ABSTRACT

TEACHEY, ANGELA LYNNE. Investigations in Conceptual Understanding of Polynomial Functions and the Impact of Mathematical Belief Systems on Achievement in an Accelerated Summer Program for Gifted Students. (Under the direction of Karen S. Norwood, Ed.D.)

This study investigated the achievement of gifted students on mathematics problems that were designed to assess both conceptual and procedural knowledge of polynomial functions, and it attempted to determine the impact of the students' mathematical belief systems on this achievement. The students were enrolled in a three-week Algebra II course at a summer program for gifted mathematics students. Data sources were belief scales, in-class examinations, and in-depth interviews. Qualitative and quantitative analyses indicated that the students were able to make a variety of connections among concepts related to polynomials and functions, and they easily applied their mathematical knowledge to real world phenomena. The participants suffered, however, from several misconceptions relating to the understanding of the roles of the independent and dependent variables in functions. They also struggled with the concept of symmetry and how it relates to polynomial functions. Statistical analyses suggested that belief systems were correlated with achievement, but the conclusions from this study were ambiguous since the correlations were unexpectedly negative. Through its identification of potential conceptual difficulties that gifted students may encounter in their learning of polynomial functions, this study suggested specific topics that teachers of gifted students should consider when planning their instructional activities.

INVESTIGATIONS IN CONCEPTUAL UNDERSTANDING OF POLYNOMIAL FUNCTIONS AND THE IMPACT OF MATHEMATICAL BELIEF SYSTEMS ON ACHIEVEMENT IN AN ACCELERATED SUMMER PROGRAM FOR GIFTED STUDENTS

by

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DEDICATION

To my family.

BIOGRAPHY

Angela Lynne Teachey was born December 11, 1971, in Wilmington, North Carolina. She was the older of two children born to Edwin Earl and Ann Price Teachey. She grew up in Rose Hill, North Carolina, and graduated from Wallace-Rose Hill High School in Teachey, North Carolina, in June 1990. Angela earned her Bachelor of Arts in Mathematics Education, awarded with highest honors, from the University of North Carolina at Chapel Hill in May 1994.

In the fall of 1994, she began teaching mathematics and computer programming at C. E. Jordan High School in Durham, North Carolina. She began coursework for her Master's of Arts in Teaching (M.A.T.) while working at Jordan High. During the 1998-99 academic year, she returned to the University of North Carolina as a full-time student and visiting faculty member in the mathematics department. She completed the M.A.T. in December 1998. In August 1999, Angela entered the Ph.D. program in Mathematics Education at North Carolina State University and served as a Graduate Teaching Assistant for Mathematics 101 for four semesters.

Beginning in summer 1998, Angela joined the Duke University Talent Identification Program (TIP) as a Summer Studies Algebra II instructor. The following summer, she became an academic coordinator and continued in that position for three consecutive summers. In August 2001, Angela joined the permanent TIP staff as a full-time academic coordinator while she completed her Ph.D. requirements. She joined the TIP research staff for the summer of 2002 in order to conduct her dissertation research.

Angela is a member of Phi Beta Kappa, Kappa Delta Pi, Phi Kappa Phi, and the Golden Key National Honor Society. As an undergraduate, she was a North Carolina Teaching Fellow. In 2001, she was named Outstanding Teaching Assistant for the Mathematics, Science, and Technology Education Department at North Carolina State University.

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As long as I can remember, I have loved learning. With each degree that I earned, I always felt as though my education was incomplete. Finally, with the help of many people, I believe that I have accomplished my educational goals. I have been surrounded by numerous loving and supportive family members, friends, colleagues, and mentors who encouraged me throughout my life.

My parents, Earl and Ann Teachey, have been a constant source of love, encouragement, and support. They instilled in me the importance of education. They always made me believe in myself, and they never allowed me to become discouraged. Without their emotional support, financial support, and editing skills, I never would have completed my formal education. I am truly blessed to have them in my life, and I love them very much.

My grandparents have always been with me, in mind and spirit. They were tremendous influences in my life, and they always believed in me and supported me. My brother and sister-in-law, Daniel and Stacey Teachey, have encouraged me and have helped me not to take life too seriously. My nephew, Price, has given me great joy, even during my most stressful days.

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INTRODUCTION

During the past several decades, the National Council of Teachers of Mathematics (NCTM) has encouraged mathematics teachers to modify their instructional strategies in order to help their students gain a deep understanding of mathematics. One of the most critical standards that NCTM has established for mathematics learning is the importance of making appropriate connections among mathematical concepts. Through these connections, NCTM contends that students' understanding will become deeper and more lasting and that students will more easily recognize the usefulness of mathematics (NCTM, 2000). Along with the importance of mathematical connections, a variety of researchers in mathematics education have found that students' beliefs about mathematics and about their own abilities in mathematics can have a strong impact on their ability to be successful in mathematics (Schoenfeld, 1985; Hackett, 1985; Betz, 1987; Fennema & Sherman,1977,1978).

In its Standard Course of Study, the North Carolina Department of Public Instruction (NCDPI) has established mathematics standards that align with the standards established by NCTM, including an emphasis on mathematical connections (NCSCOS, 1999). According to NCTM and NCDPI, acquiring an understanding of functions is one of the most important mathematics standards for the secondary grades. Since there are many valuable connections among mathematics concepts in the study of functions and since functions are so applicable in the real world, it is important to learn more about students' understandings of functions as they study this topic in their mathematics classes. One way to learn more about students' understandings of mathematical concepts is to study how they solve problems that require the use of those concepts.

Historically, mathematics educators and educational psychologists have attempted to understand the characteristics of good mathematics problem solvers. Through this understanding, they hope to develop strategies to help other students, who are not successful mathematics problem solvers, to improve their skills. In his analysis of a series of empirical studies that documented how competent mathematical problem solvers maximize the knowledge at their disposal, Schoenfeld (1985) identified four constructs that he believed contribute to students' problem solving: resources, heuristics, control, and beliefs. Hiebert and Lefevre (1986) have suggested that there are two types of mathematics understanding: procedural knowledge (rules without reasons) and conceptual knowledge (understanding the procedures and their underlying relationships to mathematical concepts). Schoenfeld indicated that students who possess adequate levels of the constructs that he identified are excellent mathematicians and problem solvers. Hiebert and Lefevre added that conceptual knowledge of the students' mathematical resources is crucial for successful problem solving. Mathematically gifted students are generally good problem solvers. The performances of gifted students on challenging mathematics problems can be rich sources of information about successful problem-solving strategies and about their conceptual understandings.

The Evolution of the Definition of Conceptual Understanding

Researchers in mathematics education have long been interested in students' abilities to obtain conceptual understanding in mathematics and under what conditions conceptual understanding can be attained. Brownell (1956) identified the discrepancy between achieving computational competence in arithmetic and constructing meaning to accompany that competence. He argued that students often perform well on standardized tests of computation but cannot complete problem-solving tasks that require more than computation. Brownell

attributed this discrepancy to inadequate instruction in mathematical meaning and advocated "meaningful habituation" (p. 133) as the ultimate goal for students. Meaningful habituation describes the automatic process, grounded in a firm basis of understanding, through which a required response is made. Brownell suggested that teachers initially help their students achieve meaningful habituation by asking them to explain *why* and *how* during varied arithmetic practice activities. Once meaning of mathematical processes has been established through varied practice, the teacher can help develop habituation through repetitive practice, which results in computational competence, economy, and permanence of skills.

Like Brownell, Van Engen (1953) differentiated between students' competence in operating on mathematical symbols (computation) and in understanding the symbols upon which they are operating; however, he used different terminology. Van Engen divided the idea of meaning in mathematics into three dimensions: (1) syntactic (the meanings of symbols related to the way they are used in relation to one another), (2) pragmatic (the meanings of symbols vary according to the given situation and the individual interpreting them), and (3) semantic (the underlying relationships among concepts represented by the symbols). In his assessment, mathematics teachers often focus on the syntactic dimension of mathematics through drill and memorization at the expense of the semantic dimension that develops *understanding.* Van Engen defined understanding as "the logical implications and sequences of thought that unite two or more statements by means of the bonds of logic" (p. 75). Abstraction and generalization were two processes that Van Engen identified as crucial in the development of understanding. Abstraction was defined by Van Engen as the classification of objects by given properties. Generalization, on the other hand, is the application of those abstracted properties to a larger class of related objects. For example, students who have abstracted the Pythagorean Theorem

should be able to use it to determine whether a triangle is a right triangle, given the lengths of its sides. The students should then be able to generalize the use of the Pythagorean Theorem to the larger class of polygons (e.g., squares and rectangles) in order to learn more about these objects.

Skemp (1976) also distinguished between two types of mathematics understanding. He used the term *relational understanding* to describe "knowing both what to do and why" (p. 9). Some students, Skemp argued, only attain what he called an *instrumental understanding* of mathematics or "rules without reasons" (p. 9). In other words, these students are able to memorize and to apply algorithms to solve certain mathematical problems, but they cannot derive the algorithms or explain why they are valid tools for solving a particular problem. Skemp indicated that, although instrumental mathematics is easier to understand and provides more immediate and apparent rewards, relational understanding is preferable because it is adaptable to new tasks (transferable), it is easier to remember, and it is intrinsically rewarding. In Skemp's theory, the relational understanding of mathematics is essential to the process of building conceptual structures (schema). Weakly constructed schema result in a breakdown of the entire problem-solving process. Skemp's theory, however, does not offer a method for determining whether students have attained a conceptual understanding of mathematics.

Hiebert and Lefevre (1986) used different terminology to describe essentially the same constructs that Skemp identified. They used the term *conceptual knowledge* to describe knowledge that is "rich in relationships" (p. 3) and *procedural knowledge* to denote the grasp of the formal language (symbol representations) and the rules or algorithms of mathematics. Conceptual knowledge is achieved by the construction of relationships among pieces of information. Either two pieces of information already stored in memory are linked, or a newly learned piece of information is linked with concepts already stored in memory. Hiebert and

Lefevre asserted that conceptual knowledge cannot be accomplished without what Brownell (1956) termed meaningful learning; however, procedural knowledge may or may not be acquired meaningfully. If procedures are learned meaningfully with appropriate connections, they are then linked to conceptual knowledge. Hiebert and Lefevre highly endorsed this linkage between the two types of knowledge because procedural knowledge learned decontextualized from relationships is restricted to a specific context and is not easily transferred to other situations. Among the benefits of linking procedural knowledge with conceptual knowledge are better retention of information, easier transferability of procedures to new contexts, and better executive control of procedures. Executive control refers to one's ability to monitor both the appropriateness of chosen procedures in a given context and the reasonableness of the solution. Hiebert and Lefevre's definition of executive control is very similar to Schoenfeld's (1985) control construct. Therefore, according to Hiebert and Lefevre, if students' mathematical resources (as defined by Schoenfeld) are learned conceptually, they gain higher levels of control as a result. Executive control also helps students to select the appropriate heuristics for the problems that they encounter.

Assessing Conceptual Knowledge

Although many mathematical psychologists and researchers have defined conceptual knowledge and have distinguished it from procedural knowledge, few have suggested methods to determine if students have acquired conceptual knowledge. Through his personal observations, Holt (1964) attempted to understand why children are often unsuccessful in school. He then offered several indicators that distinguish what he called *real learning* (conceptual knowledge) from *apparent learning* (procedural knowledge). If students have real learning of a concept, they are able to do the following:

> (1) state it in my own words; (2) give examples of it; (3) recognize it in various guises and circumstances; (4) see connections between it and other facts or ideas; (5) make use of it in various ways; (6) foresee some of its consequences; (7) state its opposite or converse (Holt, 1964, p. 104).

Inspired by Holt's list, Davis (1978) outlined two levels of understanding for four important mathematical constructs: concepts, generalizations, procedures, and facts. Students who can answer questions at Level 1 can provide examples or non-examples of a concept, can understand what a generalization or fact indicates, and can understand how a procedure works; at Level 1, students demonstrate procedural knowledge. Students at Level 2 not only can answer Level 1 questions but also can provide the characteristics of concepts in addition to their understanding of why generalizations and facts are true and why procedures work. Students at Level 2 have achieved what Hiebert and Lefevre termed conceptual knowledge.

During the 1950s and 1960s, Russian mathematical psychologist V. A. Krutetskii (1976) conducted extensive research to investigate specific mathematical abilities that are characteristic of mathematically capable students. He identified three abilities through his research on information processing skills: generalizability, flexibility of thinking, and reversibility of mental processes.

Krutetskii believed that students should be able to generalize mathematical material on two levels:

> (1) as a person's ability to see something general and *known* to him in what is particular and concrete (subsuming a particular case under a known general concept), and (2) the ability to see something general and still *unknown* to him in what is particular and concrete (to deduce the general from particular cases, to form a concept) (Krutetskii, 1976, p. 237).

According to Krutetskii, flexibility is the ability to change from one perspective to another perspective (e.g., change from an algebraic perspective to a graphical perspective). Krutetskii

defined reversibility as "an ability to restructure the direction of a mental process from a direct or reverse train of thought" (Krutetskii, 1976, p. 143).

Although the theorists who distinguished between conceptual and procedural knowledge did not provide specific guidelines, Holt, Davis, and Krutetskii have provided some ideas about specific, observable abilities through which students can demonstrate the relationships that they have made among mathematical concepts. Test questions and mathematical activities that require the use of these abilities can be designed to investigate the depth of students' conceptual knowledge.

Mathematical Belief Systems

Three of the constructs in Schoenfeld's (1985) framework can be investigated through the problem-solving activities of students: resources, heuristics, and control. The fourth component of the framework, the impact of belief systems, may not be as evident as the other three. A number of quantitative instruments have been developed for the assessment of students' mathematical self-efficacy or mathematical belief systems (Betz & Hackett, 1983; Cooper & Robinson, 1991; Fennema & Sherman, 1976; Kloosterman & Stage, 1992). Much of the research on mathematical self-efficacy and mathematical belief systems indicates a positive correlation among these constructs and mathematical achievement (Hackett, 1985; Betz, 1987; Fennema and Sherman, 1977, 1978; Kloosterman, 1991).

Study Overview

The Duke University Talent Identification Program (TIP) provides gifted middle and high school students an opportunity to take fast-paced courses in summer residential programs. Research on gifted students' success in these summer programs is a crucial component of the TIP mission in serving the gifted population. Much of the previous research conducted at TIP

has focused on gifted adolescents' self-concepts and on their academic self-efficacy (Marsh, Plucker, & Stocking, 2001; Plucker & Stocking, 2001). Since mathematics courses are a major component of the TIP curriculum, there is a need for specific research on the success of mathematics students in these fast-paced courses. No research at TIP has focused specifically on mathematics students' problem-solving skills nor on their ability to learn mathematics conceptually in the fast-paced courses. TIP's past research has also not focused on the mathematical beliefs systems of the participants in summer programs. This study promises to be extremely useful in program design and in curriculum development decisions.

The second chapter includes the conceptual framework for this study and a review of relevant literature. Since this study involves a number of facets related to mathematics learning and the achievement of gifted students, the chapter is divided into several sections. The first two sections of the literature review summarize research literature that considers students' understandings of mathematical functions and representations. The third section reviews research studies related to mathematical self-efficacy and mathematical belief systems and their impact on mathematics achievement. The fourth section considers the success of acceleration opportunities for mathematically gifted students since this study was conducted at a summer program that offers acceleration opportunities to gifted students.

The third chapter describes the participants and setting for the study. It also details the instruments used, the data collection procedures, and the analysis procedures. The final chapters are two articles that are designed for submission to research journals. These articles include the results and conclusions of the study. The first article presents the results of the quantitative portion of the study that focuses on mathematical belief systems and their effects on

achievement. The second article describes the results of the qualitative portion of the study that investigates the students' conceptual knowledge of polynomial functions.

LITERATURE REVIEW

Students often choose to take mathematics sequence classes in TIP's Summer Studies Programs in order to accelerate their studies. Acceleration, in this context, means taking a course above one's grade level. Local school officials, however, are often reluctant to accept courses completed at TIP as substitutes for taking a year-long course in the student's local school. These officials question whether TIP students are able to attain conceptual knowledge of the material in such a compressed, fast-paced version of the course. This study investigated this issue through qualitative and quantitative examinations of the roles of knowledge, strategies, and beliefs in the problem-solving activities of TIP Algebra II students. This investigation focused on the unit on polynomials, a topic at the heart of Algebra II. A conceptual understanding of polynomials is crucial for success in subsequent mathematics courses.

This chapter outlines the conceptual framework for this study that is based on Schoenfeld's (1985) framework for characterizing problem-solving behavior, on Hiebert and Lefevre's (1986) definition of conceptual knowledge, and on Krutetskii (1976) and Rachlin's (1981) abilities that mathematically capable students demonstrate. Following the conceptual framework is a review of literature on mathematics abilities and conceptual knowledge, on the role of belief systems and self-efficacy in achievement, and on the success of accelerated mathematics programs for gifted students.

Conceptual Framework

Schoenfeld (1985) developed a framework of the mathematical knowledge and behaviors that he believed were necessary to characterize problem-solving performance. In this framework, he examined what mathematical knowledge people have, what people do as they work on mathematical problems, and what accounts for their success or failure in problem-

solving attempts. The four categories in this framework are *resources, heuristics, control,* and *belief systems*. Schoenfeld defined the first category, *resources*, as the set of all facts, procedures, and skills that a person possesses prior to problem solving (i.e., the individual's "initial search space"). He noted that some of these resources may be flawed. Schoenfeld defined *heuristics* as "rules of thumb for successful problem solving, general suggestions that help an individual to understand a problem better or to make progress toward its solution" (p. 23). Arguing by contradiction and working backward are two commonly-used heuristics. According to Schoenfeld, *control* is the type of behavior that refers to how students make plans, set goals, monitor progress, assess solutions, and make necessary revisions. Finally, Schoenfeld (1994) argued that a person's *beliefs* shape his cognition and that certain beliefs may actually interfere with his ability to solve problems (e.g., there is only one correct way to solve any mathematics problem, students who understand math can solve any assigned problem in five minutes or less, and mathematics learned in school has little or nothing to do with the real world).

The first three categories of Schoenfeld's framework of mathematical knowledge and behaviors that characterize problem solving—resources, heuristics, and control—can be discussed in terms of Hiebert and Lefevre's definitions of conceptual and procedural knowledge. They used the term *conceptual knowledge* to describe knowledge that is "rich in relationships" (p. 3) and *procedural knowledge* to denote the grasp of the formal language (symbol representations) and the rules or algorithms of mathematics. Conceptual knowledge is achieved by the construction of relationships between pieces of information; either two pieces of information already stored in memory are linked, or a newly learned piece of information is linked with concepts already stored in memory. At first glance, *resources* fall under the heading

of procedural knowledge; however, if these facts and procedures are learned meaningfully through the construction of proper relationships, they would classify as conceptual knowledge (Hiebert & Lefevre, 1986). Control, as defined by Schoenfeld, is connected to conceptual knowledge. According to Hiebert and Lefevre, control is a benefit that individuals gain when they learn information conceptually. In addition, students' belief systems can provide important information about the value that they place on conceptual knowledge in the problem-solving process. Beliefs can also be an indicator of students' reliance on procedural knowledge.

The abilities of mathematically capable students identified in the research of Krutetskii (1976) and Rachlin (1981) offer some guidance in the assessment of conceptual understanding. The four abilities that they identified (ability to generalize, flexibility, reversibility, and transferability) require the construction of relationships described by Hiebert and Lefevre. Students who can generalize must be able to take a variety of examples and situations in their experiences and to link them together to make a broader statement of the relationships among them. Flexibility, by its very nature, requires the connection of pieces of knowledge so that a student can move easily from one representation to another or can connect a new representation to an existing concept. Reversibility also demands the connection of a variety of pieces of knowledge in appropriate sequence in order for students to be able to work in both forward and reverse directions. Finally, transferability requires the linkage of previously acquired knowledge to a new piece of information in an appropriate manner.

The components of the conceptual framework are included in Figure 1 below. The diagram indicates how the various pieces of the framework—Schoenfeld's (1985) framework, Hiebert and Lefevre's (1986) theory, and Krutetskii (1976) and Rachlin's (1981) ideas—fit together and interact with one another.

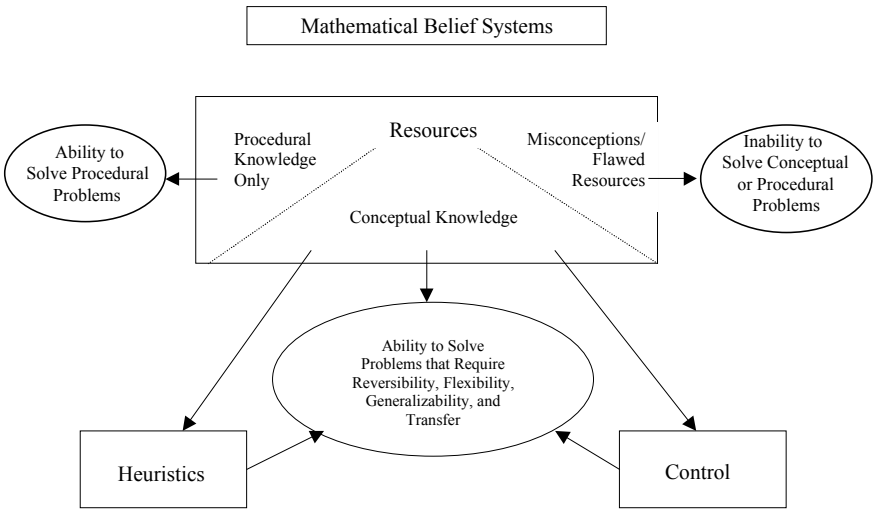

Figure 1: Visual representation of conceptual framework

Studies on Specific Mathematical Abilities and Conceptual Knowledge of Algebra

In an attempt to identify abilities of mathematically capable students, Krutetskii (1976) evaluated 201 sixth and seventh grade students' performances on algebraic, arithmetic, and geometric problem sets in four categories: information gathering skills, information processing skills, information retention, and typology. Krutetskii's research was comprehensive, spanning many years and including many different students of varying ability levels in mathematics. To distinguish between mathematically capable students and average- or low-ability students, he depended upon the input of the teachers who knew the students well. Through qualitative and quantitative methods (interviews and questionnaires, respectively), Krutetskii identified several abilities of gifted students that are related to information processing skills and understanding. These abilities are generalization, flexibility of thinking, and reversibility of mental processes.

Problem sets that Krutetskii used to test the ability to generalize included problems of a single type, problems of different types, and problems of proof. Many of these problems related to real world phenomena. He noted that gifted students were able to generalize concepts more quickly and more broadly than average students. To test for flexibility of thinking, Krutetskii

designed problems with several solutions, problems with changing content, and problems requiring the reconstruction of an operation. Gifted students, he discovered, were able to switch among processes and representations almost effortlessly while average students had a difficult time switching from one process that they had mastered to a new method or representation. Krutetskii investigated reversibility through questions that required students to employ both direct and reverse operations. Gifted students, he concluded, were able to reverse their thought processes quickly and to solve problems that required this ability almost as easily as problems that required a direct train of thought. Average students, on the other hand, generally required hints to be successful in solving reversed problems. (Krutetskii, 1976).

In his doctoral research, Rachlin (1981) investigated the extent to which college students in basic algebra were capable of demonstrating the three abilities identified by Krutetskii: generalization, flexibility, and reversibility. Rachlin divided Krutetskii's generalization ability into two separate abilities: generalization and transfer. Rachlin used the term *generalization* to refer to generalizations made within specific topics and *transfer* to refer to generalizations made between topics. He then used students' performance on the tasks requiring these abilities in order to draw conclusions about their relational understanding (as defined by Skemp, 1976). Rachlin employed a clinical approach in which he interviewed participants for 45 minutes twice a week throughout the semester. Because of the extensive time commitment required by this study, Rachlin depended upon student volunteers who were interested in participating. Eight students expressed an initial interest, but only four students, all of whom were female and classified as above average by their instructors, completed the study.

Rachlin analyzed the transcripts of the interviews in order to identify patterns of understanding and approaches to problem solving. He concluded that the students varied widely

in their abilities to complete reversibility and flexibility problems. The subjects also had difficulty with generalization and transfer problems since they tended to learn each new algebraic concept as a disjoint unit instead of connecting the concept to previous knowledge; this cast doubt on the likelihood that these students had attained conceptual knowledge of the concepts. Rachlin's sample size was very small and non-randomly selected; therefore, he admitted that his results were probably not representative of college students enrolled in basic algebra. However, this study did have the advantage of studying the same students over an extended period of time (Rachlin, 1981).

Rachlin's study provided further evidence that the abilities identified in gifted students by Krutetskii are not prevalent among all students. The subjects were college students who were enrolled in basic algebra; therefore, they had probably had a below average background in mathematics or struggled with mathematics during high school. No follow-up studies to apply Krutetskii's findings to the gifted population have been attempted.

Like Rachlin, Buck (1995) was also interested in investigating students' conceptual knowledge of algebraic concepts, particularly polynomial functions from algebraic and graphical points of view. She designed teaching episodes for a high school Algebra II class to emphasize the connections between linear and quadratic classes of polynomial functions. Then she gathered data through clinical interviews with three volunteers from the class. During the interviews, the students completed graphing activities with calculators and computer software that allowed the students to "build" polynomial functions through products of linear factors. She concluded, based on the three case studies, that the students made few connections that were indicative of conceptual knowledge with algebraic representations of linear and quadratic classes of polynomials. However, the participants did make some connections between classes of functions

and their corresponding graphical properties. Buck's sample size was small; however, she was able to complete in-depth analyses of these students' interviews. She never indicated whether the students were classified as gifted nor did she describe specific descriptions of the teaching episodes (Buck, 1995).

The research of Krutetskii, Rachlin, and Buck indicated that students vary widely in their ability to acquire conceptual knowledge of algebra. Since Krutetskii's studies were completed, very little research has been conducted to explore the prevalence of these abilities among gifted students. More research is needed to understand the problem-solving processes of gifted students and the conditions under which they are able to acquire conceptual knowledge.

Related Research on Functions and Representations

There are many aspects of conceptual knowledge to consider in the study of polynomial functions. Educational research in the learning of mathematics has revealed a variety of difficulties that students experience in their understanding of functions and in their ability to work with multiple representations of these functions.

For example, Moschkovich, Schoenfeld, and Arcavi (1993) suggested that one difficulty experienced by students in the study of functions is the inability to grasp both the process perspective and the object perspective of functions. They defined the process perspective as "linking x and y values: for each value of x, the function has a corresponding y value" and the object perspective in which "a function or relation and any of its representations are thought of as entities" (p. 71). The authors argued that mastery of both perspectives is crucial to the understanding of functions. Since the perspectives reveal characteristics of the behavior of functions in every representation, developing competency with both helps students to determine the appropriate representations within the context of a given problem.

Sfard's (1989) research led her to the same conclusion as Moschkovich, Schoenfeld, and Arcavi—that both perspectives are important for student understanding. Her research on the two perspectives indicated that process perspective (or the operational conception) must be mastered before the object perspective (or structural conception). She found that the structural conception is rarely accomplished by secondary students. As a result, they experienced difficulty consolidating a multitude of ordered pairs into a single entity and also had problems with the abstract notions of domain and range. In an experiment with 28 participants, ages ranging from 22 to 25, who were enrolled in a secondary level mathematics courses, she employed an operational approach in which the term *function* was initially used almost as a synonym for *algorithm*. Later *function* was described as a name for "the product" of an algorithm in order to move the students toward a structural conception. On a questionnaire that students completed at the end of the unit, Sfard found that the students in the experimental group made substantial movement toward a structural conception compared to students in the control group of 48 students. Unfortunately, the instructional method used for the control group was not clearly described in this study. This limited the reader's ability to determine what aspects of the operational approach were particularly successful. In addition, data were reported by comparison on the questionnaire without statistical analysis; therefore, the level of significance of the findings was in question.

Sierpinska (1992) described a variety of other difficulties that students often encounter in their attempts to understand functions. The first was the identification of *what* changes in a function. Sierpinska argued that students tend to view the independent and dependent variables as entities separate from than the phenomena they represent. They focus their attention "on the displacement itself, on the change, mindless of what was thus displaced or undergoing that

change" (p. 33). Furthermore, Sierpinska indicated that students have difficulty differentiating between the constants and the variables in a function (i.e, the given and the unknown quantities in the problems). A third difficulty students face in the understanding of functions is what Sierpinska termed "the asymmetry of the independent and dependent variables" (p. 38). The order of the variables is important in functions, and many students have difficulty realizing that the second coordinate is uniquely determined by the first and not vice-versa.

As described earlier, Moschkovich, Schoenfeld, and Arcavi (1993) argued that failure to master both the process and object perspectives of functions may inhibit students' ability to select appropriate representations for a given problem. They also argued that connections between the two perspectives allow students to switch flexibly among function representations (e.g., equations, graphs, sets of ordered pairs) and to make connections among the representations in order to solve a variety of problems. Recently, research on the use of multiple representations in the teaching of mathematics has been very prominent.

In particular, students' misconceptions related to the graphs of functions have received much attention. One common error that students make is viewing the graph as a literal picture of the problem situation (Clement, 1989; Dugdale, 1993). Instead of remembering what the axes of the graph represent and analyzing the graph accordingly, they view the graph as a diagram of the physical event. For example, students experiencing this "graph as picture" difficulty may mistake the graph in Figure 2 as a picture of the trajectory of the rocket flight rather than a graphical representation of the height of the rocket at any given second of the flight. A "picture" of the event would show that the rocket was launched vertically from the top of the tower.

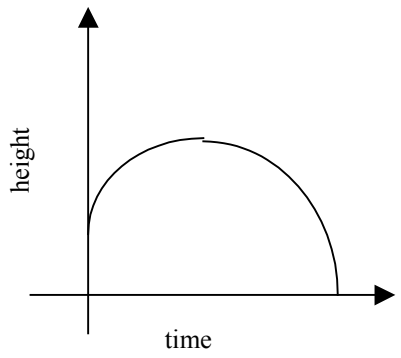

Figure 2. Height by time graph of a rocket launched vertically from the top of a tower.

The use of technology, such as graphing calculators, to produce graphical representations of functions has also resulted in a variety of student misconceptions of the graphs of functions. Goldenberg (1988) described several visual illusions, caused by graphing technology, that can lead to student misconceptions. For example, in the graphs of linear functions (in the form $y = ax + b$), changing b may appear to move the graph horizontally or vertically, depending on the shape of the viewing window. In viewing graphs of parabolas on different windows, students may believe that the parabola has changed shape. In addition, the limitations of the size of the viewing window may lead students to believe that the domain of the graphed function is bounded by the values within the window.

While some recent research has investigated the learning patterns and misconceptions of students in their study of functions, few studies have considered the gifted population. It is possible that gifted students have similar patterns of understanding and misconceptions as students in prior studies; however, gifted students may experience different challenges and successes in their understandings of polynomials and functions. To determine whether gifted students have different patterns of understanding, more research with this population is needed.

Beliefs, Self-Efficacy and Mathematics Achievement

There are many factors that contribute to students' understandings of mathematical concepts and to their ability to be successful in problem solving activities. Schoenfeld (1985) indicated that one of these factors is students' mathematical belief systems. Since the 1970s, many researchers in mathematics education have investigated the theory that mathematics self-efficacy—the belief that one's own abilities are adequate to be successful in mathematics—influences mathematics achievement. Bandura's (1977, 1982) theory viewed self-efficacy as the construct that, when addressed by interventions, increases one's perceived effectiveness in handling problematic areas in life. Other researchers, such as Hackett (1985), Betz (1987), and Fennema and Sherman (1977,1978), adapted the theory of self-efficacy to mathematics education and found that students' self-efficacy in mathematics correlated positively with their mathematics ability and negatively with their mathematics anxiety. Schoenfeld (1985) expanded the notion of mathematics self-efficacy to a broader construct called mathematical belief systems that includes not only self-efficacy but also beliefs about the nature of learning mathematics and beliefs about the importance of mathematics in society.

A variety of studies have investigated mathematical self-efficacy and its relationship to achievement. Cooper and Robinson (1991) conducted a study with 229 male and 61 female matriculating undergraduates who planned to major in a mathematics-related field at a Midwestern university that emphasized engineering and applied sciences. In this study, they investigated the relationship between mathematics self-efficacy beliefs and mathematics anxiety and performance. The investigators constructed an instrument called Mathematics Self-Efficacy (MSE – α reliability .95) that assessed students' confidence in their ability to complete successfully the ten advanced mathematics classes required of their majors. They administered

the Fennema-Sherman Mathematics Anxiety Scale (Split-half reliability r=.92) to measure students' levels of anxiety about mathematics. The quantitative score from the American College Testing Program: Mathematics Academic Test (ACT-MAT) was used to measure mathematical ability, and the Missouri Mathematics Placement Test (MMPT) was used to measure achievement. The researchers detected a moderate negative correlation between mathematics self-efficacy and anxiety ($r=-.41$, $p<.001$) and a slightly positive correlation between mathematics self-efficacy and achievement ($r=.2$, $p<.001$). These findings supported their original hypotheses; however, the researchers gathered all data simultaneously. Although the researchers were able to establish a correlation between self-efficacy and achievement, gathering all data at the same time severely limited any attempts to establish causal inference. The authors did not indicate their methods for selecting subjects, and the small number of female participants limits the possibility that this study has external validity. An important feature of this study was that it considered students who had chosen a mathematics-related major at a university that emphasized engineering and applied sciences; thus, these students likely had experienced success with mathematics and enjoyed the field of mathematics. This study, therefore, indicated that self-efficacy may be related to mathematics achievement, even if the direction of causality cannot be determined from this research design.

In a similar study, Randhawa, Beamer, and Lundberg (1993) hypothesized that self-efficacy would serve as a mediator between attitudes and achievement in mathematics. The researchers studied 117 male and 108 female students enrolled in the same twelfth-grade algebra course in three high schools in a midwestern Canadian city. They randomly selected students from these classes to participate in the study; however, they did not indicate their exact procedures of randomization. The investigators used Betz and Hackett's (1983) Mathematics

Self-Efficacy Scale that includes three subscales that measure self-efficacy in performing everyday mathematics tasks (α reliability .88), in solving mathematics academic problems (α reliability .89), and in completing high school mathematics courses (α reliability .92). To measure attitudes, they computed a composite score based on the Mathematics Attitude Inventory (α reliability .81) and the Mathematics Attitude Survey (α reliability .75). For the achievement measure, Randhawa, et al., developed a 40-item test to measure the algebra achievements of twelfth-grade students (α reliability .77, average item difficulty .42). Their structural model analysis indicated that the hypothesized model with self-efficacy mediating between attitudes and achievement was highly significant with a goodness of fit index of .955. Other analyses did not result in a better fit. This well-designed study provided evidence of a connection between self-efficacy and mathematics achievement.

Kloosterman (1991) considered a broader perspective than the role of self-efficacy when he studied the relationships between beliefs and achievement in mathematics for 233 female and 196 male seventh-graders in three lower-middle-class to upper-middle-class Indiana middle schools in small cities. He used the Fennema-Sherman (1976) confidence scale (α reliability .89) to measure self-confidence in the ability to achieve in mathematics and the Fennema-Peterson Autonomous Learning (1984) mathematics attribution scale to determine attributional style (i.e., a person's beliefs about the reasons for academic success or failure such as success is due to ability or success is due to effort). Kloosterman developed two scales to measure *effort* as a mediator of mathematical ability and *failure* as an acceptable phase in the learning of mathematics (α reliability .83 and .59 for the two scales, respectively). He administered the Stanford Mathematics Achievement Test (reliability .88 for seventh graders) to assess achievement in operations and properties of rational numbers and the Application in

Mathematics test (reliability .91 for seventh graders) to assess achievement in problem solving, geometry, measurement, and visual representations (tables, graphs). He discovered significantly positive correlations among all belief and achievement measures with two exceptions: his effort scale with the Stanford Achievement Test and his effort scale with the Applications of Mathematics test. Through a LISREL analysis, he concluded that the latent construct of beliefs was significantly positively correlated with mathematics achievement. He acknowledged, however, that the low reliability of the *failure* scale was a shortcoming of the study. Also, this study only considered seventh-graders from middle-class areas in Indiana; this limited its external validity. Kloosterman also did not indicate the classification of the students that he considered as gifted, average, or low-achieving; therefore, the applicability of these findings to gifted students is questionable.

Kloosterman and Stage (1992) developed the Indiana Mathematics Beliefs Scale in order to investigate students' belief systems. The beliefs measured by this 35 item scale are similar to the beliefs that Schoenfeld (1985) suggested can impact students' problem solving activities. This scale is designed specifically for secondary and college students. Subscales of the Indiana Mathematics Beliefs Scale measure the following beliefs using a Likert-type scale, with the Cronbach alpha reliability indicated for each subscale:

Belief 1: I can solve time-consuming mathematics problems. ($\alpha = .77$)
Belief 2: There are word problems that cannot be solved with simple step-by-step procedures.($\alpha = .67$)
Belief 3: Understanding concepts is important in mathematics. ($\alpha = .76$)
Belief 4: Word problems are important in mathematics. ($\alpha = .54$)
Belief 5: Effort can increase mathematical ability. ($\alpha = .84$)
Belief 6: Mathematics is useful in daily life. ($\alpha = .86$)

High scores on this scale indicate that the student's mathematical belief system is conducive toward success in mathematics problem solving. The authors indicated that, despite the low

reliability of the fourth belief subscale, its removal from the analysis did not affect the overall reliability of the scale (Kloosterman & Stage, 1992).

Other studies have investigated self-efficacy and beliefs predictors of mathematics achievement. Lent, Lopez, Brown, and Gore (1996), in a study of 295 college students, identified four sources of mathematics self-efficacy: 1) personal performance accomplishments on mathematics tasks or activities, 2) vicarious learning or observation of others' performances, 3) social persuasion, and 4) emotional arousal. In a similar study of 481 high school students, these researchers identified five sources of self-efficacy: the sources identified above plus adult and peer modeling of procedures.

One study targeted the gifted population and investigated whether self-efficacy among other predictors correlated with students' achievement in a high-stakes mathematics situation (i.e., the Advance Placement (AP) Calculus examination). They found a positive correlation between self-efficacy and achievement ($r=.569$, $p<.05$). The self-efficacy measure was administered after the test administration, however, because the researchers had wanted to gauge students' self-efficacy on completing the specific problems on the examination, and the AP regulations prohibited use of test items prior to test administration. The authors viewed the self-efficacy measure as more of a self-evaluation than a predictor of achievement on the test; therefore, this study provided little evidence about the causal influence of self-efficacy on the gifted students' achievement on the AP examination (Malpass, Harold F. O'Neil, & Hocevar, 1999). Another study that attempted to identify the predictors of mathematics achievement for gifted students failed to consider mathematical beliefs or self-efficacy in the analysis (Olszewski-Kubilius, Kulieke, Shaw, Willis, & Krasney, 1990).

Several research studies have positively linked mathematical self-efficacy and mathematical belief systems to mathematics achievement. All of the studies mentioned, however, consider students who are early-middle school age, late-high school age, or college age. Little mention is made of younger high school students or the gifted population. Furthermore, these studies used standardized multiple-choice tests of mathematics achievement and did not investigate the role of beliefs or self-efficacy in students' abilities to acquire conceptual knowledge. In none of these studies did the investigators target the gifted population. More research is needed in order to understand the role of self-efficacy and belief systems in mathematics achievement and in the acquisition of conceptual knowledge of mathematically gifted students.

The Success of Accelerated Mathematics Programs for Mathematically Gifted Students

The present study considers the belief systems, achievement, and conceptual understanding of gifted students in an accelerated summer program. Educational specialists disagree on the value of opportunities for acceleration, or taking classes above grade level, for mathematically gifted students. Supporters of acceleration believe that such programs benefit gifted students by enabling them to work with their mental peers on tasks commensurate with their abilities (Fox, 1974; Stanley, Lupkowski, & Assouline, 1990; Corazza, Gustin, & Edelkind, 1995). As a result, acceleration program participants are rescued from boredom and insufficient challenge. Acceleration supporters also argue that gifted students would develop better study habits and more positive self-concepts through participation in accelerated programs. Opponents of acceleration opportunities claim that mathematically gifted students may not have reached the maturity level, both socially and physically, that is required to handle acceleration. Critics also fear that acceleration programs foster elitist attitudes among participants and senses of inferiority

among students who do not qualify to participate (Kulik & Kulik, 1984). In response to these concerns, a number of researchers have investigated the effects of accelerated instruction for mathematically gifted students.

In a meta-analysis of 26 controlled studies investigating the effects of accelerated mathematics programs, Kulik and Kulik (1984) defined *accelerated* as taking classes above one's grade level. They compared the performance of accelerated students on standardized achievement tests, such as the Cooperative Mathematics Test and Sequential Tests of Educational Progress, with the performance of comparable students in non-accelerated programs (13 studies) and with older students in the classes into which these students had accelerated (13 studies). In the comparisons among students of the same age, all 13 studies indicated higher achievement among the students in the accelerated classes. There was significantly higher achievement in nine of the studies. In the comparisons of accelerated students with older students within their classes, five studies showed higher achievement among the younger students. Two of these results were statistically significant. Kulik and Kulik concluded that mathematically gifted students are able to handle the academic challenge of acceleration. The data that the studies provided about the affective outcomes related to acceleration (e.g., attitudes towards learning and subject area, participation in extracurricular school activities, effects on vocational plans) were inconsistent and inconclusive. The authors indicated that they had selected only studies that featured sound experimental design and instrumentation, although they did not describe their specific criteria for including studies in this meta-analysis. In addition, they did not provide any information about the number of students studied or the characteristics of the subjects in the studies. As a result, it is difficult to determine if these results are representative of a variety of students. The studies that they considered, however, did provide

strong evidence that gifted students were successful, as measured by standardized test scores, in accelerated courses.

Kolitch and Brody (1992) investigated the effects of acceleration for highly talented students from a different perspective. They studied 49 males and 20 females who had scored above 700 on the quantitative section of the SAT before they were 13 years old. The subjects were in their freshman year of college and had participated in mathematics acceleration programs in high school. Data were gathered through a researcher-created questionnaire. The average grade level at which these students enrolled in calculus courses was 9.49 (about 2 ½ years earlier than the typical college preparatory student who takes calculus in the twelfth grade). They had accelerated their precalculus studies through a variety of methods, including participation in summer programs, skipping or doubling up mathematics classes, and/or participating in their school's fast-paced mathematics program. Many of these students took several postcalculus courses at magnet schools or at local colleges before their senior year in high school. In addition, many participated in extracurricular mathematics activities including math teams and state and regional competitions. The authors suspected that they learned a great deal of mathematics in these activities.

Kolitch and Brody (1992) found that, with few exceptions, the students had been quite successful in all of their accelerated classes and were generally positive about their experiences in these classes. Approximately 75 percent of the students indicated that they had selected mathematics-related majors. In response to their findings, Kolitch and Brody concluded that gifted students can benefit from acceleration opportunities in middle and high school and endorsed curriculum reform that allows such acceleration. They were unable to offer specific recommendations about the extent to which course content and pace differentiation should be

allowed, however. One problem with this study was that all data were self-reported. Also, most of the reports included only percentages and descriptive information. Little statistical analysis was performed; however, the authors did report two statistically significant findings: males took calculus earlier than females, and more males chose mathematics-related majors. Although this study provided evidence that mathematically gifted students who participated in accelerated mathematics programs continue to be successful throughout their academic careers, it did not attempt to determine the depth of understanding that the participants acquired during the accelerated courses. The continued success of these students in mathematics courses, however, suggests that the accelerated programs were somewhat successful in providing the foundations that served them well in later classes.

Swiatek and Benbow (1991) completed a longitudinal study in order to determine the long-term effects of accelerated mathematics programs for mathematically gifted students. The subjects participated in a highly-accelerated program, sponsored by Johns Hopkins' Study of Mathematically Precocious Youth (SMPY), in which they completed Algebra I, Algebra II, plane geometry, analytic geometry, and trigonometry during a period of fourteen months in classes that met primarily on Saturday mornings. Participants in two sessions of this course, one fast-paced session and one self-paced session, were considered. To qualify for the first session in 1972, students had to be classified as "gifted" by their local schools and had to score in the 99th percentile on the numbers subtest of the Academic Promise Test. Sixteen students who qualified chose to participate in the first session and were classified as sixth graders to accelerated tenth graders (exact ages not indicated). To qualify for the second session, junior high school students had to score above 500 on the SAT and at or above the 48th percentile on the Educational Testing Service Cooperative Algebra I test. Twenty-eight students, classified as

seventh through accelerated ninth graders, participated in the second session. In addition, the researchers constructed a control group of 58 students who qualified for the program but did not participate or who started the program but did not finish. Due to the longitudinal nature of the study, 19 percent of the students dropped out or could not be located for the follow-up questionnaire that was mailed to them when they were approximately 23 years old. Through the data provided on this researcher-developed questionnaire, Swiatek and Benbow concluded that participants and non-participants performed well academically at all levels of their education. Participants took more mathematics and science courses than their colleges required, and most attended graduate school. Although the treatment group scored higher on all academic variables, no differences were statistically significant. The questionnaire included questions on psychosocial factors as well; however, the scale was not sufficiently reliable to draw useful conclusions. This longitudinal study consisted of only one follow-up after the students completed college. Furthermore, there is no report or speculation on the patterns of attrition, which makes any results questionable. The failure to describe the control group adequately is also a major flaw of this study. The members of the control group might have participated in other acceleration opportunities that could have accounted for the lack of significant findings in the follow-up study. In addition, no comparison was made between the participants in the fast-paced and self-paced versions of the course. Like the Kolitch and Brody study, this research did not investigate students' achievement and acquired depth of understanding during the accelerated program, but rather focused on the long-term effects.

Several researchers have been interested in the short-term and long-term achievement of gifted students in accelerated and fast-paced mathematics programs. The measures of achievement, however, have been primarily standardized tests and self-reported performance in

future courses. None of these studies have considered the problem-solving abilities or depth of understanding achieved by students in accelerated and fast-paced programs. Researchers have also not considered the impact of belief systems on student achievement.

Research Questions

Many researchers in gifted education have suggested that mathematically talented students are able to achieve at high levels in accelerated and fast-paced courses of study; however, few research studies have investigated the depth of understanding or conceptual knowledge that gifted students can achieve during the programs. Furthermore, in considering the effects of mathematics self-efficacy or mathematical beliefs on mathematics achievement, few researchers have considered the gifted population and how beliefs affect the achievement of gifted students who participated in accelerated programs.

The Duke University Talent Identification Program provides an opportunity for fast-paced acceleration during the summer by offering mathematics courses, such as Algebra II, to students who qualify for the Summer Studies Program. The course is fast-paced because the entire curriculum is completed in three weeks and is accelerated because TIP students are taking the class above grade level. This research study investigated how gifted students' knowledge and behaviors, as outlined by Schoenfeld (1985), characterize their problem-solving abilities during the polynomial function unit in Algebra II. It measured the students' mathematical belief systems in order to determine if belief scores were correlated with the students' achievement on examinations. In addition to the belief measures, the researcher analyzed responses to test and interview questions in order to determine the students' uses of resources, heuristics, and control. Some of the problems on the tests were designed to test connections among mathematical resources that Hiebert and Lefevre (1986) suggested were indicative of conceptual

understanding. Other problems were intended to test only procedural knowledge. The problems that tested conceptual knowledge required reversibility, generalizability, transfer, and flexiblity, as defined by Krutetskii (1976) and Rachlin (1981).

The following questions guided this study:

1. In the context of a three-week Algebra II course, how do TIP participants use resources, heuristics, and control to solve problems about polynomials that require them to make the connections among resources that indicate conceptual knowledge?
2. What patterns of connections and misconceptions are evident in the gifted students' attempts to solve problems about polynomials?
3. Do students' belief systems correlate with their mathematics achievement in the accelerated and fast-paced Algebra II class at TIP and also with the ability to solve problems that require reversibility, flexibility, generalization, and transfer?

The specific research methods employed to answer these questions are described in the next chapter. The methods chapter also includes more specific information about the setting and the participants and concludes with the research hypotheses that are based on the literature review and the research design.

METHODOLOGY

The purpose of this study was to investigate how gifted students who were participating in an accelerated, fast-paced summer program used resources, heuristics, and control to solve problems requiring reversibility, flexibility, generalization, and transfer. It also attempted to determine how strongly students' mathematical belief systems correlated with their problem-solving abilities and with their achievement. This chapter begins with a description of the setting, the participants, and the Duke University Talent Identification Program educational model. It concludes with an outline of the data collection and analysis procedures.

Setting and Participants

The Talent Search Model and Duke University TIP

Since the 1920s, many educational psychologists have been interested in the "giftedness" phenomenon. The definition of giftedness is fiercely debated among educators and gifted specialists. VanTassel-Baska (1984) observed that the most prevalent model for identifying gifted students is based on intelligence test scores; however, other models, that consider multiple criteria for identifying giftedness and that seek to extend the identification of gifted students into categories other than intellectual, have gained popularity in recent years. One popular and widely used model for identifying and serving gifted students is the Talent Search Identification Model. Established in 1980, The Duke University Talent Identification Program (TIP) is one of five prominent Talent Search programs in the United States. Like the other talent searches, TIP identifies students through a two-step process developed by Stanley (1990), a psychology professor at Johns Hopkins University, and his colleagues. This process includes the following two steps: (1) initially identify middle school students who scored at the 97th percentile or higher on their grade-level standardized tests in their local schools and (2) use the Scholastic Aptitude

Test (SAT) as a second test to determine the students' levels of mathematical and verbal ability (VanTassel-Baska, 1984). Once students have taken the SAT, the talent search provides local schools and parents with recommendations for providing special curricula and program opportunities to students who score at different levels on the SAT. Duke TIP, like most talent searches, offers gifted students special academic program opportunities that are not available at their local schools. In delineating some of the benefits, VanTassel-Baska noted that the talent search model (1) chooses valid and reliable instrumentation, (2) serves as a catalyst for providing specific curricular interventions designed to benefit the gifted population, and (3) leads to changes in attitude about the benefits of acceleration for gifted students.

The Duke TIP Talent Search actively recruits gifted students from a 16-state region of the southeastern United States (US). A number of students from other regions of the US and the world learn about TIP through the Internet and request to participate in TIP-sponsored programs. TIP provides a variety of programs designed for the gifted population; these include weekend enrichment programs, independent study programs, and summer residential programs. Seventh-graders who score 570 on either the verbal or quantitative portion of the SAT are eligible to participate in TIP's most well-known programs, Summer Studies Center Programs (SSP). Based at six sites located in North Carolina and in Kansas, SSPs are intense three-week residential sessions that are designed to provide participants with the equivalent of a year-long high school course, such as Algebra II or Chemistry, or a semester-long college course, such as Social Psychology or Philosophy. TIP offers two three-week sessions each summer. SSP participants are generally highly-motivated students who are seeking an opportunity to enrich or to accelerate their educational experiences in the summer. Students register to take one course for the session and attend classes for seven hours each weekday and for three hours each Saturday. Most TIP

instructors are graduate students or secondary teachers. In addition, every instructor is assigned a teaching assistant who is usually a college undergraduate or a graduate student whose major is related to the focus of the course.

During the SSPs, students live with their peers in college residence halls where they are supervised by residential counselors, who are typically college undergraduates. During the evenings and on weekends when classes are not in session, the residential staff members plan a variety of social, recreational, and cultural activities for the students. Since TIP values the social as well as academic development of participants, instructors do not assign work for students to complete after class hours. Instead, all reading, writing, and problem-solving assignments are completed during the designated class hours.

Research about gifted students is also a priority for TIP. The research department designs studies each summer that investigate a variety of characteristics of gifted students, e.g., their approach to learning and their attitudes about TIP and about their local schools. Researchers usually schedule half-hour visits to classes twice during the term to gather data, and they often interview students during class breaks. Parental consent forms for research are mailed home with students' final information packets before the summer. Students also have the opportunity during the program to decide whether or not to participate in the research studies. The present study was sponsored by the TIP research department.

Algebra II at TIP

The mathematics sequence courses are among the most popular classes offered by the TIP Summer Studies Program. Nearly every site offers the traditional high school curriculum mathematics courses: Algebra I, Algebra II, Geometry, and Precalculus. The courses follow the North Carolina Standard Course of Study (NCSCOS, 1999) and the standards endorsed by the

National Council of Teachers of Mathematics (NCTM, 1989, 2000). They use standard high school textbooks and cover approximately one chapter per day. Although TIP is a non-credit granting institution, many parents and students negotiate with their local school officials to obtain credit for the course or placement into the next level of mathematics based on the work students complete at TIP. TIP provides syllabi, course descriptions, rubric evaluations, teacher-prepared examinations, and a score from the standardized North Carolina End of Course Test as evidence of achievement in the mathematics course at TIP. Most TIP mathematics students take mathematics courses at least one year above their grade levels.

In order to take Algebra II at TIP, students must score 570 or higher on the quantitative section of the SAT and must have successfully completed Algebra I (although a number of students have completed geometry as well). Algebra II instructors employ a variety of instructional strategies daily. These include lecture, large- and small-group discussions, problem-solving sessions, data analysis opportunities, and activities with graphing calculators.

In the summer of 2002, TIP offered two sections of Algebra II during each of its three-week sessions. The two instructors taught one class per session. One instructor was a secondary mathematics teacher at a local high school. The other instructor, whose undergraduate major was mathematics, was a graduate student in operations research at a local university; he was currently teaching freshman mathematics courses at the university where he was pursuing his graduate degree. Both instructors were familiar with this research study and were willing to participate. Each class also had a teaching assistant (TA) who was an undergraduate student with a strong background in mathematics. Both TAs had teaching assistant experience from the 2001 Summer Studies Programs.

Student Participants

The participants in this study were the students who had enrolled in the four sections of Algebra II during the 2002 Summer Studies Program of the Duke University Talent Identification Program on Duke's East Campus. There were 60 students enrolled in the four sections of Algebra II (15 per section). These students had just completed the eighth, ninth, and tenth grades. These students were eligible to participate in this course because they scored 570 or better on the quantitative component of the SAT as seventh graders. The participants were not assigned to the Algebra II course by TIP. They chose to take this class from a catalog of choices. The 30 students enrolled each session were divided into the two classes based on their scores on a pretest that was administered during the class orientation meeting. The instructors and TAs graded the pretests immediately after the orientation meeting and arranged the tests from the highest to lowest score. They then selected the top two tests and randomly assigned those students to the two sections. They continued this process until all students were assigned. Moreover, they made slight adjustments to ensure that approximately the same numbers of males and females were assigned to each section. Of the 60 students in the classes, parents of 31 consented to their participation in the study (17 eighth graders, 13 ninth graders, and 1 tenth grader). There were 21 male participants and 10 female participants. The racial breakdown of the participants was 14 White, 11 Asian-American, 2 African-American, 1 Hispanic, 1 Native-American, and 3 others who chose not to classify themselves.

Data Collection

This study used four sources of data: a beliefs inventory, researcher-designed in-class examinations, clinical interviews conducted by the researcher, and classroom observations. This section describes the instruments and how they were administered. The beliefs inventory was

used to measure the students belief systems, as defined by Schoenfeld (1985). The examinations and interviews were used to assess the students' conceptual knowledge through their use of mathematical resources, heuristics, and control. The fourth data source, the researcher's notes about classroom observations and instructional strategies employed, was used for descriptive purposes to support or to refute findings from the other data sources.

Prior to the first day of instruction, students completed Kloosterman and Stage's (1992) Indiana Mathematics Beliefs Scale (see Appendix E). After the students completed the beliefs inventory, a researcher-designed pretest of Algebra II content material, including several questions that specifically cover polynomials (see Appendix B), was administered for two purposes: to divide the students into the two classes and to ensure that they were not already able to solve problems similar to the ones used in this study.

In addition to the pretest, two other in-class examinations were used: the polynomial unit examination (see Appendix B) and the final examination. These criterion-referenced examinations included free response items that assessed specific objectives (see Appendix A) from the polynomial unit (Gronlund, 1993). The polynomial unit examination included researcher-designed items that tested conceptual knowledge as measured by responses to questions requiring reversibility, flexibility, generalization, and transfer. The creation of these items was guided by the work of Krutetskii (1976) and Rachlin (1981). Reversibility questions required students to reverse their thought processes (e.g., students were asked to find a polynomial equation with given solutions rather than to compute the solutions of a polynomial equation). Flexibility questions required them to make transitions among different function representations or to reconstruct operations (e.g., given the solution of a fictional student, students were asked to determine the appropriateness of the students' work and to defend their

conclusions). Generalization problems asked students to draw conclusions about polynomials. These conclusions were based upon the experiences and problems that they had encountered in class (e.g., students were asked to determine the domain of any polynomial function after looking at specific examples in class). Transfer problems required them to use their knowledge in contexts that were different from their classroom experiences (e.g., apply knowledge to solve real world examples or to solve mathematical problems different from those encountered in class). The test also included questions that were similar to in-class examples and practice problem sets from the text. These questions were designed to assess procedural knowledge.

A researcher-designed final examination was administered at the end of the course. This test included the polynomial problems from the pretest to facilitate comparisons between the initial and final performances of the students. The researcher retained photocopies of the students' tests.

The researcher developed holistic-scoring rubrics with four-point scales to score the examinations (see Appendices C and D). Following were the general guidelines for awarding points for each problem:

4	Solution complete and correct.
3	Solution almost complete and correct, with some minor error in notation or computation.
2	Response in the right direction. Student stops halfway through or conceptual errors are present.
1	Some work correct, but with no evidence of chain of reasoning.
0	All work is wrong, meaningless, or no work shown.

Students who scored at level 3 or 4 on items designed to test conceptual knowledge were making appropriate connections among their mathematical resources. Scores lower than 3 suggested that the students were lacking some conceptual development and were operating with procedural knowledge or with misconceptions. For items designed to test procedural knowledge, scores of 3

or 4 indicated that the students had mastered the procedures. For these problems, scores lower than 3 indicated a lack of procedural knowledge and/or the presence of misconceptions. In addition to these general scoring guidelines, the researcher developed task-specific instructions for grading responses to individual test items (Thompson & Senk, 1998).

After the polynomial unit test was administered, ten students were purposefully selected using a process described by Patton (2002) as maximum variation sampling, defined as the selection of a wide variety of cases to attain variation on dimensions of interest. From the group of students who volunteered to participate in the study, the investigator used teacher input to select several students who seemed to master the material almost effortlessly, several students who were struggling, and several students whose mastery level lay between these extremes. This selection process was chosen in order to attain a breadth of understanding about common threads of problem-solving strategies and connection-making used by TIP participants in Algebra II.

The interviews occurred after class within four days of the completion of the polynomial unit test. The researcher interviewed each student individually for approximately 30 to 45 minutes. During the interviews, the researcher presented the student three or four polynomial problems that required reversibility and transfer (See Appendix F). The researcher employed the "clinical interview" method as defined by Vygotsky (see Ginsberg, 1997, p. 54). In clinical interviews, Ginsberg summarized the interviewer's role as follows: "In an attempt to gain insight into the child's thinking and learning potential, the interviewer presents problems, modifies questions if the child seems to misunderstand them, devises new problems to test hypotheses arising in the course of the interview, and challenges answers to test the strength of the child's conviction." The researcher presented each problem individually and requested that the students "think aloud" as they proceeded. As the students worked on each problem, the

researcher asked them questions in order to determine the use of resources, heuristics, and control and to determine what connections the students were making. The researcher offered probes to help students who were experiencing difficulty with a problem. The interviewer did not indicate the correctness or incorrectness of any answers; instead, she tried to determine how the student was making connections that indicated conceptual knowledge. The researcher videotaped each interview and later transcribed and analyzed these videos along with all of the students' written work on the problems. Document analysis of students' responses on test questions throughout the session provided additional support for conclusions.

Because of the nature of TIP, there was no control group for this study. Finding a comparable group of students in a regularly paced year-long Algebra II course would be a monumental task. Even honors classes at regular high schools might only contain one or two students who would qualify for TIP and would not provide enough participants for comparisons. Therefore, this study used a one-group quasi-experimental design. Most threats to internal validity were implausible at TIP. With the very brief duration of the program, the likelihood of history or maturation threats was very slim. No students left the program; therefore, attrition was not a concern. Because there was little time between tests, testing threats might have been a problem. However, responses to probing questions during the interview sessions and explanations of answers as requested on the tests would have indicated whether students truly understood the material or were just skilled at answering the test questions. The pretest ensured that the students did not have conceptual knowledge of polynomials when they began the program; therefore, it was unlikely that any explanation other than participation in the program would account for the knowledge that the students acquired.

To ensure the students' confidentiality, identification numbers were used to match students' beliefs scores with their tests. Students' names were replaced by the identification numbers on the copies of tests that the researcher retained. Furthermore, pseudonyms were used in reporting any information gathered during the interview sessions. All data, including copies of tests, videotapes, and transcripts, were stored in a locked file cabinet. Only the researcher had access to the data.

Data Analysis

Quantitative Analysis

To ensure reliability, a random sample of the researcher-designed tests was scored independently both by the researcher and by an individual trained on the rubric grading scale. The researcher compared these scores in order to determine inter-rater reliability. Regression analysis was used to determine if the scores on the belief scale correlated with scores on the polynomial unit test and with scores on the final examination. Quantitative analyses on the test scores were also used to determine if teacher effects were present since two instructors were teaching the classes.

Other variables considered as indicators of achievement in the regression analyses were gathered by TIP as part of the admissions process: gender, race, and grade level. This analysis indicated any correlation between student characteristics and mathematics achievement at TIP. Ultimately, the regression analysis attempted to determine whether positive belief systems about mathematics corresponded to higher achievement for gifted students in this accelerated, fast-paced Algebra II course.

The researcher conducted a factor analysis on items of the midterm examination for the polynomial unit in order to determine if the questions designed to test conceptual knowledge

(those requiring reversibility, flexibility, generalization, and transfer) did, in fact, measure a different construct than the questions that required only a procedural knowledge. Performance on the experimental questions (those requiring reversibility, flexibility, generalization, and transfer) that were on both the pretest and final examination were analyzed statistically in order to determine if significant improvement had occurred during the session. An additional regression analysis determined if there was a correlation between students' beliefs scores and their scores on the experimental questions.

Qualitative Analysis

By viewing the videotapes and by studying the transcriptions, the researcher analyzed the interviews and considered the following questions for each individual student:

1. What types of resources (including misconceptions), heuristics, and control is the student using to solve the problems?
2. What connections is the student making among resources that indicate conceptual knowledge of polynomials?
3. What evidence indicates that the student is unable to make connections and is, therefore, operating only with procedural knowledge?

Analysis proceeded according to a qualitative multi-case study design (Bogdan & Biklen, 1998). Each individual interview was analyzed and coded. The codes were designed according to conceptual framework, with categories to identify specific instances of the use of resources, heuristics, and control (see Appendix G). Instances of connections among various resources were also included in the coding scheme. After each individual interview was coded, the analyses of the interviews were summarized into matrix formats (see Appendix H). From the matrix analysis, common themes among the interviews were

identified and described in order to establish some generalizations about the problem-solving abilities and about the conceptual knowledge that TIP students were able to acquire during the three-week session.

The conclusions of this research study provided useful insight into gifted students' problem-solving abilities in the context of an Algebra II course taught during a three-week summer program. Through quantitative and qualitative analyses of students' performance on questions that required conceptual knowledge and procedural knowledge, the researcher was able to determine how successful the TIP course was at helping students to make the connections that indicate conceptual knowledge. Since the program's ultimate goal is to provide TIP students the deepest possible understanding during the summer sessions, such information is critical to gifted educators who design curricula and teacher training activities. In addition, this study investigated the relationship between the beliefs of gifted students and their achievement in an accelerated, fast-paced mathematics course as well as their ability to attain conceptual knowledge of the material covered in the course. This relationship has received little attention in previous research studies.

Research Hypotheses

The research literature suggests that qualitative interviewing methods are useful in revealing students' thought processes, connections, and misconceptions. The research base also provides convincing evidence that positive mathematical beliefs and self-efficacy are positively correlated with mathematical achievement. Guided by the literature review, the research hypotheses for this study were the following:

1. Individual in-depth interviews that involve problem-solving tasks requiring generalization, transfer, reversibility, and flexibility will reveal evidence of

students' levels of conceptual knowledge as demonstrated by evidence of connections in their use of resources. Evidence of use of heuristics and control will also shape the characterization of the problem-solving processes of these students.

2. A factor analysis on the items on the polynomial midterm examination will indicate that the experimental questions requiring generalization, transfer, reversibility, and flexibility measure a different level of knowledge, namely conceptual knowledge, than the questions designed to assess procedural knowledge.
3. Students' belief scores about mathematics will correlate positively with their performances on the midterm tests and the final exams and specifically, those questions designed to reflect conceptual knowledge through generalization, transfer, reversibility, and flexibility.

The final two chapters include the results and conclusions of this study in "journal-ready" form. The first article reports the results of the quantitative analysis. It reports the findings on the investigation into the relationship between belief scale and achievement on the in-class examinations that the participants completed. It also includes the results of the factor analysis on the polynomial unit test and its implications on the patterns of students' conceptual and procedural knowledge.

The second article reports the themes and conclusions that emerged through the qualitative analysis. Parts of the interviews are included in order to illustrate the themes. Discussion of possible reasons for the patterns of conceptual knowledge and misconceptions are included. Suggestions for future research directions are also indicated.

THE IMPACT OF MATHEMATICAL BELIEF SYSTEMS ON ACHIEVEMENT AND CONCEPTUAL UNDERSTANDING IN AN ACCELERATED SUMMER PROGRAM FOR GIFTED STUDENTS

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Abstract

This study investigated the achievement of gifted students on tests designed to assess both conceptual and procedural knowledge of polynomials and the impact of students' mathematical belief systems on this achievement. The students were enrolled in a three-week Algebra II course at a summer program for gifted mathematics students. Although the students showed significant improvement in their achievement with polynomials from the pretest to the final examination, a factor analysis of the polynomial unit test and a subsequent examination of the students' responses on certain test items revealed some conceptual deficiencies. Statistical analyses suggested that belief systems were correlated with achievement, but the conclusions from this study were ambiguous since the correlations were unexpectedly negative. Through its identification of potential conceptual difficulties that gifted students may encounter in their learning of polynomial functions, this study suggested specific topics that teachers of gifted students should consider when planning their instructional activities.

Introduction and Conceptual Framework

For several decades, mathematics educators and educational psychologists have attempted to identify and to understand the characteristics of good mathematical problem solvers. Mathematically gifted students are an informative group of students to study since they are generally good problem solvers. Mathematics educators have also been interested in helping students to achieve an understanding of concepts that will enable them to solve a variety of problems that are more challenging than the drill problems usually assigned to students as independent practice.

In his framework that identifies four constructs that contribute to students' problem solving abilities, Schoenfeld (1985) argued that students' beliefs about mathematics (e.g., mathematics is useful in the real world, boys are better at math than girls) may positively or negatively impact students' abilities to solve problems. In addition to belief systems, the other three constructs that Schoenfeld identified (resources, heuristics, and control) describe students' mathematical knowledge and their abilities to monitor their progress as they solve problems.

Hiebert and Lefevre (1986) distinguished between two levels of mathematical understanding. Procedural knowledge denotes the grasp of the formal language and the algorithms of mathematics while conceptual knowledge describes knowledge that is "rich in relationships" (p. 3). Relating to Schoenfeld's framework, students who have conceptual knowledge of mathematics have made appropriate connections among their mathematical resources. As a benefit of conceptual knowledge, Hiebert and Lefevre argued that students gain better control or the ability to monitor progress, to determine the appropriateness of the heuristics used, and to assess the reasonableness of solutions. They also believed that conceptual knowledge results in better retention of material.

In order to assess conceptual knowledge, the problems that students encounter on their examinations should require more than procedural knowledge. Although there is no established method to construct problems that assess conceptual knowledge, the research of Krutetskii (1976) and Rachlin (1981) offers some guidance. Krutetskii identified three mathematical abilities in information processing: generalizability, reversibility, and flexibility. Krutetskii believed that students should be able to generalize mathematical material on two levels:

> (1) as a person's ability to see something general and *known* to him in what is particular and concrete (subsuming a particular case under a known general concept), and (2) the ability to see something general and still *unknown* to him in what is particular and concrete (to deduce the general from particular cases, to form a concept) (Krutetskii, 1976, p. 237).

Rachlin (1981) later divided generalizability into two separate constructs: *generalizability* (generalizations within specific topics) and *transfer* (generalizations between topics). According to Krutetskii (1976), flexibility is the ability to change from one level of thinking about a problem to another level. Krutetskii defined reversibility as "an ability to restructure the direction of a mental process from a direct or reverse train of thought" (Krutetskii, 1976, p. 143).

Questions written to assess generalizability, transfer, flexibility, and reversibility require students to make connections among their mathematical resources. Students who can generalize must be able to take a variety of examples and situations in their experiences and to link them together to make a broader statement about the relationships among them. Flexibility, by its very nature, requires the connection of pieces of knowledge so that a student can move easily from one representation to another or can connect a new representation to an existing concept. Reversibility also demands the connection of a variety of pieces of knowledge in appropriate sequence in order for students to be able to work in both forward and reverse directions. Finally,

transferability requires the linkage of previously acquired knowledge to a new piece of information in an appropriate manner.

The present study investigated the relationship between mathematical belief systems and the problem-solving achievement as well as the attainment of conceptual knowledge of mathematically gifted students at an accelerated, fast-paced program. The Duke University Talent Identification Program (TIP) offers qualifying students an opportunity to take Algebra II during a three-week summer session. The course follows the North Carolina Standard Course of Study (NCSCOS, 1999) and the standards endorsed by the National Council of Teachers of Mathematics (NCTM, 1989, 2000). The TIP Algebra II classes use standard high school textbooks and cover approximately one chapter per day. This study focused on the polynomial functions unit in Algebra II. This unit offered many opportunities to assess conceptual knowledge through the connections that students made among various representations of polynomial functions and among the characteristics of functions and polynomials. The following research questions guided this study:

1. Do students' belief systems correlate with their mathematics achievement in the accelerated and fast-paced Algebra II class at TIP and also with the ability to solve problems that require reversibility, flexibility, generalization, and transfer?
2. What patterns of connections and misconceptions are evident in the gifted students' attempts to solve problems about polynomials?

Literature Review

Since the 1970s, many researchers in mathematics education have investigated the theory that mathematics self-efficacy, the belief that one's own abilities are adequate to be successful in mathematics, influences mathematics achievement. Bandura's (1977, 1982) theory viewed self-efficacy as the construct that, when addressed by interventions, increases one's perceived effectiveness in handling problematic areas in life. Other researchers, such as Hackett (1985), Betz (1987), Fennema and Sherman (1977,1978), and Cooper and Robinson (1991) adapted the theory of self-efficacy to mathematics education and found that self-efficacy in mathematics correlated positively with mathematics ability, as measured by standardized achievement tests, and negatively with mathematics anxiety. Similarly, Randhawa, Beamer, and Lundberg (1993) concluded that self-efficacy served as a mediator between attitudes and achievement in mathematics.

Schoenfeld (1985) expanded the notion of mathematics self-efficacy to a broader construct called mathematical belief systems that includes not only self-efficacy but also beliefs about the nature of learning mathematics and beliefs about the importance of mathematics in society. In one study that investigated belief systems similar to those described by Schoenfeld, Kloosterman (1991) concluded that a latent construct of belief systems, as measured by several scales, correlated positively with mathematics achievement. He considered seventh-graders in this study (Kloosterman, 1991).

Later, Kloosterman and Stage (1992) developed the Indiana Mathematics Beliefs Scale in order to measure students' beliefs. The beliefs measured by this 35-item scale are similar to the beliefs that Schoenfeld (1985) suggested can impact students' problem solving activities. This scale is designed specifically for secondary and college students. Subscales of the Indiana

Mathematics Beliefs Scale measure the following beliefs using a Likert-type scale, with the Cronbach alpha reliability indicated for each subscale:

Belief 1: I can solve time-consuming mathematics problems. ($\alpha = .77$)
Belief 2: There are word problems that cannot be solved with simple step-by-step procedures.($\alpha = .67$)
Belief 3: Understanding concepts is important in mathematics. ($\alpha = .76$)
Belief 4: Word problems are important in mathematics. ($\alpha = .54$)
Belief 5: Effort can increase mathematical ability. ($\alpha = .84$)
Belief 6: Mathematics is useful in daily life. ($\alpha = .86$)

High scores on this scale indicate that the student's mathematical belief system is conducive toward success in mathematics problem solving. The authors indicated that, despite the low reliability of the fourth belief subscale, its removal from the analysis did not affect the overall reliability of the scale (Kloosterman & Stage, 1992).

Although a variety of research studies have positively linked mathematical self-efficacy and mathematical belief systems to mathematics achievement, most consider students who are early-middle school age, late-high school age, or college age. Little mention is made of younger high school students or of the gifted population. Furthermore, all of these studies used standardized, multiple-choice tests of mathematics achievement and did not investigate the role of beliefs or self-efficacy in students' abilities to acquire conceptual knowledge. The present study attempted to understand the role of self-efficacy and belief systems in mathematics achievement and in the acquisition of conceptual knowledge of mathematically gifted students. Although TIP had conducted research in the past on the self-concept and academic self-efficacy of gifted students (Marsh, Plucker, & Stocking, 2001; Plucker & Stocking, 2001), none of this research focused specifically on mathematical beliefs.

Educational specialists disagree on the value of opportunities for acceleration, or taking classes above grade level, for mathematically gifted students. Supporters of acceleration believe

that such programs benefit gifted students by enabling them to work with their mental peers on tasks commensurate with their abilities (Fox, 1974; Stanley, Lupkowski, & Assouline, 1990; Corazza, Gustin, & Edelkind, 1995). As a result, acceleration program participants are rescued from boredom and insufficient challenge. Acceleration supporters also argue that gifted students would develop better study habits and more positive self-concepts through participation in accelerated programs. Opponents of acceleration opportunities claim that mathematically gifted students may not have reached the maturity level, both socially and physically, that is required to handle acceleration. Critics also fear that acceleration programs foster elitist attitudes among participants and senses of inferiority among students who do not qualify to participate (Kulik & Kulik, 1984). In response to these concerns, a number of researchers have investigated the effects of accelerated instruction for mathematically gifted students.

Several studies have investigated the short-term and long-term success of students enrolled in accelerated and fast-paced mathematics classes. Kulik and Kulik (1984) conducted a meta-analysis that indicated that students who accelerated their studies in mathematics consistently achieved significantly higher than non-accelerated students. They often achieved at a higher level than the older students enrolled in courses into which they accelerated. Other studies found that students who had been allowed the opportunity to accelerate their studies of mathematics continued to excel in mathematics throughout their academic careers, chose to take more mathematics classes than other students, and often selected mathematics-related majors in college (Kolitch & Brody, 1992; Swiatek & Benbow, 1991).

The above-mentioned research studies about the short-term and long-term achievement of gifted students in accelerated and fast-paced mathematics programs have depended primarily on standardized achievement tests and on self-reported performance in future courses. None of

these studies have considered the belief systems of gifted students, the problem-solving abilities of this population, or the depth of understanding achieved by students in accelerated and fast-paced programs.

There have been some studies that have investigated the depth of students' mathematical understanding; however, these studies have not considered the impact of students' mathematical belief systems. In an attempt to identify abilities of mathematically capable students, Krutetskii (1976) evaluated 201 sixth and seventh grade students' performances on algebraic, arithmetic, and geometric problem sets. To distinguish between mathematically capable students and average or low-ability students, he depended upon the input of the teachers who knew the students well. Through qualitative and quantitative methods (interviews and questionnaires), Krutetskii identified several abilities of gifted students: generalization, flexibility of thinking, and reversibility of mental processes. He noted that gifted students were able to generalize concepts more quickly and more broadly than average students. To test for flexibility of thinking, Krutetskii designed problems with several solutions, problems with changing content, and problems requiring the reconstruction of an operation. Gifted students, he discovered, were able to switch among processes and representations almost effortlessly while average students had a difficult time switching from one mastered process to a new method or representation. Krutetskii investigated reversibility through questions that required students to employ both direct and reverse operations. Gifted students, he concluded, were able to reverse their thought processes quickly and to solve problems that required this ability almost as easily as problems that required a direct train of thought. Average students, on the other hand, generally required hints to be successful in solving reversed problems.

In his doctoral research, Rachlin (1981) investigated the extent to which college students in basic algebra were capable of demonstrating the three abilities identified by Krutetskii: generalizability (divided into generalizability and transfer), flexibility, and reversibility. He then used students' performance on the tasks requiring these abilities in order to draw conclusions about their conceptual understanding. Rachlin employed a clinical approach in which he interviewed respondents for 45 minutes twice a week throughout the semester and analyzed the transcripts of the interviews in order to identify patterns of understanding and approaches to problem solving. He concluded that the students varied widely in their abilities to complete reversibility and flexibility problems. The subjects also had difficulty with generalization and transfer problems since they tended to learn each new algebraic concept as a disjoint unit instead of connecting the concept to previous knowledge; this cast doubt on the likelihood that these students had attained conceptual knowledge of the concepts.

Rachlin's study provided further evidence that the abilities identified in gifted students by Krutetskii are not prevalent among all students. The subjects were college students who were enrolled in basic algebra; therefore, they had probably had a below average background in mathematics or had struggled with mathematics during high school. No follow-up studies to apply Krutetskii's findings to the gifted population have been attempted.

Not only did the present study investigate the achievement of conceptual knowledge of TIP students in their unit on polynomials, but it also attempted to determine the impact of belief systems on this achievement. Mathematically gifted students may not necessarily have belief systems that are conducive toward mathematical achievement. The polynomial test questions in this study allowed students to respond freely to procedural mathematics problems in addition to more difficult problems requiring the abilities identified in the work of Krutetskii and Rachlin.

Through these problems, the researcher hoped to gain insight into the students' levels of conceptual understanding.

Methodology

Participants and Setting

The participants in this study were students enrolled in the four sections of Algebra II during the 2002 Summer Studies Program of the Duke University Talent Identification Program on Duke's East Campus. There were 60 students enrolled in the four sections of Algebra II during the two summer terms (15 students per section; 2 sections per term). These students had just completed the eighth, ninth, and tenth grades. They were eligible to participate in this course because, during the seventh grade, they scored 570 or better on the quantitative component of the SAT. The participants were not assigned to the Algebra II course by TIP. They chose to take this class from a catalog of choices. Of the 60 students in the classes, parents of 31 consented to their participation in the study (17 eighth graders, 13 ninth graders, and 1 tenth grader). There were 21 male participants and 10 female participants. The racial breakdown of the participants was 14 white, 11 Asian American, 2 African American, 1 Hispanic, 1 Native American, and 3 others who chose not to classify themselves.

Instrumentation and Procedure

This study used three sources of data: a beliefs inventory, researcher-designed in-class examinations, and classroom observations. Prior to the first day of instruction, students completed Kloosterman and Stage's (1992) Indiana Mathematics Beliefs Scale. After the students completed the beliefs inventory, a research-designed pretest of Algebra II content material, including seven questions that specifically covered polynomials (see Appendix B), was

administered for two purposes: to divide the students into the two classes and to ensure that they were not already able to solve problems similar to the ones used in this study.

In addition to the pretest, two other in-class examinations were used: the polynomial unit examination (see Appendix B) and the final examination. These criterion-referenced examinations included free response items that assessed specific objectives from the polynomial unit (Gronlund, 1993). The polynomial unit examination included researcher-designed items that tested conceptual knowledge as measured by responses to questions requiring reversibility, flexibility, generalization, and transfer. The creation of these items was guided by the work of Krutetskii (1976) and Rachlin (1981). Reversibility questions required students to reverse their thought processes (e.g., students were asked to find a polynomial equation with given solutions rather than compute the solutions of a polynomial equation). Flexibility questions required them to make transitions among different function representations or to reconstruct operations (e.g., given the solution of a fictional student, students were asked to determine the appropriateness of the students' work and to defend their conclusions). Generalization problems asked students to draw conclusions about polynomials. These conclusions were based upon the experiences and problems that they encountered in class (e.g., students were asked to determine the domain of any polynomial function after looking at specific examples in class). Transfer problems required them to use their knowledge in contexts that are different from their classroom experiences (e.g., apply knowledge to solve real world examples or to solve mathematical problems different from the ones they encountered in class). The test also included questions, designed to assess procedural knowledge, that were similar to in-class examples and practice problem sets from the text.

Unfortunately, field testing of the class examinations for reliability and validity was not possible because of the length of time between sessions of Duke TIP. Since the program is in session during the summers only, waiting an entire year between field testing and completing the study was not feasible. In constructing the test items, the researcher referred to the unit objectives (See Appendix A). She also used the research findings of Krutetskii (1976) and Rachlin (1981) in order to construct the questions that required reversibility, generalization, transfer, and flexibility. To construct the procedural items, she carefully considered Hiebert and Lefevre's (1986) definitions of conceptual and procedural knowledge. Before administering the tests, the researcher solicited the input of the TIP instructors and of other mathematics educators in order to establish face validity. Since these tests were used for the first time, an exploratory factor analysis was conducted to determine whether the test items measured what the researcher intended – conceptual or procedural knowledge.

A researcher-designed final examination was administered at the end of the course. This test included the polynomial problems from the pretest to facilitate comparisons between the initial and final performances of the students. The researcher retained photocopies of all of the students' tests.

All examinations were scored using holistic-scoring rubrics with four-point scales developed by the researcher (see Appendices C and D). Following were the general guidelines for awarding points for each problem:

4 Solution complete and correct.
3 Solution almost complete and correct, with some minor error in notation or computation.
2 Response in the right direction. Student stops halfway through or conceptual errors are present.
1 Some work correct, but with no evidence of chain of reasoning.
0 All work is wrong or meaningless or no work shown.

Students who scored at level 3 or 4 on items designed to test conceptual knowledge were making appropriate connections among their mathematical resources. Scores lower than 3 suggested that the students were lacking some conceptual development and were operating with procedural knowledge or with misconceptions. For items designed to test procedural knowledge, scores of 3 or 4 indicated that the students had mastered the procedures. For these problems, scores lower than 3 indicated a lack of procedural knowledge and/or the presence of misconceptions. In addition to these general scoring guidelines, the researcher developed task-specific instructions for grading responses to individual test items (Thompson & Senk, 1998). A second evaluator, trained by the researcher to use the rubrics, then graded a random sample of 12 of the 31 examinations in order to establish inter-rater reliability.

Discussion of Limitations and Validity

Because of the nature of TIP, there was no control group with which to compare the effects of this accelerated, fast-paced approach to the instruction of Algebra II. Finding a comparable group of students in a regularly paced, year-long Algebra II course would have been a monumental task. Even honors classes at regular high schools might have contained only one or two students who would have qualified for TIP and would not have provided enough participants for comparisons. Therefore, this study used a one-group quasi-experimental design. Because the researcher was allowed to include only those students whose parents consented, the sample size of 31 was rather small. This limited the likelihood of achieving statistically significant results with a number of variables included in the study.

Most threats to internal validity were implausible at TIP. With the very brief duration of the program, the likelihood of history or maturation threats was very slim. No participants left the program; therefore, attrition was not a concern. Because there was little time between tests,

testing threats might have been a problem. However, the explanations of answers that were requested on the tests would have indicated whether students truly understood the material or were just skilled at answering the test questions. The pretest ensured that the students did not have conceptual knowledge of polynomials when they began the program; therefore, it was unlikely that any explanation other than participation in the program would have accounted for the knowledge that the students acquired.

Results

Inter-rater Reliability

A random sample of 12 of the 31 polynomial unit tests and polynomial questions from the final examination were selected to test for inter-rater reliability. The researcher and a second evaluator, whom the researcher trained, scored this sample. Comparison of the scores showed that there was 80 to 90 percent agreement (with differences of no more than one point in cases of disagreement) on all test items except for items 3 and 5 on the polynomial unit test. Clarification of the rubric guidelines eliminated the problem for item 3, and slight adjustment to the rubric was required for item 5.

Pretest and Final Exam Performance

Out of 28 possible points, the average score on the polynomial questions on the pretest was 6.97. The students improved to an average score of 19.32 on these same questions on the final examination. This 12.35 point average improvement was highly significant ($p<.0001$) and indicated that the students' understanding of polynomials had increased as a result of their completion of the course. The final average of only 19.32 out of 28 possible points, however, suggested that the students were still lacking some concepts and perhaps had not gained complete conceptual understanding.

Regression Results with Belief Scores

Three 2 x 3 x 2 x 3 analyses of covariance (ANCOVA) were conducted on the data with the belief scale score as the concomitant variable. The factors were gender (F or M), race (White or Asian American or Other), instructor (2 instructors), and grade level (8, 9 or 10). The first dependent variable was the total score on the polynomial unit test. The second was the subscore of the questions on the polynomial unit test that required reversibility, generalizability, flexibility, and transfer. The final dependent variable was the score on the polynomial items on the final exam. Tests of interaction effects among the factors were not significant.

The first ANCOVA revealed no significant gender, race, instructor, or grade level effects ($p = .89, .77, .65, .40$, respectively) on the polynomial unit test. The belief scale score was also not significantly correlated ($\beta = -.55$, $p=.19$) but had a statistically stronger effect than the other variables.

The second ANCOVA also revealed no significant gender, race, instructor, or grade level effects ($p = .39, .96, .22, .74$, respectively) on the subscore (score on questions requiring reversibility, generalizability, flexibility, and transfer) of the polynomial unit test. The belief scale score approached significant correlation with the subscore ($\beta = -.53$, $p = .08$).

Like the first two ANCOVA's, the third revealed no significant gender, race, instructor, or grade level effects ($p = .92, .72, .62, .57$, respectively) on the score for the polynomial items on the final examination. The belief score was significantly correlated ($\beta = -.27$, $p = .02$) with the score for these items.

Perhaps the fact that TIP students were all highly gifted mathematics students served to suppress any effects that gender, race, and grade level might have on achievement in other settings. Instructor effects were minimal as well, probably because the instructors collaborated

frequently in planning both their lessons and the students' practice assignments. As a result, the students had similar experiences regardless of the instructor to whom they were assigned.

The results with the belief scale scores are more difficult to explain. As expected, the belief scale scores seem to be more strongly correlated with achievement than the other variables. A larger sample size may have produced significance on the first two ANCOVA's. The correlations are small (less than one point decrease on the tests per one point increase on the belief scale) but unexpectedly negative. A higher score on the belief scale, therefore, suggests a lower score on the tests. An examination of the students' responses on the scales revealed that their scores on similar items were fairly consistent throughout, regardless of instances of reverse coding. Thus, there have to be other factors to explain this unexpected result.

The significance may suggest that students with higher beliefs used a different approach with the problems that were unlike the ones that they encountered daily in class (i.e., the ones requiring reversibility, generalizability, flexibility, and transfer). The approach was not as successful, however, as the approaches employed by the other students.

Another possible explanation might have been the range of the belief scores. Out of 175 possible points on the belief score, the students' scores ranged from 115 to 151. Scores below 88 would have indicated mathematical beliefs systems that were not conducive toward mathematical achievement. Therefore, according to this scale, all of these students had positive beliefs about mathematics. This phenomenon might have caused the unexpected negative correlation between the belief scale scores and the test scores.

Factor Analysis

Because the polynomial unit test was created for this study, scores for the 20-item test were subjected to an exploratory factor analysis (EFA) in order to determine if the questions that

were designed to investigate conceptual knowledge actually measured a different construct from the questions designed to assess procedural knowledge. The EFA used squared multiple correlations as prior communality estimates. A principal factor method, which was employed to extract the factors, was followed by a varimax (oblique) rotation. A scree test (or graphical examination of the eigenvalues in the principal factor analysis) suggested two or three meaningful factors. When three factors were retained, very few items loaded on the third factor. Therefore, only two factors were retained for rotation. In interpreting the rotated factor pattern, a test item was considered to load on a factor if the factor loading was .35 or greater for that factor and was less than .35 for the other (See Table 1).

Five items (items 1, 2, 4, 7, 10) were problematic in the factor analysis because nearly every student answered them correctly. These items failed to load on either factor and were eliminated from the EFA in order to achieve a clearer understanding of what was occurring. Items 1, 2, 4, and 10 were designed as procedural questions. Item 7 asked students to make a generalization about the number of x-intercepts that were possible for parabolas. Most students answered this question correctly, perhaps because of emphasis that the instructors placed on this concept during class.

Using the criteria mentioned above, eight items loaded on the first factor, and four items loaded on the second factor. Test items and corresponding factor loadings are presented in Table 1. The two factors appeared to identify two levels of conceptual knowledge. The items that loaded on factor one were primarily questions that were written to test conceptual knowledge (items 5, 12, 16, 13 , and 19), with the exception of procedural items 6, 11, and 17. An examination of the students' scores on these items indicated that these procedural items were problems with which the students struggled. Item 6 required students to graph a quadratic

equation after determining a variety of points and features (e.g., intercepts, vertex, axis of symmetry). It is possible that the students approached this problem conceptually rather than procedurally, thus resulting in its loading on this factor. The other two items, 11 and 17, were designed to test factoring skills and synthetic division skills directly. These items may have loaded on the first factor because the students had more difficulty with them than the researcher had anticipated. The instructors of the course decided not to emphasize factoring since the students had covered this material in Algebra I. The synthetic division problem was also originally designed to test procedural knowledge; however, its loading on the first factor suggested that perhaps the instructors did not develop the process as thoroughly or as conceptually as the students needed in order to master this procedure.

The loadings on the second factor were items relating to solving polynomial equations or to the similar process of finding the x-intercepts or zeros of polynomial functions (items 14, 18, 9, and 8). As an examination of the student responses revealed, these items probably loaded together because of the difficulties that the students had in making the connections between these concepts. Item 3, which required students to solve a quadratic equation by completing the square and was thus related to the second factor, did not meet the indicated criteria for either factor but did load more heavily on the second factor with similar items.

Items 15 and 20 loaded onto both factors. Item 20, a conceptually-designed application problem, required a variety of skills with polynomials including solving a polynomial equation. This was perhaps the reason for its loading on the second factor. Item 15, also designed to test conceptual knowledge, asked students to reverse their processes in working with function sums. Although this item did not require solving a polynomial equation, several students approached it by setting up an equation. This approach may explain why it loaded on the second factor.

Table 1. Factor Loading for EFA

Item	Factor 1	Factor 2
Q5	0.72504	-0.17952
Q12	0.73028	0.30639
Q6	0.65695	-0.20180
Q16	0.63788	0.19751
Q11	0.61935	0.06477
Q17	0.54237	0.15619
Q13	0.48160	0.27466
Q19	0.41483	0.10529
Q14	-0.15603	0.86722
Q18	0.26619	0.57918
Q9	0.01035	0.54913
Q15	0.45577	0.57093
Q20	0.45072	0.46608
Q8	0.02024	0.38769
Q3	0.16869	0.31251

Discussion and Conclusions

This study provided encouraging evidence that TIP students were able to increase their knowledge of polynomial functions during this fast-paced summer program as indicated by the significant increase in their scores on the items that were included on the pretest and on the final exam. This achievement seemed consistent with the success that proponents of acceleration opportunities predicted for gifted students (Fox, 1974; Stanley, Lupkowski, & Assouline, 1990; Corazza, Gustin, & Edelkind, 1995). A closer examination of data, however, suggested that the students struggled on many of the polynomial unit items. Most of these questions required the students to make connections that would indicate conceptual knowledge. The students' difficulty

with these problems, therefore, suggested that they were still conceptually lacking in their knowledge of polynomials. It is possible that the fast-paced coverage of these concepts did not allow students adequate time to make the connections necessary to achieve conceptual understanding. Perhaps the foundation that was laid at TIP will develop into conceptual knowledge when the students begin to review and to apply their knowledge to new mathematical concepts in future mathematics courses.

Although the EFA did not separate the items entirely according to the intentions of the researcher in writing the items to assess procedural or conceptual knowledge, it did reveal some interesting ideas about the impact of instruction on the development of concepts. It also indicated that conceptual knowledge cannot be completely separated from procedural knowledge. As Hiebert and Lefevre (1986) suggested, procedures learned with appropriate connections become conceptual knowledge.

The EFA and an examination of students' performance on the factoring and synthetic division items on the test suggested that these processes are not necessarily easily learned by TIP students. The loading of items in the second factor suggested that students were having some difficulties making the very important connections between the solutions of polynomial equations and finding the x-intercepts of polynomial functions. For instance, item 18 required the students to solve a quartic polynomial equation. This problem would have been fairly simple had the students found the two real roots with their graphing calculators; however, few students made this connection and used this strategy. Most of these students were unable to complete the algebraic processes required.

Perhaps the students had difficulty solving polynomial equations and finding x-intercepts of polynomial functions because they did not have adequate opportunity to practice these types

of problems. More likely, as Hiebert and Lefevre (1986) might suggest, these topics were covered quickly in class without the adequate conceptual development that would have helped the students to learn this material conceptually and to retain it through the duration of the program. It is unclear whether this lack of conceptual development resulted from the time constraints of the fast-paced approach or from the instructors' belief that these gifted students do not need as much instruction on these topics because they are capable of making the connections on their own. Future training sessions and materials for mathematics instructors should indicate the importance of emphasizing and reviewing seemingly simple procedures such as factoring and synthetic division. Training should also include some suggestions for the conceptual development of these concepts.

The analyses did reveal some promising evidence that factors often related to mathematical achievement, such as gender and race, do not seem to have an impact among the TIP students. The impact of belief systems in this sample of gifted students is more difficult to interpret than in other research studies (Kloosterman, 1991; Hackett, 1985; Betz, 1987; Fennema and Sherman,1977,1978; Cooper and Robinson,1991) in which mathematical self-efficacy and positive mathematical beliefs correlated positively with mathematical achievement. The significance and near significance of correlations between TIP students' belief scale scores and the test scores and subscores suggest that beliefs systems had some impact on the mathematical achievement of TIP students, as Schoenfeld (1985) suggested. The negative correlation, however, contradicts the results of the previous literature on mathematical beliefs and self-efficacy. It is possible that some of the students might have thought that they should report beliefs that were more positive than they really felt since they were taking part of a program that expected them to enjoy and to value mathematics. This phenomenon would have limited the

effectiveness of this particular beliefs inventory in this setting. Perhaps a belief scale designed specifically for gifted population would be more effective in detecting the impact of mathematical belief systems on the achievement of these students.

This study indicated that mathematical belief systems have some impact on the achievement of these gifted individuals, but further study and possibly new instrumentation are needed to understand the belief system effects more completely. The analyses suggest that, despite promising gains in knowledge of polynomial functions between the pretest and the final examination, gifted students are not necessarily able to acquire conceptual knowledge of all aspects of polynomial functions during fast-paced algebra instruction. The conclusions of this study provide some information for mathematics educators about trends in students' deficiencies in conceptual knowledge and suggest some topics that educators should address differently in instruction.

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INVESTIGATIONS IN CONCEPTUAL UNDERSTANDING OF POLYNOMIAL FUNCTIONS IN AN ACCELERATED SUMMER PROGRAM FOR GIFTED STUDENTS

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Abstract

This study investigated ten gifted students' conceptual knowledge of polynomial functions through their problem solving performances on four interview tasks. The participants were enrolled in a three-week Algebra II course at a summer program for gifted mathematics students. The students were able to make a variety of connections among concepts related to polynomials and functions, and they easily applied their mathematical knowledge to real world phenomena. They suffered, however, from several misconceptions relating to the understanding of the roles of the independent and dependent variables in functions. They also struggled with the concept of symmetry and how it relates to polynomial functions. Through its identification of several potential difficulties that gifted students may encounter in their learning of polynomial functions, this study suggested specific topics that teachers of gifted students should consider when planning their instructional activities.

Introduction and Conceptual Framework

Each summer, many mathematically gifted students choose to accelerate their studies of mathematics at programs that offer a fast-paced coverage of the material. School administrators and guidance counselors at local schools, however, question whether the students are able to attain a conceptual understanding of mathematical concepts when presented in such a fast-paced manner. This study investigated the conceptual knowledge of polynomial functions of ten students who were enrolled in a three-week Algebra II course during the summer of 2002. Polynomials are a crucial building block as students move on to higher levels of mathematics and are, therefore, important for students to understand conceptually. The unit on polynomials provided many opportunities to investigate the connections that students were making through their manipulations of various representations of polynomial functions, through their knowledge of characteristics of polynomial functions, and through their applications of functions to real world scenarios.

Mathematics educators and educational psychologists are very interested in understanding why some students are good mathematical problem solvers while others are not. Through this understanding, they hope to develop strategies to help other students, who are not successful mathematics problem solvers, to improve their skills. Schoenfeld (1985) identified four constructs that he believed contribute to students' success in problem solving: resources, heuristics, control, and beliefs. Schoenfeld defined the first category, *resources*, as the set of all facts, procedures, and skills that a person possesses prior to problem solving. He defined *heuristics* as "rules of thumb for successful problem solving, general suggestions that help an individual to understand a problem better or to make progress toward its solution" (p. 23). According to Schoenfeld, *control* is the type of behavior which refers to the way that students

make plans, set goals, monitor progress, assess solutions, and revise plans when necessary. Finally, Schoenfeld argued that a person's *beliefs* shape his cognition and that certain beliefs may actually interfere with his ability to solve problems.

In his discussion of his *resources* category, Schoenfeld (1985) acknowledged that not all of a student's resources are mathematically sound. Educators often refer to such resources as misconceptions. Since they are part of students' resources, these misconceptions often cause confusion and incorrect solution processes and are, therefore, important for educators to identify and to address in instruction.

Other theorists, such as Hiebert and Lefevre (1986), have suggested that there are two levels of mathematics understanding: procedural knowledge (rules without reasons) and conceptual knowledge (understanding the procedures and their underlying relationships to mathematical concepts). In order to achieve conceptual knowledge, students must make appropriate connections among mathematical concepts and procedures. The National Council of Teachers of Mathematics (NCTM, 2000), in its Principles and Standards for School Mathematics, strongly endorses the development of conceptual understanding because conceptual knowledge has several advantages over knowledge that is only procedural. Mathematics learned conceptually makes more sense, is easier to remember and to apply, and makes subsequent learning easier (NCTM, 1989, 2000). Students who have attained conceptual knowledge and who possess adequate levels of the constructs identified by Schoenfeld are generally excellent mathematicians and problem solvers .

The abilities of mathematically capable students identified in the research of Krutetskii (1976) and Rachlin (1981) offer some guidance in the assessment of conceptual understanding. The four problem-solving abilities that they identified (generalizability, flexibility, reversibility,

and transferability) require the construction of relationships described by Hiebert and Lefevre. Students who can generalize must be able to take a variety of examples and situations in their experiences and to link them together to make a broader statement of the relationships among them. Flexibility, by its very nature, requires the connection of pieces of knowledge so that a student can move easily from one representation to another or can connect a new representation to an existing concept. Reversibility also demands the connection of a variety of pieces of knowledge in appropriate sequence in order for students to be able to work in both forward and reverse directions. Finally, transferability requires the linkage of previously acquired knowledge to a new piece of information.

The students in this study participated in interview sessions during which they worked on several problem solving tasks that were inspired by the research findings of Krutetskii (1976) and Rachlin (1981). The performances of these mathematically gifted students on challenging mathematics problems were rich sources of information about the connections that these students were making and the problem solving strategies that they employed. The following research questions guided this study:

1. In the context of a three-week Algebra II course, how do TIP participants use resources, heuristics, and control to solve problems about polynomials that require them to make the connections among resources that indicate conceptual knowledge?
2. What patterns of connections and misconceptions are evident in the gifted students' attempts to solve problems about polynomials?

Literature Review

The following review of literature is a summary of the research results that inspired and guided this study. Included are the findings of studies that investigated conceptual knowledge, that identified misconceptions relating to functions and their representations, and that determined the effectiveness of acceleration opportunities for mathematically gifted students. Part of this review occurred prior to the data collection and analysis; however, much of it occurred during the analysis process as the themes began to emerge.

Mathematical Abilities and Conceptual Knowledge

Through qualitative and quantitative methods with over 200 students, Krutetskii (1976) identified several mathematical abilities of gifted students that are related to information processing skills and understanding: generalization, flexibility of thinking, and reversibility of mental processes. Problem sets that Krutetskii used to test the ability to generalize included problems of a single type, problems of different types, and problems of proof. To test for flexibility of thinking, he designed problems with several solutions, problems with changing content, and problems requiring the reconstruction of an operation. Krutetskii investigated reversibility through questions that required students to employ both direct and reverse operations. Gifted students, he concluded, were able solve these problems quickly and without assistance while average ability students often required hints and longer periods of time in order for them to complete the problems.

Inspired by the work of Krutetskii, Rachlin (1981) investigated the extent to which college students in basic algebra were capable of demonstrating the three abilities identified by Krutetskii: generalization, flexibility, and reversibility. Rachlin divided Krutetskii's generalization ability into two separate abilities: generalization and transfer. Rachlin used the

term *generalization* to refer to generalizations made within specific topics and *transfer* to refer to generalizations made between topics. He used students' performance on the tasks requiring these abilities in order to draw conclusions about their conceptual understanding. Through this qualitative study, Rachlin concluded that the students varied widely in their abilities to complete reversibility and flexibility problems. The subjects also had difficulty with generalization and transfer problems since they tended to learn each new algebraic concept as a disjoint unit instead of connecting the concept to previous knowledge; this cast doubt on the likelihood that these students had attained conceptual knowledge of the concepts. Rachlin's study provided further evidence that the abilities identified in gifted students by Krutetskii are not prevalent among all students. Rachlin's subjects were college students who were enrolled in basic algebra; therefore, they probably had a below average background in mathematics or had struggled with mathematics during high school. No follow-up studies to apply Krutetskii's findings to the gifted population have been attempted.

Research on Functions and Representations

There are many aspects of conceptual knowledge to consider in the study of polynomial functions. Much educational research in the learning of mathematics has revealed a variety of difficulties that students experience in their understanding of functions and in their ability to work with multiple representations of these functions. For example, Moschkovich, Schoenfeld, and Arcavi (1993) and Sfard (1989) suggested that one difficulty that students experience in the study of functions is the inability to grasp both the process perspective and the object perspective of functions. They defined the process perspective as the "linking x and y values: for each value of x, the function has a corresponding y value" and the object perspective in which "a function or relation and any of its representations are thought of as entities" (p. 71). The authors argued that

the mastery of both perspectives is crucial to the understanding of functions. Since the perspectives reveal characteristics of the behavior of functions in every representation, developing competency with both helps students to determine the appropriate representations within the context of a given problem. Sfard's research on the two perspectives indicated that the process perspective must be mastered before the object perspective. For example, a student is given a quadratic function that represents the flight of a rocket, with time as the independent variable and height as the dependent variable. A student who has mastered the object perspective will be able to describe the entire flight, given a graph of the function. The student would also need the process perspective in order to be able to find the height of the rocket at any given time after the launch.

Sierpinska (1992) described a variety of other difficulties that students may encounter in their attempts to understand functions. She argued that students tend to view the independent and dependent variables as entities separate from the phenomena that they represent. Students focus their attention "on the displacement itself, on the change, mindless of what was thus displaced or undergoing that change" (p. 33). Sierpinska also indicated that students have difficulty differentiating between the constants and the variables in a function (i.e, the given and the unknown quantities in the problems). A third difficulty that students face in the understanding of functions is what Sierpinska termed "the asymmetry of the independent and dependent variables" (p. 38). The order of the variables is important in functions, and many students have difficulty realizing that the second coordinate is uniquely determined by the first and not vice-versa.

The ability to manipulate various representations of functions is often a stumbling block for mathematics students. Moschkovich, Schoenfeld, and Arcavi (1993) argued that connections

between the object and process perspectives allow students to switch flexibly among function representations (e.g., equations, graphs, sets of ordered pairs) and to make connections among the representations in order to solve a variety of problems. Recently, research on the use of multiple representations in the teaching of mathematics has been very prominent. In particular, misconceptions related to the graphs of functions have received much attention. One common error that students make is viewing the graph as a literal picture of the problem situation (Clement, 1989; Dugdale, 1993). Instead of remembering what the axes of the graph represent and analyzing the graph accordingly, they view the graph as a diagram of the physical event. For example, students experiencing this "graph as picture" difficulty may mistake the graph in Figure 1 as a picture of the trajectory of the rocket flight rather than a graphical representation of the height of the rocket at any given second of the flight (e.g., $h(t) = -16t^2 + 80t + 200$). A "picture" of the event would show that the rocket was launched vertically from the top of the tower.

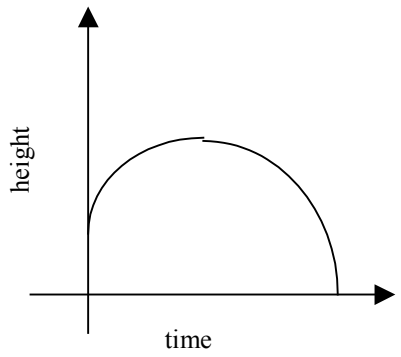

Figure 1. Height by time graph of a rocket launched vertically from the top of a tower.

Dreyfus and Eisenberg (1990) described another difficulty that students have working with visual representations of mathematical relationships. They stated that students possess a

"reluctance to process visually." Because teachers traditionally present material in a linear or sequential manner, these authors believed that students are more inclined to process mathematical concepts analytically rather than visually. In other words, teachers usually analyze the algebraic representations of functions before they produce graphical representations. Because of this approach, students tend to consider the visual aspects of a concept as peripheral to the concept itself rather than recognizing the visual aspects as characteristics of the concept. For example, the zeros of $f(x)$ are the values of x where $f(x)=0$, and, coincidentally, they are also the x-intercepts of the graph of $f(x)$. Because of this inclination to process concepts analytically, students are less likely to process mathematical concepts appropriately when they are first presented with the visual representation.

Another graphical concept that relates to polynomial functions, especially quadratics, is symmetry. Leikin, Berman, and Zaslavsky (1997) investigated the problems that in-service teachers experienced in determining whether a mathematical object is symmetrical. They discovered that the teachers' mistakes arose from several misunderstandings resulting from the way they related to aspects of the object. The first type of misunderstanding related to the object itself. The teachers either considered elements of the objects to be non-symmetrical, or they did not consider symmetry to be applicable to the object's elements (e.g., the graph of $y=x^3$ is not symmetric because elements of the object did not conform to the teachers' understandings of symmetry – they did not consider origin symmetry). The second type of misunderstanding referred to the way that the teachers related to a transformation of the object. With this misunderstanding, the teachers failed to check for an invariant property of an object with respect to the transformation (e.g., they mistakenly concluded that a parallelogram – not a rhombus - was symmetrical according to reflection with respect to a diagonal). The third type of

misunderstanding resulted from the way that the teachers related to a certain property of the object. With this misunderstanding, the teachers wrongly assumed that certain properties were necessary in order for symmetry to exist (e.g., they assumed that $y=x^3 - x^2$ was not symmetrical because it was neither an even or odd function). The final difficulty that the teachers experienced resulted from misinterpretations of the data provided. Conceivably, if teachers experience these types of misconceptions relating to symmetry, mathematics students may experience similar difficulties.

The use of technology, such as graphing calculators, to produce graphical representations of functions has also resulted in a variety of student misconceptions of the graphs of functions. Goldenberg (1988) described several visual illusions, caused by graphing technology, that can lead to student misconceptions. For example, in the graphs of linear functions (in the form $y = ax + b$), changing b may appear to move the graph horizontally or vertically, depending on the shape of the viewing window. In viewing graphs of parabolas on different windows, students may believe that the parabola has changed shape. In addition, the limitations of the size of the viewing window may lead students to believe that the domain of the graphed function is bounded by the values within the window.

While a great deal of recent research has investigated the learning patterns and misconceptions of students in their study of functions, few studies have considered the gifted population. It is possible that gifted students have similar patterns of understanding and misconceptions; however, gifted students may experience different challenges and successes in their understandings of polynomials and functions, especially in an accelerated course. In order to determine whether gifted students have different patterns of understanding, more research with this population is needed.

The Success of Accelerated Mathematics Programs for Mathematically Gifted Students

Educational specialists disagree on the value of opportunities for acceleration, or taking classes above grade level, for mathematically gifted students. Supporters of acceleration believe that such programs benefit gifted students by enabling them to work with their mental peers on tasks commensurate with their abilities (Fox, 1974; Stanley, Lupkowski, & Assouline, 1990; Corazza, Gustin, & Edelkind, 1995). As a result, acceleration program participants are rescued from boredom and insufficient challenge. Acceleration supporters also argue that gifted students would develop better study habits and more positive self-concepts through participation in accelerated programs. Opponents of acceleration opportunities claim that mathematically gifted students may not have reached the maturity level, both socially and physically, that is required to handle acceleration. Critics also fear that acceleration programs foster elitist attitudes among participants and senses of inferiority among students who do not qualify to participate (Kulik & Kulik, 1984). In response to these concerns, a number of researchers have investigated the effects of accelerated instruction for mathematically gifted students.

Several studies have investigated the short-term and long-term success of students enrolled in accelerated and fast-paced mathematics classes. Kulik and Kulik (1984) conducted a meta-analysis that indicated that students who accelerated their studies in mathematics consistently achieved at significantly higher levels than non-accelerated students. They often achieved at a higher level than the older students enrolled in courses into which they accelerated. Other studies found that students who had been allowed the opportunity to accelerate their studies of mathematics continued to excel in mathematics throughout their academic careers, chose to take more mathematics classes than other students, and often selected mathematics-related majors in college (Kolitch & Brody, 1992; Swiatek & Benbow, 1991).

The above-mentioned research studies about the short-term and long-term achievement of gifted students in accelerated and fast-paced mathematics programs have depended primarily on standardized achievement tests and on self-reported performance in future courses. None of these studies have considered the problem-solving abilities of this population or the depth of understanding achieved by students in accelerated and fast-paced programs.

Methodology

The Participants

The ten participants in this study were students in Algebra II classes at the Duke University Talent Identification Program's Summer Studies Programs (TIP), which are intense three-week residential sessions that are designed to provide participants with the equivalent of a year-long high school course. TIP participants are generally highly motivated and mathematically gifted students who are seeking an opportunity to accelerate their educational experiences. To qualify for mathematics courses offered by the Talent Identification Program (TIP), seventh grade academically gifted students scored 570 or better on the mathematics section of the Scholastic Aptitude Test (SAT).

From a pool of 20 students who volunteered to participate in the study, ten students were purposefully selected using a process described by Patton (2002) as maximum variation sampling, defined as the selection of a wide variety of cases to attain variation on dimensions of interest. The researcher used teacher input to select several students who seemed to master the material almost effortlessly, several students who were struggling, and several students between these extremes. This selection process was chosen in order to gain the greatest possible breadth of understanding of the common threads of student strategies and connection-making.

The ten students who were interviewed as part of this study came from diverse backgrounds and ability levels compared to the other students in their TIP classes. An examination of the students' pretests indicated that these students did not possess conceptual knowledge of polynomials when they began the course. All of the students, with the exception of Jason and Michael, had completed Algebra I and Geometry prior to attending TIP. Jason and Michael had both completed Algebra I only. None of the students had taken Algebra II previously. In order to ensure confidentiality, all names used are pseudonyms.

According to their instructors, Jason, Jake, and Pete all excelled in their coursework. Jason had recently finished the eighth grade and was from a suburban school district in the south central United States (US). Jake was from a midwestern urban school district. He had recently completed the ninth grade and excelled in his class. Pete, also a ninth grader, came from a southeastern suburban school district.

Julie, Michael, Eric, and Evan were all described by their instructors as middle-ability level students. Julie, an eighth grader, came from a suburban school district on the East Coast. Michael, an eighth grader, and Eric, an ninth grader, came from small cities in the Southeast. Evan had recently finished the ninth grade and was from an urban school district in the south central US.

David, Alan, and Nick's instructors indicated that they struggled with the course material. David, an eighth grader, came from a rural school district in the Midwest. Alan, a tenth grader, attended a rural school in the East. Finally, Nick was a ninth grader and lived in a small city in the south central US.

The Procedure

The researcher interviewed each of the ten students individually for approximately 45 minutes. During the interviews, she presented the student three or four polynomial problems that required reversibility, flexibility, generalization, and transfer (see Interview Protocol – Appendix F). The researcher employed the "clinical interview" method as defined by Vygotsky (see Ginsberg, 1997, p. 54). In clinical interviews, Ginsberg summarized the interviewer's role as follows: "In an attempt to gain insight into the child's thinking and learning potential, the interviewer presents problems, modifies questions if the child seems to misunderstand them, devises new problems to test hypotheses arising in the course of the interview, and challenges answers to test the strength of the child's conviction."

During the interviews, the researcher presented each problem individually and requested that the students "think aloud" as they proceeded. As the students worked on each problem, the researcher asked them questions in order to determine their use of resources and to determine what connections the students were making. She offered probes to help students who were experiencing difficulty, and she attempted not to indicate the correctness or incorrectness of any answers. The researcher also provided a graphing calculator and graph paper and indicated that the students were permitted unlimited use of these items. The interviews were videotaped and transcribed. The researcher analyzed the interview transcriptions along with all of the students' written work on the problems in the interview and on similar test problems.

The researcher employed a multi-case study design in which she analyzed each individual student's problem solving performance and then identified common themes among all of the students' work (Bogdan & Biklen, 1998). The following research questions guided the analyses of the interviews:

1. What types of resources (including misconceptions), heuristics, and control was the student using to solve the problems?
2. What connections was the student making among resources that indicate conceptual knowledge of polynomials?
3. What evidence indicated that the student was unable to make connections and was, therefore, operating only with procedural knowledge?

Each individual interview was analyzed and coded. The codes were designed according to conceptual framework, with categories to identify specific instances of the use of resources, heuristics, and control (see Appendix G). Instances of connections among various resources were also included in the coding scheme. After each interview was coded, the analyses of the interviews were summarized into matrix formats (see Appendix H). From the matrix analysis, common themes among the interviews were identified and described in order to establish some generalizations about the problem-solving abilities and about the conceptual knowledge that TIP students were able to acquire during the three-week session.

Constraints and Ethical Concerns

Because of the short time frame under which this program operated and the small amount of free time that students had, the researcher was only able to interview each student once. As a result, she chose to focus on a single unit rather than on the entire curriculum. Removing students from class for interviews would have been ill-advised because the students would have missed a great amount of instruction during their absences; this would have placed them at a disadvantage. Therefore, she interviewed students during their free time in the afternoon.

Since the participants were minors, parental consent was required before students were allowed to participate in research. The researcher, therefore, was only able to interview

volunteers whose parents had consented to their participation in research. The researcher was not the instructor-of-record for these courses nor was she the direct supervisor of the instructors. The students' performance in the interview did not impact their course evaluation in any way. The interviews consisted of mathematics problem solving activities that posed no risks to students, and the students were promised confidentiality.

Validity Issues

In order to establish rapport with the students and to understand how their instructors presented material, the researcher observed each of the two classes two or three times a week (during each of the three weeks) for an hour. The instructors introduced her to their students during the first class meeting, and she explained the research that she would be conducting throughout the term. Many of the students were familiar with TIP research and were willing to participate. During the subsequent classroom visits, the researcher was able to interact with students by assisting them while they worked on their assignments. She also spoke with the instructors and the teaching assistants daily at meals in order to learn about their lesson plans and about their general feelings about the classes.

As a TIP program coordinator, the researcher hoped that this study would reveal some evidence that these gifted students were able to achieve some conceptual knowledge of polynomial functions during the fast-paced Algebra II course. However, as a former high school Algebra II teacher and a former TIP Algebra II instructor, she recognized that the compressed time frame of the course was not conducive to the development of conceptual understanding, since this type of understanding often requires extensive problem solving with varied problems. This recognition helped her to be realistic during her data analysis.

Results

Through the analyses of the interview transcripts and the written work on the polynomial tasks, several themes of connection-making and misconceptions emerged. These themes seemed to fall naturally under three broad catagories: conceptions of polynomial functions, interpretations of real world applications of polynomial functions, and representations of polynomial functions. The following sections describe the themes under the heading of the corresponding categories along with related findings in the research literature.

Conceptions of Polynomial Functions

In general, the students possessed a number of resources—facts, algorithms, and skills—about polynomial functions that they used in working on the problems. In particular, they were knowledgeable about factoring and about polynomial operations (division, multiplication, simplification). They were not, however, always consistent in making connections among their polynomial resources.

One common connection that these students made consistently was between the definition of factor and polynomial operations. On the first task, students were asked to compute the missing linear coefficient in a cubic polynomial so that a particular binomial (x-3) would be a factor of the cubic. Most of the students immediately set up long division and began to divide the binomial into the cubic. In order to ensure that they were indeed making the connections, the researcher questioned them about their choice in strategy. Some students used a direct division approach while others employed a combination of division and multiplication in order to reach an answer.

Jason, for example, chose long division because he knew that a factor is something that divides into the dividend evenly (i.e., with a zero remainder).

Angela: ... Okay. Tell me a little bit before we stop. You used long division and you were doing some work in your head. What were your reasons for using long division?
Jason: Being a factor means it goes in evenly.
Angela: What do you mean by "goes in evenly?"
Jason: Oh, well if you're dividing something into something else, then for the thing you're dividing in to be a factor, it has to divide in evenly.
Angela: So how were you trying to make this happen?
Jason: I had to make it, *k*, so that there was no remainder when I divided. Yeah, the zero remainder makes this a factor of this (pointing at paper).

Jason's ability to explain his choice of strategies with little hesitation indicated the connections that he had made between the concept of factor and the process of long division. His success with this problem suggested that he was capable of reversing his thought processes.

Like Jason, Jake chose to use long division and later combined the process with multiplication because he became confused when his long division reached the missing coefficient.

Jake: ... How should I start? Let's see. Well, I could divide this by x-3 because if it's a factor, I'll be able to divide it in evenly.
Angela: Okay.
Jake: (writes out the division problem and works through it) $-x^2 + kx$ …hm… that could be a problem. It's going to be x-3 and that's going to be x so it'll be $(k+3)x - 6$.
Angela: Okay.
Jake: And that's going to be 2.
Angela: How do you know that's going to be 2?
Jake: Because –3 times 2 will be the –6. Otherwise, I will have a remainder, and it couldn't be a factor.

Jake also recognized the connection between the concept of factor and how division can show that a divisor is a factor if the remainder is zero. He combined multiplication with the initial division strategy, The students' abilities to connect the concepts of factors and long division to solve the first problem suggested some level of conceptual understanding of these topics.

In addition to the connections that the students made between their resources relating to the definition of factor and the operation of polynomial division, the students' conceptual knowledge was highlighted by evidence of control in their work on the first task. A number of students monitored their progress and changed their methods when their initial approach to the problem proved unsuccessful. For instance, Julie first attempted factoring by grouping on the first task. When she discovered that this approach was not yielding any solution, she changed her method and used the factor theorem (although she could not remember the name of the theorem that she was using), substituted 3 for x, set the polynomial equal to zero, and solved for k. In addition to monitoring progress, several students checked their work, either by multiplication or synthetic division, and confirmed that their solution was correct.

Also indicative of the students' conceptual knowledge of these topics was their use of a variety of heuristics in their attempts to solve the problem. Most students started the problem in a direct manner by setting up the division problem and proceeding as they did on their class examples; however, several students became confused when they reached the linear term which contained the coefficient k. They then employed different strategies. Jason, Jake, and Pete, for example, realized that they needed a zero remainder and worked backward to find k. Evan became frustrated with the direct approach, switched to a graphical representation, and experimented with different values of k on his calculator until he found the one that had an x-intercept of 3.

Although the students did show some evidence of conceptual knowledge, not all of their work with polynomial functions indicated connections. Occasionally, the students experienced one of the difficulties with functions that Sierpinksa (1992) highlighted as they viewed the independent and dependent variables as entities themselves rather than what they represented.

When this difficulty occurred, students reverted to procedural applications of polynomial knowledge. For example, David reverted to his procedural understanding of quadratic equations and the x-intercepts during his work on the second problem. In this problem, the researcher presented the students with an algebraic model for a rocket that is fired from the top of a tower and asked them to analyze the given quadratic function in order to gather as much information about the scenario as possible. Although David had already graphed the function on the calculator, he was unable to connect the variables to the phenomena that they represented, especially when questioned about the information he could gain from the x-intercept. He shifted to his procedural knowledge in order to address the question about the x-intercept:

Angela: Okay. Well, why would it be helpful to know where it touches the x-axis?
David: Well, that would make *y* equal to zero which makes it the same as *y* and so it would be zero equals $-16t^2 + 80t + 200$.
Angela: Okay. And what makes that important?
David: I think, I want to say, that if you have it at zero, then it's easier to solve for *t* because you have a value, on both sides of the equation…Then you have two variables and you want to find out what *t* is in terms of *h(t)* or *y*.
Angela: Okay. So having zero on this side, the useful thing about that is that it gives you a value on that side. To help you solve?
David: Yeah, I think so.
Angela: Okay. And does it change the problem at all to have a zero over there?
David: No… *t* is still... no, because you would still, it would still vary after you change *t* is still the same thing. You know what I'm saying? This side only varies with…no, it would not change it because when you have a function or any kind of line where you can set it equal to zero, you can just do this. I don't know why, but you just can.

In this exchange, David failed to make the connection that the x-intercept indicates the time at which the rocket lands on the ground. Although his algebraic procedures for finding the x-intercepts were correct, there were no practical connections. Furthermore, David failed to recognize that substituting a zero for *h(t)* in order to find the intercepts does affect the meaning

of the problem; he instead made an incorrect generalization that substituting zero does not change the problem. He was, as Sierpinska (1992) described, treating the variables as separate entities from the values that they represented in the context of the problem.

Pete experienced a similar difficulty on the same problem. When the researcher asked him what the axes represented, he stated, "I don't think that the x-axis has any real significance, and the y-axis is actually the *height*." Although Pete was able to connect the variable of the vertical axis to its real meaning, *height*, he was unable to connect the concept of *time* to the horizontal axis.

Another misconception that was prevalent in the students' algebraic resources was what Sierpinska (1992) described as the asymmetry of independent and dependent variables in functions. When the students encountered a polynomial function and tried to analyze it, they expressed confusion about the role of each variable. For example, in her analysis of the rocket problem, Julie stated, "... If you know the maximum height, and you plug it in then you can figure out *t*. And once you solve for *t*, then that's how you can solve for the maximum." In this problem, the *time* is the independent variable, and *height* is the dependent variable. The analytical goal was to determine the maximum height by first calculating the time at which the maximum would occur (there are a variety of methods for determining the appropriate value of the independent variable) and then by substituting that time into the equation to compute the desired value of the dependent variable. Although Julie acknowledged this goal earlier in the exchange, she became confused because of her misconception of the role of independent and dependent variables, and she engaged in circular reasoning.

David, Jake, Nick, and Alan had a similar difficulty with this problem. David, for example, stated, "Well, if you solved the equation for *t*, you could figure out the *time* and the

height because *t* equals the *time*…I'd use it to figure out the *time* after finding the *height*." He seemed to be analyzing the problem correctly at the beginning when he discussed first finding *t* and then the corresponding *height*; however, then he circled back and indicated that he would have then used the *height* to figure out the *time*. Nick very clearly exemplified this asymmetry problem when asked what useful information he could obtain about the given scenario in the second task. He stated, "The *height* and the *time*, depending on what you're given because you have one equation with two variables." Clearly, he viewed the role of these variables as interchangeable depending on the given information.

The preceding discussion of the students' conceptions of polynomial functions not only reveals evidence of some conceptual knowledge, but it also suggests that the students are conceptually lacking in their understanding of some concepts relating to functions. These misconceptions often caused the students to revert to procedural methods in their approach to the problems. The misconceptions also interfered with their abilities to solve the problems efficiently and correctly.

Interpretations of Real World Applications of Polynomial Functions

With very few exceptions, such as the previously mentioned difficulty that occurred when students disconnected the variable from the phenomenon it represented, the students were able to connect their real world experience to their understandings of the mathematical models (algebraic, tabular, or graphical) that they encountered. For example, in the second problem about the rocket launch, Jason was able to make an immediate practical connection to the equation that represented the scenario.

> Angela: ... What type of problems could you solve related to this?
> Jason: When it's going to hit the ground.
> Angela: Okay, what else?

Jason: How many seconds… before (pause) How many seconds it takes to reach the maximum height.
Angela: Okay.
Jason: The amount of seconds to reach this height again…

He completed this analysis with only the algebraic equation. Later, when questioned about features of the graph, he was again able to connect them to the real world phenomenon. He was even able to explain a feature of the graph that did not have practical significance.

Angela: All right. And then the point here (pointing). What is that?
Jason: That is how many seconds it took for the rocket to hit the ground.
Angela: Well, now this is a parabola, so what's going on with the other x-intercept in this example?
Jason: Well, it was fired 200 feet high so it was not fired from ground level. That's why this other intercept is negative.

Like Jason, Julie recognized that the vertex of the parabola represented the maximum *height* and that the rightmost x-intercept represented the *time* when the rocket landed on the ground. She explained the leftmost x-intercept differently from Jason. She expressed the discrepancy between the domain of the mathematical equation and the domain of the real world situation: "Well, the calculator doesn't do anything but graph the mathematical equation and the mathematical equation, technically speaking, [the calculator] doesn't disregard the negatives because it's impossible to have negative seconds. But, in a math equation, it's not really." Jason's explanation focused on the practical reason for the negative intercept while Julie focused on the mathematical explanation for its existence. Both explanations were appropriate and correct.

The fourth problem in the interview also required the students to apply their mathematical knowledge to a real world scenario. The problem asked the students to determine the optimal price of stereos that would ensure the greatest revenue for a retail store, given certain market conditions. The students had to develop their own equation to represent the situation and

then to produce a graph to help them with the analysis. Jake was able to develop a correct algebraic representation of the scenario and to explain its relationship to the practical phenomenon.

Jake: This is: Wait. It would be n+2 or 20, uh… 20 + 2n because original sales are 20 per week. So, it would be 20 + 2n times 70 – 5n.
Angela: Okay. Explain how you got that.
Jake: Okay. 70-5n would be… 5n would be… n would be number of 5 dollar decreases so you multiply by 5 but then you'd have to take that away from the suggested retail price. Now 20, because you'd get 2 more sales per decrease and n is the number of decreases, that means you'd get 2n.
Angela: And you multiply that together because…?
Jake: Because this is the total amount of money you get from the sales multiplied by the price.

Pete was able to develop a different, but appropriate, equation to represent the scenario. He also made a more significant connection to the real world in his explanation of why the geometric representation of the scenario, a parabola that opened down, suited the phenomenon.

Angela: Why do you think that this goes up for a little while, and then it starts to go down again?
Pete: Because here would be x=0 meaning you're selling at 70 bucks which means you'd make $1400 for the week selling 20 a week. So you get a better weekly revenue if you decrease price a bit. But then it starts going down because you decrease price by too much and you start losing profit. After a while, you start losing money for each one because you've lowered the price too much.

In addition to the clear connections that the students were making between real world scenarios and mathematical representations of these scenarios, the students also employed a variety of heuristics in their work on the second and fourth tasks. Several drew pictures of the scenario in order to visualize what was happening. Often the students broke the problem down into smaller, more manageable parts and worked on them individually. For example, on the stereo problem, Eric and Pete both worked separately to construct expressions to represent the

price per stereo and the number of stereos sold before combining the expressions to create a revenue function. The most commonly used heuristic for the second and fourth tasks was the change of the algebraic representation to another representation (graph or table) before proceeding with analysis.

Along with the variety of heuristics, the students also demonstrated control while they worked on the second and fourth tasks. They monitored their progress and were willing to make changes if their current procedures were not productive. Eric, for example, was having some difficulty adjusting his graph window for the second task so he immediately shifted to the table in order to determine how long the rocket was in the air before it struck the ground. In addition, several of the students commented that there was a superfluous piece of information, or distractor, in the fourth task. Other students expressed some "problem sense," or intuitive understanding of how the problem should appear, even if flawed resources (such as inability to remember the formula for finding the x-coordinate of the vertex) led them to an incorrect graph of the scenario. For example, Nick tried to graph the rocket function by hand, but he incorrectly computed the vertex because he could not remember the formula for find its x-coordinate. When he could not determine where the mistake had occurred, he correctly explained how the graph of the functions should look.

Overall, the participants were adept at connecting their knowledge about real world phenomena to their mathematical representations. This ability suggested that the students were operating with some conceptual knowledge. Along with the connections, the students were comfortable employing a variety of heuristics in working on the tasks that required real world applications. As Hiebert and Lefevre (1986) suggested, the fact that the students demonstrated

control over their progresses while working on these problems also suggested that they had achieved some conceptual knowledge.

Representations of Polynomial Functions

A final theme that emerged through the interview analysis was the students' use of knowledge about the characteristics of representations of polynomial functions. As described in the previous section, the students were generally able to make connections between equations and their corresponding graphs when they encountered the equation first. Most of the students immediately requested to graph the function in order to aid in their analyses of the problems. In addition, they knew what kind of graph to expect from the equation and were able to complete a rough sketch of the graph and to explain the analysis process without a careful step-by-step graphing process. For example, while working on the second task (analyzing the quadratic function $h(t) = -16t^2 + 80t + 200$), David stated, "It's going to be …a parabola because it's quadratic." Jake was even more specific about the type of parabola: "…it would probably be a parabola that opens down because of the negative 16."

When the researcher gave the students information about the graphs without the corresponding equations, however, the students experienced difficulties with their understandings about the graphical characteristics of polynomials. These difficulties seemed to occur primarily as a result of the students' tendencies to process material analytically rather than visually (Dreyfus & Eisenberg, 1990). In particular, the students possessed misconceptions relating to symmetry and polynomial functions, especially quadratics. These misconceptions emerged as the students worked on the third problem in which the researcher gave them three points that defined a unique parabola and asked them to determine what kind of curve, that they had been studying, might fit the points. Because the points themselves lacked symmetry (they

were chosen for this very purpose—see Figure 2 below), the students were unable to determine alternate ways to position a parabola to fit them. They insisted that the lowest point had to be the vertex.

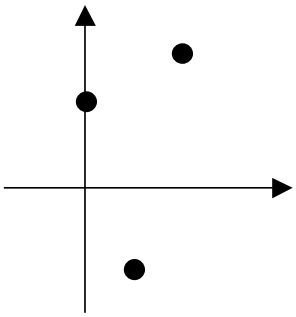

Figure 2. Graph of the three points given to students in Task 3.

Michael exhibited an example of one difficulty that the students encountered with this problem. He stumbled because the top two points were not "symmetric."

> Michael: Well, maybe it could be a parabola. I don't know.
> Angela: What makes you doubtful?
> Michael: Well, because usually parabolas are like this (draws an example over his graph). Like they'd have to be in line.
> Angela: Oh, I see. So if these two [points] aren't right next to each other, it might not be a parabola?
> Michael: Yeah, because usually parabolas look more like this (points at his example).

Michael's difficulty seemed to stem from the way that he interacted with the points. He apparently connected the points in his mind but was unable to shift his visualization of a parabola that would fit the three points without the lowest point as the vertex and the other points lying horizontal with one another. This difficulty is similar to a phenomenon described by Dreyfus and Eisenberg (1990) that they called "reluctance to process visually." They noted that students tend to consider the visual aspects of a concept as something that is peripheral to the concept. In

other words, a parabola is the graph of an equation of the form $f(x)=ax^2+bx+c$, and it just happens to be symmetric around a vertical axis of symmetry. When confronted with a visual of certain components of a parabola that do not conform to the students' concepts of symmetry, they are unable to adjust their view of the diagram in order to make it consistent with their concepts. Leikin, Berman, and Zaslavsky (1997) would have classified this symmetry problem as a difficulty in the way that the student related to the visual object presented.

Jason experienced severe difficulty with his understanding of symmetry and polynomial graphs while working on the third problem. He had a good understanding of the definition for axis of symmetry, but he inappropriately applied the idea of symmetry to all polynomial functions. Since he could not explain why the example $y=x^3$ did not have an axis of symmetry, he concluded erroneously that it was not a polynomial function.

Jason: It's not the parabola.
Angela: Why isn't it a parabola?
Jason: Because if you go up from one right here, it goes that high. So, if you go from two right here, it's only that high. They don't match up. (pointing at paper).
Angela: Okay.
Jason: Like you go from 1 to, what is this, 2? You increase by 19 on one side, but right here, by going one step, one space, you increase by 19 but on the other side you only increase by 16 in going two steps to the left. So, it does not make it even. It does not have an axis of symmetry.
Angela: Okay. Is there any way that you could think about that a little differently? You're sure that it's not a parabola?
Jason: It's not a parabola.
Angela: Is it possible that it might be some sort of polynomial?
Jason: I can't see how it could be.
Angela: What makes you think that it's not another type of polynomial?
Jason: Because all polynomials are symmetric. Except for a cube question (drawing $y = x^3$) which isn't really a polynomial function so I'm not sure. Even for ones like that (drawing a polynomial with three extrema), they would all have to have an axis of symmetry.
Angela: And describe in your own words what the axis of symmetry is.

Jason: It's like the center. The axis of symmetry is, well this sort of like an elementary definition, if you fold it in half it would line up perfectly.
Angela: Uh huh. So all parabolas, even the ones that are not… I'm sorry. I didn't understand what you were saying.
Jason: Not all, like *x* cubed. It's sort of like that (pointing at paper)
Angela: Okay.
Jason: But like *x* squared and *x* to the fourth power and the ones that go either up or down. They all have axes of symmetry.

Clearly, this problem highlighted a strong misconception to which Jason was committed—that all graphs of polynomial functions exhibit symmetry and have an axis of symmetry. He was so committed to the misconception that he failed to reconsider the concept even in the presence of a conflicting example that he produced himself.

Jake experienced a different problem with polynomials and symmetry. He correctly concluded that the points in the third problem represented a narrow parabola; however, he insisted that he could not write the equation since the parabola had no axis of symmetry.

Jake: I think it would be a very, very, very long parabola, first of all…
Angela: Oh, okay. Do you suppose you could write an equation to fit that parabola, based on the given information?
Jake: (shakes head) On this particular problem, I couldn't because there is no axis of symmetry.
Angela: So you need an axis of symmetry in order to write an equation?
Jake: Essentially. It's the only way to come up with the equation.

Jake was expressing a misconception here. He recognized that the points defined a parabola, but he failed to connect the fact that all parabolas have axes of symmetry. Because of his reliance on this misconception, he was unable to discover a method for computing the equation.

Both Jason and Jake's analyses of the third task were examples of the phenomenon identified by Dreyfus and Eisenberg (1990) in which students view visual characteristics of concepts as peripheral to the concepts themselves and, therefore, draw incorrect conclusions about the concepts. Jason erroneously concluded that a particular characteristic of quadratic

functions, namely the existence of an axis of symmetry, applied to all polynomial functions. To Jake, the presence of an axis of symmetry was a characteristic of some parabolas but not necessarily all of them.

The third interview task definitely highlighted some misconceptions related to the representations of polynomial functions. In particular, the students were unclear on the concept of symmetry and how it related to polynomials. Although they generally understood that the graphs of quadratic functions are symmetric with respect to a vertical axis, they were unable to adjust visual interpretations of the points provided to conform a parabola to include them. The students used few heuristics and control strategies while working on the third interview task, probably because of their lack of conceptual knowledge and their dependence on misconceptions.

Discussion and Conclusions

This analysis revealed several insights into TIP Algebra II students' conceptual knowledge of polynomials. Although the sample size was fairly small, the students selected came from a variety of backgrounds and achieved at different levels in their classes. Therefore, it is probable that the patterns of understanding that arose from these interviews were representative of the types of understanding that TIP students are achieving.

This study showed that TIP students were able to make fairly straightforward connections between the definition of factor and various polynomial operations and between equations and graphs. When they were able to depend on conceptual knowledge, they were also able to employ a variety of heuristics and to control (or monitor) their processes more completely. This finding was very consistent with Hiebert and Lefevre's (1986) belief that conceptual knowledge contributes to students' abilities to monitor their progress.

Perhaps the most promising and consistent connections revealed through the analyses were between mathematical models and real world applications. Because NCTM (1989, 2000) suggested that conceptual knowledge is easier for students to transfer, this finding indicated that the students had achieved conceptual understanding of some of the concepts that they had learned. The ability to apply mathematical knowledge to real world phenomena is a fundamentally important skill that many mathematics teachers hope to help their students develop and that NCTM strongly endorses. Perhaps this ability came from students' previous problem solving experience either in their TIP classes or in classes at their local schools. The TIP instructors indicated a commitment to incorporating a variety of real world problems into their lesson plans. These experiences may have contributed to the students' comfort levels in working with application problems. Therefore, it is important for TIP to encourage its mathematics instructors to continue this practice in their courses.

Despite the evidence of mathematical connections, the students experienced two major conceptual deficiencies: the difficulties with the function concepts of independent and dependent variables and the misunderstandings about polynomials and symmetry. The conceptual difficulties with the independent and dependent variables were consistent with Sierpinska's (1992) findings about difficulties that students encounter with functions. The first conceptual difficulty related to function variables was the tendency to view the variables as entities separate from the measures that they represented. This problem arose occasionally in isolated situations, perhaps due to student anxiety during the one-on-one interview setting. In general, these occasional lapses did not inhibit the students' ability to continue with other aspects of the problem. The second conceptual difficulty with functions was the students' lack of understanding of the asymmetry between the independent and dependent variables. A plausible

explanation for the occurrence of this problem was the instructional delivery of this material. The function concept was probably introduced as a variation on the concept of a mathematical equation in which the role of the variables is interchangeable. This conceptual deficiency was somewhat problematic in the students' analyses of the data and should, therefore, be addressed in the training of future mathematics instructors.

The students' misconceptions relating to symmetry and polynomial functions were also very important conceptual deficiencies that caused them many problems during the interviews. The students' misconceptions about symmetry ranged from an erroneous understanding that all polynomial graphs are symmetric around an axis of symmetry to the false conclusion that quadratic equations can only be produced for graphs of parabolas that have axes of symmetry (when in fact all parabolas have an axis of symmetry). Regardless of the type of misconception, these difficulties hampered the students' abilities to solve the third problem. In fact, none of the students were able to accomplish the ultimate goal of this task—computing the equation of the quadratic function to fit the three points. Perhaps the most logical explanation for the occurrence of these misconceptions was Dreyfus and Eisenberg's (1990) suggestion that traditional teaching methods typically focus on analytical analysis before visual analysis. These misconceptions resulted in mistakes in the way that the students related to the objects themselves, as described by Leikin, Berman, and Zaslavsky (1997). In this case, the mathematical objects to which the students had difficulty relating were their graphs of the points provided. The prevalence of the misconception is cause for concern, and perhaps mathematics instructors should be encouraged to include more visual analysis into their instructional strategies. Studies of misconceptions relating to symmetry are uncommon in the research literature, and the findings of this study indicate the need for additional research in this area.

Although TIP students experienced a variety of misconceptions that are prevalent in the research literature, they did not experience several common difficulties related to functions and representations. It is difficult to determine if they failed to encounter these difficulties because of their mathematical ability or because of the instruction that they received.

While the initial analyses of the interviews revealed some evidence of the students' interpreting "graphs as pictures," a phenomenon described by Clement (1989) and Dugdale (1993), later consideration of these instances revealed that the students did not experience any graph interpretation difficulties as a result. The researcher concluded that the problem was the students' less-than-precise verbalizations of their interpretations of the graphs instead of the true "graph as picture" phenomenon. The absence of the "graph as picture" misconception can probably be attributed to the instructors' efforts to include as many opportunities to explore real world phenomena as possible during class time.

The participants did not seem to experience any severe problems with the process or object perspectives of functions, as defined by Sfard (1989) and Moschkovich, Schoenfeld, and Arcavi (1993) because they were usually able to choose appropriate representations to analyze the scenarios presented by the researcher. These students also did not experience any of the misconceptions that Goldenberg (1988) suggested may arise from the use of technology. The students easily adjusted the windows of their graphs and did not seem to be bothered by the various visual illusions that hamper many students. It is difficult to determine whether the absence of these misconceptions was the result of classroom instructional strategies or of the students' mathematical talent. Regardless, the lack of these commonly occurring misconceptions is promising.

Overall, these gifted students were a very interesting group to study because they were so articulate and because they were very willing to participate in the study. They seemed to understand the importance of their contributions to knowledge base in gifted mathematics education. This study revealed a variety of interesting and informative insights into these gifted students' knowledge of polynomials. Further analysis and research should focus on different units of algebra and other mathematics classes in order to determine the success of various mathematics curricula for gifted students.

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APPENDICES

Appendix A

Learning Objectives for Polynomial Unit

1. Polynomial Expressions

The student will be able to:

a) Identify the degree of a polynomial
b) Perform operations (addition, subtraction, multiplication, division) on polynomials
c) Factor polynomials by a variety of methods

2. Complex Number System

The student will be able to:

a) Identify imaginary and complex numbers
b) Simplify powers of *i*
c) Perform operations (addition, subtraction, multiplication, division) on complex numbers

3. Polynomial Equations

The student will be able to:

a) Identify the degree of polynomial equations
b) Solve quadratic equations by a variety of methods (factoring, completing the square, quadratic formula, graphically with graphing calculator)
c) Solve higher order polynomial equations by a variety of methods (rational root theorem with factoring, graphically with graphing calculator)
d) Apply polynomial equations to solve real world problems

4. Functions

The student will be able to:

a) Identify a relation and a function
b) Find the domain and range of functions
c) Perform operations (addition, subtraction, multiplication, division, composition) on functions
d) Graph functions using translations and reflections
e) Use functions to model real world problems

5. Representations of Polynomial Functions

The student will be able to:

a) State characteristics of polynomial functions given various representations such as equations, graphs, and tables (discuss type of polynomial, number of intercepts, general shape, existence of extrema)
b) Compute intercepts of polynomial functions
c) Compute the vertex of a quadratic function by a variety of methods (formula, completing the square, graphically with graphing calculator)
d) State the axis of symmetry of a quadratic graph
e) Graph polynomial functions by a variety of methods (plotting points, computing points such as intercepts and vertices, transformations on monomial functions, using graphing calculators)
f) Use graphs of polynomial functions to investigate real world phenomena

Appendix B

Polynomial Test Questions (Objectives included in parentheses)

Pretest and Final Exam Questions

1. Generalization (5a)
   a) How many y-intercepts are possible for a function? How do you know?
   b) How many x-intercepts are possible for a function? How do you know?

2. Reversibility (3c)
   Write a cubic equation with real coefficients that has 4 and $3i$ as two of its solutions. Show your work, and explain your steps.

3. Flexibility – reconstruction of an operation (3b)
   Jim concluded that $x = 3$ is the solution to the equation $x^2 = 3x$. Did he solve the equation correctly? If he did, show how he solved the equation. If he did not, how should he have solved the equation?

4. Flexibility – use of a tabular representation of data (5a)
   Does the data in the following table represent a linear function? Explain your answer.

X	*Y*
-2	3
0	11
1	15
2	18
5	26

5. Generalization (4b)
   What is the domain of any polynomial function? Explain your answer.

6. Transfer to real world example (3d)
   A rocket is fired upward from ground level with an initial velocity of 96 ft. per second. If the height (h) of the rocket from the ground at t seconds after the launch can be represented by $h(t) = 96t - 16t^2$, when will the rocket land? Show your work, and explain your answer.

Polynomial Unit Test

1. Procedural question (5e)
   Determine the value of a so that $y = ax^2$ contains the point $\left(-\frac{1}{2}, 4\right)$.

2. Procedural question (2a,b,c)
   Simplify $2i(4-3i)(1+2i)$.

3. Procedural question (3b)
   Solve $4x^2 = 4x + 3$ by completing the square. Show your work.

4. Procedural question (3b)
   Solve $3x^2 - 5x = -4$ using the quadratic formula. Show your work.

5. Flexibility between graphical and algebraic representations; generalization (5e)
   a) Give the algebraic equation of the following graph. Each mark represents one unit. (Graph of $y=x^2-3$ provided)

   b) Describe the graph of any parabola of the form $y = x^2 + c$ where c is a real number.

6. Procedural question (5a,b,c,d,e)
   Sketch the graph of $y = x^2 - 10x + 24$. Indicate the direction of opening, the vertex, the axis of symmetry, and the x- and y-intercepts.

7. Generalization (5a)
   How many possible x-intercepts can a parabola have? Draw pictures to illustrate your answers.

8. Generalization/reconstruction of an operation (5a)
   Ginger stated that all cubic functions have three x-intercepts. Either explain why she is correct, or disprove her statement.

9. Reversibility (3b)
   Write a quadratic equation with real coefficients that has $3 + 5i$ as one of its solutions.

10. Procedural question (4d, 5e)
    Sketch the graphs of the following functions. Show your work.

    a) $f(x) = -x^3 - 3$ b) $f(x) = (x-2)^2 + 3$

11. Procedural question (1c)
    Factor each of the following polynomials completely.

    a) $216x^3 - 27y^9$ b) $9s^2 - 6st^2 + t^4$ c) $x^4 - 16y^4$

12. Reversibility (1c)
    Find all values of k for which $x^2 + kx - 24$ is factorable. Explain how you determined your answers.

13. Reversibility (4c)
    Write two functions $f(x)$ and $g(x)$ so that $(f \circ g)(x) = 3|x + 5| - 7$.

14. Transfer procedures from systems of linear equation to quadratic systems. (Note: the students had not attempted problems of this nature before the test.) (Substitution from previous unit, 3b)
    Solve the following system.
    $y = x^2 - 4$
    $3x - y = -6$

15. Reversibility (4c, 1b)
    If $(f + g)(x) = 3x^3 + 2x^2 - 6$ and $g(x) = x^3 - 7x$, find $f(x)$.

16. Transfer/requires geometric knowledge as well as algebraic (1b)
    If the area of a rectangle can be expressed as $2x^4 + 8x^3 - 17x^2 - 12x + 21$ and the width as $x^2 + 4x - 7$, find an expression for the length.

17. Procedural question (1b)
    Divide by synthetic division.

$$\frac{z^4 + 3z^3 - 4z^2 + 5z + 5}{z + 4}$$

18. Procedural question (1c)
    Solve $x^4 - x^3 + 7x^2 - 9x - 18 = 0$. First, list all possible rational roots. Then find all complex (real and imaginary) solutions. Show your work.

19. Transfer to real world (3d, 4e, 5f)
    An open box with a volume of 72 cm^3 can be constructed by cutting squares of equal size from the corners of a square piece of cardboard 10 cm on a side and then folding up the edges of the cardboard. To the nearest tenth, what are the possible lengths of a side of the square that is cut from each corner to achieve the desired volume? Show all of your steps.

20. Transfer to real world (3d, 4b, 4e, 5f)
The Duke Marketplace Bakery sells more loaves of blueberry bread when it reduces its prices, but changes in number of sales and prices affect the profits. The function $p = -100(x - 1.75)^2 + 300$ models the bakery's profits (p) in dollars, where x is the price of one loaf of blueberry bread in dollars. The bakery wants to sell blueberry bread at a price that maximizes profits.

a) What is the domain of the function p ? Explain your answer.
b) Find the profit for selling bread at $2.00 per loaf. Show how you reached your answer.
c) At what price per loaf will the profit be $275? Show how you reached your answer.
d) What should the bakery charge per loaf to maximize profits, and what is the maximum profit? Explain your answer.
e) What do the x-intercepts on the graph of this profit function represent?

Appendix C

Scoring Rubric for Pretest/Final Exam Questions

General Guidelines
4=Solution complete and correct
3=Solution almost complete and correct, with some minor error in notation or computation
2=Response in the right direction. Student stops halfway through or conceptual errors are present
1=Some work correct, but with no evidence of chain of reasoning
0=All work is wrong or meaningless or no work shown

No	**Specific Guidelines**
1a	4=completely correct with clear explanation (vert. line test explanation OK) 3=nearly correct with almost clear explanation 2=correct with conceptually lacking explanation 1=correct answer without explanation 0=partial correct answer without explanation/no work/meaningless
1b	4=completely correct with clear explanation 3=nearly correct with almost clear explanation 2=correct with conceptually lacking explanation/only explains in terms of polynomials and degrees 1=correct answer without explanation or with completely incorrect explanation 0=incorrect answer/no work/meaningless
2	4=completely correct with =0 to make it an equation 3=nearly correct but with one arithmetic error or forgets the =0 2=more than one arithmetic error/finds other solution and uses correct process but sets up binomials incorrectly/stops before multiplying out binomials 1=finds other solution but cannot proceed/fails to determine other solution/attempts to set up binomials but does so incorrectly 0=no work/meaningless
3	4=completely correct with appropriate solution process 3=recognition of Jim's error, but with a minor error in the solution process/says Jim solved correctly but also includes correct process and solutions 2=recognition that Jim's wrong but without correct solution process 1=does not recognize Jim's error, but with some reasonable reconstruction of process 0=completely incorrect/no work/meaningless
4	4=completely correct with clear explanation that involves slope or rate of change 3=nearly correct with almost clear explanation 2=realizes that it's not linear but with conceptually flawed explanation/makes arithmetic error that leads him/her to believe that it's linear and offers clear explanation 1=thinks it's linear with flawed explanation 0=completely incorrect/no work/meaningless
5	4=completely correct with clear explanation 3=nearly correct with almost clear explanation 2=correct answer but conceptually flawed explanation/understands domain but cannot directly address polynomial functions 1=incorrect answer but with slightly valid attempt at explanation/correct answer but no explanation 0=incorrect with no explanation/completely invalid explanation/no work/meaningless
6	4=completely correct with units indicated and clear explanation and/or solution process 3=nearly correct with almost clear explanation and/or solution process/fail to express units but clear explanation/nearly correct except a minor arithmetic error with units and clear explanation 2=more than one minor arithmetic error/correct answer but conceptually flawed explanation 1=correctly starts the process but is unable to complete and/or explain the process/correct answer with no work or explanation 0=completely incorrect process/no work/meaningless

Appendix D

Scoring Rubric for Polynomial Unit Test

General Guidelines
4=Solution complete and correct
3=Solution almost complete and correct, with some minor error in notation or computation
2=Response in the right direction. Student stops halfway through or conceptual errors are present
1=Some work correct, but with no evidence of chain of reasoning
0=All work is wrong or meaningless or no work shown

No.	Specific Guidelines
1.	4=completely correct 3=nearly correct except small arithmetic error 2=stopped after substitution/conceptual problem, e.g., switched x and y/serious error in solution process 1=tried to substitute but completely incorrectly, e.g., substitutes a value into a 0=no work/meaningless
2.	4=completely correct 3=nearly correct with no more than one arithmetic error 2=more than one arithmetic error/inability or failure to simplify powers of *i* correctly 1=starts correctly but fails to simplify (combine like terms)/more than 3 arithmetic errors/stops early/unable to multiply correctly 0=no work/meaningless
3.	4=completely correct 3=nearly correct with no more than one arithmetic error 2=more than one arithmetic error/conceptual or procedural errors/only one solution/failure to simplify 1=starts correctly but process breaks down early/stops early/uses different method 0=no work/meaningless
4.	4=completely correct 3=correct with no more than one arithmetic error 2=use of nearly correct quadratic formula/conceptual or procedural errors/only one solution/more than one arithmetic error/fails to simplify 1=uses an incorrect quadratic formula/uses different method/stops early 0=no work/meaningless
5a.	4=correct equation 3=nearly correct equation except for perhaps uses 0 in place of y, missing "y=" or using –2 (counting error) 2=one conceptual problem such as switching x and y, using $-x^2$, or using +3 instead of –3 1=multiple conceptual problems 0=no work/meaningless
5b	4=correct with direction of opening, axis of symmetry y-intercept/vertex clearly indicated 3=nearly correct with one minor omission like direction of opening or axis of symmetry 2=somewhat correct but with failure to discuss y-intercept or vertex/mentions shift UP only 1=some aspect correct but failure to discuss both y-intercept and vertex 0=no work/meaningless
6.	4=completely correct with graph and all components clearly computed and labeled 3=nearly correct with no more than one arithmetic error in one part/do not penalize further if graph works with computed information/missing "x=" for axis of symmetry 2=more than one arithmetic error/conceptual or procedural errors in computing components/error in graphing contrary to computed components/one missing component/failure to express clearly all required components 1=several missing components or missing graph/several conceptual errors 0=little or no work/meaningless

7.	4=completely correct with clear graphs that illustrate the three possibilities 3=nearly correct but with unclear graphs/uses left- or right-opening parabola for the answer "1" 2=one missing answer/inappropriate graphs 1=missing two answers or answers with no supporting graphs 0=no work/meaningless
8.	4=completely correct with clear explanation and/or proof by counterexample 3=nearly correct but without a completely clear explanation 2=correctly states that Ginger is incorrect but with conceptually flawed explanation 1=correctly states that Ginger is incorrect but with incorrect explanation or example 0=incorrect or correct "guess" with no attempt at explanation/no work/meaningless
9.	4=completely correct with =0 to make it an equation 3=nearly correct but with one arithmetic error or forgets the =0 2=more than one arithmetic error/finds other solution and uses correct process but sets up binomials incorrectly/stops before multiplying out binomials 1=finds other solution but cannot proceed/fails to determine other solution 0=no work/meaningless
10a.	4=correct graph by a valid method (plotting points/using transformations/other method explained) 3=nearly correct graph/slightly misshapen or error in computing a particular point 2=failure to correctly graph one aspect such as $-x^3$ or the y-intercept/badly misshapen graph/conceptual graphing difficulty 1=graphed wrong shape (e.g., parabola)/more than one incorrectly graphed aspect/correct started the process of working but failed to draw graph 0=no work/completely wrong graph/meaningless
10b.	4=correct graph by a valid method (plotting points/using transformations/other method explained) 3=nearly correct graph/slightly misshapen or error in computing a particular point 2=failure to correctly graph one aspect such vertex or intercept/badly misshapen graph/conceptual graphing difficulty 1=graphed wrong shape (e.g., cubic)/more than one incorrectly graphed aspect/correct started the process of working but failed to draw graph 0=no work/completely wrong graph/meaningless
11a.	4=completely factored correctly 3=nearly factored correctly with one minor error (e.g., one incorrect coefficient)/forgot to carry 27 through 2=forgot to factor GCF, but correct otherwise/incorrectly completed one piece of factoring (e.g., confused signs on the difference of cubes, one case of incorrect set of coefficients or exponents) 1=factored GCF and quit/used completely incorrect method for difference of cubes/incorrectly complete multiple pieces of factoring 0=completely incorrect factoring procedure/no work/meaningless
11b.	4=completely factored correctly 3=nearly factored correctly but with very minor error 2=factored correctly except for one procedural or conceptual error (e.g., sign or coefficient or exponent incorrect) 1=multiple procedural or conceptual errors 0= incorrect factoring procedure/no work/meaningless
11c.	4=completely factored correctly 3=nearly factored correctly with one minor error (e.g., one incorrect coefficient)/missing variable 2=factored correctly except for one procedural or conceptual error (e.g., sign or coefficent or exponent incorrect) 1=forgot to factor second time/tried to factor x^2+4y^2/multiple procedural or conceptual errors 0=incorrect factoring procedure/no work/meaningless

12.	4=correctly lists all possible k's (positives and negatives) with explanation of how they were computed 3=correctly computes all k's but explanation is unclear/leaves out one possibility 2=leaves out multiple possibilities/forgets negative possibilities/correct answers with no explanation 1=is able to compute some possibilities by guessing but with no clear process/starts a promising method but is unable to carry through 0=incorrect answers/no work/meaningless
13.	4=two correct functions 3=nearly correct functions but with careless mistake in writing one of them/confuses f and g 2=correct idea but unable to write two equations that would work 1=some evidence of understanding the goal and composition, but unable to proceed 0=incorrect method and answers/use of operation other than composition/no work/meaningless
14.	4=completely correct method with both solutions clearly stated as ordered pairs 3=correct method with one arithmetic error (don't penalize if carried through correctly) or failure to express correct solutions as ordered pairs 2=correct method with multiple arithmetic errors/committed one procedural or conceptual error/correctly computes x's but not y's 1=incorrect method/major conceptual flaws/starts correctly but stops early/no process indicated 0=no work/meaningless
15.	4=correct method and solution 3=nearly correct with one arithmetic error (if fails to subtract properly, go to 2) 2=starts correctly but with more than one arithmetic error/fails to subtract properly or backward/has answer but no work or explanation 1=adds instead of subtracts/cannot simplify correctly/stops early 0=no work/meaningless
16.	4=correct method and solution 3=nearly correct with one arithmetic error 2=knows area formula and sets up correctly but cannot divide properly/more than one arithmetic error 1=knows area formula but cannot set up division (e.g., attempts to compute by trial and error)/sets up the division but then does not proceed 0=no evidence of area formula/uses wrong operations/no work/meaningless
17.	4=correct method and solution 3=nearly correct with one arithmetic error/uses wrong variable to express quotient 2=completes synthetic division properly but errs in expressing quotient (e.g., mistakes with exponents or expressing remainder)/more than one arithmetic error 1=cannot complete synthetic division/attempts long division/completes synthetic division but does not attempt to write quotient properly 0=no work/meaningless
18.	4=completely correct with all solutions clearly specified (roots listed and evidence of proper factoring process – use of graphing calculator is acceptable) 3=nearly correct with one arithmetic error/error in simplifying complex solutions 2=multiple arithmetic errors/missing rational roots/forgot one of the complex solutions (e.g., -3i)/solutions correct but fails to list rational roots/factors correctly but doesn't list solutions 1=cannot compute complex roots/major conceptual or procedural errors 0=no work/meaningless
19.	4=completely correct with solutions explicitly stated with units indicated 3=nearly correct but missing units in solutions/correct equation with solution "off" (calculator entry error)/one arithmetic error/1 missing solution or use of an invalid solution 2=correctly sets up scenario but error in simplifying equation/arrives at correct equation but cannot proceed/multiple arithmetic errors 1=makes some correct steps but unfamiliar with volume/shows some understanding of the problem but cannot set up equation 0=no work/meaningless

20a.	4=correct domain with clear explanation (listing upper bound is okay with clear explanation) 3=correct domain with unclear explanation 2=correct idea for computing domain but not correct 1=some understanding of concept domain but unable to work within context of the problem 0=no work/meaningless
20b.	4=correct answer with clear progression of steps/calculator use OK 3=correct answer with unclear progression of steps 2=arithmetic error but clear progression of steps /missing an answer 1=errors with unclear progression of steps 0=no work/meaningless
20c.	4=correct answers with clear progression of steps/calculator use OK 3=correct answers with unclear progression of steps 2=arithmetic error but clear progression of steps/missing one answer 1=errors with unclear progression of steps/substitutes correctly but cannot compute solution/has answers but with no explanation 0=no work/meaningless
20d.	4=correct answers with clear explanation 3=correct answers with unclear explanation 2=arithmetic error but clear explanation/missing part of answer 1=errors with unclear explanation 0=no work/meaningless
20e.	4=clear explanation of the x-intercepts 3=slightly unclear articulation of the meaning of the intercepts 2=evidence of attempt to make intercepts meaningful in the context of the problem bit failure to do so 1=evidence of understanding of intercepts but unable to make them meaningful in the context of the problem 0=no work/meaningless

Appendix E

Indiana Mathematics Beliefs Scale

Directions: This survey is designed to find out your opinions about mathematics. Please read each item carefully and circle the response from the following options that best represents your feelings about that item.

SA = Strongly Agree A = Agree U = Uncertain D = Disagree SD = Strongly Disagree

1.	I study mathematics because I know how useful it is.	SA	A	U	D	SD
2.	I'm not very good at solving math problems that take a while to figure out.	SA	A	U	D	SD
3.	There are word problems that just can't be solved by following a predetermined sequence of steps.	SA	A	U	D	SD
4.	Time used to investigate why a solution to a math problem works is time well spent.	SA	A	U	D	SD
5.	Computational skills are of little value if you can't use them to solve word problems.	SA	A	U	D	SD
6.	Knowing mathematics will help me earn a living.	SA	A	U	D	SD
7.	Learning computational skills is more important than learning to solve word problems.	SA	A	U	D	SD
8.	Getting a right answer in math is more important than understanding why an answer works.	SA	A	U	D	SD
9.	Most word problems can be solved by using the correct step-by-step procedure.	SA	A	U	D	SD
10.	By trying hard, one can become smarter in math.	SA	A	U	D	SD
11.	If I can't do a math problem in a few minutes, I probably can't do it at all.	SA	A	U	D	SD
12.	Mathematics is a worthwhile and necessary subject.	SA	A	U	D	SD
13.	Memorizing steps is not that useful for learning to solve word problems.	SA	A	U	D	SD
14.	Math problems that take a long time don't bother me.	SA	A	U	D	SD

15. In addition to getting a right answer in mathematics, it is important to understand why the answer is correct.	SA	A	U	D	SD
16. Word problems are not a very important part of mathematics.	SA	A	U	D	SD
17. Working can improve one's ability in mathematics.	SA	A	U	D	SD
18. Studying mathematics is a waste of time.	SA	A	U	D	SD
19. Hard work can increase one's ability to do math.	SA	A	U	D	SD
20. Computational skills are useless if you can't apply them to real life situations.	SA	A	U	D	SD
21. It's not important to understand why a mathematical procedure works as long as it gives a correct answer.	SA	A	U	D	SD
22. Any word problem can be solved if you know the right steps to follow.	SA	A	U	D	SD
23. I feel I can do math problems that take a long time to complete.	SA	A	U	D	SD
24. A person who doesn't understand why an answer to a math problem is correct hasn't really solved the problem.	SA	A	U	D	SD
25. I find I can do hard math problems if I just hang in there.	SA	A	U	D	SD
26. Learning to do word problems is mostly a matter of memorizing the right steps to follow.	SA	A	U	D	SD
27. A person who can't solve word problems really can't do math.	SA	A	U	D	SD
28. I can get smarter in math by trying hard.	SA	A	U	D	SD
29. Mathematics will not be important to me in my life's work.	SA	A	U	D	SD
30. Math classes should not emphasize word problems.	SA	A	U	D	SD
31. It doesn't really matter if you understand a math problem if you get the right answer.	SA	A	U	D	SD
32. Word problems can be solved without remembering formulas.	SA	A	U	D	SD
33. If I can't solve a math problem, I quit trying.	SA	A	U	D	SD
34. Mathematics is of no relevance to my life.	SA	A	U	D	SD
35. Ability in math increases when one studies hard.	SA	A	U	D	SD

Appendix F

Interview Tasks and Protocol

Purpose: To explore how a TIP student uses resources, heuristics, and control to solve problems and to determine the connections he/she is making in the process.

Interview Protocol: Hi, I'm Angela. I'm so glad you decided to participate in this study! I'm going to be asking you work on some math problems about the material you've been covering in your class. I'd like for you to "think aloud" as you are working on each problem. I am not going to be evaluating whether you are right or wrong. If I stop you to ask a question, I am only trying to gain a better understanding of the way you think about math problems.

Before we start, take a look at this consent form. This is very similar to one that you and your parents completed before you arrived here this summer, but we ask you to complete another one to remind you of several things. This is a reminder to you that although we would appreciate your answers to all of the questions, if some of them are unclear to you, it's okay to ask questions or to not answer them. You may stop at any time. You won't get extra credit for doing this, and you won't get penalized for not doing it. I'll be taping this interview and transcribing it, but the information you share with me will be strictly confidential and won't be given to anyone. The work you complete during this interview will not affect your class evaluation in anyway.

The consent form says most of this again. If you want us to be able to use the information you give us, you must sign the form.

Again if you need to stop at any time, please let me know. Do you have any questions?

Pre-Interview Task:
1. What part of your TIP Algebra II class have you enjoyed most so far?
2. Have you ever been to TIP before? Is this your first TIP math class?
3. What do you like best about learning math?

Task 1: (Reversibility – Objectives 1b,c)
Determine k so that $x-3$ is a factor of x^3-4x^2+kx-6.

Probes:
- What rule/process are you using here?
- Why are you doing this?
- What is happening here?
- How is this problem different from the problems you've done in class?

Task 2: (Transfer to real world – Objectives 4e, 5a-f)
Consider the following scenario: A rocket is fired vertically from the top of a tower at a velocity of 80 feet per second. The tower is 200 feet high. The height (h) in feet of the rocket above the ground t seconds after firing is given by the formula $h(t) = -16t^2 + 80t + 200$.

Questions:
1. What are some possible ways to analyze this scenario?
2. What types of information about this scenario can you determine through your analysis?
3. Please graph this function (calculator is fine). How can we adjust the calculator screen to see the whole function?
4. What are some important features on this graph?
5. What do your axes represent?
6. (Considering intercepts, maximum, etc.) Why is this feature important?
7. What is happening here?
8. What is the maximum height that the rocket will attain? How did you determine this?
9. When does the rocket reach its maximum height? How did you determine this?
10. When will the rocket strike the ground? How did you determine this?
11. What does this other x-intercept mean?

Probes:
- What are you doing here?
- What rules are you using?

Task 3: (Reversibility – Objectives 5a-e)
Consider the following points: (0, 4), (3, 7), and (2, -12).

Questions:
1. What kind of curve might these points lie on? What type of equation would graph this type of curve? (If no possibilities are suggested, suggest linear, quadratic (parabola), etc.)
2. Can you write an equation to fit these three points? (If no ideas, suggest they write out standard form of a quadratic and see if it helps. Subtle hints/probes)
3. How would this problem have differed if you had not been given the y-intercept? (If unclear, suggest replacing (0,4) with (-1, 4)) How would you have approached this problem? Would you still be able to write an equation?

Probes:
- What are you doing here?
- Why are you using that strategy?
- What is this strategy/rule/property/process called?

Task 4: (Transfer to real world – Objectives 5a-f)
A store manager receives 100 stereos with a suggested retail price of $70. The projected sales at this price are 20 stereos per week. Based on past experience, the store manager knows that sales will increase by 2 stereos per week for every $5 decrease in price.

a) Find a function that gives the store's weekly revenue (r) as a function of the number of $5 decreases (n). Revenue is the total number of dollars, including expenses and profit, collected through sales of stereos.

Probes:
- How should you compute revenue?
- How would you express price per sterio using n?
- How would you express the number of stereos sold using n?
- How can you tie all of this together to make your equation?
- What type of equation is this?
- Does this equation make sense? Why or why not?

b) Graph this function (Calculator OK).

Probes:
- How can you adjust the calculator screen to see the more of the equation?
- What are some important features of the graph?
- What do the axes represent?
- What is happening here?
- What do the intercepts tell us?

c) What price maximizes weekly revenue?

Probes:
- How did you obtain this answer?
- Does it make sense? Why or why not?

d) What is the maximum revenue that can be collected under these conditions?

Probes:
- How did you obtain this answer?
- Does it make sense? Why or why not?

Debriefing:
Thank you so much for participating in this study. I appreciate your help! Like I mentioned earlier, I am going to be using the tapes of this problem-solving session to gain a better understanding of how TIP students think about math problems. Since our goal is to help you grow as a mathematician, this information will be valuable as we continually improve our classes and as we prepare our instructors to teach them effectively. Do you have any questions?

Appendix G

Interview Coding System

B = evidence of belief systems/motivation

R = use resources
con = evidence of connection
pro = procedural knowledge only is apparent
- LD = long division
- Syn div = synthetic division
- PolMult = polynomial multiplication
- Fac = def'n of factor/factoring strategy
- ZR = zero remainder
- RW = real world phenomenon
- G = graph
- Eq = equation
- GI = graph interpretation
- TI = table interpretation
- PrM = previous math experience
- Sl = slope
- RRT = rational root theorem
- FT = factor theorem
- Aos = axis of symmetry
- Ftn = function knowledge
- Vars = knowledge of the function of variables
- Ver = knowledge of the vertex
- Char Pol = characteristics of polynomials
- Quad = characteristics of quadratics
- x-int = knowledge about x-intercepts
- y-int = knowledge about y-intercepts
- simp = simplification procedures
- subs = substitution
- shift = function transformation
- roots = knowledge about polynomial roots
- AC = misconception/alternate concept
  - GAP = suspected graph as picture
  - IV/DV = problems with independent and dependent variables
  - Shift = inability to shift graph
  - Sym = problems with symmetry
  - Rep = difficulty changing representations

H= heuristic

- WB = Working backward
- Gr = Graphing strategy
- RsP = Restating problem
- BD = Breaking problem down into more manageable problems
- Exp = Experimenting
- Rep = Change in representation
- Rul = Ruling out options
- Pic = Uses picture to depict situation

C = evidence of control

- Ch = Checking work
- app = discussion of approach to problem
- diff = awareness of difference from typical problems
- PS = problem sense
- SW = switching methods
- Mis = recognition of a mistake
- Type = recognition of problem type
- Dis = recognizes a distractor
- Clar = asks for clarification

Appendix H

Summary Matrices for Interview Analyses

Resources Matrix

Student	Facts	Algorithms	Skills
1. Jason	Fac (84-92) Gr/Eq (129-134) PrM/Eq (270-272) Aos (319-321) Ftn (356-362) Vars (383-387)	LD (59-64) Ver (137-142) Pol Mult (445)	GI (213-225, 247-254, 468-469) PS (431-434)
2. Julie	Fac (51, 59-71, 76) RRT (77-78, 96-98) FT (106-107) RRT/FT (109-120) Char Pol (195, 294-297) RW (201-208) Ver (214-226) Char Lines (283-284)	Fac (64-69) Find x-int (93-94) Subst (98-99, 103-106) Eq (121-122) Find Ver (183-185, 233-236, 240-267) Reduce Frac (214-226)	GI (197-198)
3. David	Fac (77-79, 87-91, 99-102) Poly Gr (153-155) x-int (167-168) Sl (220-226) PrM (geometry – 233-234) y-int (370-378, 381-385)	Simp. Polys (197-207)	
4. Michael	Fac (54-61, 69-76) Pt-Sl (179-185) Quad/Poly (188-193)	Fac (51-52) y-int (97-101) Sl (167-175)	
5. Jake	Fac (79-83) See connections.	LD (62-72) Poly Mult (73-76, 85-92) x-int (146-147)	
6. Alan	Fac (66-69) Char Pol (197-203) Ftn (206-216)	Syn Div (47-60)	GI (163-170)
7. Eric	Fac (94-97) Def'n Line (218-224) PrM (218-224) Char Pol (229-240)	LD (118-129) Fac – group (72-85, 100-107) FOIL (292-301)	Tab Int (161-181) GI (302-316)
8. Nick	PrM (83-85) Fac (88-90)	Fac – group (97-101) Fac (112-127) PrM (296-307)	GI (161-165)
9. Evan	Fac (49-61) x-int/roots (92-97) Char Pol (115-129, 199-209, 218-222, 223-228)	LD (49-61) Div/Mult (73-78)	GI/Tab Int (101-113)
10. Pete	ZR/fac (51-65) Char Pol (228-230, 252-256)	LD (66-71, 72-81) Syn div (84-86) FOIL (306-309)	Shift (218-227) Eq Graph (281-293, 295-301)

Connections/Misconceptions Matrix

Student	Evidence of Beliefs	Evidence of Connections	Procedural Knowledge Only	Misconceptions
1. Jason	35-46	Fac/LD/ZR (84-92) RW (129-134, 187-189, 213-225, 231-232, 235-239, 247-254, 356-362, 431-434) Gr/Eq (142-150, 445) PrM (trig 270-272)	164-179	GAP (201-204) Sym (280-293, 305-309, 325-330)
2. Julie	40-43	Fac/RRT (77-78, 96-98,) RRT/FT (107) Fac/RRT/FT (109-120) Fac/FT (134-136) Fac/FT/root (143-145) RW (166-169, 173-177, 197-198, 201-202, 229-231, 271-273) RW/G/Eq (204-208)	132-133 214-226 240-267	IV/DV (233-236) Pol (298-304)
3. David	39-42 52-63 105-106 153-155 139 165-166	Gr/Eq (153-155, 357-359) Curve/line of best fit (235-243) RW (289-294, 331-341, 344-349)	171-182 197-207 245-263 381-385	Fac (71-75, 80-85) RW (117-120) IV/DV (127-136, 140-141) Math curve (230-233) Pol (245-263) Problem Int (280-286) y-int (360-367, 370-378) line (387-396) Gr (379-403)
4. Michael		Gr/Eq(188-193) Fac/LD (140)	97-101 108-109 128-134 152-154	Fac (60-61) Eq (103-106) GAP (123-124) x-int (141-151) Sym (195-211) Problem Int (249-256)
5. Jake	52-58	Fac/LD (62-72) RW (102-106, 185-187, 192-194, 197-209) Gr/Eq (125-126, 155-157, 171-172) Gr/RW (140-141, 143-145, 230-231, 269-277) Eq/Fac (173-177)	249-256 263-268 235-237 230-234	IV/DV (102-106) Asym (118-120) Ver (132-135) RW (138-139) GAP (140-141) Aos/Eq (161-170)
6. Alan	72-77	Fac/syn div (36-88) RW (147-150, 163-170) Ver/aos (197-202)	59-60 78-84 87-95	GAP? (129) Rep (136-138, 142-146) IV/DV (157-160)
7. Eric	37-38 47-51 87-89 249-253	Fac/LD (118-129) RW (136-137, 143-146, 161-181, 259-280,302-316, 319-329) Gr/Eq/Table (189-195, 198-203) Sym/pol (227-228) Ftn/rep (292-301)	72-85 100-107 138-141 184-187 FOIL 292-301	Terminology (110-115) Ver formula (184-187) Sym (210-214)

Student	Evidence of Beliefs	Evidence of Connections	Procedural Knowledge Only	Misconceptions
8. Nick	30-42 56-57 313-314 373	Fact/PolMult (88-90) RW (144-146, 165-169, 219-235, 338-348) Gr/eq (161-165, 275-284)	74-78 112-115 188-193	IV/DV (151-152) GAP? (165-169) Ver formula (178-181) Sym (267-271) Rep (323-324)
9. Evan	39-45	LD/fac (49-61, 115-129) LD/PolMult/fac (73-78) Gr/fac/x-int (92-97) Gr/eq (110-113) RW (143-146, 171-191) Gr/Eq/Table (199-209)	79-91 218-222	IV/DV (117-121) Shift (160-165) Sym (211-215, 223-228)
10. Pete		LD/fac/ZR (51-65, 72-81) LD/syn div (84-86) RW (152-165, 281-293, 295-301, 306-309, 320-327, 331-337) Char Pol (228-230)	66-71 168-170 198-209	Missing info (168-170) y-int (185-186) sep var from meaning (193-197) x-int (198-208) sym (214-217) eq/Pol Char (237-251)

Heuristics Matrix

Student	Restating Problem	Breaking Down Problem	Experimenting	Picture/Change Rep	Rule Out Options	Working Backward
1. Jason	373-377	393-399 462-466		268 417		66-71
2. Julie		197-199		87 281		
3. David	267-273	289-296 397-403	157-163 300-310 331-340	213-219 235-243		
4. Michael		233-245		92-94 155 176-177		
5. Jake	183-185	167-170 188-191 223-224 267-268	213-217	218-221 235-237 255-259		69-72 85-92
6. Alan			78-85	115-119 125-128		41-43
7. Eric		259-280	72-85	147-150 161-181		
8. Nick	80 139-140		83-85 88-90 103-111 289-294 323-324	245-246	253-258	
9. Evan	56-61		62-72 92-97 102-113 115-127	149-150 152-159 195-196		
10. Pete	218-227			171-181 262-268 310-317		72-81

Control Matrix

Student	Type/Problem Sense	Awareness of Sim/Diff	Recognize Mistakes/ Recognize Distractor	Checks Work	Switches Methods	Asks for Clarification
1. Jason	109-111	101-107 462-466		79-80		
2. Julie	166-169	153-154	70-71	125-126	86-87	
3. David		99-102	274 389-390			
4. Michael			246-247	63-64		
5. Jake			73-76 246-247 260-261	235-237	73-76	
6. Alan	82-84	88-89			78	97-111
7. Eric	198-203 232-236 302-316 292-301 136-137	118-129	254-258		161-181	92
8. Nick	184-187 195-208 209-218 338-348		366-369			317-319 356-358
9. Evan	69-72	132-137				
10. Pete	172-181 231-236 303-304	140-143	88-93	94-136		