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Critical analysis and overview of the nonlinear deformations, buckling and vibrations of thermally-stressed plate and shell structures of regular and irregular geometrical shapes

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ABSTRACT: In this paper brief presentation has been made of the critical analysis and overview of the nonlinear deformations, buckling and vibrations of thermally-stressed plates and shell structures of different geometrical shapes, regular or irregular, and some interesting problems have been considered.

ANALYSIS AND OVERVIEW: Nonlinear thermal deformations, buckling and vibrations including thermal stresses of elastic and elastic-plastic plate and shell structures play a crusial role in structural mechanics- particularly in aeronautics for structural analysis of high-speed spacecrafts,nuclear power reactors,pressure vessels, off-shore engineering and in many branches of engineering. The analysis becomes more complex when the plate and shell structures are made of anisotropic materials or of fibre-reinforced composites. Further complexities arise when the configurations are irregular. In modern technology there is a necessity for advanced industrial applications and the advent of new stiff, strong and light-weight composite materials offer Aerospace Engineers and Researchers a lucrative choice on designing composite structures which have high strength or of light-weighted of more sustainable property.

When the structures are subjected to severe thermal loading or placed in severe thermal evvironment,as in the case of nuclear power reactors, causing significant stretching of the middle surface,particularly when the edges are movable, the analysis has to be carried out with the help of nonlinear theory since the basic governing equations for such cases are nonlinear or sometimes highly nonlinear in character,coupled or decoupled in forms.

This branch of thermo-structural mechanics has drawn the attention of many researchers and a number of approximate and numerical or computational methods and in some cases some analytical method have been developed to study the nonlinear analysis of elastic, thermo-elastic, elastic-plastic, anisotropic and composite structures.

Although some references on the research studies of the topics may be available in scattered form, a comprehensive and critical overview of earlier works is necessary before undertaking any research project on that topic.

For investigating large deformations of plates under quasi-static and non-stationary temperature, vom Karman's coupled nonlinear partial differential differential equations [1] expressed in terms of stress function and normal displacement components have been used many authors. Since no closed form solutions is possible some authors have deduced basic differential equations in the quasi-linear forms by using Berger's approximations [2]. With notations as given in [1] von Karman's equations are given by :

$$\nabla_1^2 \nabla_1^2 F + E\alpha_t \nabla_1^2 \tau_o = E\left[(w_{,12})^2 - w_{,11}w_{,22}\right] \quad \ldots (1)$$

$$\nabla_1^2 \nabla_1^2 w + (1+\nu)\alpha_t \nabla_1^2 \tau = \frac{h}{N}\left(F_{,22}w_{,11} - 2F_{,12}w_{,12} + F_{,11}w_{,22}\right) \quad (2)$$

in which the temperature field is given by

$$T(x_1, x_2, x_3) = \tau_o(x_1, x_2) + x_3\, \tau(x_1, x_2) \quad \ldots \quad (3)$$

In the above system of equations (1) and (2) if $x_3 \tau(x_1, x_2)$ is taken equalto zero and $\tau_o(x_1, x_2) \neq 0$, one gets elastic buckling of plates when the deflections are large. For the case of circular plates corresponding equations with illustrative examples have been fully discussed by Nowacki [1].

For large deformations of heated orthotropic plates, governing equations in the von Karman sense have been derived by Biswas [3] from which the above two equations (1,2) have been deduced. The equations as given in [3] are:

$$D_x w_{,xxxx} + 2H\, w_{,xxyy} + D_y w_{,yyyy} = F_{,xx} w_{,yy} - 2F_{,xy} w_{,xy} + F_{,yy} w_{,xx} -$$

$$- \beta_1 (M_T)_{,xx} - \beta_2 (M_T)_{,yy} \quad \ldots \quad (4)$$

$$F_{,xxxx} + p^2 F_{,xxyy} + q^2 F_{,yyyy} + \lambda_1 (N_T)_{,xx} + \lambda_2 (N_T)_{,yy} =$$

$$= E_2 h \left[(w_{,xy})^2 - w_{,xx} w_{,yy}\right] \quad \ldots \quad (5)$$

with notations having their usual meaning [3].

In order to eliminate the disadvantages of von Karman's method, H.M. Berger [2] in 1955 presented two sets of nonlinear partial differential equations in the decoupled form deduced from the total energy expression by disregarding the second strain invariant. Although Berger's equations have some limitations as they fail miserably for plates with movableedge boundary, they give quite satisfactory results. Using the idea of Berger, basic governing equations for the large deflections of heated plates were derived by Basuli [4] and given in the following forms:

$$\nabla^2(\nabla^2 - \beta_1^2)\, w = \frac{1}{D}\left[q - \frac{E\,\alpha_t f(h)}{1-\nu}\, \nabla^2 T_1 \right] \quad \ldots (6)$$

$$e - (1+\nu)\,\alpha_t T_o = \beta_1^2 h^2/12 \quad \ldots (7)$$

where β_1^2 is the constant of integration and e is given by

$$e = u_{,x} + v_{,y} + (1/2)w_{,x}^2 + (1/2)w_{,y}^2 \quad \ldots (7)$$

and other notations have been explained in [3 & 4].

Using the above equations several problems of the large deformations of elastic plates under stationary and non-stationary temperature distributions have been solved [5 -8].

Berger's equations were further extended by Biswas [9] for analysing the large deflections of heated cylindrical shell and the basic equations as derived in [9] are given by :

$$\nabla^4 w - \beta^2(\nabla^2 w + 2k) + (1+\nu)\,\alpha_t \nabla^2 \tau = 0 \quad \ldots (8)$$

$$u_{,x} + v_{,y} + 2\,kw + (1/2)w_{,x}^2 + (1/2)w_{,y}^2 = \beta_1^2 h^2/12 + (1+\nu)\,\alpha_t \Upsilon_o \quad \ldots (9)$$

where k denotes the mean curvature of the middle surface.

Corresponding problem for the orthotropic case was also investigated by Biswas [10] .

Although Berger's method has been highly critised by several authors [11-12], some useful remarks on this method should be mentioned from the work of Pal [13] who critically analysed the large deformations of circular and annular circular flat plate with different boundary conditions and subjected to arbitrary symmetrical temperature considering the effect of temperature gradient through the thickness under transient heating conditions. It was shown tnere that the plates do not exhibit the EULER buckling phenomenon but to start to deflect from the beginning of heating according to temperature distribution and boundary conditions. The analysis as treated there is of considerable importance in supersonic airplanes and missiles where the deformation has no small effect on aerodynamic and aeroelastic behaviour of the structural components. It should thereforebe concluded that because of the relative simplicity of the fundamental equations as derived and presened in different papers it would become possible to attempt finding solutions of different problems of engineering purposes.

In the dynamic case, for dealing with nonlinear thermo-elastic vibrations of plate and shell structures, again von Karman's coupled nonlinear partial differential equations extended to thermal loading in the dynamic case may be expressed both in the cartesian and in the polar forms. Mansfield [14] has given the following form:

$$\nabla^2(1/h\,\nabla^2\phi) - (1+\nu)\Diamond^4(1/h,\phi) + (1/2)E[\Diamond^4(w,w)-\Diamond^4(w_o,w_o)] + E\alpha\nabla^2 T_m(x,y) = 0 \quad \ldots (10)$$

$$\nabla^2[D\nabla^2(w-w_o)] + (1+\nu)\nabla^2(D\alpha T,_z) + \rho h\, w,_{tt} = (1-\nu)\Diamond^4(D, w-w_o) + \Diamond^4(\phi, w) \quad \ldots (11)$$

where the diamond operator, $\Diamond$ (pronounced 'die) is defined by $\Diamond^4(f,g) = f,_{xx}g,_{yy} - 2f,_{xy}g,_{xy} + f,_{yy}g,_{xx}$, other notations are explained in [14].

The above equations have been used by Mansfield to discuss the large-deflection free vibrations of unsupported elliptical plates of lenticular section, the temperature distribution varying quadratically in the plane and linearly through the thickness. von Karman's plate equations in the dynamic case including the effect of thermal gradient in the forms given in [15] by A. Dalamangas who has transformed those equations into skew co-ordinates for analysing vibrations of skew plates at large amplitudes under plannar temperature distribution . In terms of skew co-ordinates these equations take the forms :

$$D\nabla^2\nabla^2 W - \frac{h}{\cos^2\phi}(W,_{\xi\xi}F,_{\eta\eta} + W,_{\eta\eta}F,_{\xi\xi} - 2W,_{\xi\eta}F,_{\xi\eta}) = 0 \quad \ldots (12)$$

$$\nabla^2\nabla^2 F - \frac{E}{\cos^2\phi}(W,_{\xi\eta}^2 - W,_{\xi\xi}W,_{\eta\eta}) + E\alpha_t\nabla^2\Theta = 0 \quad \ldots (13)$$

For a skew plate clamped along the boundary under plannar temperature distribution, stress function has been completely solved and applying Galerkin Procedure, frequency-amplitude relationship has been established.Hardening effect has been observed for variations of skew angle,aspect ratio and temperature coefficient.

von Karman's equations have further been extended to analyse the large thermal vibrations of shallow shells in the form of skew cylindrical panel under elevated temperature / 15 /. In oblique co-ordinates the governing equations can be obtained in the forms :

$$\nabla^4 F = Eh\Big[\frac{1}{\cos^2\phi}(w,_{\xi\eta}^2 - w,_{\xi\xi}w,_{\eta\eta}) - (1/R_1)\sec^2\phi(w,_{\xi\xi} - 2\sin\phi\, w,_{\xi\eta} - w,_{\eta\eta}) - (1/R_2)w,_{\xi\xi}\Big] - \nabla^2 N_T \quad \ldots (14)$$

$$D\nabla^4 w + \rho h\, w,_{tt} = \sec^2\phi(w,_{\xi\xi}F,_{\eta\eta} - 2w,_{\xi\eta}F,_{\xi\eta} - w,_{\eta\eta}F,_{\xi\xi}) - (1/R_1)\cos^2\phi[F,_{\eta\eta} - 2\sin\phi\, F,_{\xi\eta} + \sin^2\phi\, F,_{\xi\xi}] + - (1/R_2)F,_{\xi\xi} - \nabla^2 M_T \quad \ldots (15)$$

F.Buckens [16] in one of his wonderful papers analysed the vibrations of thermally-stressed plates giving special emphasis on pre-and post-buckling behaviour.

Under aerodynamic heating large amplitude free vibrations of isotropic as well as orthotropic plates have been investigated by M.C.Pal [17-18]. Using Berger's approximations, J.Mazumdar et al. [19] have carried out extensive investigation to study the vibrations of both elastic and viscoelastic plates at elevated temperature. They have obtained a general formula which links the fundamental frequency to the critical buckling temperature and the corresponding fundamental frequency of unheated plates. A criterion for thermal buckling is then presented. Using the same Berger's approximation nonlinear vibrations of heated triangular and rectangular plates have been investigated by Biswas [20-21]. It was observed that, as it should be, the nonlinear behaviour of plates due to elevated temperature is similar in nature as that of plates subjected to in-plane forces [22].

For axisymmetric circular plates nonlinear analysis in the dynamic case have been discussed by Biswas [23-24] with the help of von Karman's equations expressed in terms of displacement components and membrane stresses.

So far very few works have been reported in the literature on the large thermal bending and vibrations of sandwich plates and shells. The works of N.Kamiya [25-26] are worth mentioning in this connection. No paper other than these appears to have been reported in the literature. By extending the well-known Berger approach for single-layer isotropic elastic plates, an approximate method of analysis for the nonlinear bending of shallow sandwich shells was proposed by Kamiya. Governing field equations were derived for sandwich plates and shallow shells subjected to mechanican as well as thermal effects. Applicability of the field equations were discussed with numerical results due to temperature difference between the upper and lower faces.

It has been observed that little attention has been devoted for the nonlinear analysis of structures made of non-homogeneous materials. Non-homogenity also occurs when structures are exposed to severe thermal environment such that the elastic constants become functions of space variables. Although some works based on linear theory have been carried out by several authors [27-30], only few papers can be located where large deflections of annular plates [31] and nonlinear vibrations of doubly-curved orthotropic shallow shells under a thermal gradient have been investigated. It should be remarked that there is further scope of research in this area by the future researchers.

For irregular-shaped plates and for plates of regular polygonal shapes complex variable theory and conformal mapping technique are quite convenient. Basic governing equations are first transformed into complex co-ordinates ($z, \bar{z}$) and using the mapping function $z = f(\xi)$, $\xi = r \exp(i\theta)$, a plate shape can be mapped onto a unit circle. Such technique was first used by P.A.Laura et al.[32] and subsequently by many authors[33-37]. Large deflections and large elastic vibrations of irregular plates under thermal loading,stationary or non-stationary, have been investigated quite efficiently by this method [34-36].

As von Karman Plate or shell equations are in the coupled form having no closed-form solutions, some authors [38-42] have proposed some 'new approach' and derived new sets of nonlinear equations by considering a modified energy expression which has led the equations in decoupled forms. As in the Berger's case, although these equations give rise to quite satisfactory results in agreement with known results,they have not given any mathematical justification for 'replacement of one term by the other' in the modified energy expression. However, while Berger's method fails miserably for movable-edge boundary conditions, these equations give satisfactory results for such cases, as have been claimed by those authors. In investigating the dynamic response of shallow spherical shells under a thermal gradient, some other authors [43] derived basic governing equations following the proposation or the new approached used in [38-42]. At the end of the analysis of their results [43], they have noticed sharp deviation from the results contained in [39]. The apparent reason for such deviation of results is due to the falacious derivation of equation (7) of [39], the correct form which would be equation (11) for the limiting case of [43]. Moreover, the claim of those authors [39] that the results obtained by J.Ramachandran [44]in the analysis of the large amplitude vibrations of shallow spherical shell with concentrated mass contains some errors does not stand.

SOME OBSERVATIONS:

i) Classical von Karman equations extended in the dynamic case under thermal loading for plate or shell structures are always in the coupled forms in therms of

(a) normal displacement and stress function

(b) displacement components and membrane stresses.

As yet no closed form solutions are available- approximate or numerical methods should be adopted for their solutions. The convenient method applied in most problems considered by different authors is the application of Galerkin Procedure.

ii) basic governing equations derived in the light of Berger's approximation are very simple as they are in the decoupled form. Sufficiently accurate results have been obtained for clamped immovable edges but fails miserably for mavable-edge conditions.

iii) Basic governing equations may be derived by considering a modified energy expression in which mere replacement of terms has been made without any mathematical justification which enables one to get some 'advantages' over the above two methods.

iv) It appears from a careful study of available literature that no paper has been devoted for the nonlinear analysis of thermally-stressed plates of variable thickness.

SCOPE FOR FURTHER RESEARCH:

A brief literature study and its overview of the nonlinear deformations, buckling and vibrations of plate and shell structures reveals that considerable interest has been exhibited in the past in this area of research considering enormous practical applications of the results in structural mechanics, aeronautics, nuclear power reactors, missiles and chemical engineering. Inspite of this little attention has been devoted to the nonlinear studies of moderately thick plates and shells, sandwich plates and shells of different shapes, plates and shells of more complex geometrical shapes and for the cases variable thickness. Moreover, baring a few problems on the elastic-plastic deformations of plates and shells without thermal loading by J.Majumdar et al. [45-46], there seems to be no literature on the deformations and vibrations of elastic - plastic plate and shell structures under thermal stresses. As a result there is enough scope for research in the above areas considering the demands of the present-day sophisticated structures.

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On a new characterization of the performance of time integration schemes : application to the "alpha"- method scheme

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ABSTRACT. This work presents an alternative definition of the numerical angular frequency and numerical damping ratio, within the framework of direct time integration of second-order differential equations. The classical approach deals with the ability of the numerical schemes to deliver an accurate response in time: it is well suited to the study of the explicit schemes used in fast transient dynamic analysis. The alternative approach emphasizes the capability of the scheme to deliver an accurate resonant response. Such frequency viewpoint is helpful in the study of the implicit schemes used in civil engineering applications and allows to explain the success of the α-method.

INTRODUCTION

The solution of second-order differential equations is often computed numerically using a step-by-step time integration scheme, which advances the solution at time t_n to the solution at time $t_{n+1} = t_n + \Delta t$. The performance of such a scheme is classically described by the amount of frequency and damping distortion introduced in the solution. These quantities are found by inspecting the amplification matrix of the scheme (see e.g. [1]), that is, by deriving the expression of the numerical solution for the free vibration of a simple oscillator, and by comparing it with the analytical solution. The above mentioned distortions characterize the accuracy of the scheme and, for all of the commonly used schemes, tend to disappear as the time step tends towards zero so that the difference between the discrete numerical solution and the analytical solution tends to vanish. It is sometimes useful to study the performance of the scheme also for large values of the time step. In fact, following Hughes [1], "For unconditionally stable algorithms [...], it is often the case in large multi-degree-of-freedom applications that the Δt is chosen small enough only to accurately represent the lower modes and, as a result, Δt is very large compared with the periods of the higher modes [...]". It is thus important to know how the high frequency modes are distorted and if they can introduce a spurious contribution in the numerical response of the structure.

Large time steps are typically used in earthquake engineering applications [2]. The realistic simulation of a building usually requires a numerical model leading to a very wide spectrum of natural frequencies, whereas the earthquake loading activates the low frequency modes only. The highest modes in the structure do not participate to the response of the structure and thus do not need to be represented accurately. For this class of applications however, it is no longer the transient viewpoint exhibited by the amplification matrix which is relevant but a more frequency-related viewpoint: the external loading (accelerogram) is rather smooth and the interest is focused on the possibility for the structure to exhibit a resonant behaviour. As a matter of fact, a low-damped simple oscillator subjected to a forced vibration exhibits a transfer function with a peak for the resonance frequency. The ability for a numerical scheme to accurately predict the frequency location and the amplitude of this peak promotes the alternative definition of the frequency and damping distortion which is proposed by the author [3] and already applied to the implicit Newmark scheme [4].

The α-method scheme [5] is considered here. This scheme introduces a selective numerical damping for the high frequency modes of the structure and appears to be the reference method for seismic analyses performed in the physical space. As for the Newmark scheme, it is shown that the aforementioned time and the frequency viewpoints are identical when the time step vanishes, leading in particular to the same evaluation of the numerical damping, but that this is no longer the case when the time step becomes large.

THE SIMPLE OSCILLATOR: EXACT TIME INTEGRATION

The difference and the complementary character of the time and frequency viewpoints are first illustrated by considering the analytical response of a damped single degree-of-freedom oscillator. Its motion is described by the differential equation

$$a + 2\xi\omega v + \omega^2 d = \omega^2 f \tag{1}$$

where a, v and d are the acceleration, the velocity and the displacement, f is the external force and ξ and ω are the damping ratio and the natural angular frequency of the oscillator.

Two very typical responses are obtained when the loading f is strongly localized, ideally through a Dirac distribution, in either the time or the frequency domain: one case corresponds to the free vibration problem, where, at a given instant $t = 0$, f is suddenly released to 0, the other to the harmonic forced vibration problem where f is a sine function.

- Free vibration problem: Eq. (1) with $f(t \geq 0) = 0$, $v(0) = 0$ and $d(0) = d_0$. The analytical solution is provided by

$$d(t) = d_0 e^{-\xi\omega t}\cos(\omega_d t) \quad \text{with} \quad \omega_d = \omega(1-\xi^2)^{1/2} \quad \text{if} \quad 0 \leq \xi < 1 \tag{2}$$

  The quantity ω_d is referred to as the *damped* natural angular frequency of the oscillator. Leaving out the oscillations and keeping only the envelope, it can be said that Eq. (2) characterizes the *time transfer function* of the oscillator and leads to the first way of evaluating the damping of a structure: the *free vibration decay method* (see [6] for details).

- Forced vibration problem: Eq. (1) with $f(t) = d_f \sin(\tilde{\omega} t)$, $v(0) = 0$ and $d(0) = 0$. After a transitory regime which is damped out, the analytical solution tends towards an undamped symmetric oscillation, with angular frequency $\tilde{\omega}$ and amplitude d_a given by

$$d_a/d_f = [(1-\tau^2)^2 + (2\xi\tau)^2]^{-1/2} = D \quad , \quad \tau = \tilde{\omega}/\omega \tag{3}$$

  The ratio of the harmonic response amplitude d_a to the forcing displacement d_f is the *dynamic magnification factor* D which is referred to as the frequency transfer function of the system. Function $D(\tau)$ exhibits a peak (resonance phenomenon) if the damping ratio verifies the condition $\xi < 1/\sqrt{2}$. The maximum of D is associated with

$$\tau_r = \tilde{\omega}_r/\omega = (1-2\xi^2)^{1/2} \quad , \quad D_r = 2\xi(1-\xi^2)^{-1/2} \tag{4}$$

$\tilde{\omega}_r$ is the resonant angular frequency of the oscillator and the amplitude D_r of the dynamic magnification factor at resonance leads to a second way of evaluating the damping of the structure: the *resonant amplification method* (see [6] for details).

In summary, the damping of a structure can be considered from two viewpoints: the *time* damping and the *frequency* damping. For the (time-)continuous system defined by Eq. (1), these two types of damping coincide. They correspond to the *same* quantity ξ. It is well known that the numerical oscillator (say the result of the time integration of Eq. (1) using a numerical scheme) is not identical to the exact analytical one: its natural (or resonant) angular frequency is modified, together with the damping ratio. The following question is then raised: does the numerical damping ratio classically associated with the numerical oscillator still characterize *both* the *time* damping and the *frequency* damping?

THE SIMPLE OSCILLATOR: TIME INTEGRATION WITH THE α-METHOD SCHEME

The α-method [5]

Knowing the displacement d, the velocity v and the acceleration a at time t_n, the values of these quantities are obtained at time $t_{n+1} = t_n + \Delta t$ as

$$\begin{aligned} d_{n+1} &= d_n + \Delta t\ v_n + (\Delta t)^2\left(\frac{1}{2} - \beta\right) a_n + (\Delta t)^2 \beta\ a_{n+1} \\ v_{n+1} &= v_n + \Delta t(1-\gamma)\ a_n + \Delta t\ \gamma\ a_{n+1} \end{aligned} \tag{5}$$

and according to the verification of the time discrete equilibrium equation

$$a_{n+1} + 2\xi\omega[(1+\alpha)v_{n+1} - \alpha v_n] + \omega^2[(1+\alpha)d_{n+1} - \alpha d_n] = \omega^2[(1+\alpha)f_{n+1} - \alpha f_n] \tag{6}$$

The two parameters β and γ are chosen following

$$\beta = (1-\alpha)^2/4 \quad , \quad \gamma = (1-2\alpha)/2 \tag{7}$$

where $\alpha \in [-1/3, 0]$ allows to tune the *numerical* damping of the α-method scheme. The scheme is second-order accurate. Note that for $\alpha = 0$, the implicit Newmark scheme based on the trapezoidal rule is recovered.

Solution of the free vibration problem

It is possible to derive the expression of the numerical displacement d_n, solution of the free vibration problem. The application of the free vibration decay method, that is the comparison of this expression with the analytical one, Eq. (2), leads to the introduction of the classical quantities characterizing the performance of the numerical scheme, namely the numerical (*time*) damping ratio $\bar{\xi}$ and the numerical (*time*) damped natural angular frequency $\bar{\omega}_d$. These quantities can be simply derived by the inspection of the amplification matrix of the scheme [1] and the '*time*' term in parentheses is introduced to recall the transient character of the analysis. For small values of the time step, the distortions of ξ and ω_d are given b

$$\frac{\xi - \bar{\xi}}{\xi} = \frac{4 - 18\alpha - 33\alpha^2 - 4\xi^2(1 - 6\alpha - 12\alpha^2)}{24(1-\xi^2)}(\omega\Delta t)^2 + O(\Delta t^3) \tag{8}$$

$$\frac{\omega_d - \bar{\omega}_d}{\omega_d} = \frac{2 - 6\alpha - 9\alpha^2 - \xi^2(10 - 48\alpha - 90\alpha^2) + 8\xi^4(1 - 6\alpha - 12\alpha^2)}{24(1-\xi^2)}(\omega\Delta t)^2 + O(\Delta t^3) \quad (9)$$

These results show that, when accuracy is required, it is always possible to limit the amount of damping or frequency distortion by reducing the time step.

When $\omega\Delta t > 2$, an accurate response is not required. This case is often encountered in structural problems involving many degrees-of-freedom, where only the lower vibration modes are of interest. The highest vibration modes are processed with a large value of the time step. For these modes, the implicit numerical scheme is used on account of its unconditional stability, not of its possible accuracy. The distortions are thus no longer of interest. Note however that $\bar{\xi}$ and $\bar{\omega}_d$ tend towards a limit

$$\bar{\xi} = -\frac{(1-\xi^2)^{1/2}}{\pi}\ln\left(\frac{1+\alpha}{1-\alpha}\right) + O\left(\frac{1}{\sqrt{\Delta t}}\right) \quad (10)$$

$$\bar{\omega}_d\Delta t = \pi + O\left(\frac{1}{\sqrt{\Delta t}}\right) \quad (11)$$

Note that the limit for $\bar{\omega}_d$ corresponds to the highest reduced angular frequency (reduced cut-off angular frequency) that can be analysed for a given choice of Δt. Recall that in a digital signal with a time sampling Δt, the largest available frequency, called the cut-off frequency, is $1/(2\Delta t)$. The associated cut-off angular frequency is $2\pi \times [1/(2\Delta t)] = \pi/\Delta t$.

Indications about the variations of the time quantities $\bar{\omega}_d$ and $\bar{\xi}$ with respect to the time step can be found in Fig. 1 and Fig. 2 respectively, for various values of the damping ξ and of α.

Solution of the forced vibration problem

Although tedious calculations are required [4], it is also possible to obtain the expression of the numerical displacement d_n, solution of the forced vibration problem. This expression tends toward an undamped symmetric oscillation, with angular frequency $\tilde{\omega}$ and by analogy with the analytical solution, it is then possible to derive a numerical dynamic amplification factor $D^{num}(\omega\Delta t, \xi, \tilde{\omega}\Delta t)$. Since D^{num} presents a maximum D_r^{num} with respect to $\tilde{\omega}$, it is then possible to apply the resonant amplification method. The location and the peak value of the numerical response function of the forced oscillator are used to propose an alternative def-

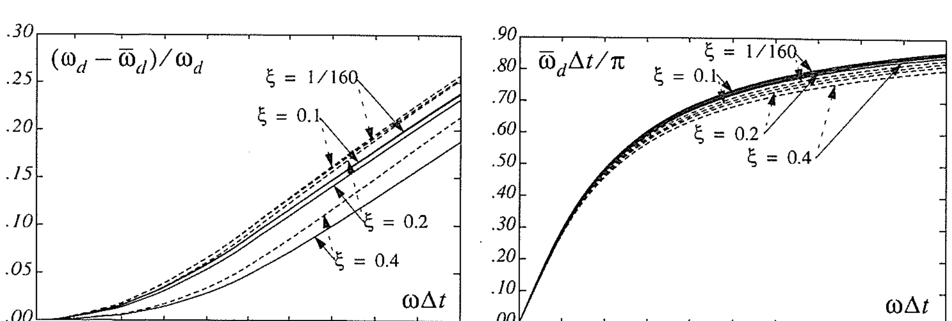

Figure 1 - α-method scheme: natural angular frequency distortion (left) and numerical damped natural angular frequency (right) (solid line $\alpha = -0.1$, dotted line $\alpha = -0.2$).

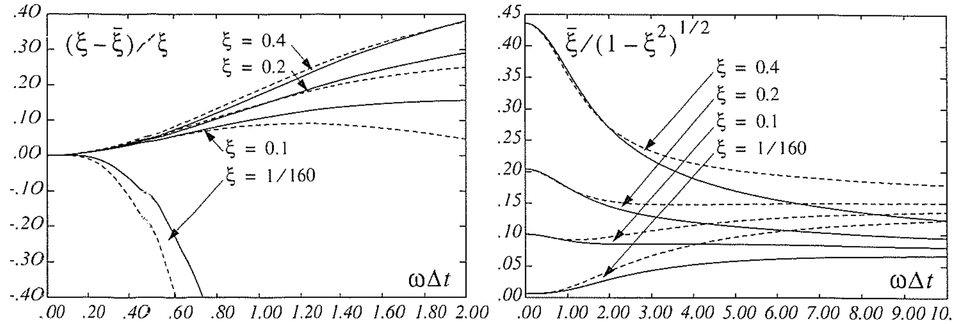

Figure 2 - α-method scheme: time damping distortion (left) and numerical time damping ratio (right) (solid line $\alpha = -0.1$, dotted line $\alpha = -0.2$).

inition of the numerical frequency and damping associated with the numerical scheme: the numerical frequency damping ratio $\bar{\xi}_r$ and the numerical resonant angular frequency $\bar{\omega}_r$.

$$D^{\text{num}}(\omega\Delta t, \xi, \bar{\omega}_r\Delta t) = D_r^{\text{num}}(\omega\Delta t, \xi) \tag{12}$$

$$2\bar{\xi}_r(1-\bar{\xi}_r^2)^{-1/2} = D_r^{\text{num}}(\omega\Delta t, \xi) \tag{13}$$

As for the time quantities $\bar{\xi}$ and $\bar{\omega}_d$, various expansion can be obtained for $\bar{\xi}_r$ and $\bar{\omega}_r$.

- Small values of the time step

$$\frac{\xi-\bar{\xi}_r}{\xi} = -2\xi\frac{1-\xi^2}{1-2\xi^2}\omega\Delta t + O(\Delta t^2) \tag{14}$$

$$\frac{\tilde{\omega}_d-\bar{\omega}_d}{\tilde{\omega}_d} = \left[\frac{1}{12}-\frac{1}{4}\alpha-\frac{3}{8}\alpha^2-\frac{1}{12}\xi^2(2-18\alpha-39\alpha^2)\right](\omega\Delta t)^2 + O(\Delta t^3) \tag{15}$$

- Large values of the time step

$$\bar{\xi}_r = \frac{1}{\sqrt{2}} + O\left(\frac{1}{\sqrt{\Delta t}}\right) \tag{16}$$

$$\bar{\omega}_r\Delta t = \cos^{-1}\left(-\frac{1-4\alpha^2-\alpha^4}{1-2\alpha^2+\alpha^4}\right) + O\left(\frac{1}{\Delta t}\right) \tag{17}$$

Indications about the variations of the frequency quantities $\bar{\omega}_r$ and $\bar{\xi}_r$ with respect to the time step can be found in Fig. 3 and Fig. 4 respectively, for various values of the damping ξ and of α.

Discussion of the results

Depending on the values of the time step, the results of the time or frequency analyses are not always similar:

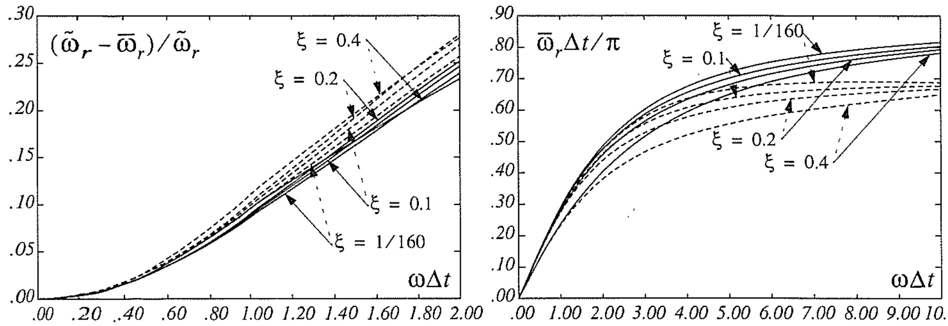

Figure 3 - α-method scheme: angular frequency distortion (left) and reduced numerical damped resonant angular frequency (right) (solid line $\alpha = -0.1$, dotted line $\alpha = -0.2$).

- For small values of the time step, that is when accuracy is particularly requested, the two analyses lead essentially to the same conclusion: the amount of distortion of the quantities under consideration (numerical damping ratios and angular frequencies) can always be controlled by reducing the time step. As can be seen from Eq. (9) and (15) and in the left part of Fig. 1 and Fig. 3, the angular frequency distortions are almost identical ($O(\Delta t^2)$). The damping distortions are different since the time damping distortion is $O(\Delta t^2)$ (Eq. (8)) whereas the frequency damping distortion is only $O(\Delta t)$ (Eq. (14)), which is also visible on the left part of Fig. 2 and Fig. 4. The distortion curves are provided for values of the reduced angular frequency $\omega\Delta t$ up to the limit of stability of the explicit central difference scheme.

- For large values of the time step, the time and the frequency analyses lead to different conclusions for both the numerical angular frequencies and damping ratios. π is no longer the limit of the numerical resonant reduced angular frequency (compare Eq. (11) and Eq. (17), or the right side of Fig. 1 and Fig. 3). This can already be considered as an advantage if the scheme is subjected to a large round-off excitation: in contrast with the time analysis, the frequency analysis shows clearly that the cut-off frequency is now unable to be close to a strongly distorted high frequency mode of *any* structure, and thus is partly protected against spurious error propagation. However, the most relevant difference concerns the numerical damping ratios. The time analysis shows that a possibly strong damping is introduced for the large frequencies in the transient response of a complex structure, and the maximum value of this damping is closely related to α (see Eq. (10) and the right side of Fig. 2). The

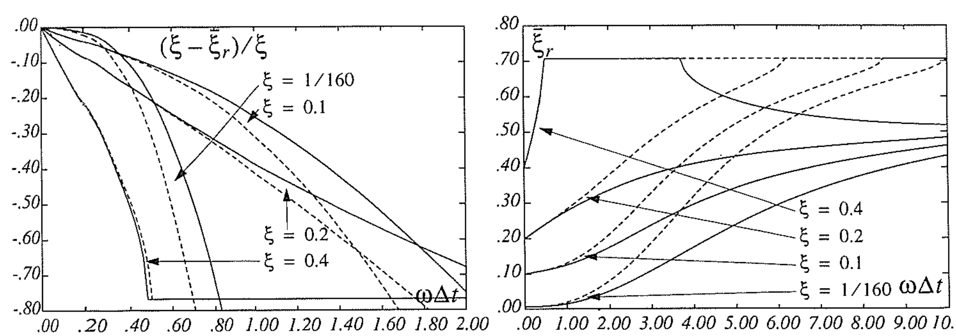

Figure 4 - α-method scheme: frequency damping distortion (left) and numerical frequency damping ratio (right) (solid line $\alpha = -0.1$, dotted line $\alpha = -0.2$).

frequency analysis shows that the α-method scheme always achieves optimal damping characteristics since, as revealed by Eq. (16), $1/\sqrt{2}$ is the maximum damping ratio reachable for a resonant system. In fact in this case the maximum value of the dynamic magnification factor is 1. This optimal property is independent of the value of ξ and α, although the transition towards this asymptotic behaviour becomes steeper and steeper for decreasing negative values of α, as illustrated in the right side of Fig. 3. This last figure shows that all curves for $\alpha = -0.2$ reach the $1/\sqrt{2}$ value of the damping for finite values of Ω. This is due to the fact that D_r^{num} tends towards 1 from below. In this range of variation ($D_r^{\text{num}} < 1$), consistently, the resonant amplification method no longer provides real solutions for ξ_r. For these cases, it was decided to affect to ξ_r the maximum value $1/\sqrt{2}$.

The results obtained with the frequency analysis allow to give a flavour of what the numerical frequency transfer function of a multi-degree-of-freedom structure could be. In fact it is easy to show that considering a unique oscillator and performing the time integration with different time steps is equivalent to considering different oscillators and performing the time integration with a unique time step. The Fig. 5 is obtained according to this equivalence. The top part of this figure presents the analytical transfer functions of a sequence of different oscillators whereas the bottom part shows the numerical counterpart of these transfer functions for a given time step. The natural undamped frequencies of the oscillator ranging from 0.1 to 50, the external angular frequency axes of Fig. 5 use a logarithmic graduation. Recalling that the cut-off angular frequency associated with the chosen time step $\Delta t = 1$ is π, the following points may be underlined.

- Contrary to a common belief, none of the resonant frequencies are filtered out by the implicit scheme. They still exist in the numerical response and may be distorted in such a way that they can be unexpectedly excited.

- Even if the analytical structure possesses well separated high resonant frequencies, their distortion close to the cut-off frequency may render them almost indistinguishable. Many modes may be spuriously triggered, even by a narrow banded excitation close to the cut-off frequency. This is particularly true when $\alpha = 0$.

- The α-method scheme acts as a filter: the high frequency modes, which may accumulate at the vicinity of the distortion limit of the scheme, are no longer amplified.

CONCLUSIONS

The study of the implicit α-method scheme shows again that the time and the frequency viewpoints are in close agreement for small values of the time step whereas clear differences appear for large values of Δt. In particular, while the limit of the numerical time damping ratio increases with $|\alpha|$, the numerical frequency damping ratio tends always towards the same optimal value whatever α may be ($-1/3 \le \alpha < 0$). This last result is important: the discussion about the 'best' choice of α cannot be based only on asymptotic values, as the behaviour of the numerical time damping would suggest. Adopting the frequency viewpoint focuses the attention on the transition frequency zone between accurate response and asymptotic optimal damping, and the type of attenuation curves presented with dotted lines in Fig. 5 may then contribute to the effective choice of α.

Given a practical problem, the choice of the time step stems from the accurate representation of the highest frequency mode of interest for the problem. Thereafter, the choice of α results from an analysis taking into account:

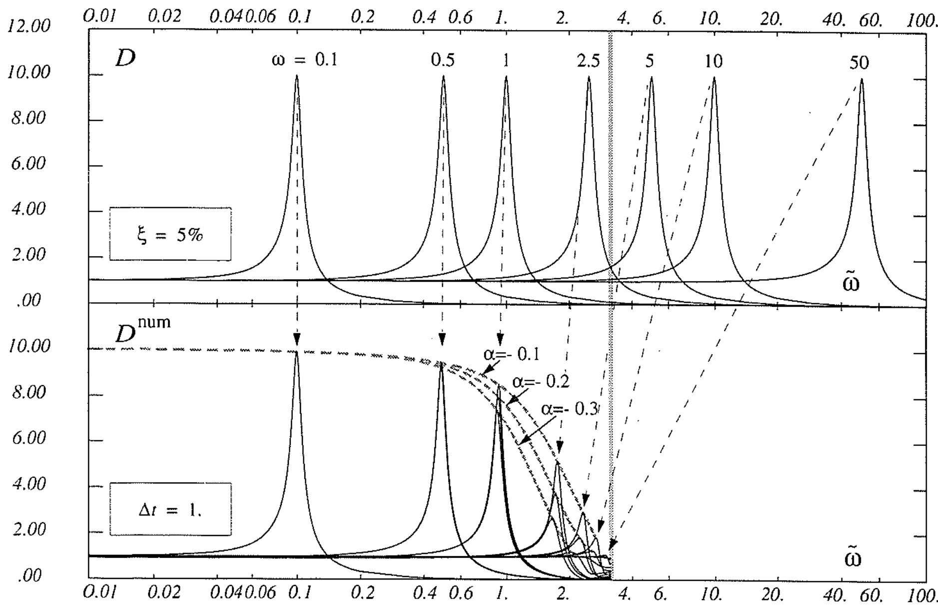

Figure 5 - α-method scheme: exact (top) and numerical (bottom) transfer functions for various oscillators and various values of α.

- the natural frequency, the damping ratio and the participation factor of the modes pertaining to the transition frequency zone,
- the spectrum of the loading in this zone.

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