# Analysis of viscoplastic plates with material degradation using influence functions

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## 1. INTRODUCTION AND SOLUTION STRATEGY

Influence functions are well-known from the computational analysis of linear elastic plates. For inelastic plates, unfortunately, this convenient Green's function method does not apply in its classical sense , because superposition of imposed loadings is not possible. However, following a complete elastic-inelastic analogy derived in /1/, /2/, /3/ for small deflections of beams and plates, the inelastic part of strain may be treated as an additional source of self-stress in the linear elastic structure with fixed (initial) stiffness. Hence , the inelastic plate is analogous to the linear elastic one, but subjected to the imposed loadings as well as to fictitious additional sources of self-stress, likewise to a given thermal loading. Accordingly, solution is split into the corresponding linear elastic part and into that due to this plastic sources:

$$\Delta\sigma_{ij} = \Delta\sigma^0_{ij} + \Delta\sigma^*_{ij}, \qquad \Delta\varepsilon_{ij} = \Delta\varepsilon^0_{ij} + \Delta\varepsilon^*_{ij}, \tag{1}$$

with $i,j=r,\vartheta$ in a polar coordinate system. $\Delta$ denotes an increment, (0) and (*) stand for the associated linear elastic solution due to imposed quasistatic loading and for the linear solution due to the inelastic part of strain $\Delta\varepsilon^N_{ij}$, respectively. Following a Maysel-type integration scheme /4/, the latter is formulated using classical unit-force influence functions. In case of an axi-symmetric problem, $r_i \leq r \leq r_a$, the deflection becomes, see /3/,

$$\Delta w^*(r) = \frac{1}{r}\left\{\int_{r_i}^{r} [\tilde{m}_r(\varrho, r)\, \Delta\varkappa^N_r(\varrho) + \tilde{m}_\vartheta(\varrho, r)\, \Delta\varkappa^N_\vartheta(\varrho)]\, \varrho\, d\varrho \right. \tag{2}$$

$$\left. + \int_{r}^{r_a} [\tilde{m}_r(\varrho, r)\, \Delta\varkappa^N_r(\varrho) + \tilde{m}_\vartheta(\varrho, r)\, \Delta\varkappa^N_\vartheta(\varrho)]\, \varrho\, d\varrho \right\}.$$

where the additional imposed curvatures $\Delta\kappa^N$ are the first moments of the distribution of sources of selfstresses over the plate thickness h

$$\Delta\varkappa_r^N = \frac{12}{h^3}\int_{-h/2}^{+h/2} \Delta\varepsilon_{rr}^N z\,dz, \qquad \text{etc.,} \tag{3}$$

and $\tilde{m}(\rho,r)$ denotes a bending moment in $\rho$ due to a unit ring force applied at $r$ in the linear elastic plate. Thus, influence coefficients for deflections and curvatures due to unit-step self-stress loadings are easily computed. In matrix notation,

$$\begin{aligned} \Delta\,\mathbf{w}^* &= [W_r^*]\,\Delta\,\boldsymbol{\varkappa}_r^N + [W_\vartheta^*]\,\Delta\,\boldsymbol{\varkappa}_\vartheta^N, \\ \Delta\,\mathbf{w}_{,r}^* &= [\Phi_r^*]\,\Delta\,\boldsymbol{\varkappa}_r^N + [\Phi_\vartheta^*]\,\Delta\,\boldsymbol{\varkappa}_\vartheta^N, \\ \Delta\,\mathbf{w}_{,rr}^* &= [M_r^*]\,\Delta\,\boldsymbol{\varkappa}_r^N + [M_\vartheta^*]\,\Delta\,\boldsymbol{\varkappa}_\vartheta^N, \end{aligned} \tag{4}$$

where

$$\Delta\,\mathbf{w}^* = [\Delta\,w^*(r_1), \ldots, \Delta\,w^*(r_l), \ldots, \Delta\,w^*(r_{p_A})]^T. \tag{5}$$

In Eq. (4), the plate area has been divided into $p_A$ rings and evaluation is done at the ring centers with radii $r_\ell$, $\ell=1,\ldots,p_A$. Components of the Green's matrices [W*], [$\phi$*] and [M*], which follow from the influence functions $\tilde{m}$, are exemplarily given in /3/ for the case of a solid plate with hinged supports.

This Green's matrices do not change during the course of calculations and are stored at the onset. What remains only, is the determination of the additional loadings $\Delta\kappa^N$. This is done in a time-stepping procedure from the given material's law, where the total strains in the fibres of the plate are evaluated using the cited Green's matrices:

$$\Delta\varepsilon_{rr}^* = -z\Delta w_{,rr}^*; \quad \Delta\varepsilon_{\vartheta\vartheta}^* = -z\frac{1}{r}\,\Delta w_{,r}^*, \tag{6}$$

and

$$\begin{aligned} \Delta\sigma_{rr}^* &= \frac{E}{1-\nu^2}\,[(\Delta\varepsilon_{rr}^* - \Delta\varepsilon_{rr}^N) - \nu(\Delta\varepsilon_{\vartheta\vartheta}^* - \Delta\varepsilon_{\vartheta\vartheta}^N)], \\ \Delta\sigma_{\vartheta\vartheta}^* &= \frac{E}{1-\nu^2}\,[(\Delta\varepsilon_{\vartheta\vartheta}^* - \Delta\varepsilon_{\vartheta\vartheta}^N) - \nu(\Delta\varepsilon_{rr}^* - \Delta\varepsilon_{rr}^N)]. \end{aligned} \tag{7}$$

Furthermore,

$$\begin{aligned} \Delta m_r^* &= -K\left[\Delta w_{,rr}^* + \frac{\nu}{r}\,w_{,r}^* + \Delta\varkappa_r^N + \nu\Delta\varkappa_\vartheta^N\right], \\ \Delta m_\vartheta^* &= -K\left[\frac{1}{r}\,w_{,r}^* + \nu\Delta w_{,rr}^* + \Delta\varkappa_\vartheta^N + \nu\Delta\varkappa_r^N\right], \quad K = Eh^3/12(1-\nu^2). \end{aligned} \tag{8}$$

During each incremental step, the computation is suitable to the individual character of the inelastic material's law and results in the increments of inelastic strains $\Delta\varepsilon^N$. Thus, by numerical integration,

$$\Delta\,\varkappa_\vartheta^N(r_l) = \frac{12}{p_z h^2}\sum_{m=1}^{p_z} \Delta\,\varepsilon_{\vartheta\vartheta}^N(r_l, z_m)\,z_m, \quad \Delta\,\varkappa_r^N(r_l) = \frac{12}{p_z h^2}\sum_{m=1}^{p_z} \Delta\,\varepsilon_{rr}^N(r_l, z_m)\,z_m, \tag{9}$$

where the plate thickness has been divided into $p_z$ laminae with center $z_m$, $m=1,\ldots,p_z$. From Eq. (9) , the status of stress and strain at the end of the considered increment is easily computed using Eqs. (4) to (8).

While in the above formulation nonlinearity is shifted to fictitious self-stress loadings $\Delta\kappa^N$ of the linear elastic plate, it is noted that the errors due to numerical approximation of this additional loadings are smoothened by integration. Because of the influence-function type formulation, no numerical differentiation of deflections, but only numerical integrations, are necessary. Because of fixing the plate stiffness during incrementation, this global influence numbers remain fixed, too, but time-stepping with respect to the local material's law may be done in a problem-oriented manner.

Application of this strategy to the Norton-Bailey law of creep has been presented in /3/. The present paper is concerned with the application to viscoplastic axi-symmetric plates, where Perzyna's well-accepted model /5/ is used. In order to study the effect of material degradation during cyclic quasistatic loading, a damage model according to Kachanov /6/ is introduced into Perzyna's constitutive equation.

## 2 CONSTITUTIVE EQUATIONS

In case of time-dependent loadings of elastoplastic structures, rate dependency of the material behaviour should be considered, even when inertia effects are negligible. Perzyna /5/ introduced a corresponding theory of viscoplasticity, where the visco-plastic strain rate $\dot{\varepsilon}^v_{ij}$ is a function of the excess stress, while the static yield stress remains rate independent:

$$\dot{\varepsilon}^v_{ij} = \frac{2k}{\Theta} < \phi(F) > \frac{\partial F}{\partial \sigma_{ij}} \tag{10}$$

with

$$F = (J_2^{1/2}/k) - 1 \tag{11}$$

and

$$< \phi(F) > = \begin{cases} \phi(F) & \ldots\ F>0 \\ 0 & \ldots\ F\leq 0. \end{cases} \tag{12}$$

k denotes the static yield stress in simple shear and $\Theta$ is a characteristic time. $J_2$ is the second invariant of the deviatoric stress tensor. An elastic stress-strain relationship is understood for the hydrostatic component.

Isotropic damage accumulation is introduced replacing the total stresses $\sigma_{ij}$ by their effective values $\overline{\sigma_{ij}}=\sigma_{ij}/(1-D)$ in the constitutive equations. D denotes an internal damage variable according to Kachanov /6/. In case of ductile fracture, D is a solution of the following evolutionary equation:

$$\dot{D} = \alpha \left(\frac{\sigma_e}{1-D}\right)^s \dot{\varepsilon}^v_e \,, \tag{13}$$

with effective values:

$$\sigma_e = (3J_2)^{1/2}, \quad \varepsilon_e^v = \left(\frac{4}{3} I_2^v\right)^{1/2}, \tag{14}$$

where $\alpha, s$ denote material parameters, and $I_2^v$ is the second invariant of the viscoplastic strains $\varepsilon_{ij}^v$.
In case of an axisymmetric problem the viscoplastic strain rates are

$$\begin{aligned} \dot{\varepsilon}_{rr}^v &= \frac{1}{\theta} < \phi((\bar{J}_2^{1/2}/k)-1) > \frac{2\sigma_{rr}-\sigma_{\vartheta\vartheta}}{3J_2^{1/2}}, \\ \dot{\varepsilon}_{\vartheta\vartheta}^v &= \frac{1}{\theta} < \phi((\bar{J}_2^{1/2}/k)-1) > \frac{2\sigma_{\vartheta\vartheta}-\sigma_{rr}}{3J_2^{1/2}}, \end{aligned} \tag{15}$$

with

$$J_2 = \frac{1}{3}(\sigma_{rr}^2 + \sigma_{\vartheta\vartheta}^2 - \sigma_{rr}\sigma_{\vartheta\vartheta}), \quad \bar{J}_2 = J_2/(1-D). \tag{16}$$

## 3 NUMERICAL SOLUTION

The first order differential equations (15) can be solved by various integration procedures, e.g. Euler's explicit method or predictor-corrector schemes. Using the Euler method for convenience, the viscoplastic strain increments in time step $\Delta t = t_{p+1} - t_p$ are:

$$\Delta\varepsilon_{rr}^v = \frac{1}{\theta} < \phi^{(p)} > \frac{2\sigma_{rr}^{(p)} - \sigma_{\vartheta\vartheta}^{(p)}}{3(J_2^{(p)})^{1/2}} \Delta t, \tag{17}$$

$\Delta\varepsilon_{\vartheta\vartheta}^v$ is analogous to Eq. (17). A superscript (p) denotes the value of the corresponding variable at time $t_p$. In a similar way the incremental form of Eq. (13) becomes:

$$\begin{aligned} \Delta D &= \alpha \left(\frac{\sigma_e^{(p)}}{1-D^{(p)}}\right)^s \Delta\varepsilon_e^v, \\ \Delta\varepsilon_e^v &= \frac{2}{\sqrt{3}} [(\Delta\varepsilon_{rr}^v)^2 + (\Delta\varepsilon_{\vartheta\vartheta}^v)^2 + \Delta\varepsilon_{rr}^v \Delta\varepsilon_{\vartheta\vartheta}^v]^{1/2} \end{aligned} \tag{18}$$

Introducing damage into the general expression for viscoplastic stress increments

$$\Delta\sigma_{rr} = \frac{E(1-D^{(p)})}{1-\nu^2} [\Delta\varepsilon_{rr} - \Delta\varepsilon_{rr}^v + \nu(\Delta\varepsilon_{\vartheta\vartheta} - \Delta\varepsilon_{\vartheta\vartheta}^v)], \tag{19}$$

extracting the linear elastic solution

$$\Delta\sigma_{rr}^o = \frac{E}{1-\nu^2}(\Delta\varepsilon_{rr}^o + \nu\varepsilon_{\vartheta\vartheta}^o), \tag{20}$$

from Eq. (19), and comparing the remaining part to Eq. (7), yields the increments of the sources of selfstresses:

$$\begin{aligned} \Delta \varepsilon^{N}_{rr} &= (1-D^{(p)})\, \Delta\varepsilon^{v}_{rr} + D^{(p)} (\Delta\varepsilon^{o}_{rr} + \Delta\varepsilon^{*}_{rr}) \\ \Delta \varepsilon^{N}_{\vartheta\vartheta} &= (1-D^{(p)})\, \Delta\varepsilon^{v}_{\vartheta\vartheta} + D^{(p)} (\Delta\varepsilon^{o}_{\vartheta\vartheta} + \Delta\varepsilon^{*}_{\vartheta\vartheta}) \end{aligned} \qquad (21)$$

Note that the introduction of damage renders $\Delta\varepsilon^N$ linearly dependent on the total strain increment $\Delta\varepsilon$. Therefore, the $\Delta\varepsilon^N$'s are computed iteratively for convenience, $\Delta\varepsilon^*$ being a linear function of the $\Delta\varepsilon^N$'s ,see Sec.1 and $\Delta\varepsilon=\Delta\varepsilon^o+\Delta\varepsilon^*$, Eq. 1.

Having calculated the increments of stress and strain in the p-th interval using the above initial strain type formulation, the procedure is started for the p+1-th time step.

## 4 RESULTS

Fig. 1 gives dimensionless moments at the center and at the boundary of a simply supported circular plate with radius a due to a constant loading $p(r,t)=p_0 H(t)$. In case of $D\equiv 0$, coincidence with the asymptotic rigid-viscoplastic solution given in /5/, p.352-359 is found. Fig. 2 shows the moment-sums $(m_r+m_\vartheta)$ as a function of $\kappa=-(w_{,rr}+\frac{1}{r}w_{,r})$ in dimensionless form for vanishing and not vanishing material damage in case of $p(r,t)=p_0 \sin\omega t$. Parameters are given in the Figures. Accuracy of the solution has been proved by changing the time step. Computational effort of the method turned out to be comparatively small.

## ACKNOWLEDGEMENT

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FIGURES: Circular elasto-viscoplastic plate of radius <u>a</u>; spatial distribution of loading $p_0$=const.

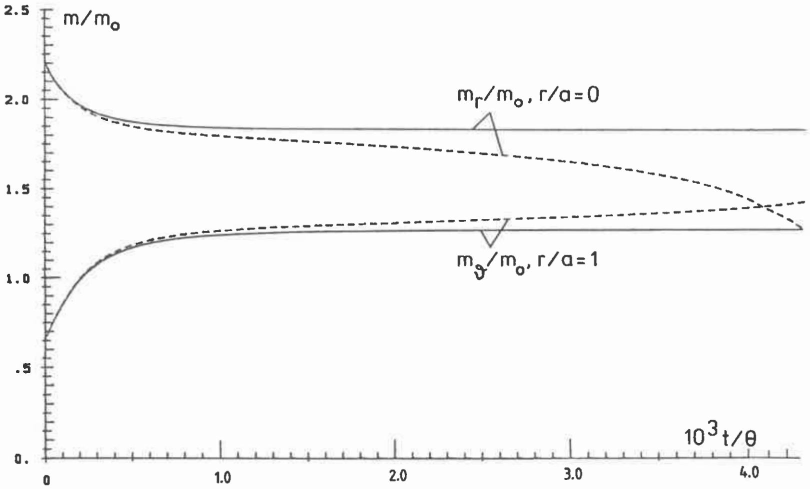

Fig. 1: Dimensionless moments $m_r/m_o$, $m_\vartheta/m_o$ in r/a=0 and r/a=1, respectively, as a function of t/Θ for step loading $p(r,t)=p_o H(t)$. $m_o=\sqrt{3}\,kh^2/4$. $p_o a^2/m_o=10$; $h/a=6.10^{-2}$; $E/\sqrt{3}\,k=4.10^3$; $\nu=0.5$ ; $\phi(F)=F$.
——— D≡0, ----- $\alpha k^s = 3.37$, s=1.8.

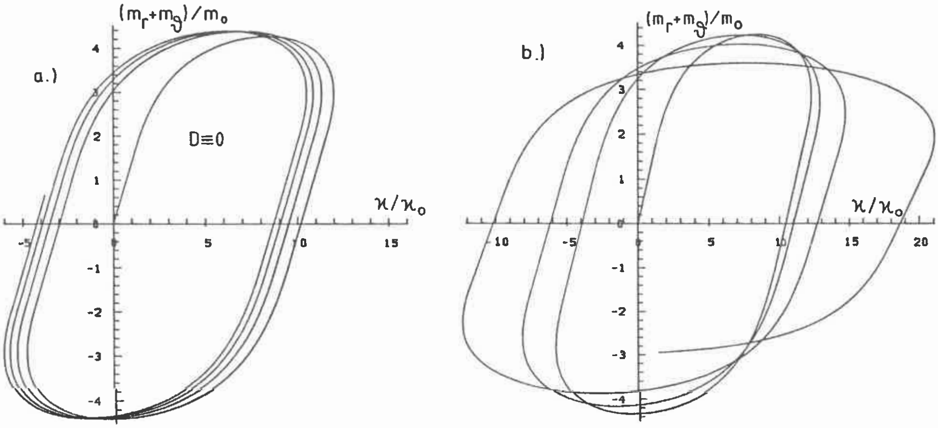

Fig. 2: Dimensionless sum of moments $(m_r+m_\vartheta)/m_o$ versus $\kappa/\kappa_o$ in r/a=0 for sinusoidal loading $p(r,t)=p_o \sin \omega t$. $m_o=\sqrt{3}\,kh^2/4$; $\kappa_o=8.4\ m_o/Eh^3$; $p_o/E=2.5.10^{-6}$; $h/a=6.10^{-2}$; $E/\sqrt{3}\,k=4.10^3$; $\nu=0.5$; $T/\Theta=3.10^{-3}$, $\omega=2\pi/T$. a.) D≡0. b.) $\alpha k^s=1.348$, s=1.8. $\phi(F)=F$.