Abstract

IYER, PRASHANT. The Complexity of Traffic Grooming in Optical Path Networks with Egress Traffic. (Under the direction of Carla D. Savage.)

We consider the problem of minimizing network costs when grooming traffic in optical networks that support Wavelength Division Multiplexing (WDM). While the general problem has been shown to be NP-Hard for a number of cost measures, there still exist restricted problems for which no complexity bound is known. In this research, we restrict our attention to traffic grooming for path networks with egress (all-to-one) traffic. This restricted model has practical significance for high speed (optical) access networks and can also lead to better bounds and approximations on more general network topologies (such as ring and star networks) that can be decomposed into path networks. Three important cost measures for this restricted model are studied.

The first cost measure is the total number of ADMs used by the solution. Minimizing this cost was known to be NP-Complete even for egress traffic without using cross connects. We show that allowing an unbounded number of wavelengths obviates the need for digital cross connects at the nodes and hence the problem remains NP-Complete even when cross connects are allowed.

The second cost measure is the number of transceivers used by the solution. We show that the problem of minimizing the number of transceivers is NP-Complete, even when restricted to egress traffic. We then develop a simple approximation scheme where the transceiver cost exceeds the minimum by at most the number of required wavelengths. Finally, we show that under certain conditions, there exist solutions that simultaneously minimize both ADM and transceiver costs.

The third cost model aims to minimize the total electronic switching in the network. For this cost measure, we develop a polynomial time algorithm to determine the cost and structure of an optimum solution when the wavelength capacity constraint is relaxed. A closed form expression to determine the minimum cost is presented for problem instances with uniform traffic. We observe that these costs provide a lower bound on the cost of solutions to problems with finite capacity. In addition, the structure of the solution for infinite capacity wavelengths is used to obtain an upper bound for instances with finite capacity having uniform unit traffic. It is already known that the problem of minimizing this cost is NP-Complete for path networks with any-to-any traffic, even when a virtual topology is already specified. We show that for networks with egress traffic, given a virtual topology, there do exist polynomial time algorithms for minimizing this cost. Finally, we present an algorithm to minimize the cost when the number of wavelengths is fixed at two.

THE COMPLEXITY OF TRAFFIC GROOMING IN OPTICAL PATH NETWORKS WITH EGRESS TRAFFIC

by
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Biography

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Prashant worked as a systems engineer for a short period at Wipro Technologies before joining Texas Instruments, India, as a software designer, in June 2000. In the Fall of 2001, he left TI to start his MS program in computer science at NC State University.

He is currently engaged to Sowmya and looks forward to sharing many many long walks with her.

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Chapter 1

Introduction

Wavelength Division Multiplexing (WDM) allows an optical fiber to carry traffic on multiple channels by assigning to each channel a unique wavelength in which the corresponding traffic is transmitted. A WDM network consists of a set of *nodes* interconnected by optical fiber *links*. Each node is equipped with a Wavelength Add-Drop Multiplexer (WADM) that can start or terminate traffic on a subset of the wavelengths supported by the network. Traffic on the remaining wavelengths at a node is said to *pass-through* that node without being processed electronically. This allows the creation of a virtual topology where the nodes are connected by *lightpaths* that may span multiple physical links. Each such lightpath between two nodes will be assigned a fixed wavelength that will optically pass-through the WADMs at intermediate nodes. Lightpaths sharing a physical link will carry their traffic on distinct wavelengths. Traffic requests are thus satisfied by routing them on a sequence of lightpaths in the virtual topology.

Practical deployments have shown that the bandwidth available in each wavelength (e.g., OC-48, OC-96) far exceeds the typical requirement per traffic request (e.g., T3 or OC-3). Hence, multiple requests can be multiplexed onto wavelengths, leading to a better utilization of available bandwidth. This grooming of low speed traffic onto lightpaths is performed by wavelength specific Add-Drop Multiplexers (ADMs) located at the nodes. The effect of traffic grooming on network costs has been the topic of much study [1, 2, 3, 4, 5, 6].

The general problem of traffic grooming to reduce network costs has been shown to be NP-Complete for a number of cost models [2, 7]. We restrict our attention to the traffic grooming problem in unidirectional, WDM path networks with static requests to a single egress node. Path networks are a simple but important class of networks primarily because most of the commonly used topologies such as ring, star and tree networks can be easily decomposed into path networks. We can thus use solutions to path networks to obtain approximations and/or bounds on the optimum solutions for the more general topologies [2]. The additional restriction of all-to-one traffic to a single egress node is a valid assumption for high speed (optical) access networks. The study of this restricted model also yields a better understanding of some of the complexity inherent in the general traffic grooming problem.

1.1 Our Contribution

We study the problem of minimizing three different network cost measures for traffic grooming in path networks with egress (all-to-one) traffic. For each cost model, we analyze the inherent complexity of the problem and derive properties of the structure of their optimum solutions. Some of these properties are then used to devise algorithms and/or obtain bounds for the cost of an optimum solution.

The first cost measure is the total number of ADMs used by the solution. Minimizing this cost was known to be NP-Complete even for egress traffic provided the nodes do not route traffic using cross connects. We show that allowing an unbounded number of wavelengths obviates the need for digital cross connects at the nodes and hence the problem remains NP-Complete even when the use of cross connects is allowed.

The second cost measure is the number of transceivers used by the solution. We show that the problem of minimizing the number of transceivers is NP-Complete, even when restricted to egress traffic. We then develop a simple approximation scheme where the transceiver cost exceeds the minimum by, at most, the number of required wavelengths, and also show that under certain conditions, there exist solutions that simultaneously minimize both the ADM as well as the transceiver costs.

The third cost model aims to minimize the total electronic switching required in the network. For this cost measure, we develop a polynomial time algorithm to determine the cost and structure of an optimum solution when the wavelength capacity constraint is relaxed. A closed form expression to determine the minimum cost is presented for problem instances with uniform traffic. We observe that these costs provide a lower bound on the cost of solutions to problems with finite capacity. In addition, the structure of the solution for infinite capacity wavelengths is used to obtain an upper bound for instances with finite capacity having uniform unit traffic. It is already known that the problem of minimizing this cost is NP-Complete for path networks with any-to-any traffic, even when a virtual topology is already specified. We show that for networks with egress traffic, given a virtual topology, there do exist polynomial time algorithms for minimizing this cost. Finally, we present an algorithm to minimize the cost when the number of wavelengths is fixed at two.

1.2 Overview

In the next chapter, we formally define the traffic grooming problem for path networks and present three important measures for the cost of a solution to the problem. The next three chapters discuss each of these cost models in detail. Chapter 3 considers the problem of minimizing the number of ADMs required by the solution. In Chapter 4, we discuss the problem of minimizing the number of transceivers. Chapter 5 considers the problem of minimizing the total electronic switching cost in the solution. Finally, Chapter 6 formulates some of the open problems, presents future directions and concludes the thesis.

Chapter 2

Problem Definition

In this chapter, we define the problem of traffic grooming for path networks. We then present a restricted version of the problem that will be studied in depth in later chapters. Finally, three different measures for network cost are presented along with concrete examples that illustrate the difference between them.

2.1 The path grooming problem

We consider a path network consisting of N nodes numbered 1 to N. There are physical optical links connecting node i to node $i+1$ for $1 \leq i < N$. Each physical link is capable of carrying traffic in at most W wavelengths and each wavelength has a capacity C expressed in an appropriate unit for network traffic (e.g. OC-3, OC-12, T3). A *lightpath* between nodes i and j is the allocation of a fixed wavelength in the series of physical links between nodes i and j such that the intermediate nodes optically pass-through traffic on this wavelength. Lightpaths sharing a physical link are assigned distinct wavelengths. The number of lightpaths traversing a link will be referred to as its *link load*. These lightpaths terminate at WADMs located at the nodes. The traffic requests are specified by a matrix of integers $R = [r_{ij}]$, $1 \leq i < j \leq N$ where each r_{ij} represents the number of units of traffic requests from node i to node j. These requests are multiplexed onto lightpaths by means of wavelength specific Add-Drop Multiplexers (ADMs) located at the nodes. In addition, we assume that each node is capable of electronically switching the traffic requests from incoming lightpaths onto outgoing lightpaths (on potentially different wavelengths) by means of digital cross-connects (DXCs).

A set of lightpaths in a path network can be viewed as a *virtual topology* imposed on the path network to route requests with improved efficiency and cost. Such a virtual topology is said to be *feasible* if it is possible to route all the traffic requests on the lightpaths in the topology such that

i) the maximum link load does not exceed the number of available wavelengths and ii) the traffic in each lightpath does not exceed the capacity of a wavelength. The traffic grooming problem in path networks is to design a virtual topology and specify a routing that satisfies all the requests so as to minimize a given cost measure. The path grooming problem can thus be formalized as follows.

Given: The number of nodes N in the path network, the number of available wavelengths W, the capacity of each wavelength C, and the traffic request matrix $R = [r_{ij}]$, $1 \leq i < j \leq N$,

find: (i) a virtual topology specified as the matrix $B = [b_{ij}]$, where b_{ij} is the number of lightpaths between nodes i and j; and (ii) a routing given by the set of integers $d_{ij}^{(sd)}$ for $1 \leq i < j \leq N$ and $1 \leq s < d \leq N$, corresponding to the number of requests from s to d that are routed through a lightpath in b_{ij},

satisfying: the capacity constraint,

$$\sum_{1 \leq s < d \leq N} d_{ij}^{(sd)} \leq C \quad \text{for all } 1 \leq i < j \leq N$$

and the wavelength constraint,

$$\sum_{1 \leq i < k \leq j \leq N} b_{ij} \leq W \quad \text{for all } 1 < k \leq N.$$

Note that the virtual topology may have multiple lightpaths between the same two nodes. However, we do not distinguish between such lightpaths when specifying the routing for the requests as $d_{ij}^{(sd)}$. The traffic grooming problem as defined in [2] requires an additional phase - to assign wavelengths to the lightpaths in the virtual topology such that no two lightpaths that share a physical link have the same wavelength. However, for path networks, it is also noted in [2] that this can always be done in linear time. This phase is hence not included in the path grooming problem defined above. An Integer Linear Program formulation for the problem can be easily derived as in [8]. However, due to the generality of available ILP solvers, this approach remains infeasible even for moderately sized networks.

2.2 Restricted path grooming

The path grooming problem can be further simplified by assuming that all traffic is destined to a single egress node at one end of the path network, say node N. This implies that the request matrix R has $r_{ij} = 0$ for $j \neq N$, and hence the set of requests can be specified by $R = \{r_i \mid 1 \leq i < N\}$, where r_i corresponds to the number of requests from node i to the egress node N. This modified version will be referred to as the *restricted path grooming problem* in the following sections.

The restricted path grooming problem is particularly common in access networks and hence the results are directly applicable to such networks. In addition, studying this restricted model not only provides better understanding of traffic grooming in general, but may also prove useful as the foundation for heuristics and approximations to the more general case.

2.3 Network cost measures

A wide number of criteria have been proposed to measure the cost of a solution to the traffic grooming problem [9, 7, 2]. This paper concentrates on the following three cost measures.

1. Total number of ADMs used in the network [7].

2. Total number of transceivers used in the network [9].

3. Total amount of electronic switching performed in the network [2].

Cost 1 assumes that the network cost is dominated by Add-Drop Multiplexers (ADMs). Although every lightpath requires one ADM at each of its endpoints, ADMs may be shared by two lightpaths at a common endpoint, provided the two lightpaths are assigned the same wavelength. Thus, in general, the lightpaths need to be assigned wavelengths before this cost can be measured for a proposed virtual topology and routing. However, it may be possible to come up with lower bounds on the number of ADMs without specifying a complete wavelength assignment. In this paper, we let k_1 represent the ADM cost of the network, when such a cost measure is available. Chapter 3 discusses this cost model in detail.

Cost 2 assumes that the network cost is dominated by transceivers. A transceiver is similar to an ADM in that it is present at the nodes and terminates lightpaths. However, unlike an ADM, transceivers cannot be shared by multiple lightpaths. In other words, every lightpath requires exactly two transceivers, one at each endpoint. Minimizing transceivers is thus identical to minimizing the number of lightpaths used by the solution. In terms of the path grooming problem defined above, the number of lightpaths used is given by $k_2 = \sum_{i,j} b_{ij}$. This cost model is discussed in detail in Chapter 4.

Cost 3 measures the amount of electronic switching done by the nodes in the network. A node has to electronically process traffic carried on the lightpaths that terminate at the node. Each time a node forwards a request from an incoming lightpath to another outgoing lightpath, it has to perform electronic switching. This cost is thus reduced by minimizing the total number of lightpath hops taken by the requests on the virtual topology. In terms of the path grooming problem defined above, the cost k_3 corresponding to this model is given by $k_3 = \sum_{i,j,s,d} b_{ij}^{(sd)}$. A detailed discussion of this cost model is presented in Chapter 5.

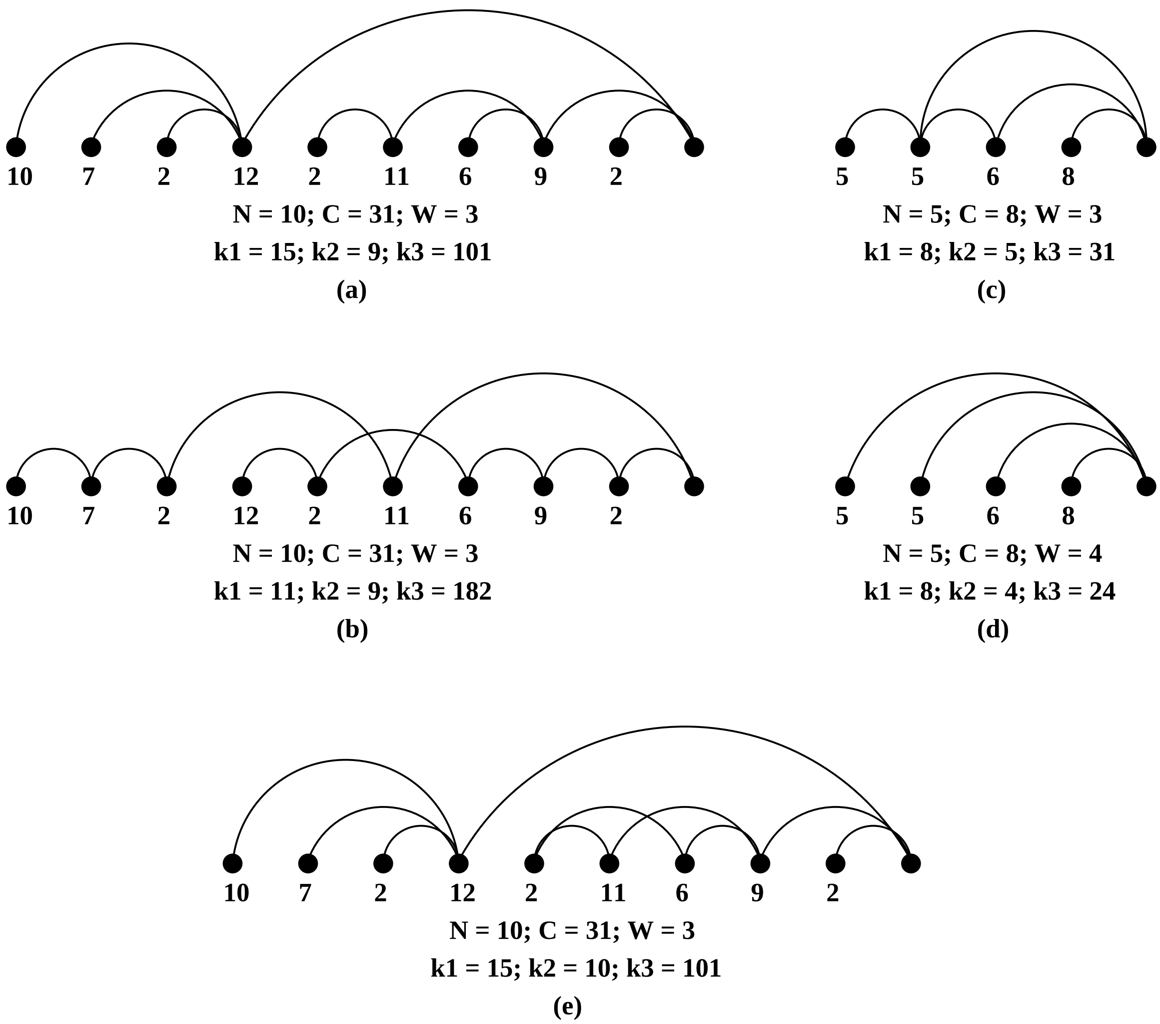

Figure 2.1: Various cost measures

Figure 2.1 shows two instances of the restricted path grooming problem. The nodes in the network are represented by dots and the number under each node corresponds to the number of requests from that node to the egress node. The nodes are assumed to be in sequential order, with node 1 at the leftmost end and the egress node, node N, at the rightmost end. The physical link between successive nodes is not shown. For each instance, we give examples of feasible virtual topologies, with the arcs representing the lightpaths created in the topology. For each topology, we indicate the minimum cost that can be achieved using the three cost models above. Note that a valid routing of requests from each node can be easily obtained for the given examples.

For the first problem instance (with $N = 10;\ C = 31;\ W = 3$), Figure 2.1a presents a solution that minimizes the total electronic switching (k_3-cost). Figure 2.1b shows another solution which requires the least number of ADMs (k_1-cost). Both of these solutions require the minimum number lightpaths (k_2-cost). For the second problem instance (with $N = 5;\ C = 8;\ W = 3$), traffic must be split at a node to achieve a valid solution. Such a solution is presented in Figure 2.1c. Figure 2.1d shows how the presence of an additional wavelength may be used to reduce the number of lightpaths.

These diagrams clearly bring out the differences between the cost models and how the design of the virtual topology affects the costs. In particular, Figure 2.1a shows that minimizing k_2 or k_3 may not minimize k_1; Figure 2.1b shows that minimizing k_1 or k_2 may not minimize k_3. Figures 2.1c and 2.1d show that minimizing k_1 may not minimize k_2 and Figure 2.1e shows that minimizing k_3 may not minimize k_2.

Another way to investigate the differences between the cost models is to determine if it is possible (and if so, under what conditions), to simultaneously minimize more than one cost measure. The problem instance in Figures 2.1a and 2.1b is an example for which costs k_1 and k_3 cannot be simultaneously minimized. In Section 4.4 we prove sufficiency conditions for simultaneously minimizing k_1 and k_2. The possibility of minimizing both k_2 and k_3 is still an open question.

The following chapters discuss each of these cost models in detail.

Chapter 3

Minimizing the Number of ADMs

In this chapter, we consider the problem of minimizing the number of ADMs (k_1-cost) in the path network. After presenting known results for this cost model in Section 3.1, we show, in Section 3.2, how wavelength assignment can be used to minimize ADM cost for a given topology and routing. Section 3.3 shows that digital cross connects cannot be used to reduce ADM costs in path networks when there is no bound on the number of available wavelengths. We conclude this Section by extending the NP-Completeness result (for minimizing k_1-cost) to the case where nodes are allowed to use cross connects.

3.1 Previous work

The problem of minimizing the ADM count in a network has been shown to be NP-Complete [7] even when restricted to path networks with an egress node. The network architecture and cost model used in [7] differs from our definition in the following two ways. First, the model in [7] does not place any resource bounds on the number of wavelengths available, whereas our model specifies an upper bound of W. Secondly, it is assumed that the nodes in the network are not equipped with Digital Cross-Connects (DXCs) and hence traffic requests cannot change wavelengths on their route to the egress node. In our problem definition, each node is equipped with DXCs, thus allowing the low rate traffic to change wavelengths by means of electronic switching.

In the absence of DXCs' and wavelength bounds, [7] shows that the problem of minimizing the number of ADMs is NP-Complete for path networks with egress node traffic. We briefly summarize the main points of this NP-Completeness proof. The optimization problem considered in [7] is to minimize the number of ADMs given an instance $\langle N, W = \infty, C, R = \{r_i \mid 0 < r_i \leq C\}\rangle$ of the restricted path grooming problem, under the assumption that no DXCs are used at the nodes. A related decision problem PATH-GROOMING-MIN-ADM is formulated as follows. Given an instance

$\langle N, W = \infty, C, R = \{r_i \mid 0 < r_i \leq C\}\rangle$, and a goal k_1, determine if there exists a solution to the restricted path grooming problem such that the number of ADMs required does not exceed k_1. Since the problem does not specify a limit on the number of wavelengths, it is argued in [7] that if $r_i \leq C$, for $1 \leq i < N$, there exists a solution with the optimum number of ADMs, that does not split the requests from a node onto more than one wavelength. In other words, there is optimum solution that uses exactly one ADM at each node i for $1 \leq i < N$, and as many ADMs at node N as there are lightpaths carrying traffic to it. We note that such a solution will always use $N - 1$ lightpaths (one from each node that has requests going to the egress node). Further, since requests are not allowed to change wavelengths, the number of wavelengths used in the solution is exactly the number of lightpaths terminating at the egress node. Thus, reducing the number of ADMs is equivalent to reducing the number of wavelengths used in the solution. This latter problem is, in turn, equivalent to the NP-Complete BIN-PACKING problem, with the number of bins corresponding to the number of wavelengths used. PATH-GROOMING-MIN-ADM is thus proved to be NP-Complete even if $r_i \leq C$.

The paper [7] also presents a polynomial time greedy algorithm that solves the problem exactly for path networks with egress nodes, when each node has the same number of requests to the egress node. For non-uniform traffic, [4] presents exact solutions when the capacity is fixed at 2, 4, and 8. It is also shown in [4] that any α-approximation for BIN-PACKING can be employed to obtain a solution to the restricted path grooming problem, that is within $(1 + \alpha)/2$ of the optimum. In all of these, it is assumed that $W = \infty$, and that the nodes do not have cross connect capability.

For instances that allow nodes to have cross connect capability, [10, 11, 12] discuss the problem of minimizing ADMs using the smallest number and size of DXCs on *ring networks*. Concrete examples in [10] show that cross connects do help reduce the number of ADMs in ring networks. [10] also derives lower bounds on the number of ADMs as a function of the number of nodes with DXCs and provides heuristic algorithms that try to reach this lower bound. The question of whether or not DXCs help reduce the ADM requirement in path networks (for both all-to-all and all-to-one traffic) has so far remained open. We provide partial answers in Section 3.3.

3.2 Wavelength assignment and ADM cost

We mentioned in Section 2.3 that in general, the k_1-cost of a solution to the path grooming problem can only be determined after assigning wavelengths to the lightpaths. This is because the number of ADMs at a node depends upon the number of different wavelengths in which lightpaths originate or terminate at that node and this number may not be the same as the number of lightpaths originating or terminating at that node. For path networks, it has been observed [8] that the problem of wavelength assignment for a virtual topology reduces to interval graph coloring, which can be solved using a greedy algorithm in linear time. Such an algorithm essentially iterates through the nodes such that for i from 1 to $(N-1)$, each lightpath originating from node i is assigned the least numbered unused wavelength currently available at that node. This version of the greedy algorithm guarantees that the total number of wavelengths used is at most the maximum number of lightpaths crossing a physical link. We now show, in Lemma 1 and Theorem 1, that by slightly modifying the wavelength selection strategy at each node, we can also achieve the lower bound on the number of ADMs required for the given virtual topology. Note that in the following argument, since the virtual topology is already specified, we do not make any assumptions regarding the use of DXCs or the presence of bounds on the number of wavelengths.

Given a virtual topology for a path network, let l_n and l'_n denote the number of lightpaths terminating and originating at node n respectively.

Lemma 1 *For path networks, given a virtual topology, it is always possible to assign wavelengths to the lightpaths such that the following conditions are simultaneously satisfied.*

1. *The total number of wavelengths used is at most the maximum number of lightpaths crossing any physical link.*
2. *The number of different wavelengths assigned to the lightpaths originating and terminating at a node n is at most max(l_n, l'_n).*

Proof. Consider the following modified version of the greedy algorithm for wavelength assignment described in the previous paragraph. For each node n, taken in sequence from node 1 to node $N-1$, we know that the wavelengths assigned to the l_n lightpaths terminating at that node are now available for re-assignment. When selecting a wavelength for one of the l'_n lightpaths originating from node n, we first select one such available wavelength that was used on a lightpath terminating at node n. When no such wavelength is available, we can start assigning any available wavelength to the remaining lightpaths. Thus, if $l_n \geq l'_n$, all lightpaths originating at node n will re-use the wavelengths assigned to lightpaths that terminate on that node - and the total number of wavelengths used at node n will be l_n. On the other hand, if $l_n < l'_n$, all wavelengths assigned to lightpaths that terminate on node n will be re-used on outgoing lightpaths - and the total number of wavelengths used at node n will be at most l'_n. Thus, the maximum number of wavelengths used at a node n is at most $\max(l_n, l'_n)$ and Statement 2 of the lemma is true.

This modification of the wavelength selection strategy clearly uses the same number of available wavelengths at each node as the previous strategy. Hence Statement 1 is still true of the modified algorithm and the theorem is proved. ■

Theorem 1 *Given a virtual topology for a path network, the following statements are true.*

1. $\sum_{n=1}^{N} max(l_n, l'_n)$ *is a lower bound on the number of ADMs required.*

2. *There exists an assignment of wavelengths to the lightpaths in the topology that achieves this lower bound on the number of ADMs.*

Proof. Given a virtual topology for a path network, we note that each of the l_n lightpaths terminating at node n traverses the physical link between nodes $(n-1)$ and n and hence must be on distinct (among the l_n lightpaths) wavelengths. Thus node n must be capable of electronically switching at least l_n wavelengths. Similarly, each of the l'_n lightpaths originating from node n traverses the physical link between nodes n and $(n+1)$ and hence must be on distinct wavelengths. Thus node n must also be capable of switching at least l'_n wavelengths. However, some lightpaths originating at node n re-use the wavelengths assigned to lightpaths that terminate at node n. We know that each node requires one ADM for each wavelength that it has to process electronically. Hence, the lower bound on the number of ADMs required at node n is $\max(l_n, l'_n)$. Summing this lower bound over all the nodes in the network, we get the lower bound on the total number of ADMs for the given virtual topology as $\sum_{n=1}^{N} \max(l_n, l'_n)$. Thus, statement 1 of the lemma is true.

From Lemma 1 we know that there exists a wavelength assignment such that for each node n, all the lightpaths that terminate and originate at node n can be assigned at most $\max(l_n, l'_n)$ wavelengths. The number of ADMs required by such an assignment achieves the lower bound as per statement 1. Thus, statement 2 is true and the lemma is proved. ■

3.3 Electronic switching and ADM cost

As we mentioned in an Section 3.1, while [10] shows that DXCs do help reduce ADM costs in ring networks, the question of whether they do so for path networks is still open. In this section we begin to analyze the effect of DXCs and finite wavelength resources on the number of ADMs required for the restricted path grooming problem. In Lemma 2, we show that when nodes are equipped with DXCs, even if there is a bound on the number of wavelengths, there exists a minimum ADM solution in which no node terminates more lightpaths that it originates. This lemma is used by Theorem 7 in Section 4.4 to compare the k_1 and k_2 costs of a solution to the restricted path grooming problem. Next, Lemma 3 shows that DXCs do not have an impact on the minimum ADM cost for restricted path grooming problems when $W = \infty$. However, we still leave open the question of whether DXCs can reduce the ADM costs in path networks with non-egress traffic or with egress traffic when a bound on the number of wavelengths is specified.

Lemma 2 *There exists a solution (using cross connects) to the restricted path grooming problem with a finite wavelength bound, that minimizes the number of ADMs such that the number of lightpaths terminating at node* $n, 1 \leq n < N$, *is at most the number of lightpaths originating at that node - i.e.,* $l_n \leq l'_n$.

Proof. Consider a solution that minimizes the number of ADMs with the property that $k_1 = \sum_{1 \leq i \leq N} \max(l_n, l'_n)$ (we know from Theorem 1 that such a solution exists) and let $n < N$ be the largest numbered node in the solution with $l_n > l'_n$.

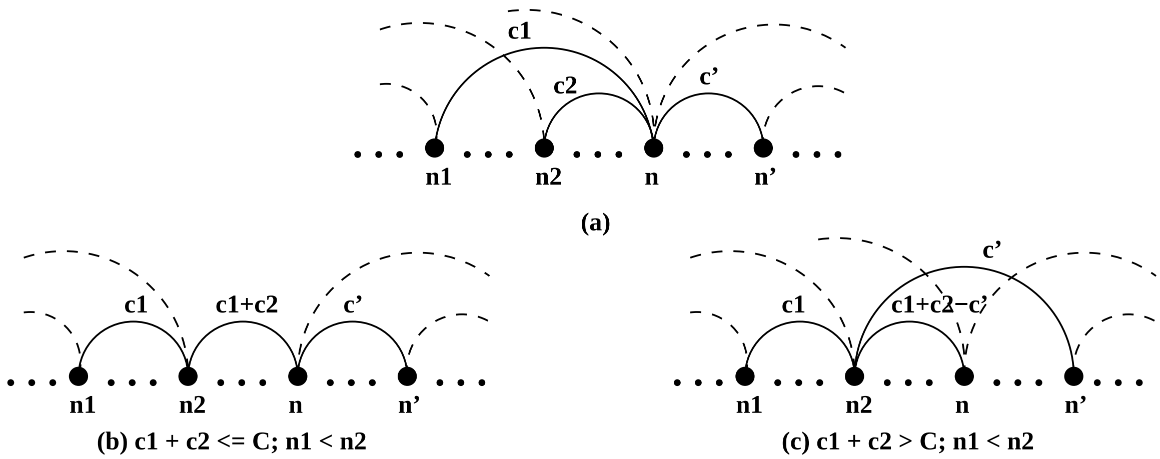

Figure 3.1: Transformations to make $l_n \leq l'_n$

In the routing specified by the solution, if there exist two lightpaths (n_1, n) and (n_2, n) with $n_1 \leq n_2$, carrying c_1 and c_2 units of traffic respectively, such that $c_1 + c_2 \leq C$, the virtual topology and the specified routing can be transformed to form a new solution as follows. If $n_1 = n_2$, one of the two lightpaths is deleted and all the $c_1 + c_2$ units of traffic is routed on the other. If $n_1 < n_2$ (see Figure (3.1b)), the lightpath (n_1, n) is replaced by a new lightpath (n_1, n_2). All the c_1 units of traffic from n_1 in the deleted lightpath is now routed first to node n_2 using the new lightpath, and then onto node n using lightpath (n_2, n). The c_2 units of traffic previously routed on lightpath (n_2, n) remains unchanged. This new solution clearly satisfies the wavelength and capacity constraints and also decreases the number of lightpaths terminating at node n by one. Since the original topology satisfied the condition $l_n > l'_n$, by Theorem 1, the new solution requires one less ADM at node n. In addition, it may need an extra ADM at node n_2, but the total ADM count will not increase and the number of lightpaths terminating at node n has decreased.

Otherwise, the routing specified by the solution is such that for every pair of lightpaths terminating at node n, the sum of the traffic on the two lightpaths exceeds the capacity C. In this case, let c_1, c_2 be the two smallest traffic on lightpaths $p_1 = (n_1, n)$ and $p_2 = (n_2, n)$, with $n_1 \leq n_2$, such that $c_1 \leq c_2$. Then, we have $c_1 + c_2 > C$. Since the incoming traffic at node n exceeds the capacity of a single wavelength, node n has at least two outgoing lightpaths i.e. $l'_n \geq 2$. Let c' be the largest traffic on these outgoing lightpaths and let one such lightpath (say p_3) with traffic c' terminate at node $n' > n$. Let c_{in} and c_{out} be the total incoming and outgoing traffic at node n. Since $c'l' \geq c_{out} \geq c_{in} \geq c_1 l$, and $l > l'$, we have $c' > c_1$ and $c_1 + c_2 - c' < c_2 \leq C$. We can now transform the virtual topology and routing to form a new solution as follows (see Figure (3.1c).) We first observe that since $c_1 + c_2 > C$, $c' \leq C$, $l'_n > 1$ and the nodes have DXCs, we can re-route traffic in the given topology such that the c' units of traffic in lightpath p_3 consists only of traffic coming into node n from n_1 or n_2. Such a re-routing clearly does not affect the k_1 cost of the solution. Now, in the new solution, lightpath p_1 is deleted and p_3 is replaced by a new lightpath $p'_3 = (n_2, n')$. If $n_1 < n_2$, a new lightpath $p'_1 = (n_1, n_2)$ is created and the c_1 units of traffic on p_1 is first routed to node n_2 through p'_1. At node n_2, the outgoing traffic $c_1 + c_2$ is split such that c' units are routed to node n' using p'_3 and the remaining $c_1 + c_2 - c'$ is routed to node n using lightpath p_2. The other requests in the network can be routed as before. This solution clearly satisfies all requests without violating the capacity and wavelength constraints. In addition, the new solution has one less lightpath terminating and originating at node n, thus decreasing both l_n and l'_n (and therefore $\max(l_n, l'_n)$). The corresponding decrease in the number of ADMs at node n (as per Theorem 1) may be offset by a requirement for an additional ADM at node n_2, but the total number of ADMs in the new solution does not increase. We also note that l_i and l'_i remain unchanged for nodes i, $n < i < N$.

The two transformations above may be repeatedly applied at node n until $l_n \leq l'_n$ is satisfied. Note that if $l'_n = 1$, the outgoing traffic at node n is at most C and hence the first condition will remain true until the number of incoming lightpaths is also reduced to 1. At this point, $l_i \leq l'_i$ is thus true for nodes i, $n \leq i < N$.

This set of transformations can now be repeated until, for every node $n < N$, we have $l_n \leq l'_n$. This completes the proof of the Lemma. ■

Lemma 3 *Given an instance of the restricted path grooming problem with $W = \infty$, there exists a minimum ADM solution such that for each non-egress node n, $l_n \leq l'_n$ and the traffic on a lightpath originating from node n is at least as much as the traffic on a lightpath assigned the same wavelength terminating at node n, if such a lightpath exists.*

Proof. From Lemma 2 we know that there exists a minimum ADM solution such that for every non-egress node n, $l_n \leq l'_n$. For such a solution, from Lemma 1, we know that there is a wavelength assignment such that for each lightpath terminating at a node $n < N$, there is a lightpath originating from that node with the same wavelength.

Consider one such minimum cost solution and wavelength assignment which does not also satisfy the conditions of this Lemma. Let n be the least numbered node for which there exist lightpaths $p_1 = (n_1, n)$ and $p_2 = (n, n_2)$ such that p_1 and p_2 are assigned the same wavelength (say λ), and in the specified routing, the traffic c_1 on p_1 is greater than the traffic c_2 on p_2. We now transform the solution so that node n no longer has such lightpaths.

First, we note that since $c_2 < c_1$, it is always possible to re-route traffic at node n, if required, such that all the requests on lightpath p_2 are from the c_1 units of traffic on p_1. Such a re-routing can be accomplished without changing the topology of the solution and without changing the total traffic on any lightpath, by exchanging the route of a non-c_1 request on p_2 with that of a c_1-request on another lightpath.

Next, since there are no bounds on the number of wavelengths available, we create a new lightpath $p_N = (n_1, N)$ (on a new wavelength, say λ_{new}) and route the c_1 units of traffic from node n_1, using this new lightpath, directly to the egress node. All lightpaths on wavelength λ terminating at node n_1 and earlier are re-assigned to the new wavelength λ_{new}. This step is required to preserve the number of ADMs at nodes $i \leq n_1$. The traffic on other lightpaths originating at node n will decrease only if they carry traffic from the c_1 units on p_1. This decrease however, does not affect the validity of the solution. Note that if the traffic on such a lightpath decreases to zero, depending on whether or not another lightpath terminates at node n on the same wavelength, this lightpath will either be deleted by a future iteration of this transformation at node n or will be completely redundant and can be removed immediately. However, the latter possibility implies the existence of a lower ADM cost solution and hence will lead to a contradiction. The resulting topology and routing are thus valid, satisfy all the requests, and maintain the $l_n \leq l'_n$ constraint. In addition, the number of wavelengths at node n in which traffic is reduced on account of switching, is decreased by one.

The effect of these changes on the cost of the solution is as follows. The new solution requires one more ADM at the egress node (due to the new wavelength), and at least one less ADM at node n due to the deletion of lightpaths p_1 and p_2. We also note that both l_n and l'_n are decreased by one, and node n has one less wavelength on which outgoing traffic is less than incoming traffic. Finally, we note that for nodes $i < n$, the condition of the Lemma continues to be satisfied.

This procedure can be applied repeatedly until we have the minimum ADM solution in which, at each node $n < N$, incoming traffic on a wavelength is at most equal to the outgoing traffic on that wavelength. Hence the Lemma is proved. ■

This Lemma can now be used to prove the following Theorem.

Theorem 2 *If $W = \infty$, digital cross connects do not help in reducing the ADM cost of a solution to the restricted path grooming problem.*

Proof. We only have to note that given a minimum ADM solution that satisfies Lemma 3, we can always change the specified routing so that all traffic coming into a node through lightpaths leave the node on the same wavelength. Thus, increase in traffic on any wavelength across a node will only be due to requests that originate from that node. As a result, the cross connect capability of a node is not employed. ■

A direct consequence of Theorem 2 is that the decision problem PATH-GROOMING-MIN-ADM, from [7], discussed previously in Section 3.1, can be reduced to a similar problem that, in addition, allows nodes to use DXCs. Since PATH-GROOMING-MIN-ADM is NP-Complete, we have the following Theorem.

Theorem 3 *The PATH-GROOMING-MIN-ADM problem remains NP-Complete even when the nodes are allowed to route traffic using Digital Cross Connects.* ■

Chapter 4

Minimizing the Number of Transceivers

In this chapter, we measure cost in terms of the number of transceivers used in the virtual topology of the proposed solution. This cost model and its justification have been discussed as cost model 2 in Section 2.3. Since every lightpath requires a distinct transceiver at each of its endpoints, minimizing the number of lightpaths results in minimizing the number of transceivers and hence the k_2-cost of the network.

In the following discussion, we continue to use the problem definition presented in Chapter 2. Specifically, nodes are assumed to be equipped with DXCs and an external bound is imposed on the number of wavelengths available. The restriction on the number of available wavelengths implies that there will be cases in which the optimum can be achieved only by splitting traffic at a node onto more than one lightpath (wavelength) even if $r_i \leq C$ for $1 \leq i < N$.

A heuristic algorithm to solve the traffic grooming problem in general networks under this cost model is presented in [9]. The paper [9] also compares the performance of the algorithm with optimum solutions for a variety of traffic patterns but the question of whether the problem was NP-Hard was left open.

4.1 Complexity results

We use an approach very similar to that of [7] described in Chapter 3, to prove that a decision version of the restricted path grooming problem is NP-Complete when minimizing the number of lightpaths. After determining a lower bound for the k_2-cost of a solution in Lemma 4, we define the decision problem PATH-GROOMING-MIN-LP and show, in Theorem 4, that it is NP-Complete.

Lemma 4 $N-1$ *is a lower bound on the number of lightpaths required to groom traffic from* N *nodes to an egress node, provided* $0 < r_i \leq C$*, for* $0 \leq i < N$*.*

Proof. If each of the $N-1$ nodes has $r_i > 0$ requests, each node i must have at least $\lceil r_i/C \rceil$ lightpaths originating from it in order to route all the requests from that node. Thus the total number of lightpaths must be at least $k_2' = \sum_{i=1}^{N-1} \lceil r_i/C \rceil$. If, in addition, $r_i \leq C$, $\lceil r_i/C \rceil = 1$ and hence the lower bound k_2' sums to $N-1$. ■

Using this lower bound, we formulate a decision problem PATH-GROOMING-MIN-LP as follows: Given N, C, W and $\{r_i\}$, determine if there exists a solution to the restricted path grooming using at most $k_2 = N-1$ lightpaths. This formulation is different from the decision problem PATH-GROOMING-MIN-ADM resulting in Chapter 3 in that W is finite and all nodes have full switching capability. However, when the target is $N-1$, and W is finite, we show that switching at a node cannot help. Nevertheless, we can use the same reduction from BIN-PACKING as in [7] to prove PATH-GROOMING-MIN-LP NP-Complete. This proof is presented in Theorem 4.

Theorem 4 *The PATH-GROOMING-MIN-LP decision problem is NP-Complete even if for each node $i < N$, $0 < r_i \leq C$.*

Proof. The problem is clearly in NP, since a candidate solution can be verified in polynomial time for correctness and in addition, its k_2-cost, i.e. the number of lightpaths used by the solution, can be verified to be less than or equal to $N-1$. We show that the problem is NP-Hard by reduction from the BIN-PACKING [13] problem.

The BIN-PACKING problem can be stated as follows. Given integers C', N', k', and a multi-set R of integers $r_i' > 0$, $0 \leq i < N'$, determine if there exists a partition of R into k' subsets such that the sum of the elements in each subset does not exceed C'.

An instance of the BIN-PACKING problem, specified by $\langle N', C', k', R = \{r_i'\}\rangle$, is transformed into an instance of the PATH-GROOMING-MIN-LP problem as follows. Let $N = N'$, $W = k'$, $C = C'$, and request $r_i = r_i'$. This transformation can clearly be done in time polynomial in N.

If the BIN-PACKING instance has a solution, then let S_j, $1 \leq j \leq k'$ be the corresponding partition of the set R. For each subset S_j, we order the elements in the subset by their index i in R. In the corresponding PATH-GROOMING-MIN-LP instance, we create a lightpath from node k to node l if r_k' and r_l' are consecutive elements in S_j. In addition, node i will have a lightpath to the egress node if r_i' is the last element in S_j. This completes the construction of the virtual topology for the PATH-GROOMING-MIN-LP instance. We now show that this topology is valid and has cost $k_2 = N-1$.

First, each node i, $1 \leq i < N$, has exactly one lightpath associated with it. Hence, the number of lightpaths used in the solution is exactly $N-1$.

Secondly, we observe that the capacity of the lightpaths terminating at the egress node is at most C, since the number of requests on these lightpaths is exactly the sum of the elements r_i' in some S_j

and hence will be at most $C' = C$. Each of the other lightpaths clearly carry less traffic and hence also satisfy the capacity constraint.

Finally, to see that the wavelength constraint of W is not exceeded, we note that each lightpath in the solution connects consecutive nodes in some S_j. Thus the set of lightpaths in the virtual topology forms a collection of *chains* leading to the egress node with one chain per subset S_j. If more than one lightpath traverses the same physical link, their endpoints must belong to different subsets and hence such lightpaths would be in different chains. Since there are at most $k' = W$ subsets, there are at most W chains and thus, at most W lightpaths crossing any given physical link.

To complete the NP-Completeness proof, we have to show that if the PATH-GROOMING-MIN-LP instance to which the problem was reduced has a solution, the BIN-PACKING problem also has a solution. We begin with the observation that at most W lightpaths can terminate at the egress node. All requests are thus routed to the egress node through these W lightpaths. Further, since there $N-1$ lightpaths, exactly one lightpath originates from each node and so, no r_i is split over more than one of these. We can thus partition the nodes into equivalence classes such that two nodes are equivalent if and only if their requests use the same final lightpath to arrive at the egress node. Each of the lightpaths can carry at most C units of traffic. This means the sum of the number of requests from all nodes in an equivalence class cannot exceed C. Finally, since $k' = W$ and $C' = C$, the equivalence classes form a partition of the set R using at most k' bins, with the sum of the elements in each bin not exceeding C'. ■

We note that the structure of the solution in the last paragraph above does not require the use of DXCs at any of the nodes. Further, the ADM cost of such a solution is given by $N - 1 + W$.

4.2 Upper bound

We now present a greedy approximation algorithm to solve the restricted path grooming problem. Let $W' = \lceil (\sum_{i=1}^{N-1} r_i)/C \rceil$ be the minimum number of wavelengths with capacity C required to obtain a valid solution that satisfies all requests. We will show that our algorithm produces a solution that exceeds the lower bound in Lemma 4 by no more than W'. This algorithm can hence be used to obtain an upper bound on the minimum number of lightpaths required to solve the restricted path grooming problem.

The procedure `PathGroomingMinLP` (Algorithm 1) takes as input an instance of the restricted path grooming problem specified by $\langle N, W, C, \mathbf{r} \rangle$. It initializes the available capacity λ_i on wavelength i to C for $1 \le i \le W$ and the number of unrouted requests r'_i from node i to r_i for $1 \le i < N$ and then calls `Assign` with $i = 1, j = 1, \mathbf{r}', \boldsymbol{\lambda}, N$ and W.

Algorithm 1 PathGroomingMinLP$(N, W, C, \mathbf{r})$

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for $i = 1$ to W **do** {Initialize $\boldsymbol{\lambda}$}
    $\lambda_i \leftarrow C$
for $i = 1$ to $N - 1$ **do** {Initialize $\mathbf{r}'$}
    $r'_i \leftarrow r_i$
Assign$(1, 1, \mathbf{r}', \boldsymbol{\lambda}, N, W)$ {See Algorithm 2}

Algorithm 2 Assign$(i, j, \mathbf{r}', \boldsymbol{\lambda}, N, W)$

1: **if** $i < N$ and $j \leq W$ **then**
2:     **if** $r'_i < \lambda_j$ **then**
3:         Lightpath$(i, i + 1)$
4:         $\lambda_j \leftarrow \lambda_j - r'_i; \quad r'_i \leftarrow 0$
5:         Assign$(i + 1, j, \mathbf{r}', \boldsymbol{\lambda}, N, W)$
6:     **else**
7:         Lightpath(i, N)
8:         $r'_i \leftarrow r'_i - \lambda_j; \quad \lambda_j \leftarrow 0$
9:         Assign$(i, j + 1, \mathbf{r}', \boldsymbol{\lambda}, N, W)$

The procedure `Assign` (Algorithm 2) is called with parameters $i < N, j \leq W, \mathbf{r}', \boldsymbol{\lambda}, N$ and W. This procedure recursively uses the capacity available in wavelengths $j \ldots W$ (as indicated by $\lambda_j \ldots \lambda_W$), to obtain a virtual topology and routing that satisfies r'_k requests from each node k for $i \leq k < N$. Each time `Assign` is called, one of the following actions will be occur.

1. If $r'_i < \lambda_j$ (Lines 2 - 5), a lightpath is created from node i to node $i + 1$ and it will carry all the r'_i requests (along with prior requests routed on wavelength j) to the node $i + 1$. All the r'_i requests are deemed routed and the available capacity λ_j is accordingly adjusted. `Assign` is then called recursively to solve for requests $r_{i+1}, \ldots, r_{N-1}$ on wavlengths $\lambda_j, \ldots \lambda_W$.

2. If $r'_i \geq \lambda_j$ (Lines 6 - 9), a lightpath is created from node i to the egress node N and this lightpath will carry λ_j requests from node i (along with prior requests routed on wavelength j) to the egress node. Node i will have $r'_i - \lambda_j$ requests unrouted and the capacity on wavelength j will be 0. `Assign` is then called recursively to solve for requests $r_i, \ldots, r_{N-1}$ on wavlengths $\lambda_{j+1}, \ldots \lambda_W$.

The procedure terminates when all requests are routed ($i = N$) or none of the wavelengths have available capacity ($j = W + 1$). In the former case, we have a valid virtual topology which satisfies all the requests and in the latter case we can conclude that the available capacity was not sufficient to route all requests.

We use the above algorithms to derive an upper bound on the number of lightpaths required to solve the restricted path grooming problem. Let $S_i = \sum_{i'=i}^{N-1} r'_{i'}$ be the sum of unrouted requests from nodes i to $N-1$ and $L_i = \sum_{i'=i}^{W} \lambda_{i'}$ be the total capacity available in wavelengths i to W. We can now prove the following result.

Lemma 5 *If $S_i \leq L_j$, then procedure `Assign` will route the remaining S_i requests using at most $N-i+W-j$ additional lightpaths.*

Proof. The lemma is proved by induction on i and j. For the base case, consider $i = N-1, j = W$. The condition $r'_{N-1} \leq \lambda_W$ implies that Case 1 (Lines 2-5) will be true. Hence one lightpath is created, followed by a call to `Assign` with $i = N$. We see from Line 1 that when $i = N$, the procedure exits without creating any lightpaths. Thus total number of additional lightpaths created is 1 and the Lemma holds true for this base case.

For the induction step, we assume that the Lemma is true for all $i' > i$ and $j' > j$. Consider a call to `Assign` with parameters $i < N$ and $j \leq W$. If Case 1 (Lines 2-5) holds, then one lightpath is created followed by a recursive call to `Assign` with parameters $i+1$ and j. Further, since λ_j is decremented by r'_i, the condition $S_i \leq L_j$ which was true at the start of the procedure now implies that $S_{i+1} \leq L_j$. Thus the inductive hypothesis can be applied and the total number of additional lightpaths created will not exceed $1 + (N-(i+1)+W-j) = N-i+W-j$.

Similarly, if Case 2 (Lines 6-9) holds, a lightpath is created followed by recursive call to `Assign` with parameters i and $j+1$. Since both r'_i is decremented by λ_j, the condition $S_i \leq L_j$ which was true at the start of the procedure now implies that $S_i \leq L_{j+1}$ and we can apply the inductive hypothesis and the number of additional lightpaths created will not exceed $1+(N-i+W-(j+1)) = N-i+W-j$. Thus the Lemma is true for $1 \leq i < N$ and $1 \leq j \leq W$. ■

The procedure `PathGroomingMinLP` solves the path grooming problem by calling `Assign` with $i = 1$ and $j = 1$. Applying Lemma 5, it is clear that if $\sum r_i \leq CW$, there is a solution that uses at most $N+W-2$ lightpaths. In addition, we note that procedure `Assign` completely uses the capacity available in wavelength i before routing requests on wavelength $i+1$. Hence the resulting solution (and its k_2-cost) would not change for different values of W provided $\sum r_i \leq CW$ remains true. Hence the k_2-cost of the solution given by `PathGroomingMinLP` will in fact not exceed $N + W' - 2$, where $W' = \lceil (\sum r_i)/C \rceil$ is the minimum number of wavelengths (each with capacity C) required to satisfy the given requests. This leads to the following upper bound on the optimal solution to the restricted path grooming problem.

Theorem 5 *There exists a solution to the restricted path grooming problem that uses at most $N + W' - 2$ lightpaths, where W' is the minimum number of wavelengths (each with capacity C) required to satisfy all the requests.* ■

4.3 Performance of the approximation algorithm

In this section, we first show that `PathGroomingMinLP` is a 2-approximation algorithm. Then, we look at some concrete examples that illustrate the behavior of the algorithm presented in the previous section. In particular, we present problem instances where the solution achieves the lower bound in Lemma 4 or the upper bound in Theorem 5, and we also study the behavior of the algorithm on uniform traffic - i.e. when each node has the same number (r) of requests to the egress node.

The following theorem provides an alternate interpretation of Theorem 5 on the worst case performance of algorithm `PathGroomingMinLP`.

Theorem 6 *Procedure `PathGroomingMinLP` is a 2-approximation algorithm for minimizing the number of lightpaths for the restricted path grooming problem.*

Proof. From the proof of Lemma 4 we know that $k_l = \sum\lceil r_i/C\rceil$ is a lower bound on the minimum number of lightpaths for a solution to the restricted path grooming problem. From Theorem 5, we know that the procedure `PathGroomingMinLP` gives a solution whose k_2-cost has the upper bound $k_u = N+W'-2$, where $W' = \lceil(\sum r_i)/C\rceil$. Thus, if k_o is the minimum k_2-cost for a problem instance, we have, $k_u = N-2+W' = N-2+\lceil(\sum r_i)\rceil/C \leq N-2+\sum\lceil r_i/C\rceil = N-2+k_l \leq N-2+k_o \leq 2k_o-1$. Thus, we get $k_u/k_o < 2$ and the theorem is proved. ■

We now consider some concrete examples illustrating the performance of `PathGroomingMinLP`. First, we note that the algorithm will achieve the lower bound as per Lemma 4 whenever the nodes

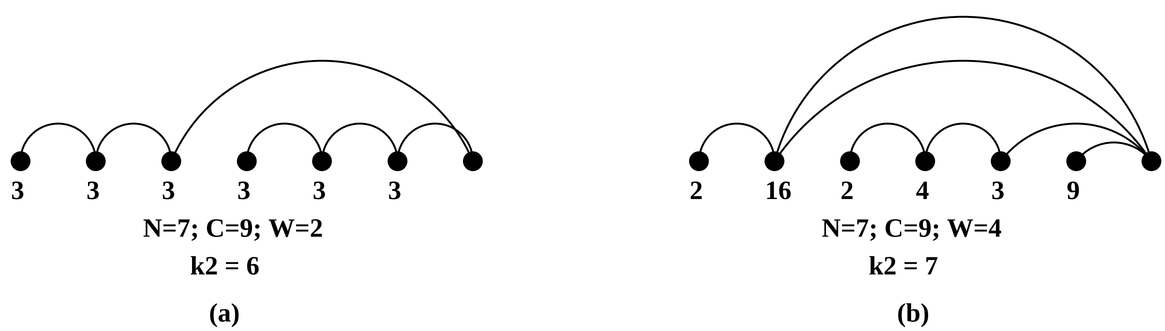

Figure 4.1: Best case inputs for `PathGroomingMinLP`

can be partitioned into sets such that each set contains consecutive nodes and the sum of the requests from nodes within a set is a multiple of the capacity C. This is illustrated in Figure 4.1(a) for uniform traffic and Figure 4.1(b) for non-uniform traffic (with some $r_i > C$).

Figure 4.2 presents a problem instance for which the algorithm performs as badly as possible, achieving the upper bound of Theorem 5. Figure 4.2(a) shows the solution given by algorithm `PathGroomingMinLP` with cost $k_2 = N + W - 2 = 9$ and Figure 4.2(b) shows a solution achieving

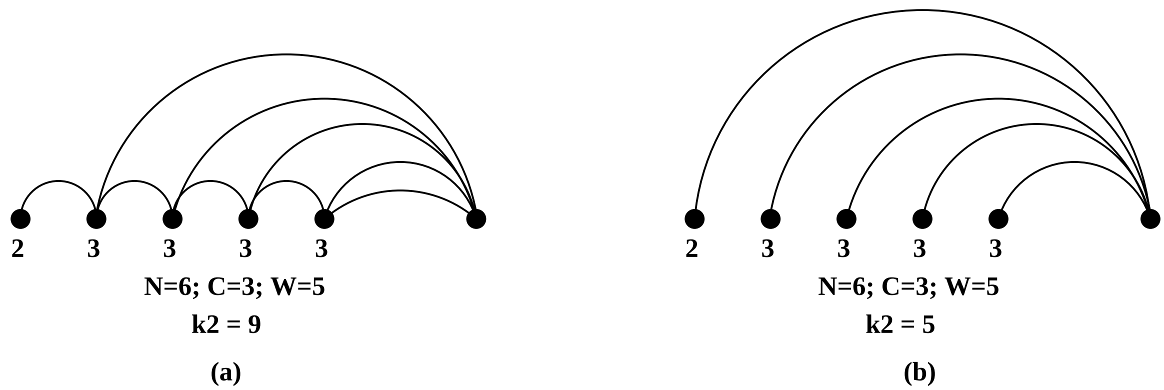

Figure 4.2: Worst case input for `PathGroomingMinLP`

the lower bound with cost $k_2 = N - 1 = 5$. This example can be generalized as follows. Given any $N > 1$ and $C > 1$, the path grooming problem specified by $\langle N, W = N-1, C, R = \{r_i\}\rangle$ such that $r_1 = C - 1$ and $r_i = C$ for $1 < i < N$ will have a solution with cost $k_2 = N - 1$ whereas the `PathGroomingMinLP` algorithm will yield a solution with cost $k_2 = N + W - 2$.

Figure 4.3: Sub-optimal input for `PathGroomingMinLP` with uniform traffic

For uniform traffic, the algorithm may or may not lead to an optimum solution. Figure 4.3 shows that even with minimum W, the algorithm does not always yield an optimum solution. This example can be generalized for any $r > 1$ as follows. Given any $r > 1$, the path grooming problem specified by $\langle N = 3 + C, W = 2 + r, C = 2r - 1, R = \{r_i = r\}\rangle$ will have a solution with cost $k_2 = 3r$ whereas the `PathGroomingMinLP` algorithm will yield a solution with cost $k_2 = 3r + 1$.

4.4 Comparing ADM and transceiver costs

From the examples presented in Section 2.3 we know that there are solutions (to the restricted path grooming problem) that minimize one of these costs but not the other. The NP-Completeness proof in Section 4.1 implies that some solutions that require only $N-1$ lightpaths also require only $N-1+W$ ADMs and thus achieve the lower bound for both costs. In this section we study some other conditions under which there exist solutions that minimize both the ADM and lightpath costs of the network.

Theorem 7 *If an instance of the restricted path grooming problem specified by $\langle N, W, C, R = \{r_i\}\rangle$ satisfies the condition $0 < r_i \leq C$ for $1 \leq i < N$, then, there exists a solution that simultaneously minimizes both $k1$ and $k2$.*

Proof. If an instance of the restricted path grooming problem satisfies the condition $0 < r_i \leq C$ for $1 \leq i < N$, then the number of lightpaths used in the solution i.e., the k_2 cost of a solution is given by $k_2 = (N-1) + s$, where s is the total number of traffic splits. Since $(N-1)$ is fixed for a particular instance, a solution minimizes k_2 if and only if it minimizes s.

Now we consider the number of ADMs used by a solution. In the following argument, we show that if $r_i \leq C$ for $1 \leq i < N$, there is an optimum solution whose k_1-cost is given by $k_1 = (N-1) + W + s$, where s is the number of traffic splits. Since N and W are fixed, this cost is optimum if and only if s is minimized. From the previous paragraph, we know that such a solution also minimizes the k_2-cost and is hence simultaneously optimum for both the cost models.

Recall that a solution to the restricted path grooming problem specifies both a virtual topology that satisfies all requests and a routing of the requests on the topology. Given the virtual topology specified by a minimum k_1-cost solution, let l_n and l'_n be the number of lightpaths terminating and originating at node n $(1 \leq n \leq N)$ respectively. From Theorem 1 and Lemma 2, we know that there is a minimum k_1-cost solution for which $l_n \leq l'_n$ is true for all non-egress nodes $n < N$. Such a solution requires exactly l'_n ADMs for each non-egress node n. The k_1-cost of this solution is given by $k_1 = (N-1) + s + w$, where w is the number of lightpaths terminating at the egress node and s is the number of traffic splits ($s = \sum_{n=1}^{N-1}(l'_n - 1)$).

For the optimum solution satisfying these properties, if $w < W$ and $s > 0$, we show that it is always possible to reduce s by creating a new lightpath terminating on the egress node. Let n be the largest numbered non-egress node with $l'_n > 1$.

We first show that no physical link across nodes $i \geq n$ uses all W wavelengths. Since n is the largest node with $l'_n > 1$, all non-egress nodes $i > n$ have $l'_i = 1$. If the link between nodes $i-1$ and i (where $n < i < N$) has W lightpaths, it follows that the link between i and $i+1$ also has W lightpaths because exactly one of these lightpaths must terminate at node i (note that $l_i \leq l'_i = 1$) and exactly one lightpath must originate from it. This however implies (by induction) that W lightpaths cross the link between $N-1$ and N contradicting our assumption that $w < W$ lightpaths terminate at the egress node. Hence, no physical link across nodes $i \geq n$ uses all W wavelengths.

Now, consider a wavelength assignment that achieves the optimal k_1 value specified above. If, in this wavelength assignment, the traffic on wavelengths that terminate at node n does not decrease (due to electronic switching), we can route all the requests originating from node n (r_n), directly to the egress node on a new wavelength. All other lightpaths originating from this node will now have to carry exactly as much traffic as the corresponding lightpaths terminating at the node on the same wavelength. (If no other lightpath terminates at the node on the same wavelength, the originating lightpath is clearly redundant and can be deleted.) Since there is no change in the total traffic on each of these wavelengths, each lightpath pair (n_1, n) and (n, n_2) on the same wavelength can be replaced by a single lightpath (n_1, n_2) on that wavelength. As a result, node n will now require just one ADM (as opposed to the previous $l'_n > 1$ ADMs) and the egress node will require an additional ADM for the new wavelength. Thus, the total k_1 cost is not increased.

On the other hand, if the wavelength assignment is such that traffic on a wavelength (say λ) that terminates at node n is decreased due to electronic switching, we can use the transformation described in Lemma 3 to delete the two lightpaths that terminate λ at node n (and thus decrease both l_n and l'_n by at least one), and use the new wavelength to create a lightpath that terminates at the egress node. This new solution reduces s by one, increases w by one, preserves the condition $l_n \leq l'_n$, and maintains the k_1 cost.

This procedure can be repeated until either $s = 0$ or $w = W$. Thus, the k_1-cost can be minimized by using all W ADMs at the egress node and then minimizing the number of splits s. However, as we noted earlier, a solution that minimizes s also achieves the optimum k_2 cost. We conclude that the theorem is true. ∎

For $r_i > C$, we do not as yet have a proof that there exist solutions that simultaneously minimize both k_1 and k_2.

Chapter 5

Minimizing the Electronic Switching Cost

In this chapter, we consider the dominant cost of the network to be the total amount of electronic switching in the network. As mentioned in Section 2.3, electronic switching is required at a node for each incoming request that has to be re-transmitted on an outgoing lightpath. Hence, the total electronic switching cost of a solution can be measured (as first presented in [8]) by the sum of the number of lightpath changes in the virtual topology route assigned to each traffic request. In the following discussion, we will use a slightly different measure for the total electronic switching cost. We define the cost k_3 as the sum of number of lightpaths used in the virtual topology route assigned to each request. Note that since the number of lightpaths in a route is always one more than the number of lightpath changes in the route, the difference in the two cost measures is just the total number of traffic requests and minimizing the latter cost will automatically minimize the former.

It is proved in [14] that the problem of minimizing total electronic costs in path networks is NP-complete for any-to-any traffic. In addition, [14] shows that it is in fact NP-Complete to determine whether a valid routing that satisfies all the requests and meets the capacity constraints exists when the virtual topology in the path network is fixed. In this chapter, we prove complexity results and present algorithms for the problem of minimizing k_3 for path networks with egress node (i.e., all requests terminate at the single egress node). Specifically, in Section 5.1, we show that, in contrast to the case for general path grooming, given a virtual topology, we can find a minimum k_3-cost routing for the requests in polynomial time. Then, in Section 5.2 we present a polynomial time dynamic programming algorithm that solves the path grooming problem when the wavelength capacity is not bounded. In Section 5.3, we discuss how the results for infinite capacity can be used to derive upper bounds on the optimum solutions for unit traffic with finite capacity and also present an algorithm for non-uniform requests when the number of available wavelengths is restricted to two.

5.1 Complexity results

In [14], it is proved that the problem of minimizing k_3 for a path network is NP-Complete for any-to-any traffic even when a virtual topology is already specified. We now show that the restricted path grooming problem (with egress traffic) reduces to the problem of minimizing cost in a weighted network flow problem. The latter can be solved in polynomial time by a number of standard algorithms. The NP-Completeness of restricted path grooming when no virtual topology is specified, is however, still an open question.

The *minimum cost flow problem* is defined in [15] as follows. Let $G = (V, E)$ be a directed network defined by a set V of n *vertices* and a set E of m *directed edges*. Each edge $(i, j) \in E$ has an associated *cost* $c_{i,j}$ that denotes the cost per unit flow on that edge. This cost varies linearly with the amount of flow. Each edge $(i, j) \in E$ is also associated with a *capacity* u_{ij} that denotes the maximum amount that can flow on that edge and a *lower bound* l_{ij} that denotes the minimum amount that must flow on the edge. Each vertex $i \in V$ is associated with an integer number b_i representing its supply/demand. The decision variables in the minimum cost flow problem are edge flows and the flow on edge $(i, j) \in E$ is represented by x_{ij}. The optimization model for the minimum cost flow problem can now be formulated as follows. Minimize

$$\sum_{(i,j)\in E} c_{ij} x_{ij}$$

subject to

$$\sum_{\{j:(i,j)\in E\}} x_{ij} - \sum_{\{j:(j,i)\in E\}} x_{ji} = b_i \text{ for all } i \in V,$$

$$l_{ij} \leq x_{ij} \leq u_{ij} \text{ for all } (i,j) \in E,$$

where $\sum_{i=1}^{n} b_i = 0$.

Theorem 8 *When a virtual topology is already specified, the k_3-cost of a restricted path grooming problem can be minimized in polynomial time.*

Proof. Given a virtual topology for an instance of the restricted path grooming problem, the complete solution requires a routing of the requests using the given topology. In order to find a minimum k_3-cost routing in polynomial time, we find a minimum cost flow in the following network flow problem.

We construct a network $G = (V, E)$ where V is the set of vertices and E is the set of directed edges in the network. Let $l(e)$ and $u(e)$ be the lower and upper limits of the flow on edge $e \in E$. Let $c(e)$ be the cost per unit flow in edge e. Finally, let $b(v)$ be the supply/demand at vertex $v \in V$.

The network is constructed from the given restricted path grooming instance and virtual topology as follows.

$$V = \{v_i \mid 1 \leq i \leq N\}$$
$$E = \{(i,j) \mid (i,j) \text{ is a lightpath in the virtual topology}\}$$
$$l(e) = 0,\ u(e) = C,\ c(e) = 1\ \forall e \in E$$
$$b(v_i) = r_i,\ b(v_N) = -\sum r_i\ 1 \leq i < N$$

Let x_{ij} be the flow on each edge $(i,j) \in E$. The corresponding network flow problem is thus to minimize $\sum_{(i,j)\in E} x_{ij}$ such that

$$\forall j\ \sum_i x_{ij} - \sum_i x_{ji} = r_j$$
$$\forall i\ \forall j\ (i,j) \in E \Rightarrow x_{ij} \leq C$$

A minimum cost solution to the network flow problem can be directly mapped to a routing for the requests in the restricted path grooming problem. Note that the upper bound C on the edges of network ensure that the capacity of the lightpaths in the virtual topology is not exceeded. In addition, since the flow satisfies the supply/demand at each vertex, all the requests are guaranteed to be routed to the egress node in the corresponding routing for the virtual topology. Finally, assigning a cost of one per unit flow for the edges in the network flow problem ensures that the sum of the flows on each edge is minimized. This, in turn, implies that the sum of the number of lightpaths used by each request (i.e. cost k_3) is minimized.

To conclude the proof, we only have to note that [15] presents a number of polynomial time algorithms to solve the minimum cost flow problem including the *enhanced capacity scaling algorithm* (Section 10.7 in [15]) which runs in time $O(m\log(n)(m+n\log(n))$, where n is the number of vertices in G and m is the number of edges. Briefly, this algorithm iteratively looks for augmenting paths in residual networks to increase the flow until all the supply/demands have been satisfied. Capacity scaling is used in multiple phases to provide guarantees on the minimum amount of flow increase per iteration. "Enhanced" capacity scaling uses additional tricks, including a careful choice of the scaling factor in each phase. ■

5.2 Wavelengths with unlimited capacity

In this section, we consider traffic grooming for path networks with single egress node under the assumption that wavelengths have infinite capacity.

5.2.1 Motivation

The assumption of unbounded capacity might not be true in practical systems. However, an analysis of this case will result in a better understanding of the nature of traffic grooming for path networks. Specifically, solutions to instances with unbounded capacity provide a lower bound on the k_3-cost of solutions to the corresponding instances with a finite capacity. In addition, polynomial time algorithms for problems with unbounded capacity might prove useful in deriving exact or approximate solutions and bounds for problems with finite capacity.

5.2.2 Structure of optimal solutions

In order to obtain an optimal sub-structure for the dynamic programming solution, we note the following properties for any instance of the restricted path grooming problem with infinite capacity.

First, since each node is assumed to have at least one request to the egress node, each node will have at least one lightpath originating from it. Further, unbounded wavelength capacity implies that given a valid virtual topology, requests can always be satisfied by routing on the shortest path (in the virtual topology) from the source node to the egress node, thus minimizing cost k_3. Hence, the problem reduces to designing a topology that minimizes the sum of the shortest path lengths for each request to the egress node, under the assumption that the routing will employ these paths.

Traffic at a node is said to be *split* if the requests from that node are routed on multiple lightpaths originating from that node. In general, traffic splitting may be necessary to obtain a solution that satisfies all constraints. However, if the wavelength capacity is unlimited, we have the following lemma.

Lemma 6 *If the wavelength capacity is unlimited, there exists a minimum k_3-cost solution to the path grooming problem for networks with a single egress node, that does not split traffic.*

Proof. Consider an optimum solution to an instance of the restricted path grooming problem. If the solution splits traffic at some node, let node i be the one closest to the egress node and assume its traffic is split onto $k > 1$ lightpaths. Let these lightpaths terminate at nodes $i_1, i_2, \ldots, i_k$ and let i_j be one of these nodes that has the shortest additional number of hops to the egress node. Since there is no limit on the capacity of each lightpath, all traffic at node i (including the incoming traffic on lightpaths that terminate at node i), can be routed on lightpath (i, i_j) and from node i_j, follow the shortest path to the egress node. Such a routing clearly does not split traffic at any node k for $i \leq k < N$. It also satisfies the wavelength constraint since no new lightpaths have been created. In addition, this change in routing does not increase the k_3-cost of the solution because of the way node i_j was selected. This process can be continued until we arrive at an optimum solution that does not split traffic. ■

Two lightpaths (i, j) and (i', j') with $i < i'$ in a virtual topology are said to *intersect* if and only if $i < i' < j < j'$. The following Lemma shows that intersecting lightpaths do not help to reduce the amount of electronic switching if the wavelengths have unlimited capacity.

Lemma 7 *If the wavelengths have unlimited capacity, there exists a minimum k_3-cost solution to the restricted path grooming problem that does not split traffic at any node and does not have intersecting lightpaths.*

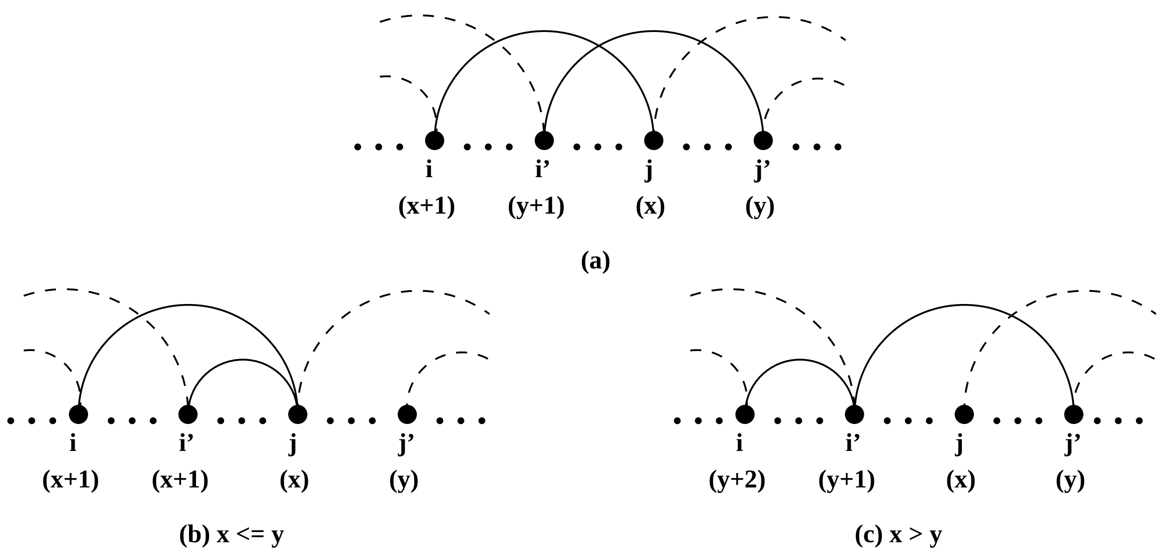

Figure 5.1: Intersecting lightpaths

Proof. Consider an optimum solution to an instance of the restricted path grooming problem with unlimited capacity. By Lemma 6, we can assume that the solution does not split traffic at any node. Let (i, j) and (i', j') be any two intersecting lightpaths in the solution such that $i < i' < j < j'$ as shown in Figure 5.1(a). Let r'_i and $r'_{i'}$ be the total number of outgoing requests at node i and i' respectively. Let x and y be the minimum additional number of lightpaths to the egress node from nodes j and j' respectively. Thus, the contribution of r'_i and $r'_{i'}$ to the cost of the solution is $k = r'_i(x + 1) + r_{i'}(y + 1)$.

Now, if $x \leq y$, we can adjust the virtual topology as shown in Figure 5.1(b) by changing lightpath (i', j') to (i', j) so that all the $r'_{i'}$ requests are routed on the sequence of lightpaths from node j to the egress node. Such a transformation is valid since capacity is unlimited and the new lightpath occupies the wavelength of the deleted lightpath between nodes i' and j. The contribution of the requests r'_i and $r'_{i'}$ to the cost of the transformed solution is given by $k' = r'_i(x + 1) + r'_{i'}(x + 1)$. Since $x \leq y$, we have $k' \leq k$. It is also clear that the remaining cost of the solution is unaffected by this transformation.

On the other hand, if $x > y$, we can adjust the virtual topology as shown in Figure 5.1(c) by changing lightpath (i, j) to (i, i'). Such a transformation is also valid and the resulting contribution to the cost of the transformed solution is given by $k'' = r'_i(y+2) + r'_{i'}(y+1)$. Since $x \geq y+1$, we have $k'' \leq k$.

Both of these changes essentially reduce the span of a lightpath so as to eliminate the intersection. Hence, no new intersections are introduced, resulting in a decrease in the total number of intersections. We also note that the transformation does not introduce any traffic splits. Repeating the process until no more lightpaths intersect will thus produce an optimum solution that does not have intersecting lightpaths and does not split traffic at any node. ■

5.2.3 The dynamic programming algorithm

We now present a recursive definition for the minimum k_3-cost of a solution to the restricted path grooming problem when the wavelength capacity is unbounded. This definition is then applied in Algorithm 3 to develop a polynomial time dynamic programming solution.

Consider an instance $\langle N, W, C = \infty, R = \{r_i\}\rangle$ of the restricted path grooming problem. Let $C_{i,j,w}$ with $i \leq j$ and $w \leq W$, represent the minimum k_3-cost of a solution that satisfies only the traffic requests from nodes $i \ldots j$ using at most w wavelengths (each having unlimited capacity). The minimum k_3-cost for a solution that satisfies the requests from all the nodes is thus represented as $C_{1,N-1,W}$. We now have the following theorem.

Theorem 9 *The minimum k_3-cost of a solution to the restricted path grooming problem with unlimited wavelength capacity that satisfies only the requests from node i up to node j is given by:*

$$C_{i,j,w} = \begin{cases} 0 & \textit{if} \quad j - i < 0 \\ \sum_{m=i}^{j} r_m & \textit{if} \quad 0 \leq j - i < w; \\ \sum_{m=i}^{j} r_m * (j - m + 1) & \textit{if} \quad w = 1; \\ \min_{k=i}^{j} C_{i,k-1,w} + S_{i,k} + C_{k+1,j,w-1} & \textit{otherwise;} \end{cases}$$

Proof. The theorem is proved by induction on $j - i$ and w.

For the base cases, if $j - i < w$, each of the nodes can have a direct lightpath to the egress node and hence, the cost is just the sum $S_{i,j} = \sum_{m=i}^{j} r_m$ of the requests from each node. If $w = 1$, there is only one possible solution - to have lightpaths connecting consecutive nodes $i, (i+1), \ldots, j$ in a chain followed by a lightpath from j to the egress node - and the k_3-cost of this solution is $S'_{i,j} = \sum_{m=i}^{j} r_m * (j - m + 1)$.

For the induction step we note that by Lemma 7, there exists an optimum solution that does not split traffic and has no intersecting lightpaths. For such a solution, let k be the least numbered node with a direct lightpath to the egress node. We can now divide the problem into two parts that can be solved independently. One of these satisfies requests from nodes i to $k-1$ using at most w wavelengths, routing them to node k first and then to the egress node through the direct lightpath from node k to node N. The second subproblem satisfies the requests from nodes $k+1$ to j using at most $w-1$ wavelengths. By the inductive hypothesis, the minimum k_3-cost for each of these subproblems is given by $C_{i,k-1,w}$ and $C_{k+1,j,w-1}$ respectively. When these two solutions are combined, requests from nodes i to k now have an additional lightpath to traverse - the direct lightpath from node k to the egress node - and hence the total cost of the solution will increase by one for each such request. Thus, the k_3-cost of the solution, considering only requests from nodes i to j with w wavelengths and using node k to divide the problem, is the sum $C_{i,k-1,w} + S_{i,k} + C_{k+1,j,w-1}$. This cost can now be minimized by determining the value of k for which the sum is minimum. ■

Algorithm 3 PathGroomingMinES($N, W, \mathbf{r}$)

for $i = 1$ to $N-2$ **do** {Initialize S_{ij} and $C_{i,j,1}$}
    for $j = i+1$ to $N-1$ **do**
        $S_{ij} = \sum_{k=i}^{j}(r_k)$
        $C_{i,j,1} = \sum_{k=i}^{j} r_k * (k - j + 1)$

for $w = 2$ to W **do** {Initialize $C_{i,j,w}$ for all $j - i < w$}
    for $i = 1$ to $N - w$ **do**
        for $j = i+1$ to $i + w - 1$ **do**
            $C_{i,j,w} = S_{ij}$

for $w = 2$ to W **do** {Main Loop}
    for $i = N - w$ down to 1 **do**
        for $j = i + w$ to N **do**
            $C_{i,j,w} = \infty$
            for $k = i$ to $j - w + 1$ **do**
                $C_{i,j,w} = \min(C_{i,j,w},\ C_{i,k-1,w} + S_{ik} + C_{k+1,j,w-1})$

Thus, appropriately choosing k by recursively solving the resulting sub-problems, we can obtain an optimum solution to the problem. However, because of the overlapping sub-structure, this would take an exponential amount of time. Instead, the recursive formulation lends itself to a dynamic programming algorithm that computes $C_{i,j,w}$ for all $1 \leq i \leq j \leq N$ and increasing w, using previously computed results, until the solution to the entire network, i.e., $C_{1,N-1,W}$ is determined. The procedure `PathGroomingMinES` (Algorithm 3) presents the details. This algorithm can be easily modified to keep track of the minimum cost topology itself. A Maple implementation of the algorithm, along with the necessary modifications to derive the virtual topology, is presented in Appendix A. For the sample run given in Appendix A, Figure 5.2 shows the resulting virtual topology. Assuming $W \leq N$, the running time of algorithm can be shown to be $O(N^4)$, where N

is the number of nodes in the path network. (If $W > N$, case (2) of the recurrence applies and the running time is $O(N)$.)

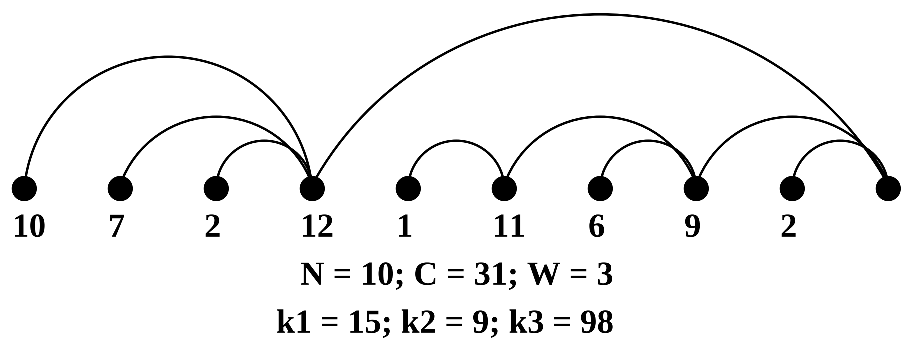

Figure 5.2: Sample output when $C = \infty$

5.2.4 Uniform traffic

In this section, we apply the dynamic programming algorithm from Section 5.2.3 to derive a closed form expression for the minimum k_3-cost of a solution to the restricted path grooming problem with infinite wavelength capacity and uniform traffic requests. In order to simply the derivation, we initially assume uniform unit traffic. However, since wavelength capacity is unbounded, these results hold true (with obvious changes) for any uniform traffic.

When each node has unit traffic to the egress node, the minimum k_3-cost of a solution, as given by Theorem 9, can be simplified as follows. First, we note that the cost would now depend on the number of nodes n and the number of available wavelengths w - in particular, it is independent of the exact two nodes i and j of the sub-problem. Hence, we can define $C_{n,w}$ as the minimum k_3-cost for grooming unit requests from n nodes using w wavelengths of unbounded capacity to the egress node. Thus, the minimum k_3-cost for the entire network is given $C_{N-1,W}$. Further, the summations defining the base cases in Theorem 9 can now be easily evaluated. The resulting recursive formulation for $C_{n,w}$ is thus,

$$C_{n,w} = \begin{cases} 0 & \text{if} \quad n < 0 \\ n & \text{if} \quad 0 \leq n \leq w; \\ \binom{n+1}{2} & \text{if} \quad w = 1; \\ \min_{i \leq k \leq j-1}\{C_{k-1,w} + k + C_{n-k,w-1}\} & \text{otherwise;} \end{cases} \tag{5.1}$$

Given a solution to the restricted path grooming problem for unit traffic with unlimited capacity, we define the value d_i, for each node $i < N$, to be the number of lightpaths in the shortest route for the request from node i to the egress node N. Thus d_i is the cost of routing the request from node i in the specified solution, and the total cost of the solution is $k_3 = \sum_{i=1}^{N-1} d_i$. Let $H_k = |\{i \mid 1 \leq i < N \;\wedge\; d_i = k\}|$ be the number of nodes in the solution whose request can be routed on a sequence of k lightpaths.

We now have the following lemma.

Lemma 8 *For a solution to the restricted path grooming problem with w wavelengths having unlimited capacity and unit traffic request from $n > 0$ nodes, if the solution does not split traffic and does not have intersecting lightpaths then,*

$$H_k \leq \binom{w+k-1}{k}$$

Proof. For the basis of the induction proof, we consider the two cases $k = 1$ and $w = 1$ separately.

For $k = 1$, we note that with w wavelengths available, there are at most w nodes that can have a direct lightpath to the egress node. Hence, the number of nodes with $d_i = 1$ is clearly less than or equal to w. Further, since $\binom{w+1-1}{1} = w$, the lemma is satisfied for this base case.

For $w = 1$, the only possible solution is to have a lightpath between each pair of consecutive nodes and to route requests on this chain of lightpaths. Hence, for each $i, 1 \leq i \leq n$, there is exactly one node j with $d_j = i$. Since $\binom{1+k-1}{k} = 1$, the lemma is satisfied for this base case.

For the inductive step, consider a solution that does not split traffic and does not have intersecting lightpaths. Let m be the least numbered node with a direct lightpath to the egress node. The solution can now be viewed as a recursive combination of two sub-problems - the first one with unit requests from $(m-1)$ nodes using all w wavelengths, corresponding to nodes $1, 2, \ldots, (m-1)$, and the second one with unit requests from $(N-1-m)$ nodes using up to $(w-1)$ wavelengths, corresponding to nodes $(w+1), (w+2), \ldots, (N-1)$. Then, H_k is maximized by independently maximizing the number of nodes with $d_i = (k-1)$ for the first sub-problem and $d_i = k$ for the second sub-problem. By the inductive hypothesis, H_k is thus bounded by

$$H_k \leq \binom{w+k-2}{k-1} + \binom{w-1+k-1}{k} = \binom{w+k-1}{k}$$

Hence the Lemma is true. ∎

Now, in order to solve the recurrence in Equation 5.1, we introduce a term $N_{k,w}$ defined as follows:

$$N_{k,w} = \sum_{1 \leq j \leq k} \binom{w+j-1}{j} = \binom{w+k}{k} - 1 \tag{5.2}$$

Since $N_{k,w}$ increases strictly monotonically with k, any positive integer n can be expressed uniquely for any $w \geq 1$, as $n = N_{k,w} + r$, where k is such that $N_{k,w} \leq n < N_{k+1,w}$ and $0 \leq r < (N_{k+1,w} - N_{k,w})$.

Applying Lemma 8 to the total k_3-cost of the solution gives us the following result.

Corollary 1 *Given n and w, let k and r be such that $n = N_{k,w} + r$ and $0 \leq r < (N_{k+1,w} - N_{k,w})$. Then, the k_3 cost of a solution to the restricted path grooming problem with w wavelengths of unlimited capacity and unit traffic from n nodes, that does not split traffic and does not have intersecting lightpaths, has the following lower bound.*

$$k_3 \geq w\binom{w+k}{k-1} + r(k+1) \tag{5.3}$$

Proof. The k_3-cost of a solution is given by

$$k_3 = \sum_{j \geq 1} jH_j$$

By Lemma 8, this sum is bounded by

$$k_3 \geq \sum_{1 \leq j \leq k} j\binom{w+j-1}{j} + r(k+1) \tag{5.4}$$

The sequence of steps for simplifying the above summation uses the following basic properties of binomial coefficients derived in [16]

$$k\binom{r}{k} = r\binom{r-1}{k-1} \qquad \text{integer } k \tag{5.5}$$

$$(r-k)\binom{r}{k} = r\binom{r-1}{k} \qquad \text{integer } k \tag{5.6}$$

$$\sum_{k \leq n}\binom{r+k}{k} = \binom{r+n+1}{n} \qquad \text{integer } n \tag{5.7}$$

Using these properties, we have,

$$\begin{aligned}
k_3 &\geq \sum_{1 \leq j \leq k} j\binom{w+j-1}{j} + r(k+1) && \text{from Eq. (5.4)} \\
&= \sum_{1 \leq j \leq k} (w+j-1)\binom{w+j-2}{j-1} + r(k+1) && \text{using Eq. (5.5)} \\
&= \sum_{1 \leq j \leq k} w\binom{w+j-1}{j-1} + r(k+1) && \text{using Eq. (5.6)} \\
&= w\sum_{0 \leq j < k}\binom{w+j}{j} + r(k+1) \\
&= w\binom{w+k}{k-1} + r(k+1) && \text{using Eq. (5.7)}
\end{aligned}$$

Hence the Corollary is proved. ∎

We now show that the lower bound in Equation (5.3) can always be achieved.

Theorem 10 *For the restricted path grooming problem with w wavelengths having unlimited capacity and unit traffic request from $n = N_{k,w} + r$ nodes, where $N_{k,w} \le n < N_{k+1,w}$ and $0 \le r < (N_{k+1,w} - N_{k,w})$, there exists a solution with k_3-cost given by $C_{n,w} = w\binom{w+k}{k-1} + r(k+1)$.*

Proof. For the basis of the induction proof, we consider the two cases $k = 0$ and $w = 1$ separately.

If $k = 0$, we have $N_{k,w} = 0$. The range for n is thus $N_{0,w}$ to $N_{1,w} - 1$. From equation (5.2), this range for n is $\binom{w}{0} - 1 = 0$ to $\binom{w+1}{1} - 1 - 1 = w - 1$. For each of these cases we have $n < w$ and hence the cost $C_{n,w}$, as per the recursive definition in equation (5.1) is n itself. Further, $w\binom{w}{-1} + r = 0 + r = n$. Hence the theorem is true for this base case.

For $w = 1$, the only possible solution is to have a lightpath between each pair of consecutive nodes and to route requests on this chain of lightpaths. Since $N_{k+1,1} - N_{k,1} = 1$ and r must satisfy $0 \le r < (N_{k+1,1} - N_{k,1})$, we have $r = 0$ and $n = N_{k,1} = k$. Now, from equation (5.1) if $n > 0$, $C_{n,1} = \binom{n+1}{2}$. Further, $\binom{k+1}{k-1} + r(k+1) = \binom{n+1}{n-1} = \binom{n+1}{2}$ and hence the theorem is also true for $w = 1$.

For the inductive step, we have $k > 0$ and $w > 1$. From the recursive definition in equation (5.1) and Corollary 1, we know that equation (5.8) below holds for any $l, 1 \le l < (N_{k,w} + r)$.

$$w\binom{w+k}{k-1} + r(k+1) \le k_3 = C_{N_{k,w}+r,w} \le C_{l,w} + l + C_{N_{k,w}+r-l,w-1} \tag{5.8}$$

Selecting a value l_m for l as follows,

$$l_m = \begin{cases} \binom{w+k-1}{k-1} & \text{if } 0 \le r < \binom{w+k-1}{k+1} \\ \binom{w+k-1}{k-1} + r - \binom{w+k-1}{k+1} & \text{if } \binom{w+k-1}{k+1} \le r < \binom{w+k}{k+1} \end{cases} \tag{5.9}$$

we can now show that the lower bound in Corollary 1 is achieved.

For case (i), we have $l_m = \binom{w+k-1}{k-1}$. The total cost as per the recursive definition in equation (5.1) is given by:

$$\begin{aligned} C(n,w) &= C(l_m - 1, w) + l_m + C(n - l_m, w-1) \\ &= C(N_{k-1,w}, w) + N_{k-1,w} + 1 + C(N_{k,w} + r - N_{k-1,w} - 1, w-1) \\ &= w\binom{w+k-1}{k-2} + \binom{w+k-1}{k-1} + C\left(\binom{w-1+k}{k} - 1 + r, w-1\right) \\ &= w\binom{w+k-1}{k-2} + \binom{w+k-1}{k-1} + (w-1)\binom{w-1+k}{k-1} + r(k+1) \\ &= w\binom{w+k}{k-1} + r(k+1) \end{aligned}$$

The above derivation uses the inductive hypothesis to evaluate the right hand expression. The final value matches the lower bound in Corollary 1.

For case (ii), we have $l_m = \binom{w+k-1}{k-1} + r - \binom{w+k-1}{k+1}$. The total cost as per the recursive definition in equation (5.1) is given by:

$$
\begin{aligned}
C(n,w) &= C(l_m - 1, w) + l_m + C(n - l_m, w-1) \\
&= C\left(N_{k-1,w} + r - \binom{w+k-1}{k+1}, w\right) + \binom{w+k-1}{k-1} + \\
&\quad r - \binom{w+k-1}{k+1} + C\left(N_{k,w} + r - N_{k-1,w} - 1 - r + \binom{w+k-1}{k+1}, w-1\right) \\
&= w\binom{w+k-1}{k-2} + (k)\left(r - \binom{w+k-1}{k+1}\right) + \binom{w+k-1}{k-1} + r - \binom{w+k-1}{k+1} + \\
&\quad C\left(\binom{w-1+k}{k} - 1 + \binom{w+k-1}{k+1}, w-1\right) \\
&= w\binom{w+k-1}{k-2} + (k)\left(r - \binom{w+k-1}{k+1}\right) + \binom{w+k-1}{k-1} + r - \binom{w+k-1}{k+1} + \\
&\quad (w-1)\binom{w+k-1}{k-1} + (k+1)\binom{w+k-1}{k+1} \\
&= w\binom{w+k}{k-1} + r(k+1)
\end{aligned}
$$

This derivation also uses the inductive hypothesis to evaluate the right hand side. Note that the values of r in each case lies within the proper limits. The final value corresponds to the lower bound in Corollary 1 and hence the theorem is proved. ■

Since the capacity of lightpaths is unlimited, the above theorem easily generalizes for instances with uniform traffic of $r > 1$ requests from each node.

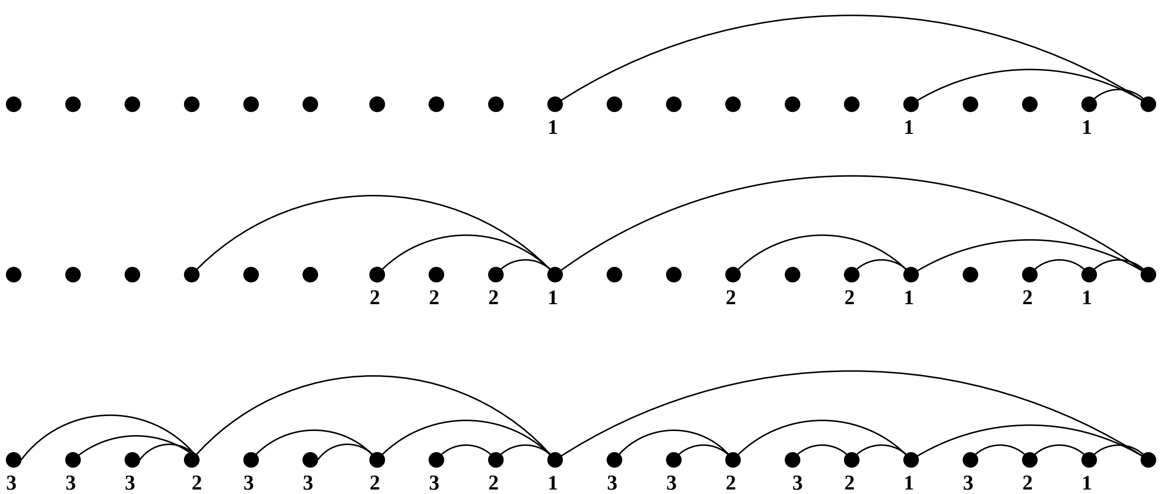

Figure 5.3: Unit traffic with unbounded capacity

Figure 5.3 shows the structure of the resulting topology for unit requests with $N = 20$ nodes and

$W = 3$. For each node $i < N$, the value of d_i is indicated just below the node.

5.3 Wavelengths with finite capacity

In this section we look at problem instances with a finite bound on the capacity of wavelengths.

5.3.1 Uniform unit traffic

This section uses the results from Section 5.2.4 to derive an upper bound on the minimum k_3-cost of a solution to the restricted path grooming problem with uniform unit traffic and bounded wavelength capacity.

For uniform unit traffic, an instance of the restricted path grooming problem is specified by the tuple $\langle N, C, W \rangle$. Clearly, a valid solution to the problem exists if and only if $(N-1) \leq CW$. If $(N-1) = CW$, one possible solution can be as follows. We first divide the $N-1$ non-egress nodes into W groups of C consecutive nodes each. Let these groups be numbered sequentially such that group 1 is closest to the egress node N and group W is the farthest. Then, for each group i, $1 \leq i \leq W$, we construct the solution to route all traffic to the highest numbered node in that group using i wavelengths of unbounded capacity. This solution is constructed using the recursive formulation in equation (5.1). Note that since each group has exactly C units of traffic, the actual load on the lightpaths created within each group does not exceed C. Finally, a lightpath is created from the egress node N to the highest numbered node in each group. This lightpath carries all the C units of traffic from that group to the egress node. Figure 5.4 shows an example of such a solution for $N = 16, C = 5, W = 3$. As with Figure 5.3, the value of d_i is shown just below each node.

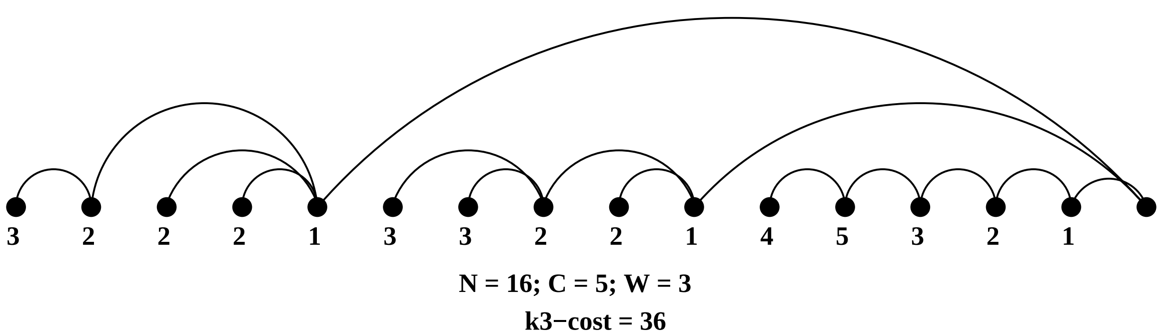

Figure 5.4: Unit traffic with bounded capacity

If $(N-1) < CW$, then we can simply construct the corresponding solution for an instance $\langle N' = CW+1, C, W \rangle$ as described above, and then remove the $N'-N$ nodes having the largest d_i (along with the associated lightpaths) to obtain a solution for the instance $\langle N, C, W \rangle$.

The k_3 cost of these solutions provide an upper bound on the optimum cost of a solution to the problem. We note that the described solution does not require more than one lightpath per non-egress node and none of the lightpaths created cross each other. Among solutions that satisfy these conditions, it can be easily shown that the solution proposed above is optimal. The question of whether there exist other solutions with strictly lower cost is, however, still an open problem.

5.3.2 When $W = 2$ and $r_i \leq C$

In this section we describe an algorithm to minimize the k_3-cost of a solution to the restricted path grooming problem for finite capacity wavelengths when the number of wavelengths is restricted to two and all the requests satisfy the condition $r_i \leq C$.

This algorithm is based on a recursive function F_n defined for each non-egress node n. This function takes four parameters - $c1, c2, d1$ and $d2$ - and returns the minimum additional k_3-cost of routing all the traffic from nodes 1 to n onto the egress node, under the assumption that $c1$ and $c2$ are the remaining capacities available on the two wavelengths at node n and $d1$ and $d2$ are the additional number of nodes ($> n$, including the egress node) that terminate these wavelengths respectively. The value of $F_{N-1}(C, C, 1, 1)$ will thus be the minimum k_3 cost of the entire solution. Briefly, the recursive algorithm works as follows.

For the base case, we consider node 1. The value of F_1 is determined for each possible combination of $(c1, c2, d1, d2)$ as follows. If $d1 \leq d2$, all traffic from r_1 up to $c1$ units is routed on wavelength 1 and the remaining, if any, are routed on wavelength 2. Otherwise ($d1 > d2$), all traffic from r_1 up to $c2$ units are first routed on wavelength 2 and the remaining is routed on wavelength 2. The value returned by F_1 is just the sum of the products of the traffic from r_1 on two wavelengths with the corresponding distances $d1$ and $d2$.

For each node $1 < n < N - 1$, the values of F_n can be determined based on the values of F_{n-1} as follows. Since there are only two wavelengths available, we note that node n may either terminate one of these two wavelengths or it may terminate both wavelengths. The algorithm looks at each of these three possibilities and selects the one which provides the least increase in the k_3-cost of the solution. For each possibility, this increase in k_3-cost is determined based on the parameters $c1, c2, d1$ and $d2$ as follows.

1. If $r_n \leq c1$, then node n might terminate only wavelength 1 and all of its requests can be routed on that wavelength. The increase in k_3-cost will then be $r_n * d1 + F_{n-1}(c1 - r_n, c2, d1 + 1, d2)$.

2. If $r_n \leq c2$, then node n might terminate only wavelength 2 and all of its requests can be routed on that wavelength. The increase in k_3-cost will then be $r_n * d2 + F_{n-1}(c1, c2 - r_1, d1, d2 + 1)$.

3. The final option, which is always feasible, is to terminate both wavelengths at node n. In this case, we first obtain the value $k' = F_{n-1}(C, C, 1, 1)$. Then, all the incoming traffic at node n, say $S(n)$, is routed with least cost, given $c1, c2, d1$ and $d2$, as in the case for F_1. The increase in k_3-cost will then be k' added to the sum of the products of the traffic from $S(n)$ on each wavelength with the corresponding distances $d1$ and $d2$.

We note that in the worst case, this algorithm determines the value of F_n for each node, for each combination of the parameters $c1, c2, d1$ and $d2$. Hence, the running time of the algorithm, when implemented using dynamic programming techniques, is bounded by $O(N^3C^2)$. An attempt to generalize this approach for any value of W yields an algorithm that is exponential in W, because the number of subsets of wavelengths that a node may terminate (and hence, the number of options to consider) is of the order $O(2^W)$. The detailed algorithm for $F_n(c1, c2, d1, d2)$, including the modifications necessary to determine the structure of the resulting topology, is presented as a Maple program in Appendix B. Figure 5.5 shows the topology of the solution obtained for the sample run given in the Appendix.

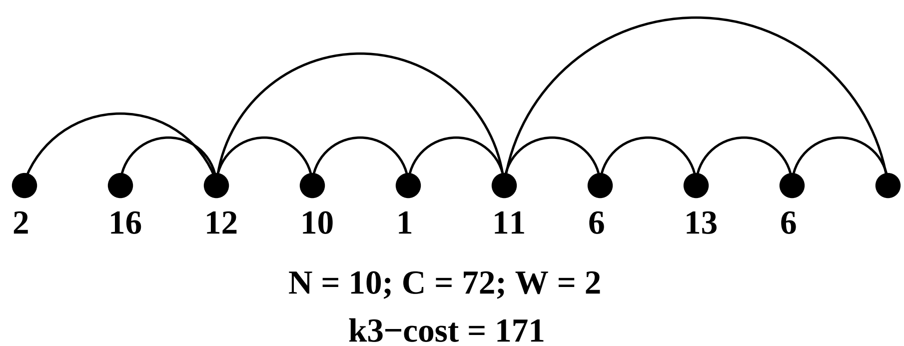

Figure 5.5: Sample output when $W = 2$ is fixed

Chapter 6

Conclusion and Future Work

In this chapter, we summarize the results obtained so far and list some of the open questions and possible directions for future research.

6.1 Summary of results

We considered the problem of traffic grooming in WDM path networks with all-to-one traffic with respect to minimizing three important cost measures.

The first cost measure was the total number of ADMs used by the solution. Minimizing this cost was known to be NP-Complete even for the restricted case, provided DXCs were not used. For this model, we showed that allowing an unbounded number of wavelengths obviated the need for cross connects at the nodes and hence the problem remains NP-Complete even when the nodes are equipped with DXCs.

The second cost measure considered was the number of transceivers used by the solution. We showed that the problem of minimizing the number of transceivers is NP-Complete, even when restricted to egress traffic. We then developed a simple approximation scheme for this cost measure that exceeds the optimum cost by at most the number of required wavelengths. Finally, we showed that under certain conditions, there exist solutions that simultaneously achieve the optimum value for both ADM and transceiver costs.

The third cost model aimed to minimize the total electronic switching in the network. For this cost measure, we developed a polynomial time algorithm to determine the cost and structure of an optimum solution when the wavelength capacity constraint is relaxed. A closed form expression to determine the minimum cost was derived for problem instances with uniform traffic. These costs provide a lower bound on the cost of solutions to problems with finite capacity. In addition, the structure of the solution for infinite capacity wavelengths was used to obtain an upper bound for instances with finite capacity having uniform unit traffic. It was already known that the problem of minimizing this cost is NP-Complete for path networks with any-to-any traffic, even when a virtual topology is already specified. We showed that for networks with egress traffic, given a virtual topology, there do exist polynomial time algorithms for minimizing this cost. Finally, we presented an algorithm to minimize the cost when the number of wavelengths is fixed at two.

6.2 Open problems and future directions

The following is a partial list of interesting questions, in the topics covered so far, that still remain unresolved.

- Do cross connects help reduce the number of ADMs in the path network with egress traffic, when a limited number of wavelengths is available?
- Does there exist a polynomial time algorithm to minimize the electronic switching cost of a solution to the restricted path grooming problem? Or is the problem NP-Hard?
- Can traffic splitting and traffic routing (using the cross connects) help reduce the electronic switching cost for the restricted path grooming problem? Can they do so when nodes have uniform or unit requests?
- What are the necessary and sufficient conditions to ensure that a solution to the restricted path grooming problem can simultaneously minimize a given subset of the three cost measures?

This thesis can also be used as the basis for future research on some of the following problems.

First, we have only considered three of a variety of cost models. Network costs may be characterized by a number of other schemes that include, for example, minimizing the maximum amount of switching at a node, minimizing the cost of other network components, and minimizing a combination of other cost measures. For each of these schemes, path networks may provide a relatively simple and useful framework for analyzing the problem.

For the problem of minimizing the number of transceivers, it may be possible to devise heuristics using existing work on bin packing that are better than the 2-approximation algorithm presented here.

The unbounded capacity model used to derive lower bounds for the k_3-cost may also be applicable for deriving lower bounds for other traffic models such as all-to-all, uniform all-to-all traffic etc. Since the actual solution, even under this assumption, only uses a finite capacity on each wavelength, this method can be used to derive an approximation algorithm based on iteratively solving for different values W until the actual capacity used matches the capacity available on the wavelengths. Such an approach would be quite feasible, especially if the cost of wavelengths is relatively low.

Finally, we note that research on path networks is only the first step for obtaining useful results that can be applied towards more general network topologies. Using these results to solve problems on more practical network topologies and traffic models is still an exciting research area with numerous potential applications.

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Appendix A

Maple 8.0 program for `PathGroomingMinES`

```
> restart;
> S := proc(i, j) global r;
  option remember;
  if   i>j  then return 0
  elif i=j  then return r[j]
  else           return S(i,j-1) + r[j]
  end if;
  end proc:
> S2 := proc(i, j) global r;
  option remember;
  if   i > j then return 0
  elif i = j then return r[j]
  else            return S2(i,j-1) + S(i,j) end if;
  end proc:
> C := proc(i, j, w)
  option remember;
  global r, LP;
  local Ct, Cm, k;
  if j - i < w then return S(i, j);
  elif w = 1 then return S2(i, j);
  else
  Cm := infinity;
  for k from i to j-1 do
  Ct := C(i, k-1, w) + S(i, k) + C(k+1, j, w-1);
  if Ct < Cm then Cm := Ct; LP[i, j, w] := k; end if
  end do;
  return Cm
  end if
  end proc:
> soln := proc(lp, i, j, w)
  if j - i < 0 then return
  elif j - i < w then seq([k,j+1], k=i..j)
  elif w = 1 then seq([k,k+1], k=i..j)
  else soln(lp, i, lp[i,j,w]-1, w),soln(lp, lp[i,j,w]+1, j,
  w-1),
  [lp[i,j,w], j+1];
  end if;
  end proc:
```

Sample Run.

```
> N := 10:  rnd := rand(1..16):  r := [seq(rnd(), i=1..N-1)];
  forget(C); forget(S); LP := 0:  st := time():
  Cost := C(1, N-1, 3); `Time taken (sec)` := time()-st;
  `Lightpaths:`; soln(LP, 1, N-1, 3);
```

$$r := [10, 7, 2, 12, 1, 11, 6, 9, 2]$$

$$Cost := 98$$

$$Time\ taken\ (sec) := 0.070$$

$$Lightpaths :$$

$$[1, 4], [2, 4], [3, 4], [5, 6], [7, 8], [6, 8], [9, 10], [8, 10], [4, 10]$$

Appendix B

Maple 8.0 program for Fn($c1, c2, d1, d2$)

```
> restart;
> S := proc (N) global r; option remember;
  if N=1 then return r[1] else return S(N-1)+r[N] end if;
  end proc:
> F := proc (n, c1, c2, d1, d2) option remember;
  global r, C; local k, op, opt, kt;
  if n=1 then
  if d1 <= d2 then
  if r[1] <= c1 then return (r[1]*d1, 1);
  else return (c1*d1 + (r[1]-c1)*d2, 3);
  end if;
  else
  if r[1] <= c2 then return (r[1]*d2, 2);
  else return ((r[1]-c2)*d1 + c2*d2, 3);
  end if;
  end if;
  else
  # option 3 - choose to split traffic
  (kt,opt) := F(n-1, C, C, 1, 1);
  if d1 <= d2 then
  if S(n) <= c1 then k := kt + S(n)*d1;
  else k := kt + c1*d1 + (S(n)-c1)*d2;
  end if;
  else
  if S(n) <= c2 then k := kt + S(n)*d2;
  else k:= kt + (S(n)-c2)*d1 + c2*d2;
  end if;
  end if;
  op := 3;
  # option 1 - choose to use only c1
  if r[n] <= c1 then
  (kt,opt) := F(n-1, c1-r[n], c2, d1+1, d2);
  kt := kt + r[n]*d1;
  if kt<k then k := kt; op := 1 end if;
  end if;
  # option 2 - choose to use only c2
  if r[n] <= c2 then
  (kt,opt) := F(n-1, c1, c2-r[n], d1, d2+1);
  kt := kt + r[n]*d2;
  if kt<k then k := kt; op := 2 end if;
  end if;
  return (k,op);
  end if;
  end proc:
```

```
> soln := proc (n, c1, c2, d1, d2, n1, n2)
  global C, r; local k, op;
  (k,op) := F(n,c1,c2,d1,d2);
  if n=1 then
  if op=1 then [1,n1];
  elif op=2 then [1,n2];
  else [1,n1], [1,n2];
  end if;
  else
  if op=1 then soln(n-1,c1-r[n],c2,d1+1,d2,n,n2),[n,n1];
  elif op=2 then soln(n-1,c1,c2-r[n],d1,d2+1,n1,n),[n,n2];
  else soln(n-1, C, C, 1, 1, n, n), [n,n1], [n,n2];
  end if;
  end if;
  end proc:
```

Sample Run:

```
> R:= 16:  cap:= n->(n*R/2):  rnd:= rand(1..R): N:= 10; C:= cap(N-1);
  r:= [seq(rnd(), i=1..(N-1))]; forget(F); forget(S); st:= time():
  (t1,t2):= F(N-1, C, C, 1, 1):  Cost:= t1; `Time taken`:= time()-st;
  `Lightpaths:`;
> soln(N-1,C,C,1,1,N,N);
```

$$N := 10$$
$$C := 72$$
$$r := [2, 16, 12, 10, 1, 11, 6, 13, 6]$$
$$Cost := 171$$
$$Time\ taken := 0.080$$
$$Lightpaths :$$
$$[1, 3], [2, 3], [3, 4], [3, 6], [4, 5], [5, 6], [6, 7], [6, 10], [7, 8], [8, 9], [9, 10]$$