SOME ESTIMATION PROBLEMS OF A NONLINEAR SUPPLY MODEL IMPLIED BY A MINIMAX DECISION RULE

by

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Institute of Statistics
Mimeo Series No. 365
May 1963

ERRATA SHEET

Page 11, Equation 3.15 should read

$$Y = \frac{\alpha q^{\beta}[(\bar{X} + \frac{R}{2})^{\gamma} - (\bar{X} - \frac{R}{2})^{\gamma}]}{R} \quad .$$

Change page number of page 42 to page 40.

TABLE OF CONTENTS

LIST OF TABLES

Page

APPENDIX TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

Supply analysis has been a field of interest for many years in economics. Considerable attention has been given to the determination of what changes would be profitable for firms to make in relation to changes in factor and product prices.

Agriculture continues to face apparent problems of resource misallocation. For this reason supply analysis is particularly needed. Furthermore, aggregation of individual firm supply responses is necessary before many results useful for policy and forecasting purposes can be derived.

One of the major problems in the application of supply theory is that of specifying the correct, or at least a useful, relation between the constructs of the theory and the variables that can actually be observed. In time series supply analysis one form of this problem is that of specifying the relation between observable events and the prices which farmers expect to pay for their inputs and the prices which farmers expect to receive for their outputs. With this view in mind, this dissertation examines the estimation problems inherent in one possible relation between quantity supplied of a product and observable prices of inputs and outputs.

A basic datum for all supply analyses is the production function which summarizes the technological possibilities for transforming inputs into outputs. The production function may either be explicit or implicit in the analysis, but it is always present in one way or another. The conventional production function is generally expressed as an implicit functional relationship between all outputs and all variable inputs, i.e.

$$f(y_1, y_2, \ldots, y_n; x_1, x_2, \ldots, x_m) = 0$$

where $y_1, y_2, \ldots, y_n$ are the outputs and $x_1, x_2, \ldots, x_m$ are quantities of inputs. Given a production function for the firm of the above form, it is possible to derive outputs in terms of given prices of the outputs and the inputs.

In summary, this dissertation considers the development of a nonlinear supply function from a given production model, incorporating in the process the concept of minimizing maximum loss, rather than maximizing (expected) profit, and the principle of aggregation. The primary object of this paper will be the investigation of the estimation problems involved in the model to be developed. Forecasting aspects of the model will not be dealt with in this study.

CHAPTER II

REVIEW OF LITERATURE AND RELATED STUDIES

The problem of aggregating supply functions for individual firms and single commodities into supply functions for groups of firms and groups of commodities has been explored mainly from a theoretical point of view. Theil (1954) has developed an approach to the problem which may be characterized as follows: Suppose we are given a microtheory (theory of the individual firm's supply) and some simple aggregates of the microvariables (sums of output levels and input levels). Then a macromodel analogous to the micromodel which relates the aggregates is assumed. The parameters in the macromodel are then estimated.

Another approach to supply analysis that has been used by many is the linear programming approach. This is simply the derivation of a supply function for the individual firm from a linear programming production function. Applications of the technique have been made by Knutson and Cochrane (1958), Toussaint (1956), McPherson and Farris (1958), and Tompkin (1958). The device of programming with variable prices is discussed by Heady and Candler (1958). Plaxico (1958) has presented a comprehensive discussion of the approaches to aggregation problems in linear programming.

Recently emphasis has been placed on time series supply analysis. Nerlove (1961) discusses the different time series supply models that have been used.

A difficulty that raises problems in time series supply analysis as well as other approaches to supply analysis is uncertainty. It is clear that if uncertainty affects how farmers should behave with respect to production plans and investment, it must also affect how they actually behave, although perhaps not in exactly the same way. A basic and useful construct in this connection as discussed by Nerlove (1958) is the notion of certainty equivalence. In the certainty equivalent to a problem of decision-making under uncertainty, each uncertain variable is replaced by one or more variables the values of which if expected with certainty would lead to the same solution as that which would be obtained by treating the decision problem in its full generality. This notion of certainty equivalence underlies time series supply analysis although not always explicitly. Models of expectation formation as illustrated by Nerlove (1961) are essentially methods of arriving at certainty equivalents for uncertain future prices.

Girschick and Haavelmo (1953) and Tintner (1958) illustrate the use of the method of simultaneous equations in deriving simple linear aggregate agricultural demand and supply functions. Tintner related supply to current farm prices, national income, and a cost factor. Girschick and Haavelmo related aggregate farm production to farm prices, to farm prices lagged one year and to time. Cromarty's (1961) econometric model for the U. S. agriculture is a comprehensive model which interrelates the supply, demand, and prices for farm commodities. Hildreth and Jarrett (1955) studied the feed-live-stock economy where prices, feeds, production and others were considered for all classes of livestock as an aggregate.

Individual supply models have been derived and used on hogs, beef, milk, eggs, corn, cotton, tobacco and potatoes. A summary of these studies has been made by Knight (1961). These were all linear models.

An aggregate time series supply model that has been used and recommended by many research workers is the distributed lag model. It seems that the concept of the distributed lag was first introduced by Fisher (1925). Koyck (1954), Nerlove (1958a), and Kline (1958) among others, discuss several aspects of distributed lags.

CHAPTER III

THE MODEL AND ESTIMATION OF THE PARAMETERS

A simple production function with a single input, W, and a single output, Y, shall be considered in deriving the model. The derivation below of the minimax input follows essentially that of Hildreth's (1957).

3.1 Development of Minimax Input

Let X be the price of the output, known to lie between a lower price limit X_2 and an upper limit X_3. Denote by q the price of the input (assumed known) at the time the input is decided. Let the production function be

$$Y = W^{\delta} \tag{3.1}$$

and let the net revenue be

$$NR = XY - qW . \tag{3.2}$$

Equating the derivative to zero

$$\frac{dNR}{dW} = \delta X W^{\delta-1} - q = 0 \quad \text{and} \tag{3.3}$$

$$W_0 = \left(\frac{X\delta}{q}\right)^{\frac{1}{1-\delta}} . \tag{3.4}$$

Equation (3.4) gives the optimal input, W_0, for any known price. This is the firm's derived demand function for W. Using equations (3.4) and (3.2), optimal revenue is given by

Optimal net revenue = ONR

$$= X\left(\frac{X\delta}{q}\right)^{\frac{\delta}{1-\delta}} - q\left(\frac{X\delta}{q}\right)^{\frac{1}{1-\delta}}$$

$$= X(X\delta)^{\frac{\delta}{1-\delta}} q^{-\frac{\delta}{1-\delta}} - (X\delta)^{\frac{1}{1-\delta}} q^{\left(1-\frac{1}{1-\delta}\right)}$$

$$= (1-\delta)\, \delta^{\frac{\delta}{1-\delta}} X^{\frac{1}{1-\delta}} q^{-\frac{\delta}{1-\delta}} \quad . \tag{3.5}$$

The loss corresponding to a pair (X,W) is given by the difference between optimal net revenue given by equation (3.5) and net revenue for W given by equation (3.2); thus,

$$L = (1-\delta)\left(\frac{\delta}{q}\right)^{\frac{\delta}{1-\delta}} X^{\frac{1}{1-\delta}} - XW^{\delta} + Wq \quad . \tag{3.6}$$

Note that

$$\frac{dL}{dX} = \left(\frac{\delta X}{q}\right)^{\frac{\delta}{1-\delta}} - W^{\delta} \tag{3.7}$$

and if

$$X = \delta^{-1}\, qW^{1-\delta} \tag{3.8}$$

then $L = 0$ and $\frac{dL}{dX} = 0$. Thus, for smaller X, L is decreasing and for larger X, L is increasing. For a given W, say $W = W'$, the loss function will look something like the figure below.

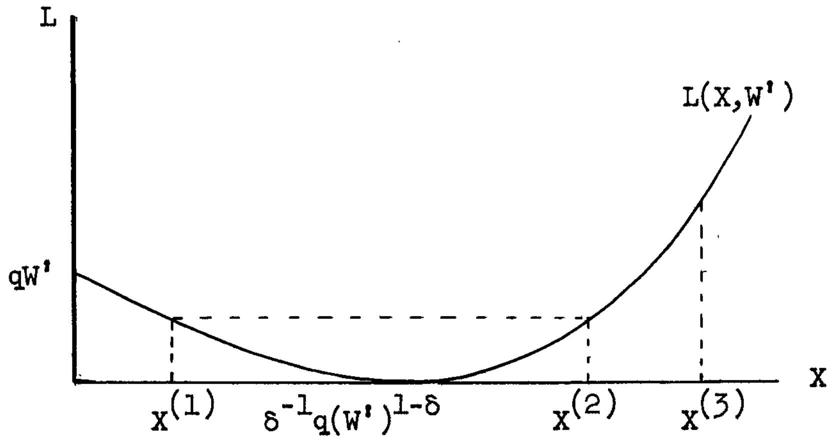

Figure 1. Loss function for a given production input, W'

Inspection of the curve tells us that

(a) if $X_2 = X^{(2)}$ and $X_3 = X^{(3)}$ then $X = X_3$ where X denotes the admissible value, between the lower and the upper prices, that maximizes the loss,

(b) if $X_2 = X^{(1)}$ and $X_3 = X^{(3)}$ then $X = X_3$,

(c) if $X_2 = X^{(1)}$ and $X_3 = X^{(2)}$ then $X = X_2$ or X_3 .

Suppose we now consider a curve $L(X,W_1)$ where $W_1 < W'$ but is sufficiently close such that $\delta^{-1}qW_1^{1-\delta}$ is very close to $\delta^{-1}q(W')^{1-\delta}$. To the left of $\delta^{-1}qW_1^{1-\delta}$, $L(X,W_1) < L(X,W')$ and to the right of $\delta^{-1}q(W')^{1-\delta}$, $L(X,W_1) > L(X,W')$. This is the same as saying that W_1 is better than W' for low prices and worse for high prices. If $X_2 = X^{(1)}$, $X_3 = X^{(2)}$ then $X = X_3$ for $W = W_1$ and $L(X_3,W_1) > L(X_3,W')$. Similarly if we take $W = W_2$, W_2 slightly higher or greater than W' then $X = X_2$ for $W = W_2$ and $L(X_2,W_2) > L(X_2,W')$. From these considerations then it is seen that there is a critical region value of W say $\hat{W}$ such that $X = X_3$ for any $W < \hat{W}$ and

$X = X_2$ for any $W > \hat{W}$. Furthermore, to increase or decrease W increases the maximum loss and $\hat{W}$ is therefore the minimax input. The maximum loss may be found from equation (3.5) by setting $W = \hat{W}$. We characterize $\hat{W}$ by

$$L(X_2,\hat{W}) = L(X_3,\hat{W}) \tag{3.9}$$

and to solve for $\hat{W}$ we set (assuming same X's)

$$(1-\delta)\left(\frac{\delta}{q}\right)^{\frac{\delta}{1-\delta}} X_2^{\frac{1}{1-\delta}} - X_2\hat{W}^{\delta} + q\hat{W}$$
$$= (1-\delta)\left(\frac{\delta}{q}\right)^{\frac{\delta}{1-\delta}} X_3^{\frac{1}{1-\delta}} - X_3\hat{W}^{\delta} + q\hat{W}$$

and after simplification, we obtain

$$\hat{W}^{\delta} = q^{-\frac{\delta}{1-\delta}}(1-\delta)\,\delta^{\frac{\delta}{1-\delta}}\,\frac{\left(X_3^{\frac{1}{1-\delta}} - X_2^{\frac{1}{1-\delta}}\right)}{X_3 - X_2}\quad . \tag{3.10}$$

A more rigorous proof of the existence of a minimax under conditions considered here, is illustrated by Wald (1950).

3.2 Model for an Individual Firm or Period

The minimax supply model for an individual firm can now be obtained by substituting equation (3.10) into equation (3.1) which, after simplification, yields

$$Y_i = q^{-\frac{\delta_i}{1-\delta_i}}(1-\delta_i)\,\delta_i^{\frac{\delta_i}{1-\delta_i}}\,\frac{X_3^{\frac{1}{1-\delta_i}} - X_2^{\frac{1}{1-\delta_i}}}{X_3 - X_2}\ , \tag{3.11}$$

$$i = 1,2,\ldots,n\ .$$

For characterizing the individual firm decision under the minimax rule, equation (3.11) is sufficient. However, interest here is not to be focused on the individual firm but on the aggregate response of a group of firms given changes in X_2 and X_3. Therefore, we must aggregate over n firms (or periods), say, to obtain the aggregate supply function.

3.3 The Aggregate Model

If we assume that the aggregate function has the same form as the individual relationship and that only the parameters change, then we can write equation (3.11) as,

$$Y = \alpha\, q^{\beta} \frac{(X_3^{\gamma} - X_2^{\gamma})}{X_3 - X_2}, \quad \alpha, \gamma > 0, \quad \beta < 0 \qquad (3.12)$$

and take equation (3.12) as our minimax supply function.

It is apparent from equation (3.12) that we have an essentially nonlinear model. Since the estimation of the parameters in essentially nonlinear models is generally not simple, many researchers may shy away from such a model. It is then the object of this dissertation to investigate the estimation problems involved in the model given by equation (3.12).

The importance of the model in equation (3.12) lies in the fact that the parameter γ when estimated will lead to an estimate of the supply elasticity. This will be illustrated below.

Starting with the model

$$Y = \alpha \, q^{\beta} \; \frac{(X_3^{\gamma} - X_2^{\gamma})}{X_3 - X_2}$$

we make the following transformations,[1]

$$\frac{X_3 + X_2}{2} = \bar{X} \tag{3.13}$$

and

$$X_3 - X_2 = R$$

where $\bar{X}$ is the mid-range of the prices and R is the range. Solving equations (3.13) simultaneously we obtain

$$X_3 = \bar{X} + \frac{R}{2} \tag{3.14}$$

and

$$X_2 = \bar{X} - \frac{R}{2} \, .$$

Substituting equations (3.14) to equation (3.12) and simplifying, we have

$$Y = \frac{\alpha \, q^{\beta} \, [(1 + \frac{R}{2})^{\gamma} - (\bar{X} - \frac{R}{2})^{\gamma}]}{R} \tag{3.15}$$

which can be written in the form

$$Y = \frac{\alpha \, q^{\beta} \, [\bar{X}^{\gamma} \, (1 + \frac{R}{2\bar{X}})^{\gamma} - \bar{X}^{\gamma} \, (1 - \frac{R}{2\bar{X}})^{\gamma}]}{R} \tag{3.16}$$

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[1]This was suggested by S. Reutlinger in a personal communication.

and dividing both numerator and denominator by $\bar{X}$ we get

$$Y = \frac{\alpha q^{\beta} \bar{X}^{\gamma-1}[(1 + \frac{R}{2\bar{X}})^{\gamma} - (1 - \frac{R}{2\bar{X}})^{\gamma}]}{R/\bar{X}} . \tag{3.17}$$

A logarithmic transformation of equation (3.17) and defining $R/\bar{X} = V$ yields

$$\log Y = \log \alpha + \beta \log q + (\gamma-1) \log \bar{X}$$
$$+ \log [(1 + \frac{V}{2})^{\gamma} - (1 - \frac{V}{2})^{\gamma}] - \log V . \tag{3.18}$$

Taking the partial derivative of log Y with respect to $\bar{X}$

$$\frac{1}{Y} \frac{\partial Y}{\partial \bar{X}} = \frac{\gamma-1}{\bar{X}}$$

or

$$\frac{\partial Y}{\partial \bar{X}} \cdot \frac{\bar{X}}{Y} = \gamma-1 . \tag{3.19}$$

But the left hand side of equation (3.19) is the expression for supply elasticity, hence $(\gamma-1)$ is the supply elasticity.[2]

3.4 Estimation of the Parameters of the Model

In any estimation problem it is always desirable to be able to obtain "best estimators" of the parameters. Certain criteria which must be fulfilled, can be set up in determining what function will

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[2]Since $\bar{X}$ is not independent of V, the elasticity $(\gamma-1)$ must be interpreted carefully. The partial elasticity with respect to $\bar{X}$ refers to the response in Y due to variation in price when accompanied by a proportional variation in range to leave V unchanged.

yield a "best estimator" of a given parameter. There are of course cases where it may not be possible to find an estimator which possesses the requirements asked for because in some cases a "best estimator" does not exist. It is however not intended to discuss in this dissertation all of the connotations involved in the problem of a "best estimator". A few widely accepted properties will serve as a guide to the solution of the estimation problems to be encountered later.

The estimation method proposed here will yield maximum likelihood estimators, c.f. Stevens (1951). Properties of maximum likelihood estimators are well known--included are asymptotic efficiency, consistency (asymptotic unbiasedness), and sufficiency if sufficiency exists. However, small sample properties of maximum likelihood estimators must be investigated in each circumstance. The purpose of the remainder of this thesis is to investigate the maximum likelihood procedure proposed here for unbiasedness and small sample efficiency.

3.4.1 The Estimation Method. From here on the following notations will be used;

q = price of input at time t

X_3 = upper price of output at time t

X_2 = lower price of output at time t and

any primed expression shall denote the common logarithm of the expression, e.g., $X' = \log X$.

If it is assumed that the errors of the model in equation (3.12) are multiplicative, a logarithmic transformation makes the errors additive. Such a transformation yields:

$$Y' = \alpha' + \beta\, q' + (X_3^\gamma - X_2^\gamma)' - (X_3 - X_2)' + e' \quad . \qquad (3.20)$$

For the discussion in this chapter it will be further assumed that the errors in equation (3.20) are normally distributed with expectation zero and common variance, σ^2.

The estimates will be obtained by maximizing the log likelihood function which is equivalent to maximizing the likelihood of the function. For a sample of size n , the log likelihood, L, is,

$$L = -\frac{n}{2} \ln (2\Pi\sigma^2) \qquad (3.21)$$

$$- \frac{1}{2\sigma^2} \Sigma[Y' - \alpha' - \beta\, q' - (X_3^\gamma - X_2^\gamma)' + (X_3 - X_2)']^2 \; .$$

The maximum of L is obtained by differentiating equation (3.21) with respect to each of the unknown parameters α', β, and γ, setting each partial derivative equal to zero and solving the set of simultaneous equations thus obtained. These are given below neglecting σ^2.

$$\frac{\partial L}{\partial \alpha'} = \Sigma[Y' - \alpha' - \beta\, q' - (X_3^\gamma - X_2^\gamma)' + (X_3 - X_2)']$$

$$\frac{\partial L}{\partial \beta} = \Sigma\, q'[Y' - \alpha' - \beta\, q' - (X_3^\gamma - X_2^\gamma)' + (X_3 - X_2)'] \qquad (3.22)$$

$$\frac{\partial L}{\partial \gamma} = \Sigma\, \frac{1}{X_3^\gamma - X_2^\gamma} (X_3^\gamma X_3' - X_2^\gamma X_2')[Y' - \alpha' - \beta\, q' - (X_3^\gamma - X_2^\gamma)' + (X_3 - X_2)']$$

and from equations (3.22) we obtain the normal equations for the maximum likelihood estimates by equating the equations of (3.22) to zero and replacing the parameters by corresponding estimates.

In further discussion of the estimation problem it will be convenient to define a new variable

$$Z = \frac{X_3^{\gamma}X_3' - X_2^{\gamma}X_2'}{X_3^{\gamma} - X_2^{\gamma}} \quad . \tag{3.23}$$

The normal equations are nonlinear in the estimate of the parameter γ and, as it commonly happens in these cases, will not have explicit solutions for the values of the estimators which make the log likelihood, and consequently the likelihood, a maximum. However we may, c.f. Stevens (1951), use the asymptotic variance-covariance matrix to improve on inefficient estimates by an iterative procedure.

The Hessian of the log likelihood, replacing Y' by its expectation, and neglecting σ^2, is derived below making the substitution from equation (3.23).

$$-\frac{\partial^2 L}{\partial\alpha'^2} = n. \quad -\frac{\partial^2 L}{\partial\alpha'\partial\beta} = \Sigma q' \; .$$

$$-\frac{\partial^2 L}{\partial\alpha'\partial\gamma} = \Sigma \; \frac{1}{(X_3^{\gamma} - X_2^{\gamma})}\,(X_3^{\gamma}X_3' - X_2^{\gamma}X_2') = \Sigma Z \; .$$

$$-\frac{\partial^2 L}{\partial\beta\partial\alpha'} = \Sigma q'. \quad -\frac{\partial^2 L}{\partial\beta^2} = \Sigma(q')^2 \; . \tag{3.24}$$

$$-\frac{\partial^2 L}{\partial\beta\partial\gamma} = \Sigma \; \frac{q'}{X_3^{\gamma} - X_2^{\gamma}}\,(X_3^{\gamma}X_3' - X_2^{\gamma}X_2') = \Sigma q'Z \; .$$

$$-\frac{\partial^2 L}{\partial\gamma\partial\alpha'} = \Sigma \frac{X_3^\gamma X_3' - X_2^\gamma X_2'}{X_3^\gamma - X_2^\gamma} = \Sigma Z .$$

$$-\frac{\partial^2 L}{\partial\gamma\partial\beta} = \Sigma \frac{q'(X_3^\gamma X_3' - X_2^\gamma X_2')}{X_3^\gamma - X_2^\gamma} = \Sigma q'Z .$$

$$-\frac{\partial^2 L}{\partial\gamma^2} = \Sigma \left[\frac{X_3^\gamma X_3' - X_2^\gamma X_2'}{X_3^\gamma - X_2^\gamma}\right]^2 = \Sigma Z^2 .$$

The procedure then is to substitute estimates a_0', b_0, and c_0, for α', β, and γ respectively, into these formulae and compute $\delta a'$, δb, and δc, additive corrections to the inefficient estimates using the following (which is essentially a multiple regression procedure):

Call:

$$N_1 = \Sigma Y' - na_0' - b_0\Sigma q' - \Sigma(X_3^{c_0} - X_2^{c_0})' + \Sigma(X_3 - X_2)' ,$$

$$N_2 = \Sigma q'Y' - a_0'\Sigma q' - b_0\Sigma(q')^2 - \Sigma q'(X_3^{c_0} - X_2^{c_0})' + \Sigma q'(X_3 - X_2)'$$

$$N_3 = \Sigma Z_0 Y' - a_0'\Sigma Z_0 - b_0\Sigma Z_0 q' - \Sigma Z_0(X_3^{c_0} - X_2^{c_0})' + \Sigma Z_0(X_3 - X_2)' , \tag{3.25}$$

where

$$Z_0 = \frac{X_3^{c_0} X_3' - X_2^{c_0} X_2'}{X_3^{c_0} - X_2^{c_0}} .$$

Then

$$\begin{bmatrix} n & \Sigma q' & \Sigma Z_0 \\ \Sigma q' & \Sigma (q')^2 & \Sigma q' Z_0 \\ \Sigma Z_0 & \Sigma q' Z_0 & \Sigma Z_0^2 \end{bmatrix} \begin{bmatrix} \delta a' \\ \delta b \\ \delta c \end{bmatrix} = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} . \qquad (3.26)$$

Solving for $\delta a'$, δb, and δc, we obtain improved estimates by:

$$a_1' = a_0' + \delta a', \quad b_1 = b_0 + \delta b, \quad \text{and} \quad c_1 = c_0 + \delta c .$$

One iteration is usually sufficient to obtain asymptotically efficient estimates. The process is repeated until $\delta a'$, δb, and δc are zero or near-zero.

The inverse of the coefficient matrix for the corrections in equations (3.26) has intrinsic interest, Kendall and Stuart (1961), for multiplied by σ^2 and with parameters instead of estimates, it would become the asymptotic variance-covariance matrix of our estimates. Thus, the present form multiplied by an estimate of σ^2, say the residual sum of squares divided by n - 3, provides consistent estimates of the asymptotic variances and covariances.

3.4.2 Procedures in Obtaining Initial Estimates. The remaining question for the implementation of this technique of improving estimates is a systematic procedure of obtaining starting values a_0', b_0, and c_0. Let us then look into this.

The model in equation (3.12) can be written as

$$Y = \frac{\alpha q^{\beta} X_3^{\gamma}}{X_3 - X_2} \; [1 - (\frac{X_2}{X_3})^{\gamma}] \tag{3.27}$$

in which form it can be noted that the ratio $\frac{X_2}{X_3}$ must play an important role in determining the value of Y. We shall consider here two cases: One when the value of the ratio $\frac{X_2}{X_3}$ tends to one and the other when its value tends to zero. Since X_2 is less than or at most equal to X_3 then the ratio $\frac{X_2}{X_3}$ is less than or at most equal to one. The ratio $\frac{X_2}{X_3}$ will tend to one if the lower and upper prices for any time period do not differ greatly, i.e., when the lower and upper prices tend to the same value. If such is the case then, $Y = \frac{0}{0}$ in equation (3.27) which is indeterminate. Using L'Hospital's rule to determine the value of Y, we differentiate both the numerator and the denominator of the right hand side of equation (3.27) with respect to X_3 say, resulting in

$$Y = \alpha\gamma q^{\beta} X_3^{\gamma-1} \;=\; \lambda q^{\beta} X_3^{\gamma-1} \; , \quad \text{say.} \tag{3.28}$$

Differentiating with respect to X_2 will produce an identical result. A logarithmic transformation of equation (3.28) yields

$$Y' = \lambda' + \beta q' + (\gamma-1)\, X_3' \; , \tag{3.29}$$

and the parameters in this model can be estimated by the usual least squares technique. Since $\lambda' = (\alpha\gamma)'$ then α can be estimated in a two stage procedure, i.e., we estimate λ and γ first then estimate α using the relationship $\lambda' = (\alpha\gamma)'$. Such estimates can be used as the starting values for obtaining the improved estimates.

Now consider the other case. If for any time period the lower and upper prices of the output are greatly different, i.e., the upper price is much higher than the lower price, then the ratio $\frac{X_2}{X_3}$ will be small and for large values of γ the ratio $(\frac{X_2}{X_3})^{\gamma}$ will tend to zero and hence negligible, and equation (3.27) is simplified thus,

$$Y = \frac{\alpha q^{\beta} X_3^{\gamma}}{X_3 - X_2} \quad , \tag{3.30}$$

or in terms of the logarithmic model as

$$Y' = \alpha' + \beta q' + \gamma X_3' - (X_3 - X_2)' \quad , \tag{3.31}$$

in which form again we can estimate the parameters also by least squares and the estimates can be used in obtaining the improved estimates.

3.5 Problems of Estimation

There are problems that may be encountered in the estimation of the parameters of the model in equation (3.12) using the proposed iteration scheme because of the nature of the elements of the matrices in equation (3.26). A close look at these expressions suggests that the corrections to the initial estimates are primarily functions of Z, a, b and c. The Z's are in turn functions of the X's (lower and upper prices) and c. If the spread of the X's is large or if c were large the Z's may vary widely from cycle to cycle of the iteration process and hence may cause the corrections to be inefficient or to slow and prevent convergence. Variability in a and b may also seriously affect the corrections or convergence. For purposes of estimating γ only, the variability due to a and b would be eliminated if they were known; if this is so, we delete the first two rows and first two columns of the matrices in equation (3.26).

If the ratios $\frac{X_2}{X_3}$ were constant or near-constant for all Y''s, i.e., as the coefficient of variation of the ratios approaches zero, the model in equation (3.12) becomes an approximately linear model after the logarithmic transformation, hence, estimates obtained by the iterative scheme may be little, if any, more efficient than the least squares estimates from the linear model. Also, the estimates from the linear model will be more convenient to obtain.

Most of the estimation problems posed here are mathematically intractable. For this reason we shall make use of empirical studies to investigate these problems. The empirical results are discussed in the next chapter.

CHAPTER IV

RESULTS AND DISCUSSION

In this chapter we shall determine the effects on the iterative estimation procedure proposed in the preceding chapter of the following factors:

(a) number of parameters estimated,

(b) range of the lower and of the upper prices (range of X's),

(c) size of the parameter estimated,

(d) coefficient of variation of the ratios $\frac{X_2}{X_3}$, and

(e) coefficient of variation of the Y''s.

Without loss of generality in estimating the parameter of interest, γ, we shall utilize a simplified form of the model in equation (3.12) to determine the effects of the factors above. The simplified form is

$$Y = \alpha \frac{X_3^{\gamma} - X_2^{\gamma}}{X_3 - X_2} . \tag{4.1}$$

We shall look into the effects of the factors in question on the asymptotic efficiency of the estimates of γ obtained by the iterative estimation procedure, and on the efficiency of estimates obtained by the same scheme in small samples.

4.1 Asymptotic Efficiency

The asymptotic variances (AV) of the estimates of α and γ in equation (4.1) obtained by the iterative scheme, can be obtained from

$$C^{-1}\sigma^2 = \begin{bmatrix} n & \Sigma Z \\ \Sigma Z & \Sigma Z^2 \end{bmatrix}^{-1} \sigma^2 , \tag{4.2}$$

where the matrix C is the same coefficient matrix in equation (3.26) with the second row and second column deleted (since the simplified model in equation (4.1) does not involve q), and

$$Z = \frac{X_3^\gamma X_3' - X_2^\gamma X_2'}{X_3^\gamma - X_2^\gamma}$$

as defined in equation (3.23). Since we are primarily concerned with the parameter γ we shall consider only the asymptotic variance of the estimate of γ, $AV(\hat{\gamma})$. Note that $AV(\hat{\gamma})$ is a function of the X's, γ and σ^2. In discussing the effects of any of the factors, unless stated otherwise, the other factors are to be assumed constant.

4.1.1 Number of Parameters Estimated. Let $AV(\hat{\gamma}|\alpha)$ be the asymptotic variance of the estimate of γ when α is assumed known and only γ is estimated, and $AV(\hat{\gamma}, \hat{\alpha})$ the asymptotic variance of the estimate of γ when both α and γ are estimated. From equation (4.2) we have

$$AV(\hat{\gamma}|\alpha) = \frac{\sigma^2}{\Sigma Z^2} \quad \text{and}$$

$$AV(\hat{\gamma}, \hat{\alpha}) = \frac{\sigma^2}{\Sigma(Z - \bar{Z})^2} \quad .$$

Since $\Sigma Z^2 \geq \Sigma(Z - \bar{Z})^2$ then

$$AV(\hat{\gamma}|\alpha) \leq AV(\hat{\gamma}, \hat{\alpha}) \quad . \tag{4.3}$$

4.1.2 Range of X's. Let r_1 be the range of X_3 and r_2 the range of X_2 for a given set of X's. Let us form a new set of X's by multiplying the X's in the given set by a factor $g > 1$. The ranges of the new X's will be gr_1 and gr_2. Now denote by Z_o the Z-values for the original X's and by Z_N the Z-values for the new set of X's. It can be shown that

$$Z_N = g' + Z_o \, .$$

Now

$$AV(\hat{\gamma}|\alpha)_o = \frac{\sigma^2}{\Sigma Z_o^2} \quad \text{for the set of X's with ranges } r_1 \text{ and } r_2$$

and

$$AV(\hat{\gamma}|\alpha)_N = \frac{\sigma^2}{\Sigma Z_N^2} \quad \text{for the set of X's with ranges } gr_1 \text{ and } gr_2 .$$

Since $g > 1$ then $g' > 0$ and $\Sigma Z_o^2 < \Sigma Z_N^2$, thus

$$AV(\hat{\gamma}|\alpha)_o > AV(\hat{\gamma}|\alpha)_N \quad . \tag{4.4}$$

When both α and γ are estimated it can be shown that since $Z_N = g' + Z_o$, then $\Sigma(Z_o - \bar{Z}_o)^2 = \Sigma(Z_N - \bar{Z}_N)^2$ and

$$AV(\hat{\gamma}, \hat{\alpha})_o = AV(\hat{\gamma}, \hat{\alpha})_N \, . \tag{4.5}$$

4.1.3 <u>Size of γ</u>. First note that Z can be written as

$$Z = X_3' - \frac{(\frac{X_2}{X_3})^{\gamma} (\frac{X_2}{X_3})'}{1 - (\frac{X_2}{X_3})^{\gamma}} \, .$$

Since $\frac{X_2}{X_3} < 1$ then $k = \frac{(\frac{X_2}{X_3})^{\gamma} (\frac{X_2}{X_3})'}{1 - (\frac{X_2}{X_3})^{\gamma}} < 0$; k tends monotonically to zero as γ increases. Let Z_I be the Z-values for $\gamma = \gamma_i$ and Z_J be the Z-values for $\gamma = \gamma_j$, where $\gamma_i < \gamma_j$, then

$$AV(\hat{\gamma}|\alpha)_{\gamma=\gamma_i} = \frac{\sigma^2}{\Sigma Z_I^2} \quad \text{for } \gamma = \gamma_i \text{ and}$$

$$AV(\hat{\gamma}|\alpha)_{\gamma=\gamma_j} = \frac{\sigma^2}{\Sigma Z_J^2} \quad \text{for } \gamma = \gamma_j .$$

Since $Z_J < Z_I$,

$$AV(\hat{\gamma}|\alpha)_{\gamma=\gamma_i} < AV(\hat{\gamma}|\alpha)_{\gamma=\gamma_j} , \quad \gamma_i < \gamma_j . \tag{4.6}$$

When α and γ are estimated, it can be shown that if the ratios $\frac{X_2}{X_3}$ = constant

$$AV(\hat{\gamma}, \hat{\alpha})_{\gamma=\gamma_i} = AV(\hat{\gamma}, \hat{\alpha})_{\gamma=\gamma_j} . \tag{4.7}$$

However, when the ratios $\frac{X_2}{X_3}$ are variable we have the relationship

$$\Sigma(Z_I - \bar{Z}_I)^2 = \Sigma(Z_J - \bar{Z}_J)^2 + \Delta$$

where

$$\Delta = (n-1) [\text{Variance } (k_I) + 2 \text{ Cov } (k_I, X_3')] .$$

Thus

$$AV(\hat{\gamma}, \hat{\alpha})_{\gamma=\gamma_i} \leq AV(\hat{\gamma}, \hat{\alpha})_{\gamma=\gamma_j}, \quad \text{if } \Delta \geq 0 . \tag{4.8}$$

Although no proof can be shown that in general $\Delta \geq 0$, worked out examples have shown this to be true.

4.1.4 Variability of the Ratios X_2/X_3. No definite comparisons of the effects of differences in the magnitude of this factor can be shown here for the case when only γ is estimated. When both α and γ are estimated we note from the results in section 4.1.3 that

$$\Sigma(Z_C - \bar{Z}_C)^2 = \Sigma(Z_V - \bar{Z}_V)^2 + \Delta$$

where Z_C are the Z-values when $\frac{X_2}{X_3}$ are constant, Z_V are the Z-values when $\frac{X_2}{X_3}$ are variable and

$$\Delta = (n-1)\ [\text{Variance}\ (k_v) + 2\ \text{Cov}\ (X_3', k_v)]\ .$$

Therefore

$$AV(\hat{\gamma}, \hat{\alpha})_C \leq AV(\hat{\gamma}, \hat{\alpha})_V \quad \text{if} \quad \Delta \geq 0\ . \tag{4.9}$$

Again no proof can be shown that in general $\Delta \geq 0$ but, as stated previously, worked out examples have shown this to be true.

4.1.5 Coefficient of Variation of the Y''s. It is apparent that since the $AV(\hat{\gamma})$ for any case is the ratio of σ^2 to a sum of squares, a decrease in the coefficient of variation of the Y''s can result in a smaller $AV(\hat{\gamma})$ only if σ^2 is decreased. If σ^2 were constant the $AV(\hat{\gamma})$ will remain the same no matter what the coefficient of variation of the Y''s may be.

4.1.6 Summary. We can now summarize briefly the effects of the various factors on the asymptotic efficiencies of the estimates of γ.

(a) The efficiency when both α and γ are estimated is less than when only γ is estimated assuming α known.

(b) When only γ is estimated, the efficiency is less when the ranges of the X's are small than when the ranges are large. When both α and γ are estimated the efficiencies do not differ for small or large ranges.

(c) When only γ is estimated, the efficiency decreases as γ increases. When both α and γ are estimated and the ratios $\frac{X_2}{X_3}$ are variable, a necessary and sufficient condition for the efficiency of the estimate of a smaller γ to be greater than that of the estimate of a larger γ is that $(n-1)\,[\mathrm{Var}(k_I) + 2\,\mathrm{Cov}(k_I, X_3'))] > 0$, where k_I are the k-values for the smaller γ. When the ratios $\frac{X_2}{X_3}$ are constant and both α and γ are estimated the efficiencies of the estimates for small and large values of γ are the same.

(d) No definite comparison could be made of the effect of variability in the ratios $\frac{X_2}{X_3}$ when only γ is estimated. When both α and γ are estimated a necessary and sufficient condition for the efficiency of the estimate when the ratios $\frac{X_2}{X_3}$ are variable to be greater than that when the ratios $\frac{X_2}{X_3}$ are constant is that $(n-1)\,[\mathrm{Var}(k_V) + 2\,\mathrm{Cov}(k_V, X_3')] > 0$, where k_V are the k-values when the ratios are variable.

(e) The efficiencies of the estimates increase when the coefficient of variation of the Y''s decrease only if σ^2 is decreased.

4.2 Sampling Investigations

We shall make use of the results from empirical studies to determine the effects of the same factors considered in Section 4.1 on the efficiency of the estimates obtained by the iterative estimation scheme in small samples.

The empirical studies consisted of obtaining random samples from artificial populations and using these samples for the estimation of the parameters. Four simulation models, corresponding to four sets of X's, were considered to make possible the determination of the effects of the factors in question. The nature of the models will be discussed below.

4.2.1 Generating Artificial Populations and Samples. The samples were generated by adding normally distributed errors with mean zero and variance one to the equation

$$Y' = \alpha' + (X_3^{\gamma} - X_2^{\gamma})' - (X_3 - X_2)' \quad . \tag{4.10}$$

The same set of errors was used for each model. Twenty-five samples of size 12 were generated for each population. Three populations, corresponding to three values of γ, were considered for each simulation model. All samples were generated by the IBM 1410.

4.2.2 Nature of Simulation Models. The nature of the models used in the empirical investigations can be summarized as follows:

	Model 1	Model II	Model III	Model IV
Coefficient of variation of X_2/X_3, (%)	6.7	21.08	21.08	21.08
Range of X's:				
X_2	3.48	8.55	8.55	34.20
X_3	4.54	7.37	7.37	29.48
Size of γ	2, 4, 8	2, 4, 8	2, 4, 8	2, 4, 8
Coefficient of variation of Y', (%)				
$\gamma=2$	51.22	11.94	51.22	51.22
$\gamma=4$	17.95	11.94	17.95	17.95
$\gamma=8$	7.97	11.94	7.97	7.97
Values of α'				
$\gamma=2$	0.42766	6.98351	0.60206	0
$\gamma=4$	1.25265	4.56976	1.80618	0
$\gamma=8$	3.25183	0	4.21442	0

The X's for Model I were obtained from Appendix Table 1 by taking twenty-two-year lags of the prices from 1925 to 1958 and dividing these by ten to attain a small spread in the X's. The X's used for Models II and III are the same. These were obtained from Appendix Table 2 for moving three-year-periods from 1916 to 1929. The X's for Model IV were obtained by multiplying those used in Model III by four to attain a wide spread in the X's. Appendix Table 3 shows the X's for the four models.

For each model two estimates of γ were obtained; one by assuming α known and estimating γ only and the other by estimating both α and γ. In both cases the iterative estimation procedure was used to obtain the estimates. The initial values used for starting the iterations were the least squares estimates of α and γ from the equation

$$Y' = \alpha' + \gamma X_3' - (X_3 - X_2)' \quad . \tag{4.11}$$

4.2.3 Sampling Results. The estimates of γ obtained by the iterative procedure for the four simulation models are given in Appendix Tables 4 and 5. The initial estimates used for the iterations are shown in Appendix Table 6. A summary of the biases, sampling variances, asymptotic variances, and mean square errors of the estimates is given in Table 1. The sampling variances were obtained by calculating the variance of the 25 estimates of γ for each population in each model. The asymptotic variances were obtained using equation (4.2) with $\sigma^2 = 1$. Each mean square error is the sum of the square of the bias and the variance of an estimate.

Table 1. Means, biases, sampling and asymptotic variances and mean square errors of iterated estimates of γ for the four simulation models

	Population	Value of γ	Mean	Bias	Sampling variance	Asymptotic variance	Mean square error
Model I							
α known	1	2	2.40	0.40	0.0360	0.0397	0.1960
	2	4	4.64	0.64	0.0520	0.0446	0.4616
	3	8	9.22	1.22	0.0652	0.0459	1.5536
α unknown	1	2	4.48	2.48	71.9982	115.3800	78.1486
	2	4	6.80	2.80	96.7521	121.6249	104.5921
	3	8	11.55	3.55	114.3470	122.5941	126.9495
Model II							
α known	1	2	2.27	0.27	0.0563	0.0513	0.1292
	2	4	4.35	0.35	0.0758	0.0605	0.1984
	3	8	8.55	0.55	0.0842	0.0674	0.3867
α unknown	1	2	4.06	2.06	7.5329	6.7137	11.7765
	2	4	6.40	2.40	9.1506	8.5925	14.9106
	3	8	12.37	4.37	14.2693	9.6862	33.3662
Model III							
α known	1	2	2.27	0.27	0.0563	0.0513	0.1292
	2	4	4.35	0.35	0.0758	0.0605	0.1984
	3	8	8.55	0.55	0.0842	0.0674	0.3867
α unknown	1	2	4.06	2.06	7.8033	6.7137	12.0469
	2	4	7.10	3.10	13.1180	8.5925	22.7280
	3	8	12.57	4.57	14.6649	9.6862	35.5498
Model IV							
α known	1	2	1.87	-0.13	0.0189	0.0237	0.0358
	2	4	3.73	-0.27	0.0224	0.0265	0.0953
	3	8	7.41	-0.59	0.0200	0.0284	0.3681
α unknown	1	2	4.77	2.77	4.7247	6.7137	12.3976
	2	4	7.66	3.66	8.8852	8.5925	22.2808
	3	8	14.32	6.32	13.8131	9.6862	53.7555

We shall now look into the effects of the various factors in question on the efficiencies of the estimates obtained by the iterative estimation procedure. Since as could be noted in Table 1, the estimates for all models are highly biased we will utilize the mean square errors in comparing the relative efficiencies of the estimates. Hence from here on efficiency will denote the ratio of the mean square errors of two estimates of γ, multiplied by 100%.

4.2.3.1 Number of Parameters Estimated. The efficiencies of the estimates when γ was estimated simultaneously with α relative to the estimates when γ was estimated by assuming α known are summarized below for each parametric value of γ.

Size of γ	Model I	Model II	Model III	Model IV
2	0.25	1.10	1.07	0.28
4	0.44	1.33	0.87	0.43
8	1.22	1.16	0.97	0.07

It is readily seen that γ was much more efficiently estimated when α was assumed known.

4.2.3.2 Range of X's. The estimates for Model III (small range of X's) were found to be less efficient than the Model IV (wide range of X's) estimates when α was assumed constant. When γ was estimated with α the Model III estimates were as efficient as the Model IV estimates for $\gamma = 2$ and $\gamma = 4$, and slightly more efficient when $\gamma = 8$. The efficiencies of the Model III estimates relative to the Model IV estimates are shown below for each parameter value when α was known and when α was unknown.

	$\gamma = 2$	$\gamma = 4$	$\gamma = 8$
α known	27.71	48.03	95.22
α unknown	102.91	98.03	135.33

4.2.3.3 Coefficient of Variation of the Ratios X_2/X_3. The estimates when the coefficient of variation of the ratios X_2/X_3 was small (Model I) were found to be much less efficient than the estimates when the coefficient of variation of the ratios X_2/X_3 was large (Model III) whether α was known or unknown. The efficiencies of the Model I estimates relative to the Model III estimates were 65.91, 42.97 and 24.89 for $\gamma = 2, 4,$ and 8 respectively when α was known and 15.41, 21.73, 31.28 for $\gamma = 2$, $\gamma = 4$ and $\gamma = 8$ respectively, when α was unknown.

4.2.3.4 Size of γ and Coefficient of Variation of Y''s. It is apparent from the results of Model II in which the coefficient of variation of the Y''s was held constant for all parametric values of γ that the efficiencies of the estimates decreased monotonically as γ increased whether α was known or unknown. Except perhaps for sampling variation, the estimates became more efficient as the coefficient of variation of the Y''s decreased. This was noted from the results of Models II and III when α was unknown. These two models could not be compared to determine the effect of the coefficient of variation of the Y''s when α was known because the same estimates were obtained for the two models. This was expected since the same set of errors was used for all models (for convenience in programming the empirical studies) and when α was known the two models had the same Y''s.

4.3 Asymptotic Versus Small Sample Results

If the effects of the factors under consideration on the efficiency of the estimates from small samples are compared with the effects on the asymptotic estimates discussed in Section 4.1, one can easily see that there are no differences in the effects. We shall now determine the efficiencies of the small sample estimates relative to the asymptotic estimates.

A resumé of the efficiencies of the small sample estimates relative to the asymptotic estimates is shown below for all the four models.

	Model I		Model II		Model III		Model IV	
Value of γ	α known	α unknown	α known	α unknown	α known	α unknown	α known	α unknown
2	20.25	147.64	39.70	57.01	39.70	55.72	66.20	54.15
4	9.65	116.28	30.52	57.62	30.52	37.80	27.81	38.56
8	2.96	96.57	17.42	29.03	17.42	27.24	7.71	18.02

The efficiencies above were obtained by taking the ratios of the asymptotic variances of the estimates to their respective mean square errors. In general the small sample estimates were much less efficient. In Model I, however, when α was unknown, efficiencies of more than 100% were obtained. These, however, can be attributed to sampling variation. Chi-square tests showed that for these cases the small sample variances (and/or the mean square errors) of the estimates do not differ significantly from their asymptotic variances.

4.4 Other Results and Discussion

One of the results obtained in this study that would not be commonly expected to occur is concerned with the estimates for Model I when α was unknown. In this case, negative estimates of γ were obtained. Investigations to determine the cause of these results proved to be unsuccessful.

The variances of the estimates in Model I where negative estimates were obtained were calculated using the actual estimates. If the negative values were taken to be zero, both the variances and mean square errors of the estimates were not significantly reduced.

In determining the efficiencies of the small sample estimates relative to the asymptotic estimates it will be recalled that we utilized the mean square errors since the estimates were highly biased in all cases. However if one were to adjust the estimates for bias the efficiencies of the small sample estimates relative to the asymptotic estimates will be of respectable magnitude. This can be seen by comparing the sampling variances to their respective asymptotic variances. If each estimate was adjusted by adding or subtracting a constant (the expected bias) the variance of the adjusted estimates would remain the same as the variance of the unadjusted estimates. The efficiencies of the small sample estimates, adjusted for bias, relative to the asymptotic estimates are shown below.

	Model I		Model II		Model III		Model IV	
Value of γ	α known	α unknown	α known	α unknown	α known	α unknown	α known	α unknown
2	110.28	160.25	91.12	89.12	91.12	86.04	125.40	142.09
4	85.77	125.71	79.81	93.90	79.81	65.50	118.30	96.70
8	70.40	107.21	80.05	67.88	80.05	66.05	142.00	70.12

In all cases where the efficiencies were greater than 100% chi-square tests showed that the sampling variances did not differ significantly from their respective asymptotic variances. Hence the occurrence of the efficiencies greater than 100% can be attributed to sampling fluctuations.

4.5 Convergence and Initial Estimates

The iteration program for the IBM 1410 was written in such a way that if the estimates for two successive cycles were identically the same to at least three significant figures the iteration was stopped. By doing this the corrections to the estimates would be of the order 10^{-3} which for our purposes was sufficient. Rarely was it necessary to iterate more than four cycles. As might be expected more iterations were required to attain convergence for larger values of γ and when both α and γ were estimated. Convergence was attained in all cases.

As mentioned earlier, the initial estimates for starting the iterations were obtained using equation (4.11). To simplify the estimation program for the IBM 1410, the machine was instructed to use the same initial estimates of γ in estimating γ when α was unknown as those used when γ was estimated assuming α known.

CHAPTER V

SUMMARY AND CONCLUSIONS

In this paper we have considered the development of a mathematical model which relates the aggregate supply of a commodity to the price of the input used in producing the commodity and to the lower and upper prices of the commodity. Accordingly, the observed aggregate supply (Y) can be expressed as

$$Y = \alpha\, q^{-\beta}\, \frac{(X_3^{\gamma} - X_2^{\gamma})}{X_3 - X_2}$$

where α is a scaling factor, β is a production constant, $(\gamma-1)$ is a measure of the elasticity of supply, q is the price of the input, X_3 is the upper price of Y and X_2 is the lower price of Y.

Methods of estimating the parameters were investigated. An iterative procedure was developed. Methods of obtaining systematic initial estimates to use as starting values in the iteration were also developed. Problems involved in utilizing the iterative scheme in estimating the parameters were investigated. Empirical studies were conducted to determine the effects on the efficiency of the estimates obtained by the iterative scheme of: the spread in the lower and in the upper prices, the number of parameters estimated, the coefficient of variation of the ratios $\frac{X_2}{X_3}$ and the size of the parameter estimated. In the empirical investigations the model

$$Y = \alpha\, \frac{X_3^{\gamma} - X_2^{\gamma}}{X_3 - X_2}$$

was used instead of the full model. There was no loss of generality in using this in estimating the parameter of interest, γ.

Four simulation models were considered to determine the effects of the factors. Artificial populations and samples were generated for each model and used for the estimation of γ by the iterative procedure.

The effects of the various factors considered on the efficiency of the sampling estimates were found to be the same as the effects of the same factors on the asymptotic estimates. The effects on the asymptotic estimates were determined algebraically.

For both the asymptotic and small sample estimates it was found that the estimates of γ obtained by the iterative scheme were

(a) more efficient when γ was estimated by assuming α constant than when γ was estimated simultaneously with α,

(b) less efficient when the ranges of the prices were small when only γ was estimated and slightly more efficient when the ranges of the prices were large when both α and γ were estimated,

(c) more efficient when the coefficient of variation of the ratios X_2/X_3 was large, whether or not α was known,

(d) more efficient when the value of γ estimated was small whether or not α was known, and

(e) more efficient when the coefficient of variation of the Y''s was small.

The estimates obtained in the sampling studies were highly biased. The efficiencies of the small sample estimates relative to the asymptotic estimates were low in most cases. However when the estimates were adjusted for bias the small sample estimates were efficient. A few efficiencies greater than 100% were obtained but these were determined to be due to sampling variation.

Negative estimates of γ were obtained in one model when γ was estimated simultaneously with α . Investigations to determine the cause of this proved unsuccessful. Further investigation of this point may be worthwhile. A study of the mean square prediction errors of the model might shed light on this question. It would be of interest to determine how the mean square prediction errors with negative estimates compare with the mean square prediction errors when no negative estimates are obtained.

Since γ was much more efficiently estimated by assuming α constant than when γ was estimated simultaneously with α , it might be better to guess the value of α , hold it constant and estimate γ only, instead of iterating on both α and γ . It remains to be determined how far off the guess of α can be from its true value and still obtain efficient estimates. Collateral studies on this, using uniformly distributed errors with mean zero and variance 0.01, showed that efficient estimates were obtained when α was guessed to be one and one-half times its true value.

A study which considers more samples than were considered here would provide more stable estimates of the sampling variances and contribute to still more meaningful comparisons with the asymptotic properties of the estimates.

Finally, one might consider the full model and determine the effects of the factors discussed here, on the estimates obtained by the iterative procedure. Modifications on the iteration scheme may have to be taken into account to allow for negative estimates and accelerate convergence.

LIST OF REFERENCES

Anderson, R. L. and T. A. Bancroft. 1952. Statistical Theory in Research. McGraw-Hill Book Co., Inc., New York.

Cromarty, W. A. 1959. An econometric model for United States agriculture. J. Amer. Stat. Assoc. 54:556-574.

Fisher, I. 1925. Our stable dollar and the so-called business cycle. J. Amer. Stat. Assoc. 21:179.

Fisher, R. A. 1922. On the mathematical foundations of theoretical statistics. Phil. Trans. Roy. Soc., London, Ser. A. 222:309-368.

Girschick, M. A. and T. Haavelmo. 1953. Statistical analysis of the demand for food: Examples of simultaneous equation of structural equations. Studies in Econometric Method, (W. C. Hood and T. C. Koopmans, Eds.):92-111. John Wiley and Sons, Inc., New York.

Griliches, Z. 1961. A note on serial correlation bias in estimates of distributed lags. Econometrica 29:65-73.

Hartley, H. O. 1961. The modified Gauss-Newton method for the fitting of nonlinear regression functions by least squares. Technometrics 3:269-280.

Heady, E. O. and W. Candler. 1958. Linear Programming Methods. Iowa State College Press, Ames, Iowa.

Hildreth, C. 1957. Problems of uncertainty in farm planning. J. Farm Econ. 39:1430-1441.

Hildreth, C. and F. G. Jarrett. 1955. A Statistical Study of Livestock Production and Marketing. John Wiley and Sons, Inc., New York.

Kendall, M. G. and A. Stuart. 1961. Advanced Theory of Statistics. Vol. 2. Charles Griffin and Co., Ltd., London.

Klein, L. R. 1958. The estimation of distributed lags. Econometrica 26:553-565.

Knight, D. A. 1961. Evaluation of time series as data for estimating supply parameters. Agricultural Supply Functions (Heady et al., Eds.):74-104. Iowa State University Press, Ames, Iowa.

Knudson, A. C. and W. W. Cochrane. 1958. A supply function for flax at the firm level. J. Farm Econ. 40:117-123.

Koyck, L. M. 1954. Distributed Lags and Investment Analysis. North Holland Publishing Co., Amsterdam.

McPherson, W. W. and J. E. Faris. 1958. Price "mapping" of optimum changes in enterprises. J. Farm Econ. 40:821-834.

Nerlove, M. 1958a. Distributed lags and demand analysis. U. S. Dept. Agr. Handbook No. 141, U. S. Government Printing Office, Washington.

Nerlove, M. 1958b. The Dynamics of Supply: Estimation of Farmer's Response to Price. The Johns Hopkins Press, Baltimore, Md.

Nerlove, M. 1961. Time series analysis of the supply of agricultural products. Agricultural Supply Functions (Heady, et al., Eds.). Iowa State University Press, Ames, Iowa. P. 42.

Nerlove, M. and W. Addison. 1958. Statistical estimation of long-run elasticities of supply and demand. J. Farm Econ. 40:861-880.

Plaxico, J. 1958. Aggregation of supply concepts and firm supply functions, in Farm Size and Output Research: A Study in Research Methods. Southern Cooperative Ser. Bul. 56:76-94. Oklahoma Agr. Experiment Station, Stillwater, Oklahoma.

Scarborough, J. B. 1958. Numerical Mathematical Analysis. 4th Ed., The Johns Hopkins Press, Baltimore, Md.

Stevens, W. L. 1951. Asymptotic regression. Biometrics 7:240-267.

Theil, H. 1954. Linear Aggregation of Economic Relations. North Holland Publishing Co., Amsterdam.

Tintner, G. 1958. Scope and methods of econometrics: Illustrated by applications to American agriculture. Statistical and Social Inquiry Society of Ireland, 18:161-175.

Tompkin, J. R. 1958. Response of farm production unit as a whole to prices. J. Farm Econ. 40:115-128.

Toussaint, W. D. 1958. Programming optimum firm supply functions in Farm Size and Output Research: A Study in Research Methods, Southern Cooperative Ser. Bul. 56:62-72. Oklahoma Agr. Experiment Station, Stillwater, Oklahoma.

Wald, A. 1950. Statistical Decision Functions. John Wiley and Sons, Inc., New York.

Appendix Table 1. Average wholesale prices of Philippine sugar, for crop years 1925-1936 and 1947-1959

Crop year	Average wholesale price per ton (pesos)
1925	108.30
1926	102.13
1927	111.62
1928	112.72
1929	98.81
1930	126.32
1931	122.05
1932	106.08
1933	116.68
1934	102.92
1935	116.04
1936	133.59
1947	207.90
1948	196.20
1949	211.06
1950	223.55
1951	214.86
1952	225.29
1953	240.31
1954	235.72
1955	218.49
1956	220.49
1957	233.36
1958	241.58
1959	235.41

Source: National Economic Council, Manila

Appendix Table 2. Supply and price of rice in the Philippines, 1916 to 1941 and 1946 to 1955

Year	Supply (in sacks of 56 kilos clean rice)	Price of rice per sack (pesos)
1916	14,041,800	6.45
1917	17,045,650	6.50
1918	21,537,460	10.00
1919	18,142,130	15.75
1920	18,851,750	15.95
1921	22,221,810	15.00
1922	22,909,690	14.00
1923	23,517,390	14.00
1924	23,904,040	12.00
1925	25,094,140	10.00
1926	25,626,930	9.16
1927	25,456,660	7.15
1928	26,235,270	7.36
1929	27,273,890	8.58
1930	26,508,280	6.13
1931	25,533,730	4.58
1932	24,357,450	4.12
1933	24,757,910	4.67
1934	27,062,290	4.00
1935	23,462,970	5.01
1936	23,167,560	6.38
1937	29,354,070	5.43
1938	26,868,990	6.33
1939	28,147,980	6.33
1940	28,078,710	5.57
1941	27,806,670	5.74
1946	21,490,280	43.68
1947	25,694,110	27.69
1948	27,710,280	27.42
1949	31,369,240	26.99
1950	30,071,240	21.09
1951	32,287,080	25.49
1952	33,948,000	24.18
1953	36,362,920	18.72
1954	37,528,820	18.94
1955	38,339,140	20.17

Source: National Economic Council, Manila

Appendix Table 3. Lower and upper prices used in generating samples for the simulation models

Model I		Model II and Model III		Model IV	
Lower	Upper	Lower	Upper	Lower	Upper
10.83	20.79	6.45	10.00	25.80	40.00
10.21	19.62	6.50	15.75	26.00	63.00
11.16	21.11	10.00	15.00	40.00	60.00
11.27	22.36	15.00	15.95	60.00	63.80
9.88	21.49	14.00	15.95	56.00	63.80
12.63	22.53	14.00	15.00	56.00	60.00
12.20	24.03	12.00	14.00	48.00	56.00
10.61	23.57	10.00	14.00	40.00	56.00
11.67	21.85	9.16	12.00	36.64	48.00
10.29	22.06	7.15	12.00	28.60	48.00
11.60	23.34	7.15	9.16	28.60	36.64
13.36	24.16	7.15	8.58	28.60	34.32

Appendix Table 4. Iterated estimates of γ for Model I and Model II

Model I						Model II					
α known			α unknown			α known			α unknown		
Population number and value of γ						Population number and value of γ					
1(2)	2(4)	3(8)	1(2)	2(4)	3(8)	1(2)	2(4)	3(8)	1(2)	2(4)	3(8)
2.41	4.77	9.39	15.51	19.45	26.86	2.44	4.60	8.81	4.32	7.90	13.28
2.27	4.61	9.06	-3.45	-2.73	0.71	2.16	4.20	8.38	3.27	1.31	7.13
2.04	4.16	8.99	-4.26	-3.54	-2.19	1.91	3.96	8.11	8.64	12.00	17.57
2.11	4.23	8.93	-5.84	-5.11	-3.76	1.94	3.99	8.21	2.73	6.37	12.12
2.59	4.68	9.22	-0.02	0.86	9.65	2.38	4.46	8.70	2.62	5.95	11.52
2.26	4.59	9.02	-0.08	2.72	17.90	2.15	4.13	8.32	3.28	1.30	7.30
2.44	4.75	9.10	-0.25	0.85	7.62	2.18	4.30	8.47	1.59	3.71	9.92
2.94	5.09	9.84	11.80	15.25	22.09	2.80	4.91	9.12	2.13	5.43	11.07
2.27	4.40	9.02	-2.25	-1.52	5.62	2.02	4.04	8.24	5.72	9.50	15.77
2.57	4.65	9.22	7.98	9.50	13.93	2.39	4.46	8.70	6.07	9.26	14.66
2.52	4.87	9.56	13.47	17.06	20.82	2.54	4.65	8.88	6.80	10.22	15.70
2.16	4.29	8.91	11.56	18.45	25.58	1.98	4.04	8.22	2.77	5.75	11.50
2.34	4.67	9.21	-3.12	-2.39	-1.04	2.36	4.46	8.65	3.10	6.57	12.64
2.28	4.41	9.03	-1.30	-0.58	1.86	2.03	4.05	8.26	1.72	5.85	12.05
2.38	4.73	9.09	19.15	24.69	22.42	2.17	4.22	8.41	1.21	1.32	6.13
2.54	4.86	9.58	7.94	8.92	14.66	2.56	4.71	8.90	1.44	4.57	10.57
2.39	4.72	9.12	-2.88	-2.16	-0.81	2.17	4.21	8.38	12.86	7.88	22.45
2.30	4.56	9.04	-0.51	3.21	8.85	2.12	4.17	8.33	6.45	10.30	15.82
2.42	4.76	9.38	-0.22	0.72	8.04	2.43	4.58	8.81	3.89	7.54	12.58
2.43	4.76	9.40	-1.49	-0.76	0.34	2.45	4.58	8.81	4.77	8.01	13.75
2.22	4.35	8.94	-3.22	-2.50	-1.30	2.00	4.03	8.22	1.81	5.84	11.75
2.39	4.71	9.05	19.71	25.19	26.92	2.16	4.20	8.38	4.49	7.93	13.34
2.51	4.92	9.56	2.88	11.15	18.36	2.53	4.67	8.87	6.55	9.64	15.59
2.59	4.88	9.64	9.63	11.26	15.74	2.59	4.71	8.96	1.50	4.31	9.49
2.59	4.69	9.24	21.20	22.03	29.76	2.40	4.49	8.67	1.65	1.44	5.55

Appendix Table 5. Iterated estimates of γ for Model III and Model IV

Model III						Model IV					
α known			α unknown			α known			α unknown		
Population number and value of γ						Population number and value of γ					
1(2)	2(4)	3(8)	1(2)	2(4)	3(8)	1(2)	2(4)	3(8)	1(2)	2(4)	3(8)
2.44	4.60	8.81	4.40	7.87	13.77	1.96	3.84	7.50	5.88	8.92	15.31
2.16	4.20	8.38	3.25	1.19	8.31	1.80	3.64	7.33	1.78	3.53	9.25
1.91	3.96	8.11	7.89	11.75	18.89	1.63	3.50	7.16	7.89	12.09	20.06
1.94	3.99	8.21	2.44	6.12	13.42	1.66	3.53	7.20	3.33	7.15	14.18
2.38	4.46	8.70	2.44	6.28	13.26	1.96	3.79	7.47	4.02	7.62	14.14
2.15	4.13	8.32	1.29	1.09	8.39	1.80	3.64	7.33	9.66	2.68	9.25
2.18	4.30	8.47	6.41	4.16	10.54	1.85	3.68	7.41	2.37	5.34	11.22
2.80	4.91	9.12	2.10	5.01	12.25	2.17	4.06	7.75	3.80	6.62	12.95
2.02	4.04	8.24	5.69	9.45	16.76	1.72	3.60	7.28	5.53	9.32	17.00
2.39	4.46	8.70	6.07	9.28	15.76	1.96	3.78	7.47	5.85	9.42	16.42
2.54	4.65	8.88	6.40	10.24	16.53	2.00	3.91	7.56	6.17	10.33	17.23
1.98	4.04	8.22	1.88	5.27	12.23	1.68	3.54	7.22	3.28	6.94	13.21
2.36	4.46	8.65	2.22	6.89	13.86	1.94	3.79	7.47	3.81	7.47	14.92
2.03	4.05	8.26	1.82	5.36	13.41	1.72	3.60	7.28	3.06	6.32	13.58
2.17	4.22	8.41	1.21	11.79	7.27	1.85	3.67	7.37	2.93	4.70	9.35
2.56	4.71	8.90	1.74	4.56	11.54	2.01	3.92	7.56	3.66	6.10	12.63
2.17	4.21	8.38	13.14	16.87	2.36	1.84	3.67	7.37	9.77	16.67	24.44
2.12	4.17	8.33	6.52	9.91	16.94	1.78	3.62	7.32	6.94	9.94	18.30
2.43	4.58	8.81	3.89	7.55	13.97	1.97	3.83	7.50	4.61	8.16	15.01
2.45	4.58	8.81	4.45	8.10	14.99	1.96	3.84	7.50	4.71	8.37	15.57
2.00	4.03	8.22	1.97	5.42	12.57	1.71	3.57	7.25	3.20	6.34	13.53
2.16	4.20	8.38	4.90	7.96	13.78	1.84	3.66	7.36	5.81	8.30	14.68
2.53	4.67	8.87	6.17	9.59	16.83	2.00	3.91	7.56	5.95	9.99	17.17
2.59	4.71	8.96	1.47	4.25	10.22	2.03	3.94	7.58	3.68	5.80	10.99
2.40	4.49	8.67	1.65	1.67	6.39	1.96	3.80	7.48	1.61	3.33	7.64

Appendix Table 6. Initial estimates of γ used for the iterative estimation procedure

Model I			Models II and III			Model IV		
Population								
1	2	3	1	2	3	1	2	3
2.36	4.76	9.41	1.88	4.17	8.52	1.73	3.67	7.42
2.01	4.41	9.06	1.48	3.77	8.11	1.51	3.45	7.20
1.77	4.17	8.82	1.25	3.54	7.89	1.36	3.31	7.06
1.81	4.21	8.86	1.27	3.57	7.91	1.38	3.33	7.08
2.27	4.67	9.31	1.78	4.07	8.41	1.67	3.62	7.36
2.00	4.40	9.05	1.47	3.76	8.10	1.50	3.44	7.19
2.11	4.51	9.16	1.60	3.89	8.23	1.57	3.52	7.27
2.68	5.08	9.73	2.23	4.52	8.86	1.93	3.87	7.62
1.90	4.30	8.95	1.38	3.68	8.02	1.44	3.39	7.14
2.27	4.67	9.32	1.78	4.07	8.42	1.67	3.62	7.37
2.45	4.85	9.50	1.98	4.27	8.62	1.78	3.73	7.48
1.84	4.24	8.89	1.30	3.60	7.94	1.40	3.34	7.10
2.26	4.65	9.30	1.77	4.06	8.41	1.66	3.61	7.36
2.34	4.30	8.95	1.38	3.67	8.01	1.44	3.39	7.14
2.08	4.47	9.12	1.54	3.83	8.17	1.54	3.49	7.24
2.46	4.86	9.51	1.98	4.27	8.62	1.79	3.73	7.48
2.07	4.47	9.12	1.60	3.89	8.24	1.56	3.50	7.25
1.99	4.39	9.04	1.48	3.78	8.12	1.50	3.44	7.19
2.35	4.75	9.40	1.87	4.16	8.51	1.72	3.67	7.42
2.36	4.76	9.41	1.88	4.18	8.52	1.73	3.67	7.42
1.87	4.27	8.92	1.34	3.64	7.98	1.42	3.37	7.12
2.07	4.47	9.12	1.55	3.85	8.19	1.54	3.59	7.24
2.45	4.84	9.50	1.98	4.27	8.62	1.78	3.73	7.48
2.49	4.89	9.54	2.01	4.30	8.65	1.80	3.75	7.50
2.28	4.68	9.33	1.76	4.06	8.40	1.67	3.61	7.36

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