.322	2.446	2.527	2.678
75	2.392	2.542	2.688	2.776	2.950
80	2.401	2.543	2.684	2.772	2.945
85	2.407	2.548	2.684	2.765	2.926
90	2.415	2.552	2.685	2.771	2.934
95	2.422	2.556	2.685	2.763	2.921
100	2.612	2.760	2.901	2.986	3.158

APPENDIX 8.5

Enumeration of Partitions for the X^2 Statistic

It is necessary to enumerate the ways in which n objects can be put into k cells in order to compute the probability of obtaining a given X_k^2 for a sample of size n. It is sufficient to list the distinct partitions weighted by the number of ways they can occur since $p_1 = \dots = p_k = 1/k$. The systematic generation of the distinct arrangements is accomplished by an algorithm due to Lehmer (1964). It is contained in the subroutine PARTIT listed below. The arguments of PARTIT are n, the number of objects, and k, the number of cells. DLN, the logarithms of factorials, is carried from the main program to CHISQV, which computes the probability of each partition. CHISQV is also listed below.

The power study used sample sizes of 2, 5, 10, 15, 20 and 50. For n of 2, 5, 10, 15 and 20 the distinct partitions were enumerated and their probabilities computed. The partitions were ordered on their X^2 values. Critical values were chosen from a listing of the X^2 values and cumulative probabilities. Because of the discreteness of X^2, tests of exactly α ($\alpha = .1$, $.05$ and $.01$) rarely occurred so a randomization strategy was used to obtain exact size α tests to compare with the other statistics. For example, for $n = 6$ and $k = 10$ the following table was produced.

Partition	X^2_{10}	Probability	Cumulative Probability
1 1 1 1 1 1 0 0 0 0	4.	.1512	.1512
2 1 1 1 1 0 0 0 0 0	7.333	.4536	.6048
2 2 1 1 0 0 0 0 0 0	10.667	.2268	.8316
2 2 2 0 0 0 0 0 0 0	14.	.0108	.8424
3 1 1 1 0 0 0 0 0 0	14.	.1008	.9432
3 2 1 0 0 0 0 0 0 0	17.333	.0432	.9864
3 3 0 0 0 0 0 0 0 0	24.	.0009	.9873
4 1 1 0 0 0 0 0 0 0	24.	.0108	.9981
4 2 0 0 0 0 0 0 0 0	27.333	.00135	.99945
5 1 0 0 0 0 0 0 0 0	37.333	.00054	.99999
6 0 0 0 0 0 0 0 0 0	54.	.00001	1.

Let N_A be the number of samples drawn (in a power comparison) with X^2 values ≤ 10.667 and let N_B be the number ≤ 14. Then

$$N_{.1} = N_A + (N_B - N_A)\left(\frac{.9 - .8316}{.9432 - .8316}\right)$$

is the randomization estimate of the number of samples rejected by an $\alpha = .1$ test.

The number of distinct arrangements of n things in 10 or 20 urns increases rapidly as n increases.

		Number of Distinct Arrangements	
		k = 10	k = 20
n Objects	10	42	42
	20	530	627
	30	3,590	5,507
	40	16,928	35,251
	50	62,740	181,271
	60	195,491	791,131
	70	533,975	3,034,232
	80	1,314,972	10,474,462
	90	2,977,866	
	100	6,292,069	

For n = 50 and k = 10 and 20 the enumeration problem is formidable so the critical values were obtained by simulation. Ten thousand random samples of size 50 were drawn from U(0,1) and their X^2 values were ordered and percentage points were printed out. A randomization procedure was still required to obtain exact α tests.

```
      SUBROUTINE PARTIT(N,K,DLN)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION II(100),DLN(200)
      M = 1
      DO 1 I=1,K
    1 II(I) = 0
    2 J = M - 1
      IF(J)3,3,4
    3 II(1) = N
      GO TO 8
    4 DO 5 I=1,J
    5 II(I) = 1
    6 IS = 0
      DO 7 I=1,J
    7 IS = IS + II(I)
      II(M) = N - IS
    8 CALL CHISQV(N,K,M,II,DLN)
      L = M - 1
    9 IF(L)14,14,10
   10 IF(II(M) - II(L) - 1)11,11,12
   11 L = L - 1
      GO TO 9
   12 JJ = M - 1
      IKK = II(L) + 1
      DO 13 I=L,JJ
   13 II(I) = IKK
      GO TO 6
   14 M = M + 1
      IF(M - K)15,15,16
   15 IF(M - N)2,2,16
   16 RETURN
      END PARTIT
```

```
      SUBROUTINE CHISQV(N,K,M,II,DLN)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION II(100), A(100), DLN(200), IJ(100)
      PROB = 0.0
      DO 1 I=1,K
      IJ(I) = II(I)
    1 A(I) = II(I)
      AN = N
      AK = K
      EXP = AN/AK
      XSQ = 0.0
      DO 2 I=1,K
    2 XSQ = XSQ + (A(I) - EXP)**2
      XSQ = XSQ/EXP
      PROB = DLN(N) + DLN(K) - AN*DLOG(AK)
      DO 3 I=1,M
      IKL = IJ(I)
    3 PROB = PROB - DLN(IKL)
      IF(K - M)5,5,4
    4 IKL = K - M
      PROB = PROB - DLN(IKL)
      IF(M - 1) 12, 12, 5
    5 I = 1
      KL = M - 1
      KM = M
    6 J = I + 1
      IKL = 1
    7 IF(IJ(I) - IJ(J))8,9,8
    8 J = J + 1
      GO TO 11
    9 IKL = IKL + 1
      DO 10 IK=J,KL
      JK = IK + 1
   10 IJ(IK) = IJ(JK)
      KL = KL - 1
      KM = KM - 1
      IF(J - KM)7,7,11
   11 PROB = PROB - DLN(IKL)
      I = I + 1
      IF(I - KM)6, 12, 12
   12 PROB = DEXP(PROB)
      WRITE(51,13) XSQ,PROB,(II(I),I=1,M)
   13 FORMAT(1H ,2E20.10,8X,20I4/1H ,32I4/1H ,32I4/,1H ,16I4)
      RETURN
      END CHISQV
```

APPENDIX 8.6

Attaining the Minimum Q_n

The test statistic Q_n does not attain its minimum value when the $U_1, \ldots, U_n$ cut off equal spacings. Let

$$K = \sum_{i=1}^{n+1} Y_i^2 + \sum_{i=1}^{n} Y_i \, Y_{i+1} - \lambda\left[\sum_{i=1}^{n+1} Y_i - 1\right] \qquad (8.2)$$

where $Y_i = U_{(i)} - U_{(i-1)}$, $U_{(0)} = 0$ and $U_{(n+1)} = 1$.

To find the minimum we take the partial derivatives of k with respect to the Y_i and set them equal to zero.

$$\frac{\partial k}{\partial \underline{Y}} = \begin{bmatrix} 2 & 1 & 0 & . & . & . & 0 \\ 1 & 2 & 1 & & & & \\ 0 & 1 & 2 & 1 & & & \\ . & & & . & & & \\ . & & & & . & & \\ . & & & & & . & 1 \\ 0 & & & & & 1 & 2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ . \\ . \\ . \\ Y_n \end{bmatrix} - \lambda \begin{bmatrix} 1 \\ 1 \\ 1 \\ . \\ . \\ . \\ 1 \end{bmatrix} = A\underline{Y} - \lambda\underline{1} = 0 \qquad (8.3)$$

or

$$A\underline{Y} = \lambda \underline{1} \quad (8.4)$$

Therefore

$$\underline{Y} = \lambda A^{-1}\underline{1} \quad (8.5)$$

We know that

$$\sum_{i=1}^{n+1} Y_i = 1 \quad (8.6)$$

Thus

$$\underline{1}'\underline{Y} = \lambda\ \underline{1}'\ A^{-1}\underline{1} \quad (8.7)$$

or

$$\lambda = 1/1'\ A^{-1}\underline{1} \quad (8.8)$$

and

$$\underline{\lambda} = A^{-1}\underline{1}/\underline{1}'\ A^{-1}\underline{1} \quad (8.9)$$

Now from Uppuluri and Carpenter (1969) we know that

$$A^{-1} = \begin{bmatrix} n+1 & -n & (n-1) & -(n-2) & \cdots & & \pm 1 \\ -n & 2n & -2(n-1) & 2(n-2) & & & \pm 2 \\ (n-1) & -2(n-1) & 3(n-1) & -3(n-2) & & & \pm 3 \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \pm 2 & & & & & 2n & -n \\ \pm 1 & \cdots & & & & -n & n+1 \end{bmatrix} \qquad (8.10)$$

The (1, n+1) element is positive if n is an even integer and negative if n is odd. The sums of the rows of n have a closed form. Let $K = n + 1$. Then

(K+1) x (SUM OF ROWS)

Row	General Formula	K Even	K Odd
1	$1\left[\frac{K+1}{2}\right] - (K)\left[\frac{1}{2}\right]$	$\frac{K}{2}$	$\frac{K+1}{2}$
2	$2\left[\frac{K}{2}\right] - (K-1)\left[\frac{2}{2}\right]$	1	0
3	$3\left[\frac{K-1}{2}\right] - (K-2)\left[\frac{3}{2}\right]$	$\frac{K-2}{2}$	$\frac{K+1}{2}$
4	$4\left[\frac{K-2}{2}\right] - (K-3)\left[\frac{4}{2}\right]$	2	0
5	$5\left[\frac{K-3}{2}\right] - (K-4)\left[\frac{5}{2}\right]$	$\frac{K-4}{2}$	$\frac{K+1}{2}$
6	$6\left[\frac{K-4}{2}\right] - (K-5)\left[\frac{6}{2}\right]$	3	0
7	$7\left[\frac{K-5}{2}\right] - (K-6)\left[\frac{7}{2}\right]$	$\frac{K-6}{2}$	$\frac{K+1}{2}$
8	$8\left[\frac{K-6}{2}\right] - (K-7)\left[\frac{8}{2}\right]$	4	0

(8.11)

The matrix A^{-1} is centro symmetric so that the sum of row i, $i \leq K$ is the same as the sum of row $K + 1 - i$. For example, for $n = 8$, to attain the minimum

$$Y_i = \begin{cases} 1/5 & i = 1, 3, 5, 7, 9 \\ 0 & i = 0, 2, 4, 8 \end{cases} \qquad (8.12)$$

and the minimum value of $Q_8 = .2$. For $n = 9$, to attain the minimum

$$30\, Y_i = \begin{cases} 5 & i = 1,\ 10 \\ 1 & i = 2,\ 9 \\ 4 & i = 3,\ 8 \\ 2 & i = 4,\ 7 \\ 3 & i = 5,\ 6 \end{cases} \tag{8.13}$$

and the minimum value of $Q_9 = .183$.

The percentage of Q_n values less than Q_n value of $(2n+1)/(n+1)^2$ for equal spacings can be estimated from the percentage points tabulated by Monte Carlo methods.

n	% $Q_n \leq (2n+1)/(n+1)^2$
2	17
3	8
4	4
5	2
6	1
7	< 1
8	< 1
9	< 1

(rounded to the nearest percentage point)

APPENDIX 8.7

Power Study Tables

Tables 8.4 - 8.15 display the percent rejected for various families of alternative distributions and probabilities of rejection under the null hypothesis. The left value for each test statistic is computed without the aid of the Durbin (1961) transformation while the right value was obtained after the Durbin transformation had been applied.

ORNL-DWG 74-13235

TABLE 8.4

Family I
Percent Rejected for $\alpha = .1$

	i	X^2_{10}•		X^2_{20}		D		W^2		U^2		A		G		Q		QM	
n=2	1	16	14	13	12	24	19	25	19	15	14	33	25	25	20	26	21	32	23
	2	25	21	21	17	37	29	39	30	22	19	52	41	39	32	41	34	51	41
	3	34	28	28	24	48	38	50	40	29	25	66	56	49	41	52	44	64	54
	4	41	34	35	30	54	46	57	48	36	31	74	66	56	49	60	52	72	64
n=5	1	25	17	21	15	35	23	33	24	22	17	48	30	31	22	36	24	37	24
	2	48	33	41	29	60	42	63	45	39	29	77	58	51	39	60	45	59	44
	3	65	48	57	43	74	59	77	61	53	40	89	76	65	52	73	60	72	59
	4	75	61	69	55	82	69	85	71	62	51	95	86	73	63	81	71	77	67
n=10	1	39	21	35	19	52	29	57	31	33	20	67	37	37	23	46	27	47	26
	2	73	48	70	45	83	59	86	63	63	42	94	74	65	46	77	55	66	54
	3	89	69	88	68	94	79	95	82	81	62	99	91	81	64	90	74	85	72
	4	94	83	94	82	98	90	98	90	90	75	100	97	89	76	95	86	88	81
n=15	1	52	25	49	22	66	34	70	37	44	23	80	42	42	24	53	28	56	27
	2	89	60	87	57	93	72	95	75	80	53	98	84	73	51	86	62	87	62
	3	98	84	97	82	99	90	99	92	93	76	100	97	89	73	96	83	94	83
	4	99	94	99	93	100	96	100	97	97	88	100	99	95	84	99	92	95	91
n=20	1	65	30	60	26	78	41	52	44	56	27	89	49	49	26	60	31	62	30
	2	96	72	95	68	98	81	98	84	90	62	100	91	82	57	92	68	91	67
	3	99	92	99	91	100	95	100	96	98	85	100	99	94	78	98	88	96	87
	4	100	98	100	98	100	99	100	99	100	94	100	100	98	90	100	96	97	94
n=50	1	96	50	95	46	98	64	99	71	97	45	100	75	70	31	84	39	88	40
	2	100	97	100	96	100	98	100	99	100	92	100	100	97	75	100	87	100	89
	3	100	100	100	100	100	100	100	100	100	99	100	100	100	95	100	99	100	99
	4	100	100	100	100	100	100	100	100	100	100	100	100	100	99	100	100	100	100

ORNL-DWG 74-13236

TABLE 8.5

Family I
Percent Rejected for α = .05

	i	X^2_{10}		X^2_{20}		D		W^2		U^2		A		G		Q		QM	
n=2	1	8	7	9	7	16	12	17	13	8	7	24	17	17	13	17	13	21	14
	2	12	10	16	13	29	22	30	24	14	12	43	33	29	23	30	24	36	28
	3	17	14	24	19	40	31	42	32	20	17	58	48	39	32	42	34	45	38
	4	21	17	31	26	47	39	49	41	27	23	68	59	48	40	50	43	50	44
n=5	1	15	10	15	10	23	14	27	16	13	10	38	21	20	14	25	16	26	15
	2	33	22	34	22	47	31	51	35	27	20	69	48	39	28	47	33	47	33
	3	48	34	52	37	64	48	67	51	41	29	84	69	52	40	62	48	59	47
	4	59	46	64	49	74	59	76	61	51	40	91	80	63	51	72	59	65	55
n=10	1	29	13	26	12	40	20	45	22	23	12	58	27	25	14	33	17	35	17
	2	64	39	63	36	74	48	79	52	52	32	90	66	51	33	65	42	65	43
	3	84	61	84	61	89	70	91	74	72	51	98	87	69	51	82	63	77	61
	4	92	77	93	78	95	83	97	86	83	66	99	95	80	64	91	77	81	72
n=15	1	42	16	38	14	53	24	59	27	33	15	71	33	29	14	39	18	44	18
	2	84	51	82	47	88	61	91	66	71	42	97	78	60	38	76	49	79	51
	3	96	79	96	77	97	84	98	87	89	66	100	95	80	60	92	73	90	74
	4	99	91	99	90	99	93	100	95	96	82	100	99	90	74	97	86	92	85
n=20	1	55	19	49	17	67	30	72	34	43	18	83	40	35	16	46	21	51	20
	2	93	61	92	59	96	73	97	77	83	51	99	86	70	43	84	56	85	56
	3	99	87	99	86	99	92	100	94	96	78	100	98	88	67	96	80	92	80
	4	100	96	100	97	100	98	100	98	99	90	100	100	95	81	99	92	94	89
n=50	1	94	40	92	34	96	52	98	60	85	34	99	65	55	20	72	26	81	29
	2	100	94	100	93	100	96	100	98	100	87	100	99	93	63	99	78	99	83
	3	100	100	100	100	100	100	100	100	100	99	100	100	99	90	100	97	100	97
	4	100	100	100	100	100	100	100	100	100	100	100	100	100	98	100	100	100	100

ORNL-DWG 74-13237

TABLE 8.6

Family I
Percent Rejected for α = .01

	i	X^2_{10}		X^2_{20}		D		W^2		U^2		A		G		Q		QM	
n=2	1	2	1	2	1	7	5	7	5	2	2	11	7	6	5	7	4	7	4
	2	3	2	3	3	17	12	18	13	5	4	27	20	16	12	17	12	14	11
	3	3	3	5	4	26	20	27	21	9	7	42	33	24	19	26	20	20	16
	4	4	3	6	5	34	27	36	28	14	12	54	45	32	26	35	28	23	20
n= 5	1	6	3	5	3	10	5	12	6	4	2	21	9	8	5	10	6	11	5
	2	18	10	15	9	26	16	32	19	11	7	51	32	19	13	26	17	26	16
	3	31	19	28	18	41	27	48	32	19	13	72	53	29	20	43	29	37	27
	4	42	30	39	28	52	38	59	43	27	20	83	68	38	29	53	41	42	34
n=10	1	13	4	11	4	20	8	25	10	8	4	37	13	9	5	14	6	16	5
	2	44	22	43	19	52	28	60	33	29	15	80	48	27	16	39	22	41	21
	3	68	43	69	42	74	50	80	56	50	30	94	76	44	29	61	40	55	38
	4	82	61	82	61	85	66	89	72	66	45	98	89	58	41	75	55	61	50
n=15	1	23	6	20	5	31	10	37	13	15	6	52	16	11	5	18	6	23	7
	2	62	32	67	30	73	40	80	47	49	23	92	62	34	17	51	27	60	31
	3	89	61	90	62	91	67	94	73	74	46	99	88	55	35	75	50	76	55
	4	96	80	96	82	97	83	98	87	87	65	100	96	71	50	87	67	81	69
n=20	1	35	8	32	7	41	13	51	17	22	7	66	22	15	6	21	8	28	8
	2	84	45	83	41	86	51	91	59	65	31	98	73	44	22	60	31	67	34
	3	97	77	97	76	97	79	98	85	88	59	100	94	69	44	84	57	81	61
	4	99	92	99	93	99	92	100	95	96	79	100	99	83	61	94	76	85	75
n=50	1	85	22	82	17	87	30	93	39	67	17	97	45	27	7	44	11	62	13
	2	100	88	100	85	100	89	100	93	98	71	100	97	77	36	93	54	96	66
	3	100	99	100	99	100	99	100	100	100	96	100	100	95	70	100	88	99	92
	4	100	100	100	100	100	100	100	100	100	100	100	100	99	89	100	98	99	98

ORNL-DWG 74-13238

TABLE 8.7

Family II
Percent Rejected for $\alpha = .1$

	i	X^2_{10}		X^2_{20}		D		W^2		U^2		A		G		Q		QM	
n=2	1	13	8	11	9	2	9	2	9	13	9	1	9	1	10	1	10	1	10
	2	17	8	13	9	0	10	0	10	17	8	0	12	0	13	0	14	0	12
	3	19	7	14	9	0	10	0	11	20	8	0	12	0	15	0	16	0	13
	4	22	6	15	8	0	9	0	9	22	7	0	11	0	17	0	19	0	13
n=5	1	18	8	15	9	6	8	4	8	27	8	2	9	3	10	1	11	2	10
	2	25	8	19	9	5	7	3	7	47	8	1	8	4	12	0	13	3	12
	3	31	8	24	8	5	7	2	6	63	8	0	8	5	13	0	15	4	12
	4	36	9	28	9	5	9	2	7	74	10	0	9	7	17	0	19	4	14
n=10	1	26	9	21	10	11	10	9	9	47	9	5	9	10	10	4	11	4	11
	2	45	12	34	11	19	17	18	17	81	15	12	16	23	15	9	17	4	13
	3	61	16	45	15	27	31	31	30	95	25	24	28	41	24	17	28	4	18
	4	73	21	54	17	36	45	45	45	99	36	36	41	56	36	27	40	4	22
n=15	1	37	11	29	11	19	14	17	14	65	12	15	14	19	11	13	12	7	11
	2	67	19	49	16	38	34	43	35	95	27	44	33	48	22	34	25	8	17
	3	84	30	66	24	56	57	71	60	100	46	73	56	73	40	59	45	7	24
	4	94	44	79	32	72	77	87	79	100	65	89	75	88	57	79	65	7	31
n=20	1	48	13	36	12	27	19	27	18	78	15	28	18	28	13	21	14	8	12
	2	63	28	65	22	57	50	68	51	99	39	74	49	67	30	59	35	6	19
	3	96	48	84	35	81	79	92	80	100	66	95	78	90	54	87	62	5	28
	4	99	65	94	49	93	93	98	94	100	83	99	93	98	73	97	82	4	33
n=50	1	92	31	83	27	69	47	85	52	99	37	92	52	69	25	73	29	18	17
	2	100	81	100	68	99	95	100	97	100	88	100	97	99	72	100	81	14	36
	3	100	98	100	93	100	100	100	100	100	99	100	100	100	95	100	98	8	44
	4	100	100	100	99	100	100	100	100	100	100	100	100	100	100	100	100	5	44

ORNL-DWG 74-13239

TABLE 8.8

Family II
Percent Reject for $\alpha = .05$

	i	X^2_{10}		X^2_{20}		D		W^2		U^2		A		G		Q		QM	
n=2	1	7	4	6	4	0	4	0	4	6	4	0	5	0	5	0	5	0	5
	2	8	4	8	4	0	5	0	5	8	4	0	6	0	7	0	7	0	6
	3	10	4	9	3	0	5	0	6	10	4	0	7	0	8	0	9	0	7
	4	11	3	11	3	0	5	0	5	11	3	0	6	0	8	0	10	0	7
n=5	1	10	4	9	4	2	4	1	4	17	4	0	4	1	5	0	6	1	5
	2	14	4	13	4	1	3	0	3	31	3	0	4	1	6	0	7	1	6
	3	19	4	16	4	1	3	0	2	45	4	0	3	1	7	0	8	1	6
	4	23	5	20	4	1	4	0	2	56	5	0	4	1	9	0	10	1	8
n=10	1	16	5	12	5	4	5	2	5	33	5	1	5	4	5	1	6	2	6
	2	31	6	22	6	6	10	4	9	67	8	1	9	11	9	3	10	1	7
	3	47	9	31	8	9	20	7	19	87	15	3	18	22	16	6	18	1	10
	4	59	13	40	10	11	31	11	30	96	24	5	27	34	26	11	28	1	12
n=15	1	24	6	18	5	8	7	5	8	51	6	4	7	10	6	6	6	3	6
	2	50	11	35	9	17	23	16	24	88	17	14	22	31	14	19	16	3	10
	3	71	19	51	15	27	45	34	46	98	34	32	42	56	29	40	34	3	15
	4	86	30	66	21	41	65	55	67	100	52	54	63	76	45	61	53	2	19
n=20	1	34	6	24	6	12	11	10	11	66	9	10	11	16	7	12	8	4	6
	2	70	16	49	13	30	37	35	39	97	27	40	37	51	21	43	25	3	11
	3	90	31	72	23	54	68	69	70	100	53	74	67	80	43	75	52	2	17
	4	97	48	85	35	73	86	88	89	100	74	92	86	94	63	92	73	1	22
n=50	1	86	21	70	16	47	35	62	40	98	26	76	40	55	15	60	19	10	10
	2	100	71	99	54	94	91	99	94	100	81	100	94	98	61	99	72	7	25
	3	100	96	100	87	100	100	100	100	100	98	100	100	100	91	100	97	4	32
	4	100	100	100	98	100	100	100	100	100	100	100	100	100	99	100	100	2	33

ORNL-DWG 74-13240

TABLE 8.9

Family II
Percent Rejected for $\alpha = .01$

	i	X^2_{10}		X^2		D		W^2		U^2		A		G		Q		QM	
n=2	1	1	1	1	1	0	1	0	1	1	1	0	1	0	1	0	1	0	1
	2	2	1	2	1	0	1	0	1	2	1	0	1	0	1	0	1	0	1
	3	2	1	3	1	0	1	0	1	2	1	0	1	0	2	0	2	0	1
	4	2	1	3	1	0	1	0	1	2	1	0	1	0	2	0	2	0	1
n=5	1	3	1	2	1	0	1	0	1	4	1	0	1	0	1	0	1	0	1
	2	5	1	3	1	0	0	0	0	8	0	0	1	0	1	0	1	0	1
	3	7	0	4	1	0	0	0	0	13	0	0	1	0	2	0	2	0	1
	4	9	1	5	1	0	0	0	0	19	1	0	1	0	3	0	3	0	2
n=10	1	4	1	3	1	0	1	0	1	12	1	0	1	0	1	0	1	0	1
	2	11	1	8	1	0	2	0	2	35	2	0	2	1	3	0	3	0	2
	3	19	2	12	2	0	6	0	6	59	5	0	5	3	6	0	6	0	3
	4	27	4	17	3	0	11	0	10	77	9	0	9	5	11	0	11	0	3
n=15	1	8	1	6	1	1	2	0	2	25	1	0	2	2	1	1	2	0	1
	2	24	3	14	2	2	9	0	10	66	6	0	8	8	5	3	6	0	2
	3	42	5	24	4	3	21	1	23	89	15	0	20	20	13	10	17	0	4
	4	59	10	35	7	4	38	2	40	97	27	1	34	37	25	21	31	0	6
n=20	1	13	2	8	1	1	3	1	3	38	2	0	3	4	2	2	2	0	1
	2	39	5	23	4	3	16	2	19	85	11	2	17	23	9	14	11	0	3
	3	67	13	42	8	8	41	8	46	98	30	6	41	52	24	38	29	0	5
	4	84	24	60	14	14	64	19	69	100	50	17	63	74	44	62	51	0	7
n=50	1	63	7	42	5	12	16	13	20	92	11	24	20	29	6	32	8	2	3
	2	99	45	93	28	56	75	79	81	100	61	93	81	90	39	94	53	2	10
	3	100	85	100	65	91	98	99	99	100	93	100	99	100	79	100	90	0	14
	4	100	97	100	88	99	100	100	100	100	99	100	100	100	95	100	99	0	16

ORNL-DWG 74-13241

TABLE 8.10

Family III
Percent Rejected for $\alpha = .1$

	i	X^2_{10}		X^2_{20}		D		W^2		U^2		A		G		Q		QM	
n=2	1	12	14	12	12	18	19	19	19	14	14	29	25	25	20	27	21	30	24
	2	16	20	14	17	23	29	24	30	20	18	47	42	39	31	42	33	49	41
	3	19	27	17	24	28	38	29	39	27	25	60	55	48	40	52	43	61	54
	4	23	34	21	29	32	45	33	47	34	31	69	65	56	48	60	52	69	63
n=5	1	14	15	14	14	18	20	17	21	18	15	33	27	27	20	34	22	34	21
	2	24	29	24	24	27	38	26	39	33	25	60	53	46	35	56	40	57	40
	3	34	43	35	39	36	54	34	57	49	37	77	73	61	49	72	56	72	56
	4	42	56	45	51	43	65	41	67	58	46	86	84	70	58	80	66	80	65
n=10	1	19	17	20	16	19	23	19	24	25	17	39	29	30	20	39	22	40	22
	2	41	39	43	37	36	50	36	53	55	34	75	67	50	38	70	47	73	48
	3	59	60	64	60	52	71	52	73	75	53	91	86	73	56	86	67	87	67
	4	72	77	78	77	63	84	63	85	86	68	97	95	83	69	93	80	93	80
n=15	1	25	18	25	17	23	25	22	27	34	18	46	32	33	20	43	23	47	22
	2	55	47	58	45	44	57	45	61	69	40	85	73	62	41	76	50	87	53
	3	78	75	83	74	65	82	67	84	90	64	97	93	80	62	92	73	94	77
	4	89	89	92	90	79	93	82	94	96	81	99	99	91	77	98	87	98	89
n=20	1	31	21	31	18	25	28	24	29	41	19	52	34	35	19	45	22	50	22
	2	68	56	73	54	53	65	56	69	83	46	92	80	67	43	83	53	87	56
	3	89	83	92	83	77	89	80	90	96	73	99	97	86	66	95	78	97	81
	4	96	94	98	95	89	96	92	97	99	88	100	99	94	81	99	91	99	92
n=50	1	60	27	64	26	40	35	46	41	75	25	81	46	44	18	59	22	68	25
	2	97	84	99	84	89	88	94	92	99	74	100	96	86	53	96	67	98	75
	3	100	99	100	99	99	99	100	100	100	96	100	100	98	83	100	93	100	96
	4	100	100	100	100	100	100	100	100	100	100	100	100	100	95	100	99	100	100

ORNL-DWG 74-13242

TABLE 8.11

Family III
Percent Rejected for α = .05

	i	X^2_{10}		X^2_{20}		D		W^2		U^2		A		G		Q		QM	
n=2	1	6	7	7	7	12	12	12	12	7	7	21	17	17	13	18	13	19	15
	2	8	10	10	13	17	22	18	23	12	11	38	34	29	23	32	24	34	28
	3	10	14	13	19	22	31	23	32	19	17	52	47	39	32	42	33	42	37
	4	11	17	17	25	27	38	28	39	25	22	61	58	47	39	50	42	47	43
n=5	1	8	9	8	8	9	12	10	14	11	9	22	18	18	12	22	14	23	13
	2	14	18	17	18	16	27	16	30	23	17	47	44	34	24	44	29	45	29
	3	22	30	27	33	24	43	21	46	37	27	68	65	49	37	60	44	60	44
	4	28	41	37	46	30	54	25	57	47	36	79	77	58	46	70	55	68	53
n=10	1	12	11	12	9	11	15	11	16	17	10	27	21	19	11	26	14	29	14
	2	30	29	33	28	22	38	22	42	44	24	65	57	42	26	57	35	62	37
	3	49	52	54	53	33	60	38	64	65	42	85	81	59	43	76	55	80	56
	4	63	69	70	71	43	75	44	79	79	57	94	92	73	56	87	70	87	70
n=15	1	16	11	16	10	13	16	13	18	23	10	33	23	21	11	29	13	35	14
	2	44	38	47	35	28	46	28	50	59	29	76	65	46	27	64	37	72	41
	3	69	67	75	66	46	73	48	77	88	53	94	89	68	48	85	62	90	67
	4	84	85	89	85	62	88	66	90	93	72	99	97	82	64	94	78	95	82
n=20	1	21	11	21	11	14	18	14	20	30	11	38	24	22	11	32	13	38	14
	2	57	44	62	48	36	54	37	59	74	35	86	72	52	30	72	40	79	44
	3	83	76	87	76	61	82	64	83	93	63	98	94	76	52	91	67	94	73
	4	93	91	96	92	77	94	81	95	98	82	100	99	89	71	97	84	97	87
n=50	1	50	19	52	17	25	25	29	31	64	16	68	36	29	10	43	13	56	16
	2	96	78	97	76	76	82	84	87	99	63	100	94	74	39	91	53	96	64
	3	100	98	100	98	96	99	99	99	100	93	100	100	95	72	99	87	100	93
	4	100	100	100	100	100	100	100	100	100	99	100	100	99	90	100	97	100	99

ORNL-DWG 74-13243

TABLE 8.12

Family III
Percent Rejected for α = .01

	i	X^2_{10}		X^2_{20}		D		W^2		U^2		A		G		Q		QM	
n=2	1	1	1	1	1	4	5	5	5	1	2	9	7	6	5	7	5	6	5
	2	2	2	2	3	9	12	10	12	4	4	22	20	16	12	17	12	13	11
	3	2	3	3	4	14	19	14	20	8	7	35	32	24	19	26	20	18	16
	4	2	3	3	5	18	27	19	28	12	11	46	44	32	26	34	27	21	20
n=5	1	2	3	2	2	3	5	3	5	3	2	9	8	6	4	9	5	10	4
	2	5	8	5	7	6	13	6	15	9	5	27	27	16	10	24	15	25	14
	3	9	17	10	15	9	24	10	28	17	11	47	49	26	19	40	26	37	25
	4	13	26	15	25	11	34	12	39	25	17	61	64	35	26	51	36	45	32
n=10	1	4	4	4	3	3	6	3	7	6	3	10	10	7	3	11	4	12	5
	2	13	15	16	14	8	20	8	24	23	10	41	39	20	11	32	15	38	18
	3	26	34	33	34	14	39	12	45	44	23	69	67	34	22	52	32	58	34
	4	39	52	48	53	19	57	16	62	60	37	85	85	48	33	68	46	69	47
n=15	1	5	3	6	3	4	6	4	8	9	3	13	10	7	3	12	4	17	5
	2	23	20	28	20	11	26	10	31	37	14	53	47	22	11	37	17	52	22
	3	46	47	56	50	21	53	18	59	66	33	84	80	41	25	63	37	77	46
	4	66	70	76	73	32	73	30	78	83	53	95	94	58	39	80	55	87	64
n=20	1	8	4	9	3	4	6	4	8	13	4	16	11	8	3	12	4	19	5
	2	35	28	42	26	13	31	12	39	53	18	66	55	28	13	43	18	60	24
	3	65	62	74	63	28	63	27	70	82	43	92	88	51	29	72	41	84	52
	4	83	83	90	85	45	83	45	88	94	65	99	97	70	47	87	63	92	71
n=50	1	28	8	31	6	7	10	7	14	42	6	39	18	10	3	18	4	34	5
	2	88	62	92	61	42	63	48	72	95	43	98	86	47	17	72	28	90	42
	3	99	95	100	96	79	94	87	97	100	82	100	100	80	44	96	65	99	82
	4	100	100	100	100	95	99	98	100	100	96	100	100	95	71	100	89	100	96

ORNL-DWG 74-13244

TABLE 8.13

Family IV
Percent Rejected for $\alpha = .1$

	i	X^2_{10}		X^2_{20}		D		W^2		U^2		A		G		Q		QM	
n=2	1	14	15	12	13	25	18	25	18	14	15	26	18	29	14	29	11	23	13
	2	15	19	12	14	32	23	33	23	16	19	33	21	40	16	40	12	28	14
	3	18	21	14	15	38	29	39	29	19	21	39	27	50	21	50	16	34	18
	4	21	23	15	16	41	36	42	36	21	23	43	33	58	27	59	20	37	22
n=5	1	17	16	15	14	24	26	21	27	28	21	29	25	42	18	48	15	32	12
	2	25	22	20	19	32	43	28	45	47	34	38	40	65	29	71	21	40	12
	3	32	29	25	23	40	59	34	61	63	46	49	55	80	42	85	32	47	14
	4	36	34	28	27	43	71	36	73	75	56	56	66	88	54	92	42	50	15
n=10	1	26	20	21	17	31	33	26	34	47	26	36	32	52	23	61	20	39	10
	2	45	35	32	27	48	63	41	64	80	50	57	61	83	46	90	40	47	11
	3	61	48	45	36	60	83	56	84	94	68	74	82	95	67	98	62	50	11
	4	74	60	56	45	70	92	68	93	98	81	85	92	99	81	100	77	49	11
n=15	1	38	24	29	20	40	39	34	42	65	41	46	32	62	27	72	26	43	11
	2	67	49	50	36	64	76	61	79	95	78	75	59	93	60	97	57	53	13
	3	85	67	67	50	78	93	81	94	100	83	91	94	99	83	100	81	54	13
	4	94	80	79	63	88	98	92	99	100	93	97	98	100	93	100	93	53	12
n=20	1	49	28	36	23	47	46	41	48	78	35	54	47	69	31	79	29	48	11
	2	83	59	66	45	77	85	78	87	99	72	87	87	97	70	99	68	56	11
	3	96	82	84	65	91	98	95	98	100	92	98	98	100	91	100	91	53	10
	4	99	93	93	78	97	100	99	100	100	98	100	100	100	98	100	98	51	8
n=50	1	92	53	83	43	80	70	88	76	99	56	92	77	92	50	97	52	69	15
	2	100	95	100	88	100	99	100	100	100	97	100	100	100	94	100	96	73	15
	3	100	100	100	99	100	100	100	100	100	100	100	100	100	100	100	100	68	12
	4	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	61	9

ORNL-DWG 74-13245

TABLE 8.14

Family IV
Percent Rejected for $\alpha = .05$

	i	X^2_{10}		X^2_{20}		D		W^2		U^2		A		G		Q		QM	
n= 2	1	7	8	7	8	15	10	15	10	7	8	16	10	16	7	16	5	13	7
	2	8	9	8	9	20	12	20	12	8	9	20	12	22	8	22	6	15	7
	3	9	11	9	10	25	16	25	16	9	10	25	16	29	12	30	8	19	9
	4	10	12	11	12	29	20	30	20	11	12	30	20	37	14	37	10	21	12
n=5	1	9	9	9	8	14	17	14	18	16	13	17	16	29	10	33	8	19	6
	2	14	13	13	12	21	30	19	32	30	22	25	27	50	19	55	13	24	6
	3	20	18	17	16	28	45	24	47	45	32	34	40	67	30	72	21	28	7
	4	23	21	20	19	30	56	27	59	57	41	39	51	78	41	82	29	31	7
n=10	1	16	13	13	10	19	22	15	24	33	17	22	22	39	14	48	12	26	5
	2	31	24	22	17	33	49	25	52	67	33	39	48	73	34	81	29	33	5
	3	47	35	32	24	43	72	34	75	87	56	56	71	90	55	95	49	35	5
	4	60	46	41	32	51	85	42	87	96	70	69	84	97	71	99	66	34	5
n=15	1	25	15	18	11	27	28	20	31	52	21	30	29	48	18	59	16	31	6
	2	51	34	35	23	46	65	38	68	89	51	56	66	86	48	93	45	39	7
	3	71	52	52	36	61	87	57	89	98	74	77	87	98	74	99	72	39	6
	4	85	67	66	48	73	96	73	97	100	87	90	96	100	88	100	87	39	6
n=20	1	34	16	23	14	31	33	25	36	66	24	37	35	56	21	69	20	36	6
	2	70	42	51	31	59	76	55	79	97	60	72	78	93	60	98	58	43	6
	3	90	67	72	50	78	95	79	96	100	85	92	95	99	86	100	86	41	5
	4	98	83	85	65	88	99	93	99	100	95	98	99	100	95	100	96	38	4
n=50	1	86	40	71	29	64	58	70	65	98	44	82	67	85	39	94	41	58	8
	2	100	91	99	79	97	98	100	99	100	93	100	99	100	91	100	93	64	9
	3	100	100	100	97	100	100	100	100	100	98	100	100	100	98	100	100	58	7
	4	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	52	5

ORNL-DWG 74-13246

TABLE 8.15

Family IV
Percent Rejected for α = .01

	i	X^2_{10}		X^2_{20}		D		W^2		U^2		A		G		Q		QM	
n=2	1	1	2	1	2	4	2	4	2	1	2	4	3	4	1	4	1	3	1
	2	2	2	2	2	5	3	5	3	2	2	6	3	5	2	5	1	3	2
	3	2	2	2	2	7	4	7	4	2	2	8	4	7	2	7	2	4	2
	4	2	2	2	2	8	4	8	4	2	2	10	5	8	3	8	2	4	3
n=5	1	3	3	2	2	5	5	5	6	4	3	6	5	10	3	12	2	5	1
	2	5	5	3	3	8	11	6	13	7	6	9	9	21	6	23	4	6	1
	3	8	7	4	4	11	19	8	22	14	10	13	17	34	11	37	7	7	1
	4	9	9	5	5	12	28	8	30	19	14	15	23	46	17	50	11	7	1
n=10	1	5	4	4	3	7	9	5	10	13	6	8	9	18	5	23	4	9	1
	2	11	7	7	5	13	25	9	28	36	17	15	23	46	16	55	13	11	1
	3	19	13	12	8	20	46	12	49	60	30	23	43	70	31	79	26	12	1
	4	28	20	17	12	24	63	14	66	77	43	31	58	85	47	91	40	12	1
n=15	1	9	4	6	3	10	12	7	14	25	8	11	7	25	12	34	6	13	1
	2	24	12	13	8	20	40	12	44	66	28	23	39	65	27	78	25	18	2
	3	42	23	24	14	30	68	18	71	89	49	39	66	89	52	95	49	18	1
	4	60	36	35	22	38	85	25	88	98	68	56	84	97	72	99	70	18	1
n=20	1	12	5	8	4	11	14	7	17	38	10	13	16	34	9	43	8	17	1
	2	40	20	27	12	27	51	17	57	85	36	35	54	81	40	89	37	21	1
	3	67	40	42	23	40	81	30	86	98	65	60	83	97	70	99	68	20	1
	4	85	59	60	35	52	94	44	96	100	83	79	95	100	87	100	87	18	1
n=50	1	63	18	42	11	29	33	24	41	92	23	46	43	67	22	82	24	36	2
	2	99	75	92	52	75	92	84	95	100	81	95	95	99	80	100	84	44	3
	3	100	97	100	86	96	100	99	100	100	98	100	100	100	98	100	99	39	2
	4	100	100	100	98	100	100	100	100	100	100	100	100	100	100	100	100	33	1

APPENDIX 8.8

The Conditional Probability Integral Transformation for Simple Linear Regression

The subroutine listed below computes the V_j's defined in Section 6.2 in an iterative fashion, beginning with j equal to four.

```
      SUBROUTINE LRCPIT(N,X,Y,V)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION X(200),Y(200),V(200)
      SX = X(1) + X(2) + X(3)
      SY = Y(1) + Y(2) + Y(3)
      SXY = X(1)*Y(1) + X(2)*Y(2) + X(3)*Y(3)
      SXX = X(1)*X(1) + X(2)*X(2) + X(3)*X(3)
      SYY = Y(1)*Y(1) + Y(2)*Y(2) + Y(3)*Y(3)
C
C     V(J) = (C**.5)*A/(B - A*A)**.5
C
      DO 1 J=4,N
      C = J - 3
      C = DSQRT(C)
      SX = SX + X(J)
      SY = SY + Y(J)
      SXY = SXY + X(J)*Y(J)
      SXX = SXX + X(J)*X(J)
      SYY = SYY + Y(J)*Y(J)
      AJ = J
      CONST = AJ*SXX - SX*SX
      CONST = 1.0/CONST
      A = CONST*(SXX*SY - SX*SXY + X(J)*(AJ*SXY - SX*SY))
      A = Y(J) - A
      ZZ = SY*(SXX*SY - SX*SXY) + SXY*(AJ*SXY - SX*SY)
      SJ = SYY - CONST*ZZ
      B = SJ*(1.0 - CONST*(SXX - 2.0*SX*X(J) + AJ*X(J)*X(J)))
      D = B - A*A
      E = DSQRT(D)
      V(J) = C*A/E
    1 CONTINUE
      RETURN
      END LRCPIT
```