BOUNDS TO PLASTIC STRAINS AND DISPLACEMENTS IN DYNAMIC SHAKE-DOWN OF WORKHARDENING STRUCTURES

G. MAIER, E. VITIELLO*

Istituto di Scienza e Tecnica delle Costruzioni, Politecnico di Milano, I-20133 Milano, Italy

*(*At present: Department of Civil Engineering, University of California, Berkeley, California 94720, U.S.A.)*

SUMMARY

In structural elasto-plasticity, safety traditionally means that plastic strains and, hence, the total plastic work, are bounded asymptotically in time. This circumstance is referred to as adaptation or shakedown. However, even if adaptation is predicted to occur, various events such as the following ones, make the calculated safety factor $s > 1$ seriously inaccurate: (a) the structure becomes eventually unserviceable and/or the theory unreliable because of excessive displacements; (b) the plastic deformation capacity of material of structural elements is eventually exceeded. Methods of checking these occurrences are always useful in order to guarantee safety, and become particularly important when strain-hardening and/or geometry changes are taken into consideration.

The task of ensuring that no critical event of the above types will occur, can be pursued by determining upper bounds on some meaningful plastic displacements (i.e. displacements due to plastic deformations) and/or on plastic strains, and comparing them to some critical values.

In the dynamic range, the time dependence of external actions must be known for the shakedown analysis; therefore it is in principle possible to assess safety by following step-by-step the plastodynamic evolution of the system. However, a far simpler and more practical way is represented, as in the quasi-static range, by the direct evaluation of the safety factor and the subsequent use of upper bounds on suitable plastic strains and/or displacement.

In this paper various methods are proposed for the determination of upper and lower bounds on the values that plastic displacements, plastic strain components and other linear or non-linear functions of the plastic multipliers, eventually attain when an elastic work-hardening system shakes down under periodic external actions (loads and dislocations) in the presence of significant dynamic effects. All procedures proposed presume merely the linear elastic dynamic analysis of a fictitious structure and are amenable to well-known numerical techniques in mathematical programming (linear, quadratic, or nonlinear). This circumstance, which confers an operative value to the theory developed, rests on the assumed discretization of the structures, piecewise linearization of the yield loci and of the hardening rules. It would be worthwhile, particularly from the theoretical standpoint, to remove the restrictions and tackle the same problem by a broader approach.

The simple illustrative example included in this paper seems to indicate that fairly stringent and, hence, useful upper bounds can be achieved with a reasonable amount of calculations (and far smaller than that required by the otherwise inevitable evolutive study of the whole plastodynamic response).

1. Introduction

Melan's theorem on quasi-static shakedown of elastic perfectly plastic structures has been extended to the dynamic range by Ceradini in [1] and further generalized to some classes of hardening structures by Maier [2] [3].

Starting from the above previous work, the present paper establishes bounding techniques for various quantities after dynamic adaptation. Techniques of this type are useful in many situations for a realistic assessment of safety. Bounding procedures in the quasi-static shakedown theory have been proposed by Vitiello [4], Maier [5], Ponter [6], König et al. [7] [8], at various levels of generality.

2. Governing relations

Consider first a two-dimensional continuum discretized in a finite element model through piecewise linearization of the displacement field (fig. 1). Strains $\tilde{\underline{\varepsilon}}^i \equiv [\varepsilon_x^i\ \varepsilon_y^i\ \gamma_{xy}^i]$ and stresses $\tilde{\underline{\sigma}}^i \equiv [\sigma_x^i\ \sigma_y^i\ \sigma_{xy}^i]$ are constant throughout each triangular element i and are related to the side elongations $\tilde{\underline{q}}^i \equiv [q_1^i\ q_2^i\ q_3^i]$ (generalized "natural" strains) and, respectively, to the corresponding selfequilibrated forces $\tilde{\underline{Q}}^i = [Q_1^i\ Q_2^i\ Q_3^i]$ (generalized "natural" stresses) through nonsingular contragradient linear transformations [9]. Therefore, instead of the material behavior described by the $\underline{\sigma}$ versus $\underline{\varepsilon}$ law, the "finite element behavior" described by a fully similar $\underline{Q}^i$ versus $\underline{q}^i$ law can be referred to in setting up the governing relations.

The plastic behavior of the element will be described by means of y yield planes, which define the instantaneous polyhedrical elastic domain and translate by yielding, as indicated schematically in fig. 2 (for simplicity in two components). To this aim, express the element plastic strain vector $\underline{p}^i$ as a linear combination, through nonnegative plastic multipliers $\underline{\lambda}^i$, of y constant vectors taken as columns of a matrix $\underline{N}^i$:

$$\underline{p}^i = \underline{N}^i \underline{\lambda}^i, \qquad \dot{\underline{\lambda}}^i \geq \underline{0} \tag{1}$$

and define y plastic potentials (or yield functions) linear in both $\underline{Q}^i$ and $\underline{\lambda}^i$:

$$\underline{\varphi}^i = \tilde{\underline{N}}^i \underline{Q}^i - \underline{H}^i \underline{\lambda}^i - \underline{k}^i \leq \underline{0} \tag{2}$$

The hardening matrix $\underline{H}^i$ governs the evolution of the yield surface defined by Eq. (2) when plastic flow occurs ($\dot{\underline{\lambda}}^i \neq \underline{0}$). E. g. $\underline{H}^i = H^i \tilde{\underline{N}}^i \underline{N}^i$ corresponds to kinematic hardening according to Prager's rule, as indicated in fig. 2 (Maier[2]). Plastic flow may occur according to a yield mode j ($\dot{\lambda}_j^i > 0$) only if the relevant yield plane contains the stress point ($\varphi_j^i = 0$); moreover, plastic flow ($\dot{\lambda}_j^i > 0$) and local unloading with respect to the relevant mode ($\dot{\varphi}_j^i < 0$) are mutually esclusive. These two circumstances can be expressed by the combinatorial relations:

$$\tilde{\underline{\varphi}}^i \dot{\underline{\lambda}}^i = 0, \qquad \tilde{\dot{\underline{\varphi}}}^i \dot{\underline{\lambda}}^i = 0 \tag{3}$$

The elastic deformations of the element is represented by

$$\underline{e}^i = \underline{E}^{i\,-1} \underline{Q}^i \tag{4}$$

where $\underline{E}^i$ denotes the symmetric positive definite matrix of elastic stiffness.

Let us define the block-diagonal matrices:

$$\underline{E} \equiv \text{diag}\,(\underline{E}^i),\ \underline{H} \equiv \text{diag}\,(\underline{H}^i),\ \underline{N} \equiv \text{diag}\,(\underline{N}^i) \tag{5}$$

index i running over the number m of elements which form the structural model. Analogously, collect all vectors $\underline{e}^i$, $\underline{p}^i$, $\underline{\lambda}^i$, $\underline{\varphi}^i$, $\underline{k}^i$ for the m elements, taken in the same order, in supervectors $\underline{e}$, $\underline{p}$, $\underline{\lambda}$, $\underline{\varphi}$, $\underline{k}$. Using these symbols the preceding relations Eqs. (1)-(4) can be condensed for i = 1. . . m in the following matrix relations which describe elementwise the constitutive laws for the whole (disassembled) system:

$$\underline{E}\ \underline{e} = \underline{Q} \tag{6}$$

$$\underline{p} = \underline{N}\ \underline{\lambda}, \qquad \dot{\underline{\lambda}} \geq \underline{o} \tag{7}$$

$$\underline{\varphi} = \tilde{\underline{N}}\ \underline{Q} - \underline{H}\ \underline{\lambda} - \underline{k} \leq \underline{o} \tag{8}$$

$$\tilde{\underline{\varphi}}\,\dot{\underline{\lambda}} = o \qquad \dot{\tilde{\underline{\varphi}}}\,\dot{\underline{\lambda}} = o \tag{9}$$

If the m-vector $\underline{r}$ (t) defines a distribution of imposed strains (such as thermal strains) then:

$$\underline{q} = \underline{e} + \underline{p} + \underline{r} \tag{10}$$

Let $\underline{u}$ and $\underline{F}$ denote the n-vectors of the nodal displacements and corresponding external forces, respectively. Deformations are assumed not to affect equilibrium. Geometric compatibility and equilibrium can be expressed in the form:

$$\underline{q} = \underline{C}\ \underline{u} \tag{11}$$

$$\tilde{\underline{C}}\ \underline{Q} + \underline{I}\ \ddot{\underline{u}} + \underline{V}\ \dot{\underline{u}} = \underline{F} \tag{12}$$

where $\underline{C}$ represents the compatibility matrix (of rank n); $\underline{I}$ and $\underline{V}$ denote the inertia and viscous matrices, both symmetric positive-definite. The initial conditions will be indicated as:

$$\underline{u}\,(t = o) = \underline{u}_o, \qquad \dot{\underline{u}}\,(t = o) = \dot{\underline{u}}_o \tag{13}$$

The dynamic evolution of the structural models under a given history of external actions $\underline{F}(t)$, $\underline{r}(t)$ is governed by the relation set Eqs. (6)-(13).

By suitable re-interpretations of the symbols, the same relations are readily seen to govern also other classes of discrete structures, such as: threedimensional continua discretized in constant strain tetrahedrical finite elements (then vectors $\underline{q}^i$ and $\underline{Q}^i$ have 6 components) (Argyris, 9); trusses loaded on the joints only and lumped-compliance models of beams and frames (then each deformable element exhibits one natural stress and strain component and two yielding modes). Thus a convenient matrix description and a fairly broad coverage are provided for the subsequent developments.

3. Shakedown criteria

For the classes of structures considered in Sec. 2, shakedown is characterized by the fact that all plastic multipliers are bounded in time, i. e. their asymptotic values $\underline{\lambda}_\infty$

are finite:

$$\underline{\lambda}_\infty \equiv \lim_{t \to \infty} \underline{\lambda}(t) < \underline{\infty} \tag{14}$$

The meanings of some notions and terminology to be used later, are specified below.

By "associate elastic system" we will mean the structure assumed unlimitedly elastic by suppressing all yielding modes. We will call "fictitious response" the motion of the associate elastic system under the given external actions $\underline{F}(t)$, $\underline{r}(t)$ for suitable initial conditions $\underline{u}^*_o$, $\dot{\underline{u}}^*_o$, in general different from the given ones. The relevant quantities will be marked by asterisks: $\underline{u}^*(t)$ is governed by the differential equation

$$\underline{S}\,\underline{u} + \underline{V}\,\dot{\underline{u}} + \underline{I}\,\ddot{\underline{u}} = \underline{F} + \underline{\tilde{C}}\,\underline{E}\,\underline{r} \tag{15}$$

where $\underline{S} = \underline{\tilde{C}}\,\underline{E}\,\underline{C}$ is the symmetric, positive-definite elastic stiffness of the associate elastic system; the fictitious elastic stresses $\underline{Q}^{e*}(t)$ flow from $\underline{u}^*(t)$ through:

$$\underline{Q}^e = \underline{E}\,(\underline{C}\,\underline{u} - \underline{r}) \tag{16}$$

Selfstresses due to imposed strains $\underline{r}$ in the elastic associate system in quasi-static conditions can be derived substituting Eq. (15) for $\underline{I} = \underline{V} = \underline{O}$ and $\underline{F} = \underline{O}$ in Eq. (16):

$$\underline{Q}^s = \underline{Z}\,\underline{r}, \text{ with } \underline{Z} = \underline{E}\,\underline{C}\,\underline{S}^{-1}\underline{\tilde{C}}\,\underline{E} - \underline{E} \tag{17}$$

The hypothesis:

$$\text{matrix } \underline{A} \equiv \underline{H} - \underline{\tilde{N}}\,\underline{Z}\,\underline{N} \text{ symmetric, positive-semidefinite} \tag{18}$$

plays a key role in what follows. It rules out some hardening rules and ensures overall stability (in the sense of nonnegative second order work required by any incremental process), whatever set of yielding modes may become active, [2].

The dynamic shakedown analysis of the structures considered here, can be based on the following theorem:

(a) When the hypothesis (18) holds, adaptation will occur under given external actions $\underline{F}(t)$, $\underline{r}(t)$, if and only if some constant plastic multiplier vector $\underline{\lambda}'$, a time t' and some initial conditions $\underline{u}^*_o$, $\dot{\underline{u}}^*_o$ can be found such that for any $t > t'$:

$$\underline{\tilde{N}}\,\underline{Q}^{e*}(t) - \underline{A}\,\underline{\lambda}' - \underline{k} \leq \underline{o}\,, \tag{19}$$

$\underline{Q}^{e*}(t)$ being the stress history in the fictitious process defined by $\underline{u}^*_o$, $\dot{\underline{u}}^*_o$.

The proof of this statement was given by Maier [3], as a generalization to hardening structure of the basic dynamic shakedown theorem (Ceradini [1]).

Let us define the "vector $\underline{M}'$ of the maximum projections" on outward normals to all yield planes j of all elements i :

$$\underline{\tilde{M}}' \equiv [\ldots \underline{M}^i_j \ldots] \text{ where } \underline{M}^i_j \equiv \max_{t \geq t'} \underline{\tilde{N}}^i_j\,\underline{Q}^{e*}_i(t) \tag{20}$$

Making use of $\underline{M}'$, inequality (19) can be replaced by the equivalent one:

$$\underline{M}' - \underline{A}\,\underline{\lambda}' \leq \underline{k} \tag{21}$$

4. Regions in the $\underline{\lambda}$ space which contain $\underline{\lambda}_\infty$.

Consider a structure for which hypothesis (18) holds and a scalar $\alpha > 1$, a vector $\underline{\lambda}'$ and initial conditions $\underline{u}^*_o \underline{\dot{u}}^*_o$ have been found such that:

$$\underline{M}_o - \underline{A}\,\underline{\lambda}' \leq \frac{1}{\alpha}\,\underline{k} \tag{22}$$

$\underline{M}_o$ being calculated according to (20) for t' = o. By virtue of theorem (a) the structure is guaranteed to shakedown under the given dynamic loading, i. e. the safety factor s is greater than 1.

The following inequality can be proved to hold under the above assumptions:

$$(1 - \frac{1}{\alpha})\,\underline{\tilde{k}}\,\underline{\lambda}_\infty \leq \chi + \frac{1}{2}\,\underline{\tilde{\lambda}}'\underline{A}\,\underline{\lambda}' - \tag{23}$$
$$-\frac{1}{2}\,(\underline{\tilde{\lambda}}_\infty - \underline{\tilde{\lambda}}')\,\underline{A}\,(\underline{\lambda}_\infty - \underline{\lambda}') - \int_o^\infty (\underline{\tilde{\dot{u}}} - \underline{\tilde{\dot{u}}}^*)\,\underline{V}\,(\underline{\dot{u}} - \underline{\dot{u}}^*)\,dt$$

where:

$$\chi = \frac{1}{2}\left[(\tilde{Q}^e - \tilde{Q}^{e*})\,E^{-1}(Q^e - Q^{e*})\right]_{t=o} + \frac{1}{2}\,(\underline{\tilde{\dot{u}}}_o - \underline{\tilde{\dot{u}}}^*_o)\,\underline{I}\,(\dot{u}_o - \dot{u}^*_o) \tag{24}$$

The proof of inequality (23), omitted here for brevity, is rooted in the proof of theorem (a).

The inequality still holds if the nonnegative integral in its r. h. s. is dropped; thus (23) becomes:

$$\left[(1 - \frac{1}{\alpha})\,\underline{\tilde{k}} - \underline{\tilde{\lambda}}'\underline{A}\right]\underline{\lambda}_\infty + \frac{1}{2}\,\tilde{\lambda}_\infty A\,\lambda_\infty \leq \chi \tag{25}$$

If on the r. h. s. of (23), besides the integral, also the third term is dropped, an inequality linear in $\underline{\lambda}_\infty$ is obtained:

$$(1 - \frac{1}{\alpha})\,\underline{\tilde{k}}\,\underline{\lambda}_\infty \leq \chi + \frac{1}{2}\,\underline{\tilde{\lambda}}'\,\underline{A}\,\underline{\lambda}' \tag{26}$$

As a consequence of the constitutive laws, inequalities (7) (8), vector $\underline{\lambda}_\infty$ must satisfy the further inequalities:

$$\underline{M}_o - \underline{A}\,\underline{\lambda}_\infty \leq \underline{k}, \qquad \underline{\lambda}_\infty \geq \underline{o} \tag{27}$$

In fact, because of the adaptation and the damping effects, the real and the fictitious motions tend to coincide asymptotically; therefore $\underline{\lambda}_\infty$ will certainly fulfil a relation like (27) for $\underline{M}'$ calculated over $t \geq t'$, being t' a sufficiently large time; since clearly $\underline{M}_o \geq \underline{M}'$, also (27) will be fulfilled.

Let $\Lambda(\underline{\lambda}')$ and $\Lambda_\ell(\underline{\lambda}')$ denote the domains defined in the $\underline{\lambda}$ space by Eqs. (25) (27) and by Eqs. (26) (27), respectively, when $\underline{\lambda}_\infty$ is replaced by a variable vector $\underline{\lambda}$. It is easy to observe that:

(α) Λ_ℓ contains Λ, since the linear constraint (26) is less stringent than the nonlinear one (25);

(β) both Λ and Λ_ℓ are bounded and convex.

5. Bounding procedures

Consider the h-th component of the plastic displacement vector $\underline{u}^p_\infty$ at shakedown; it can be expressed as a linear function of $\underline{\lambda}_\infty$:

$$u^p_{\infty_h} = \underline{\tilde{b}}\,\underline{\lambda}_\infty \tag{28}$$

where $\underset{\sim}{\tilde{b}}$ represents the h-th row of matrix $\underline{S}^{-1}\underline{\tilde{C}}\ \underline{E}\ \underline{N}$, as readily seen from (7) and (15). The following upper bounds U on $u^p_{\infty h}$ can be generated on the basis of the results of Sec. 4.

(a) $U_o = \min_{\underline{\lambda}'} \max_{\underline{\lambda}} \underline{\tilde{b}}\ \underline{\lambda}$, subject to:

$$\underline{\lambda} \in \Lambda(\underline{\lambda}'), \quad \underline{M} - \underline{A}\,\underline{\lambda}' \leq \frac{1}{\alpha}\underline{k} \qquad (29)$$

(b) By referring to the relaxed constraint (26) the min -max problem reduces to a sequence of a quadratic and a linear programming problem, since $\underline{\lambda}'$ affects Λ_ℓ only through the nonnegative scalar $\psi \equiv \underline{\tilde{\lambda}}'\underline{A}\,\underline{\lambda}'$:

$$\psi_o \equiv \min \frac{1}{2}\ \underline{\tilde{\lambda}}'\underline{A}\,\underline{\lambda}' \ , \text{ subject to: } \underline{\bar{M}} - \underline{A}\,\underline{\lambda}' \leq \frac{1}{\alpha}\underline{k} \qquad (30)$$

$$U_1 = \max_{\underline{\lambda}} \underline{\tilde{b}}\ \underline{\lambda} \ , \text{ subject to: } \underline{\lambda} \in \Lambda_\ell(\psi_o) \qquad (31)$$

The dual to quadratic program (30) has sign constraints only and can be advantageously used to obtain ψ_o .

(c) If the dual to the linear program (31) is formed the value its objective assumes for only of its feasible vectors provides an upper bound U_2.

(d) If a vector $\underline{\lambda}'$ complying with the linear inequality (22) is determined, (29) reduces to a convex maximization with a single nonlinear constraint, which still supplies an upper bound U_3 .

Other bounding procedures can be easely established by paths of reasoning similar to the preceding ones. In general the more stringent the bound the larger the computational effort. Lower bounds can be similarly derived: e. g. the most stringent will be attained as in (29), but by solving a max min, instead of a min max problem.

All the procedures for plastic displacements after adaptation apply inaltered for any other linear function of $\underline{\lambda}_\infty$ such as measures of plastic deformation ($\underline{\tilde{P}}\ \underline{u}^p_\infty$, $\underline{P}$ being fixed fictitious loads), plastic strains, selfstresses or a single plastic multiplier.

6. Special cases

The choice of the initial condition $\underline{u}^*_o$, $\underline{\dot{u}}^*_o$ (which affect $\underline{M}_o$) and of the scalar α was constrained only by inequality (22). This selection might be obtimized in order to obtain the best bound, but this is in general a difficult task.

However, for periodic external actions, i. e. when

$$\underline{F}\ (t) = \underline{F}\ (t+T),\ \underline{r}\ (t) = \underline{r}\ (t+T),\ \text{for any } t \qquad (32)$$

the best choice can be proved to be the following one, uniquely determined "a priori": α = safety factor s; $\underline{u}^*_o$, $\underline{\dot{u}}^*_o$ = the particular initial conditions, say $\underline{\bar{u}}^*_o$, $\underline{\dot{\bar{u}}}^*_o$,which make periodic the fictitious response throughout $t \geq o$.

For undamped systems ($\underline{V} = \underline{o}$) under periodic actions the fictitious process to be considered in the bounding procedures turns out to be the forced vibration part alone (indipendent from the initial conditions) of the solution of Eq. (15) with $\underline{V} = \underline{o}$.

For perfectly plastic systems ($\underline{H} = \underline{o}$), in all the preceding relations the role of the variables $\underline{\lambda}'$ can be played by selfstresses $\underline{Q}^{s'}$ and matrix $\underline{Z}$ need not be calculated. In fact, through (17) (18) and the virtual work principle:

$$\underline{A}\,\underline{\lambda}' = \underline{\tilde{N}}\,\underline{Q}^{s'}, \qquad \underline{\tilde{\lambda}}'\underline{A}\,\underline{\lambda} = \underline{\tilde{p}}'\,\underline{Q}^{s'} = -\,\underline{\tilde{Q}}^{s'}\underline{E}^{-1}\underline{Q}^{s'} \tag{33}$$

The vector $\underline{Q}^{s'}$ is constrained by the self-equilibrium condition $\underline{\tilde{C}}\,\underline{Q}^{s'} = \underline{o}$, so that the redundants can be taken as the only independent variables.

7. Examples

The two degrees-of-freedom beam model represented in fig. 3-a has masses and compliances lumped in the plastic hinges a, b, c, d. These exhibit kinematic hardening, with $H_{11} = E\,10^{-1}$, E being the elastic stiffness. The frequency of the variable load is half the lower natural frequency of the model, $\omega = \frac{1}{2}\,\omega_1$; it has been assumed $F = 4/9\ k\,l$, k being the original yield moment. The actual value $\lambda^{b}_{1\infty}$ at shakedown of the plastic multiplier for the positive yielding mode in hinge b (i. e. the total amount of sagging plastic rotation in b) has been obtained for several load factor ξ by step integration. In fig. 3-b the actual $\lambda^{b}_{1\infty}$ is compared to the upper bound on it obtained by solving numerically the minimax problem (a) of Sec. 5.

The two-storey frame represented in fig. 4-a has been studied with distributed elastic deformability and with plastic deformability confined in 10 critical sections (Corradi and Nova, [10]. It has been assumed $\omega = \frac{1}{2}\,\omega_1$, $Fl = k$, elastic bending stiffness $EI = 10\,Fl$. Failure occurs due to incremental collapse for $s = 1.97$ in the absence of hardening, for $s = 3.49$ in the presence of kinematic hardening. In both cases the technique (b) of Sec. 5 has been employed for various ξ to generate bounds on the horizontal plastic displacements at shakedown in the nodes a and b. Fig. 4-b shows the resulting upper bounds compared to the static elastic displacements in the same nodes; the static elastic displacement u^{E}_{st} at the joint a for $\xi = 1$ has been assumed = 1.

8. Nomenclature

Underlined letters denote matrices and column-vectors. $\underline{O}$ is a matrix or vector whose entries are zeros. A tilde $\sim$ indicates transpose. All other symbols are defined where they are used for the first time.

9. References

[1] CERADINI, G., "Sull'adattamento dei corpi elastoplastici soggetti ad azioni dinamiche", Giorn. Genio Civile, May 1969, pp. 239-250.

[2] MAIER, G., "A matrix structural theory of piecewise linear elastoplasticity with interacting yield planes", Meccanica, Vol. 5, 1970, pp. 54-66.

[3] MAIER, G., "Shakedown of plastic structures with unstable parts", Techn. Report ISTC, Milan, June 1971, and J. Eng. Mech. Div. Proc. ASCE, Oct. 1972, pp. 1322-1327.

[4] VITIELLO, E., "Upper bounds to plastic strains in shakedown of structures subjected to cyclic loads", 1^ AIMETA Congress, Udine, July 1971, and Meccanica, Vol. 7, n. 3, 1972, pp. 205-213.

[5] MAIER, G., "A shakedown matrix theory allowing for workhardening and second-order geometric effects", Proc. Symp. on Foundations of Plasticity, Warsaw, 1972, pp. 417-432.

[6] PONTER, A.R.S., "An upper bound on the small displacements of elastic perfectly plastic structures", J. Appl. Mech., Vol. 39, E, No. 4, pp. 959-963.

[7] KÖNIG, J., "Shakedown deflections. A finite element approach", Proc. Bulgarian Acad. Sci., Theor. Appl. Mech., III, No. 2, Sofia, 1972, pp. 65-69.

[8] BRZEZINSKI, R., KÖNIG, J.A., "Evaluation of shakedown deflections of framed structures by linear programming", Proc. Symp. Plastic Analysis of Structures, Jassy, Sept. 1972, pp. 101-116.

[9] ARGYRIS, J.H., "Continua and discontinua", Proc. Conf. Matrix Methods in Struct. Mech., Wright-Patterson A.F.B., Ohio, 1967, p. 12.

[10] CORRADI, L., NOVA, R., "A comparative study of bounding techniques in shakedown of elastoplastic structures", in preparation.

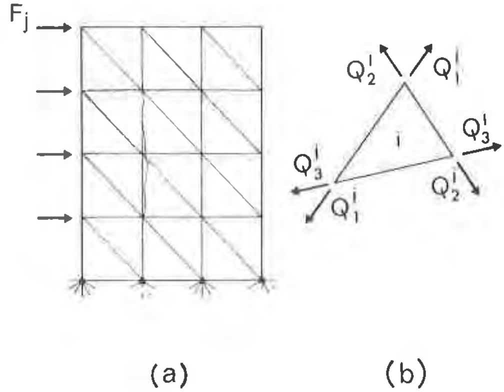


fig. 1

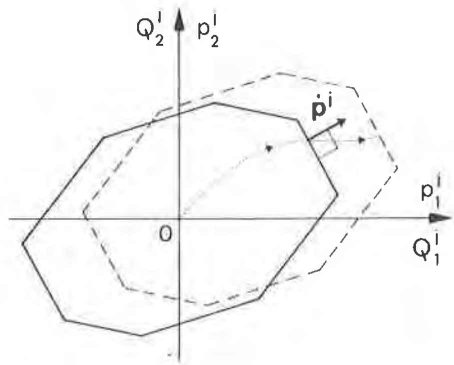


fig. 2

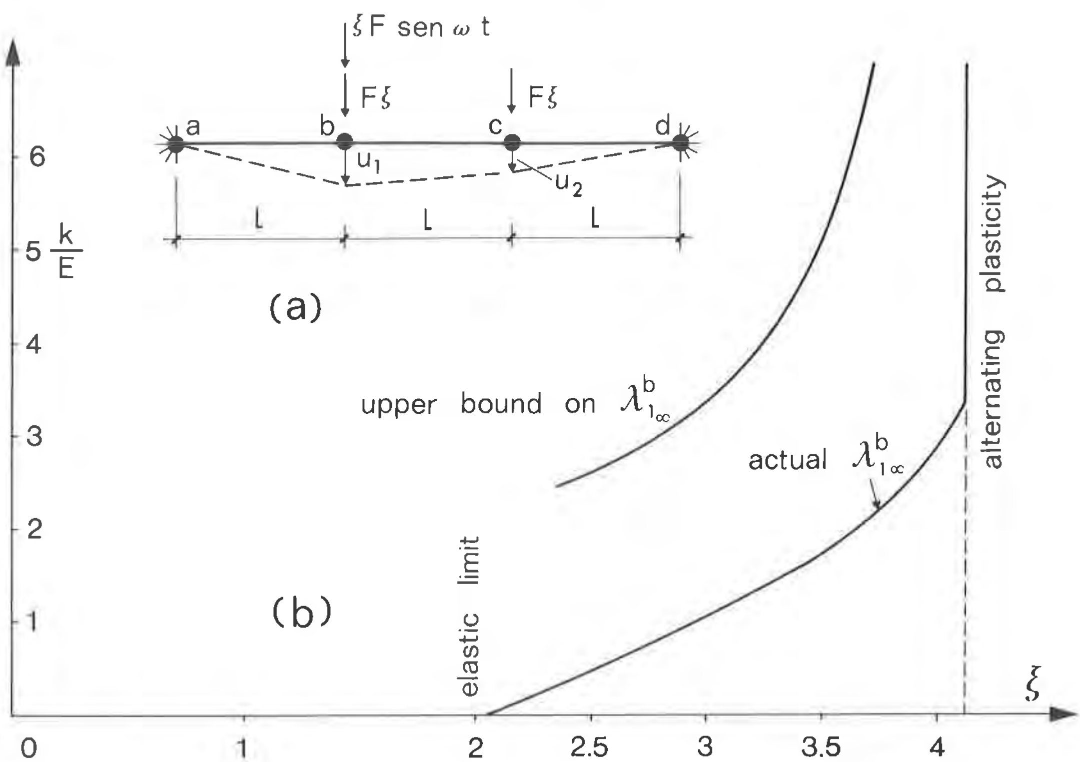
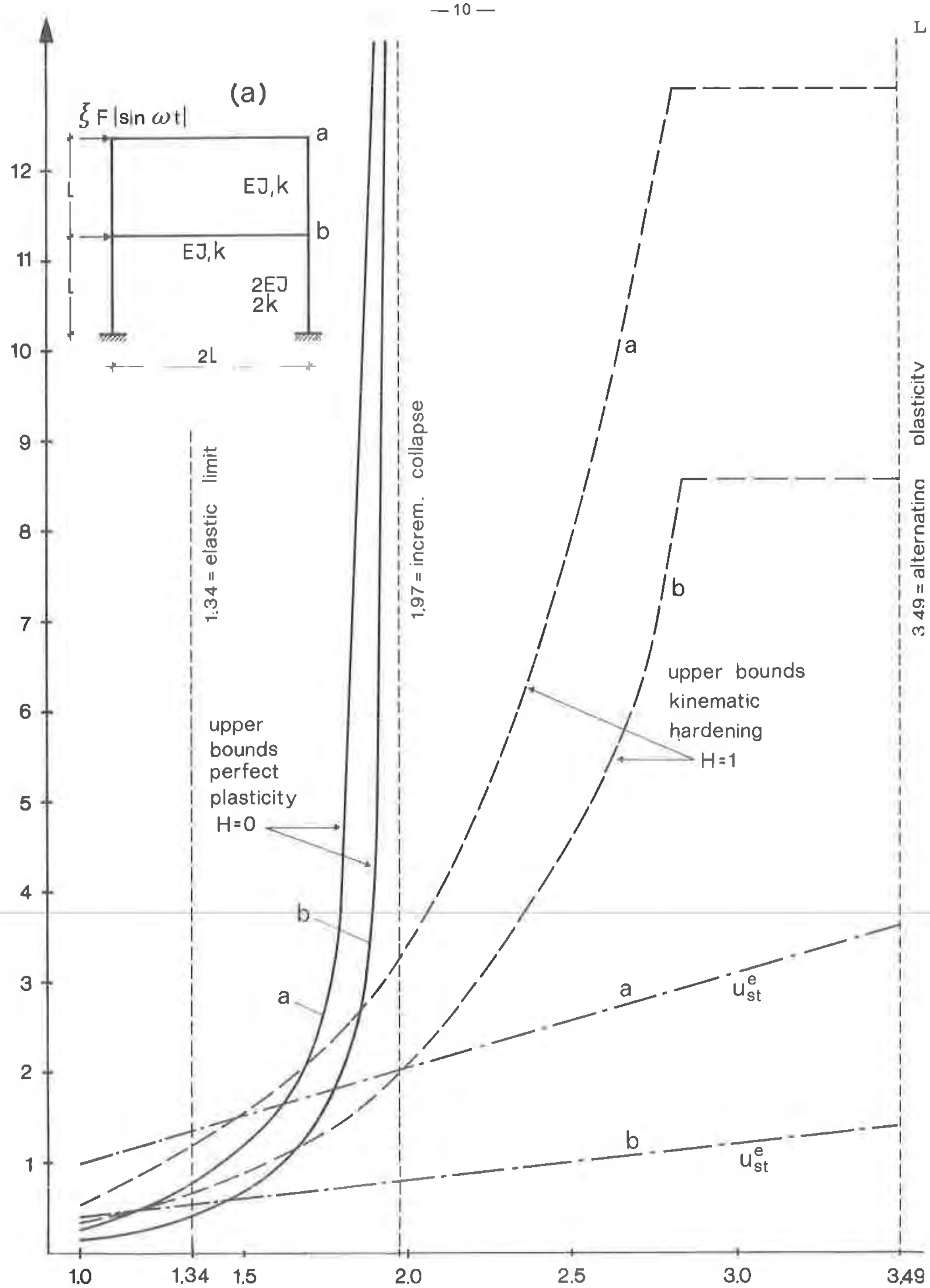


fig. 3



(a)
ξ F |sin ωt|
a
b
EJ,k
EJ,k
2EJ
2k
L
L
2L
12
11
10
9
8
7
6
5
4
3
2
1
1.34 = elastic limit
1.97 = increm. collapse
3.49 = alternating plasticity
upper bounds perfect plasticity H=0
upper bounds kinematic hardening H=1
a
b
u_{st}^{e}
u_{st}^{e}
1.0
1.34
1.5
2.0
2.5
3.0
3.49

fig. 4