UNIVERSITY OF NORTH CAROLINA

Department of Statistics
Chapel Hill, N. C.

MONOTONICITY OF THE POWER FUNCTIONS OF SOME TESTS FOR A PARTICULAR KIND OF MULTICOLLINEARITY? AND UNBIASEDNESS AND RESTRICTED MONOTONICITY OF THE POWER FUNCTIONS OF THE STEP DOWN PROCEDURES FOR MANOVA AND INDEPENDENCE

by

C. G. Khatri

April 1964

Contract No. AF-AFOSR-84-63

This research was supported by the Mathematics Division of the Air Force Office of Scientific Research.

Institute of Statistics
Mimeo Series No. 388

MONOTONICITY OF THE POWER FUNCTIONS OF SOME TESTS FOR A PARTICULAR KIND OF MULTICOLLINEARITY, AND UNBIASEDNESS AND RESTRICTED MONOTONICITY OF THE POWER FUNCTIONS OF THE STEP DOWN PROCEDURES FOR MANOVA AND INDEPENDENCE

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C. G. Khatri
University of North Carolina and Gujarat University

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1. Introduction:

The problems of a particular kind of multicollinearity of means defined by Roy [7] and extended by the author [4, 5] to that of regression coefficients were reduced to a single form [5] in the following way.

Let the joint density function of the elements of the random matrices $\underset{\sim}{Y}$: $(p+q)\times n$, $(n \geq p+q)$ and $\underset{\sim}{X}$: $(p+q)\times m$ be

(1) $$MN(\underset{\sim}{Y};\ \underset{\sim}{0},\ \underset{\sim}{\Sigma})\quad MN(\underset{\sim}{X};\ \underset{\sim}{\beta}\underset{\sim}{Z},\underset{\sim}{\Sigma})$$

where

(2) $$MN(\underset{\sim}{A}:\ r\times s;\ \underset{\sim}{B}:\ r\times s,\ \underset{\sim}{C}:\ r\times r) = (2\pi)^{-\frac{1}{2}rs}\,|\underset{\sim}{C}|^{-\frac{1}{2}s}\exp[-\tfrac{1}{2}\operatorname{tr}\underset{\sim}{C}^{-1}(\underset{\sim}{A}-\underset{\sim}{B})(\underset{\sim}{A}-\underset{\sim}{B})'],$$

$\underset{\sim}{Z}$: $u\times m$ is a known matrix of rank $u(\leq m)$, $\underset{\sim}{\Sigma}:(p+q)\times(p+q)$ is an unknown covariance matrix and $\underset{\sim}{\beta}$: $(p+q)\times u$ is an unknown regression matrix. Let us write $\underset{\sim}{X}' = (\underset{\sim}{X}_1'\ \ \underset{\sim}{X}_2')$, $\underset{\sim}{Y}' = (\underset{\sim}{Y}_1'\ \ \underset{\sim}{Y}_2')$,

$$\underset{\sim}{\Sigma} = \begin{pmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \underset{\sim}{\Sigma}_{12}' & \underset{\sim}{\Sigma}_{22} \end{pmatrix} \quad\text{and}\quad \underset{\sim}{\beta} = \begin{pmatrix} \underset{\sim}{\beta}_1 \\ \underset{\sim}{\beta}_2 \end{pmatrix},$$ where $\underset{\sim}{\Sigma}_{11}$: $p\times q$, $\underset{\sim}{\Sigma}_{22}$: $q\times q$, $\underset{\sim}{\Sigma}_{12}$: $p\times q$,

$\underset{\sim}{\beta}_1$: $p\times u$, $\underset{\sim}{\beta}_2$: $q\times u$, $\underset{\sim}{X}_1$: $p\times m$, $\underset{\sim}{X}_2$: $q\times m$, $\underset{\sim}{Y}_1$: $p\times m$ and $\underset{\sim}{Y}_2$: $q\times m$.

Then, to test the hypothesis of multicollinearity, H_o, given by

(3) $$H_o:\ (\underset{\sim}{\beta}_{1.2} = \underset{\sim}{\beta}_1 - \underset{\sim}{\Sigma}_{12}\underset{\sim}{\Sigma}_{22}^{-1}\underset{\sim}{\beta}_2 = \underset{\sim}{0})$$

against the alternative $H(\underset{\sim}{\beta}_{1.2} \neq \underset{\sim}{0})$, let us consider the following three test procedures:

(4) the likelihood ratio test [3, 5] whose acceptance region is

$$\prod_{i=1}^{p} (1 + c_i) \leq \lambda_1 \text{ , a constant ,}$$

(5) the trace test [3] whose acceptance region is $\sum_{i=1}^{p} c_i \leq \lambda_2$, a constant;

and

(6) the maximum root test [3, 4] whose acceptance region is $c_1 \leq \lambda_3$, a constant, where $c_1 \geq c_2 \geq \ldots \geq c_p$ are the characteristic (ch.) roots of $\underset{\sim}{S}_h \underset{\sim}{S}_e^{-1}$, $\underset{\sim}{S}_h$ and $\underset{\sim}{S}_e$ being the matrices of sums of products due to hypothesis and due to error, respectively, with

$$(7) \qquad \underset{\sim}{S}_h = [\underset{\sim}{X}_1 - \underset{\sim}{Y}_1\underset{\sim}{Y}_2'(\underset{\sim}{Y}_2\underset{\sim}{Y}_2')^{-1}\underset{\sim}{X}_2]\ [\underset{\sim}{I} + \underset{\sim}{X}_2'(\underset{\sim}{Y}_2\underset{\sim}{Y}_2')^{-1}\underset{\sim}{X}_2]^{-1}[\underset{\sim}{X}_1 - \underset{\sim}{Y}_1\underset{\sim}{Y}_2'(\underset{\sim}{Y}_2\underset{\sim}{Y}_2')^{-1}\underset{\sim}{X}_2]'$$

and

$$(8) \qquad \underset{\sim}{S}_e = \underset{\sim}{Y}_1\underset{\sim}{Y}_1' - \underset{\sim}{Y}_1\underset{\sim}{Y}_2'(\underset{\sim}{Y}_2\underset{\sim}{Y}_2')^{-1}\ \underset{\sim}{Y}_2\underset{\sim}{Y}_1' \text{ .}$$

For computational purposes, we may note that (7) can be rewritten as

$$(7') \qquad \underset{\sim}{S}_h = (\underset{\sim}{X}_1+\underset{\sim}{Y}_1)(\underset{\sim}{X}_1+\underset{\sim}{Y}_1)' - (\underset{\sim}{X}_1+\underset{\sim}{Y}_1)(\underset{\sim}{X}_2+\underset{\sim}{Y}_2)'[(\underset{\sim}{X}_2+\underset{\sim}{Y}_2)(\underset{\sim}{X}_2+\underset{\sim}{Y}_2)']^{-1}(\underset{\sim}{X}_2+\underset{\sim}{Y}_2)(\underset{\sim}{X}_1+\underset{\sim}{Y}_1)' - \underset{\sim}{S}_e \text{ .}$$

We consider, in this paper, the step down procedures for two stages only. The results for more than two stages are similar. The step down procedures for MANOVA (refer J. Roy [6]) can be stated as follows:

Let the first stage parameters be $\underset{\sim}{\beta}_2$ and the second stage parameters be $\underset{\sim}{\beta}_{1.2} = \underset{\sim}{\beta}_1 - \underset{\sim}{\Sigma}_{12}\ \underset{\sim}{\Sigma}_{22}^{-1}\ \underset{\sim}{\beta}_2$. Then, the null hypothesis for MANOVA is

$$(9) \qquad H_o\ \underset{\sim}{\beta} = \underset{\sim}{0}) = [H_{01}(\ \underset{\sim}{\beta}_2 = \underset{\sim}{0})] \bigcap\ [H_{02}(\ \underset{\sim}{\beta}_{1.2} = \underset{\sim}{0})] \text{ ,}$$

and the alternative hypothesis is $H(\underset{\sim}{\beta} \neq \underset{\sim}{0}) = [H_1(\ \underset{\sim}{\beta}_1 \neq \underset{\sim}{0})] \bigcup [H_2(\ \underset{\sim}{\beta}_{1.2} \neq \underset{\sim}{0})]$. We propose below only three test criteria for testing (9), but many more can be stated depending on likelihood, trace and maximum root tests:

(10) An intersection procedure based on the likelihood ratio test, at each stage, whose acceptance region is the intersection of $\prod_{j=1}^{q} (1 + c_{1,j}) \leq \lambda_{1,1}$ and $\prod_{i=1}^{p} (1 + c_i) \leq \lambda_1$, $\lambda_{1,1}$ and λ_1 being constants;

(11) an intersection procedure based on the trace test, at each stage, whose acceptance region is the intersection of $\sum_{j=1}^{q} c_{1,j} \leq \lambda_{1,2}$ and $\sum_{i=1}^{p} c_i \leq \lambda_2$, $\lambda_{1,2}$ and λ_2 being constants; and

(12) an intersection procedure based on the maximum root test, at each stage, whose acceptance region is the intersection of $c_{1,1} \leq \lambda_{1,3}$ and $c_1 \leq \lambda_3$, $\lambda_{1,3}$ and λ_3 being constants, where $c_{1,1} \geq c_{1,2} \geq \cdots \geq c_{1,q}$ are the characteristic roots of $(\underset{\sim}{X}_2\underset{\sim}{X}_2')(\underset{\sim}{Y}_2\underset{\sim}{Y}_2')^{-1}$ and $c_1 \geq c_2 \geq \cdots \geq c_p$ are the ch. roots of $\underset{\sim}{S}_h\underset{\sim}{S}_e^{-1}$ with $\underset{\sim}{S}_h$ and $\underset{\sim}{S}_e$ as defined in (7) and (8).

Now, the step down procedures at two stages for the independence of sets as considered by Roy and Bargmann [8] can be summarized as below:

Let $\underset{\sim}{X}' = (\underset{\sim}{X}_1' \;\; \underset{\sim}{X}_2' \;\; \underset{\sim}{X}_3')$, $\underset{\sim}{X}: (p_1 + p_2 + p_3) \times n$, $\underset{\sim}{X}_1: p_1 \times n$, $\underset{\sim}{X}_2: p_2 \times n$ and $\underset{\sim}{X}_3: p_3 \times n$ have a joint density function as

$$\text{MN}(\underset{\sim}{X};\ \underset{\sim}{0}, \underset{\sim}{\Sigma}) \qquad (13)$$

where

$$\underset{\sim}{\Sigma} = \begin{pmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} & \underset{\sim}{\Sigma}_{13} \\ \underset{\sim}{\Sigma}_{12}' & \underset{\sim}{\Sigma}_{22} & \underset{\sim}{\Sigma}_{23} \\ \underset{\sim}{\Sigma}_{13}' & \underset{\sim}{\Sigma}_{23}' & \underset{\sim}{\Sigma}_{33} \end{pmatrix} \begin{matrix} p_1 \\ p_2 \\ p_3 \end{matrix}$$
$$\qquad\quad p_1 \qquad p_2 \qquad p_3$$

is positive definite. Here the first stage parameters are $\underset{\sim}{\Sigma}_{12}$ and the second stage parameters are $(\underset{\sim}{\Sigma}_{13}$ and $\underset{\sim}{\Sigma}_{23})$. That is

(14) $H_0(\Sigma_{ij} = 0 \text{ for } i \neq j) = H_{01}(\Sigma_{12} = 0) \cap H_{02}(\Sigma_{13} = 0, \Sigma_{23} = 0)$

against the alternative $H(\Sigma_{ij} \neq 0 \text{ for } i \neq j) = [H_1(\Sigma_{12} \neq 0)] \cup [H_2(\Sigma_{13} \neq 0, \Sigma_{23} \neq 0)]$.

As before, we give below three test procedures:

(15) An intersection procedure based on the likelihood ratio test, at each stage, whose acceptance region is the intersection of $\prod_{j=1}^{p_2} (1 + \omega_{1,j}) \leq \lambda_{1,4}$ and $\prod_{i=1}^{p_3} (1 + \omega_{2,j}) \leq \lambda_{2,4}$, $\lambda_{1,4}$ and $\lambda_{2,4}$ being constants;

(16) An intersection procedure based on the trace test, at each stage, whose acceptance region is the intersection of $\sum_{j=1}^{p_2} \omega_{1,j} \leq \lambda_{1,5}$ and $\sum_{j=1}^{p_3} \omega_{2,j} \leq \lambda_{2,5}$, $\lambda_{1,5}$ and $\lambda_{2,5}$ being constants, and

(17) an intersection procedure based on the maximum root test, at each stage, whose acceptance region is the intersection of $\omega_{1,1} \leq \lambda_{1,6}$ and $\omega_{2,1} \leq \lambda_{2,6}$, $\lambda_{1,6}$ and $\lambda_{2,6}$ being constants

where $\omega_{1,1} \geq \omega_{1,2} \geq \cdots \geq \omega_{1,p_2}$ are the ch. roots of $S_{1,h} S_{1,e}^{-1}$ and $\omega_{2,1} \geq \omega_{2,2} \geq \cdots \geq \omega_{2,p_3}$ are the ch. roots of $S_{2,h} S_{2,e}^{-1}$ with

(18) $S_{1,h} = X_2 X_1'(X_1 X_1')^{-1} X_1 X_2'$ and $S_{1,e} = X_2 X_2' - S_{1,h}$,

and

(19) $S_{2,h} = X_3(X_1' \; X_2') \left[\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} (X_1' \; X_2') \right]^{-1} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} X_3'$ and $S_{2,e} = X_3 X_3' - S_{2,h}$.

Results on the monotonicity of the test procedures when $q=0$ and $p_3 = 0$ in the above cases were first established by Roy and Mikhail [9, 10], Srivastava [11] and then by a different method due to Das Gupta, Anderson and Mudholkar [1] and Anderson and Das Gupta [2]. In this paper, we prove the monotonicity of the test procedures for multicollinearity and restricted monotonicity and unbiasedness of the step down procedures for MANOVA and independence.

The orders of the identity matrix I and the null matrix 0 will be understood by the context.

2. Multicollinearity and step down procedures for MANOVA:

Lemma 1: The ch. roots of $S_h S_e^{-1}$ and $(X_2 X_2')(Y_2 Y_2')^{-1}$ are invariant under the transformations $(G_1 X_1 + G_2 X_2)\ \Delta_1$, $G_3 X_2\ \Delta_1$, $(G_1 Y_1 + G_2 Y_2)\ \Delta_2$ and $G_3 Y_2\ \Delta_2$ where G_1: pxp and G_3: qxq are non-singular, Δ_1: mxm and Δ_2: nxn are orthogonal matrices and G_2: pxq.

Proof: Let $R_1 = (G_1 X_1 + G_2 X_2)\ \Delta_1$, $R_2 = G_3 X_2\ \Delta_1$, $V_1 = (G_1 Y_1 + G_2 Y_2)\ \Delta_2$ and $V_2 = G_3 Y_2\ \Delta_2$. Then

$$R_2'(V_2 V_2')^{-1} R_2 = \Delta_1' X_2'(Y_2 Y_2')^{-1} X_2 \Delta_1 \ ,$$

$$R_1 - V_1 V_2'(V_2 V_2')^{-1} R_2 = G_1 X_1 \Delta_1 + G_2 X_2 \Delta_1 - (G_1 Y_1 + G_2 Y_2) Y_2'(Y_2 Y_2')^{-1} X_2 \Delta_1$$

$$= G_1 [X_1 - Y_1 Y_2'(Y_2 Y_2')^{-1} X_2] \Delta_1$$

and

$$V_1 V_1' - V_1 V_2'(V_2 V_2')^{-1} V_2 V_1' = (G_1 Y_1 + G_2 Y_2)[I - Y_2'(Y_2 Y_2')^{-1} Y_2](G_1 Y_1 + G_2 Y_2)'$$

$$= G_1 [Y_1 Y_1' - Y_1 Y_2'(Y_2 Y_2')^{-1} Y_2 Y_1'] G_1' \ .$$

Moreover, the ch. roots of AB are the ch. roots of BA except for some zero ch. roots. (See Roy [7]). Using these results, the lemma 1 is obvious.

Lemma 2: There exist matrices U: pxm , W: px(n-q), R: qxm and V: qxn having a joint density function

$$(20) \qquad MN(U;\ \theta,\ I)\ \ MN(W;\ 0,\ I)\ \ MN(R;\ \eta_2,\ I)\ \ MN(V;\ 0,\ I)\ ,$$

where θ: pxm $= (\theta_{ij})$, $\theta_{ij} = 0$ for $i \neq j$, θ_{ii}^2 $(i = 1, 2, \ldots)$ are the possibly non-zero ch. roots of * $\eta_{1,ij} = 0$ for $(i \neq j)$ and $\eta_{1,ii}^2$ $(i=1,2,\ldots)$ are the possibly nonzero ch. roots of $(\beta_{1.2} Z Z' \beta_{1.2}' \Sigma_{1.2}^{-1})$, $\Sigma_{1.2} = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{12}'$, and $\eta_2 \eta_2'$ is a diagonal matrix with diagonal elements α_i^2 which are the possibly nonzero ch. roots of

* $\eta_1 [I + R'(V V')^{-1} R]^{-1} \eta_1'$, $\eta_1 : p \times m = (\eta_{1,ij})$,

$(\beta_2 ZZ' \beta_2' \Sigma_{22}^{-1})$. Moreover, c_i's, the ch. roots of $S_h S_e^{-1}$, are the ch. roots of $(UU')(WW')^{-1}$ and $c_{1,j}$'s, the ch. roots of $(X_2 X_2')(Y_2 Y_2')^{-1}$, are the ch. roots of $(RR')(VV')^{-1}$.

Proof: Let $\Sigma = TT'$ where $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ such that $T_1^2 = \Sigma_{1.2}$, $T_3^2 = \Sigma_{22}$, $T_2 = \Sigma_{12}\Sigma_{22}^{-\frac{1}{2}}$, T_1 and T_3 are symmetric matrices. Let $\delta' = (T^{-1}\beta Z)' = (\delta_1' \;\; \delta_2')$, δ_1: pxm and δ_2: qxm. Then $\delta_1 = T_1^{-1}\beta_{1.2}Z$ and $\delta_2 = T_3^{-1}\beta_2 Z$. Then we can write δ_1 and δ_2 as (refer [3]),

(20) $$\delta_1 = \Gamma_1 \eta_1 \Delta_1' \quad \text{and} \quad \delta_2 = \Gamma_2 \eta_2 ,$$

where Γ_1: pxp, Γ_2: qxq, Δ_1: mxm are orthogonal matrices, $\eta' = (\eta_1' \;\; \eta_2')$, $\eta_1 = (\eta_{1,ij})$: pxm, $\eta_{1,ij} = 0$ for $i \neq j$ and $\eta_{1,ii}^2$ are the possibly nonzero ch. roots of $\delta_1'\delta_1 = Z'\beta_{1.2}'\Sigma_{1.2}^{-1}\beta_{1.2}Z$ or $\Sigma_{1.2}^{-1}\beta_{1.2}ZZ'\beta_{1.2}'$, $\eta_2 = \eta_3\Delta$: qxm, Δ: mxm is an orthogonal matrix and $\eta_3 = (\eta_{3,ij})$, $\eta_{3,ij} = 0$ for $i \neq j$ and $\alpha_i^2 = \eta_{3,ii}^2$ are the possibly nonzero ch. roots of $\eta_2\eta_2'$ or $\delta_2\delta_2'$ or $\beta_2 ZZ' \beta_2' \Sigma_{22}^{-1}$. Let us apply the transformation

(21) $$R_o = \begin{pmatrix} R_1 \\ R \end{pmatrix} = \begin{pmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{pmatrix}' T^{-1} X \Delta_1 \quad \text{and} \quad V_o = \begin{pmatrix} V_1 \\ V \end{pmatrix} = \begin{pmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{pmatrix}' T^{-1} Y.$$

The jacobian of the transformation is $J(X, Y; R_o, V_o) = |\Sigma|^{\frac{1}{2}(m+n)}$ and by lemma 1, the ch. roots of $S_h S_e^{-1}$ are invariant. The joint density function of R_o and V_o can be written as

(22) $$MN(V_o; 0, I) \quad MN(R_o; \eta, I) ,$$

where $\eta' = (\eta_1' \;\; \eta_2')$.

The matrix $I - V'(VV')^{-1}V$ is idempotent of rank $(n-q)$. Then we can write $I - V'(VV')^{-1}V = \Delta_3 \Delta_3'$ where Δ_3': $(n-q) \times n$ is orthonormal (i.e. $\Delta_3' \Delta_3 = I$) such that $\Delta_3' V' = 0$. Let $\Delta_2 = V'(VV')^{-\frac{1}{2}}$. Then $(\Delta_3 \ \ \Delta_2)$ is an orthogonal matrix. Using the transformation

(23) $$(W_1 \ \ W_2) = V_1(\Delta_3 \ \ \Delta_2) \quad ,$$

the joint density function of R_1, R, V, W_1 and W_2 can be written as

(24) $$MN(W_1;\ 0,\ I)\ MN(W_2;0,I)\ \ MN(R_1;\ \eta_1,I)\ \ MN(R;\eta_2,I)\ \ MN(V;0,I) \ ,$$

and c_i's, the ch. roots of $S_h S_e^{-1}$ are the ch. roots of $L\{I+R'(VV')^{-1}R\}^{-1} L'(W_1W_1')^{-1}$, $L = R_1 - W_2(VV')^{-\frac{1}{2}} R$ and $c_{1,j}$'s, the ch. roots of $(X_2X_2')(Y_2Y_2')^{-1}$ are the ch. roots of $(RR')(VV')^{-1}$. Now from (24), we may note that the distribution of R_1 and W_2 are independent normal. Hence the distribution of L is normal with $E(L) = \eta_1$. To find the covariance matrix, let ℓ_i $(i = 1, 2, \ldots, p)$ be the i-th column of L'. Then it is easy to see that covariance matrix of ℓ_i is $I + R'(VV')^{-1}R$ and covariance matrix between ℓ_i and $\ell_{i'}$ is zero. Hence the distribution of L is

(25) $$MN[L';\ \eta_1',\ I + R'(VV')^{-1} R] \ .$$

Using the transformation $L[I + R'(VV')^{-1} R]^{-\frac{1}{2}} = U_1$, we get the joint distribution of U_1, W_1, R and V as

(26) $$MN(W_1;0,I) \quad MN(U_1;\xi,I) \quad MN(R;\eta_2,I) \quad MN(V;0,I)$$

where $\xi = \eta_1[I + R'(VV')^{-1}R]^{-\frac{1}{2}}$, and c_i's, the ch. roots of $S_h S_e^{-1}$ are the ch. roots of $(U_1U_1')(W_1W_1')^{-1}$.

Let us write (refer [3]) ,

(27) $$\underset{\sim}{\xi} = \underset{\sim}{\Gamma}_3 \, \underset{\sim}{\theta} \, \underset{\sim}{\Delta}_5$$

where $\underset{\sim}{\Delta}_5$: mxm and $\underset{\sim}{\Gamma}_3$: pxp are orthogonal matrices, and $\underset{\sim}{\theta}$: pxm = (θ_{ij}), $\theta_{ij} = 0$ for $i \neq j$ and θ_{ii}^2 are the possibly nonzero ch. roots of $\underset{\sim}{\xi}\underset{\sim}{\xi}'$.

Finally, using the transformation

(28) $$\underset{\sim}{U} = \underset{\sim}{\Gamma}_3' \, \underset{\sim}{U}_1 \, \underset{\sim}{\Delta}_5' \quad \text{and} \quad \underset{\sim}{W} = \underset{\sim}{\Gamma}_3' \, \underset{\sim}{W}_1 ,$$

we can write (26) as mentioned in the lemma 2. Thus, lemma 2 is established.

Theorem 1: An invariant test of multicollinearity for which the acceptance region is convex in each column vector of $\underset{\sim}{U}$ for each set of fixed $\underset{\sim}{W}$, fixed values of the other column vectors $\underset{\sim}{U}$ has a power function which is monotonically increasing in each $r_i = \eta_{1,ii}$.

Proof: It follows from theorem 3 of Das Gupta, Anderson and Mudholkar [1] that for given $\underset{\sim}{R}$ and $\underset{\sim}{V}$, the conditional probability of the acceptance region monotonically decreases in each θ_{ii}^2 $(i = 1, 2, \ldots, m)$ are the possibly nonzero ch. roots of $(\underset{\sim}{\eta}_1' \, \underset{\sim}{\eta}_1) \, \{\underset{\sim}{I} + \underset{\sim}{R}'(\underset{\sim}{V}\underset{\sim}{V}')^{-1} \underset{\sim}{R}\}^{-1}$, or $\underset{\sim}{P}(\underset{\sim}{\eta}_1' \, \underset{\sim}{\eta}_1)\underset{\sim}{P}$ where $\underset{\sim}{P} = \{ \underset{\sim}{I} + \underset{\sim}{R}'(\underset{\sim}{V}\underset{\sim}{V}')^{-1}\underset{\sim}{R} \}^{-\frac{1}{2}}$ is a symmetric matrix. Let $\underset{\sim}{\eta}_1^*$ be the matrix obtained from $\underset{\sim}{\eta}_1$ by changing the diagonal elements of $\underset{\sim}{\eta}_1$ from $r_i = \eta_{1,ii}$ to $\eta_{1,ii}^* = r_i^*$ where $r_i^* \geq r_i$ $(i = 1, 2, \ldots)$. Then $\eta_1^* \eta_1^{*\prime} - \eta_1 \eta_1'$ is positive semi-definite at least, and, since $\underset{\sim}{P}$ is a non-singular symmetric matrix, we get $\underset{\sim}{P} \, \underset{\sim}{\eta}_1^* \, \underset{\sim}{\eta}_1^{*\prime} \, \underset{\sim}{P} - \underset{\sim}{P} \, \underset{\sim}{\eta}_1 \, \underset{\sim}{\eta}_1' \, \underset{\sim}{P}$ to be positive semi-definite, and so $\theta_{ii}^* \geq \theta_{ii}$ for each i, and conversely if $\theta_{ii}^* \geq \theta_{ii}$, then $r_i^* \geq r_i$. Hence, for any fixed $\underset{\sim}{R}$ and $\underset{\sim}{V}$, the conditional probability of the acceptance region decreases monotonically in each r_i. The theorem 1 now follows from the fact that the marginal distribution $\underset{\sim}{R}$ and $\underset{\sim}{V}$ does not contain any r_i.

Now let W_k be the sum of all different products of $(1+c_1)$, $(1+c_2), \ldots,$ $(1+c_p)$ taken k at a time $(k = 1, 2, \ldots, p)$. Then by using theorem 1 above and the lemmas 2 and 3 of [1], we have the following corollaries:

Corollary 1: The maximum root test given by (6) has a power function which is monotonically increasing in each r_i.

Corollary 2: A test having the acceptance region $\sum_{k=1}^{p} a_k W_k \leq \mu$, $a_k \geq 0$, has a power function which is monotonically increasing in each r_i.

Corollary 3: The power function of the likelihood ratio test given by (4) increases as each r_i increases.

Corollary 4: The power function of the trace test given by (5) increases as each r_i increases.

Theorem 2: An invariant test for which the acceptance region is sectionwise convex at each stage of the step down procedure for MANOVA is unbiased and moreover is monotonically increasing with respect to the ch. roots of the parameters at the i-th stage when the parameters of (i-1) stages are fixed and all the parameters above the i-th stage are zero. [This type of monotonicity will be called the restricted monotonic property.]

This follows from theorem 1, lemma 2 and theorem 3 of [1] for the two stages of the step down procedure for MANOVA. The generalization is immediate.

Let W_{1j} be the sum of all different products of $(1+c_{1,1}), \ldots,$ $(1+c_{1,q})$ taken j at a time $(j = 1, 2, \ldots, q)$. Then using theorem 2, we get the following corollaries:

Corollary 5: A test having the acceptance region $\sum_{k=1}^{p} a_k W_k \leq \mu$, and $\sum_{j=1}^{q} b_j W_{1,j} \leq \mu$, $(a_k \geq 0,\ b_j \geq 0)$ of the step down procedure for MANOVA at two stages is unbiased and has the restricted monotonic property.

Corollary 6: The test procedures given by (10), (11) and (12) are unbiased and each has the restricted monotonic property.

It may be noted that the regions described above are all, in fact, not only conditionally sectionwise convex, but also conditionally sectionwise ellipsoids, so that, for proving the above results, we could as well have used Roy and Mikhail [9, 10].

3. Step down procedure for independence:

Let $\Sigma_{3.1,2} = \Sigma_{33} - (\Sigma'_{13}\ \Sigma'_{23}) \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma'_{12} & \Sigma_{22} \end{pmatrix}^{-1} \begin{pmatrix} \Sigma_{13} \\ \Sigma_{23} \end{pmatrix}$, $\Sigma_{2.1} =$

$\Sigma_{22} - \Sigma'_{12}\ \Sigma_{11}^{-1}\ \Sigma_{12}$, $E = \Sigma_{2.1}^{-\frac{1}{2}}\ \Sigma_{12}\ \Sigma_{11}^{-\frac{1}{2}}$ and

$$F = \Sigma_{3.1,2}^{-\frac{1}{2}}(\Sigma'_{13}\ \Sigma'_{23}) \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{pmatrix}^{-1} \begin{pmatrix} \Sigma_{11}^{\frac{1}{2}} & 0 \\ 0 & \Sigma_{2.1}^{\frac{1}{2}} \end{pmatrix}.$$

Now, we use the transformation $Y_1 = \Sigma_{11}^{-\frac{1}{2}} X_1$, $Y_2 = \Sigma_{2.1}^{-\frac{1}{2}} X_2$ and $Y_3 = \Sigma_{3.1,2}^{-\frac{1}{2}} X_3$ in (13). Since the ch. roots of $S_{i,h} S_{i,e}^{-1}$ (i = 1, 2) given by (18) and (19) are invariant, we replace Y_i by X_i (i = 1,2,3), and then write the joint density function of X_1, X_2 and X_3 as

$$(29) \quad MN(X_1;\ 0, I)\ \ MN(X_2;\ EX_1,\ I)\ \ MN[X_3;\ F\begin{pmatrix} X_1 \\ X_2 \end{pmatrix},\ I] \quad ,$$

where E: $p_2 x p_1$ and F: $p_3 x(p_1+p_2)$ are defined above.

Since $I - (X'_1\ X'_2) \left[\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} (X'_1\ X'_2)\right]^{-1} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ and $I - X'_1(X_1X'_1)^{-1}X_1$

are idempotent matrices of ranks $(n - p_1 - p_2)$ and $(n - p_1)$ respectively, we can write them respectively as $\Delta_1\ \Delta'_1$ and $\Delta_3\ \Delta'_3$ where $\Delta'_1:(n-p_1-p_2)xn$ and $\Delta'_3:(n-p_1)xn$ are orthonormal (i.e. $\Delta'_1\ \Delta_1 = I$ and $\Delta'_3\ \Delta_3 = I$)

such that $\Delta_1'(X_1' \; X_2') = 0$ and $\Delta_3' X_2' = 0$. Let us write $\Delta_2'(X_1' \; X_2') \left[\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} (X_1' \; X_2')\right]^{-\frac{1}{2}}$ and $\Delta_4 = X_1'(X_1 X_1')^{-\frac{1}{2}}$. Then $(\Delta_1 \; \Delta_2)$ and $(\Delta_3 \; \Delta_4)$ are orthogonal matrices. Now transform X_2 and X_3 by

(30) $\quad (W_1 \; W_2) = X_2(\Delta_3 \; \Delta_4)$ and $(V_1 \; V_2) = X_3(\Delta_1 \; \Delta_2)$.

Since the transformations are conditional in X_2, we have the jacobian of the transformation $J(X_2, X_3; W_1, W_2, V_1, V_2) = J(X_2; W_1, W_2) \; J(X_3; V_1, V_2) = 1$, and so the density function of X_1, W_1, W_2, V_1 and V_2 can be written as

(31) $\quad MN(X_1; 0, I) \;\; MN(W_1; 0, I) \;\; MN(W_2; E(X_1X_1')^{\frac{1}{2}}, I) \; MN(V_1; 0, I) \; MN(V_2; FG, I)$,

where $G = \begin{pmatrix} X_1X_1' & (X_1X_1')^{\frac{1}{2}} W_2' \\ W_2(X_1X_1')^{\frac{1}{2}} & W_2W_2' + W_1W_1' \end{pmatrix}^{\frac{1}{2}}$, the ch. roots of $S_{1,h}S_{1,e}^{-1}$ are the ch. roots of $(W_2W_2')(W_1W_1')^{-1}$, and the ch. roots of $S_{2,h}S_{2,e}^{-1}$ are the ch. roots of $(V_2V_2')(V_1V_1')^{-1}$.

Let us write

(32) $$E(X_1 \; X_1')^{\frac{1}{2}} = \Gamma_2 \, \eta_1 \, \Gamma_1 \quad \text{and} \quad FG = \Gamma_3 \, \eta_2 \, \Gamma_4 \quad ,$$

where Γ_1: $p_1 x p_1$, Γ_2: $p_2 x p_2$, Γ_3: $p_3 x p_3$ and Γ_4: $(p_1+p_2)x(p_1+p_2)$ are orthogonal matrices, $\eta_1 = (\eta_{1,ij})$: $p_2 \times p_1$, $\eta_{1,ij} = 0$ for $i \neq j$ and $\eta_{1,ii}^2$ are the possibly nonzero ch. roots of $[(X_1X_1')E'E]$ and $\eta_2 = (\eta_{2,ij})$: $p_3 \times (p_1+p_2)$, $\eta_{2,ij} = 0$ for $i \neq j$ and $\eta_{2,ii}^2$ are the possibly nonzero ch. roots of $(F'FG^2)$. Now, we apply the transformation

(33) $\quad U_2 = \Gamma_2' \, W_2 \, \Gamma_1'$, $U_1 = \Gamma_2' \, W_1$, $U_3 = \Gamma_3' \, V_1$ and $U_4 = \Gamma_3' \, V_2 \, \Gamma_4'$.

Since the transformations are conditional in $\underset{\sim}{W}_1$ and $\underset{\sim}{W}_2$, we get the jacobian of the transformation as $J(\underset{\sim}{W}_1, \underset{\sim}{W}_2, \underset{\sim}{V}_1, \underset{\sim}{V}_2; \underset{\sim}{U}_1, \underset{\sim}{U}_2, \underset{\sim}{U}_3, \underset{\sim}{U}_4)$ $= J(\underset{\sim}{W}_1, \underset{\sim}{W}_2; \underset{\sim}{U}_1, \underset{\sim}{U}_2)\ J(\underset{\sim}{V}_1, \underset{\sim}{V}_2; \underset{\sim}{U}_3, \underset{\sim}{U}_4) = 1$. Hence the joint density function of $\underset{\sim}{X}_1, \underset{\sim}{U}_1, \underset{\sim}{U}_2, \underset{\sim}{U}_3$ and $\underset{\sim}{U}_4$ is

(34) $\quad MN(\underset{\sim}{X}_1; \underset{\sim}{0}, \underset{\sim}{I}) \quad MN(\underset{\sim}{U}_1; \underset{\sim}{0}, \underset{\sim}{I}) \quad MN(\underset{\sim}{U}_2; \underset{\sim}{\eta}_1; \underset{\sim}{I}) \quad MN(\underset{\sim}{U}_3; \underset{\sim}{0}, \underset{\sim}{I}) \quad MN(\underset{\sim}{U}_4; \underset{\sim}{\eta}_2, \underset{\sim}{I})$,

where the ch. roots of $\underset{\sim}{S}_{1,h}\underset{\sim}{S}^{-1}_{1,e}$ are the ch. roots of $(\underset{\sim}{U}_2\underset{\sim}{U}_2')(\underset{\sim}{U}_1\underset{\sim}{U}_1')^{-1}$ and the ch. roots of $\underset{\sim}{S}_{2,h}\ \underset{\sim}{S}^{-1}_{2,e}$ are the ch. roots of $(\underset{\sim}{U}_4\underset{\sim}{U}_4')\ (\underset{\sim}{U}_3\underset{\sim}{U}_3')^{-1}$. On account of (34), we may note the following which are derivable by the similar arguments as in Section 2.

(35) If $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \underset{\sim}{\Sigma}_{11}, \underset{\sim}{\Sigma}_{12}$ and $\underset{\sim}{\Sigma}_{22}$ are fixed, then the power of the test due to the ch. roots of $(\underset{\sim}{U}_4\underset{\sim}{U}_4')(\underset{\sim}{U}_3\underset{\sim}{U}_3')^{-1}$ increases with each $\eta_{2,ii}$ and hence due to each ch. root of $\underset{\sim}{\Sigma}^{-1}_{3.1,2}(\ \underset{\sim}{\Sigma}_{33} - \underset{\sim}{\Sigma}_{3.1,2})$.

(36) If $\underset{\sim}{\Sigma}_{13} = \underset{\sim}{0}$ and $\underset{\sim}{\Sigma}_{23} = \underset{\sim}{0}$, and $\underset{\sim}{X}_1$ is fixed, then the power of the test due to the ch. roots of $(\underset{\sim}{U}_2\underset{\sim}{U}_2')(\underset{\sim}{U}_1\underset{\sim}{U}_1')^{-1}$ increases with each $\eta_{1,ii}$ and hence due to each ch. root of $\underset{\sim}{\Sigma}^{-1}_{2.1}(\ \underset{\sim}{\Sigma}_{22} - \underset{\sim}{\Sigma}_{2.1})$.

From (35) and (36), we have the following theorem.

<u>Theorem 3</u>: An invariant test for which the acceptance region is sectionwise convex at each stage of the step down procedure for independence is unbiased and has the restricted monotonic property.

Let $W_{1,j}$ be the sum of all different products of $(1 + \omega_{1,1})$, $(1 + \omega_{1,2}), \ldots, (1 + \omega_{1,p_2})$ taken j at a time $(j = 1,2,\ldots,p_2)$ and $W_{2,k}$ the sum of all different products of $(1 + \omega_{2,1}), (1 + \omega_{2,2}), \ldots,$ $(1 + \omega_{2,p_3})$ taken k at a time $(k = 1, 2, \ldots, p_3)$. Then using theorem 3, we have the following corollaries:

Corollary 7: A test having the acceptance region $\sum_{j=1}^{p_2} b_j W_{1,j} \leq \mu_1$ and $\sum_{k=1}^{p_3} a_k W_{2,k} \leq \mu_2$ ($b_j \geq 0$, $a_k \geq 0$) of the step down procedure for independence at two stages is unbiased and has the restricted monotonic property.

Corollary 8: The test procedures given by (15), (16) and (17) are unbiased and each has the restricted monotonic property. The same remarks (involving "the conditionally sectionwise ellipsoidal nature of the regions") apply here as at the end of section 2.

4. Acknowledgement: I am extremely thankful to Professor S. N. Roy for suggesting the step down procedures and for helpful discussions with him in the course of preparation of this paper.

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