The Energetic Approach of Elastoplastic Fracture Mechanics Applied to the Problem of Unloading

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ABSTRACT

In this paper, a cleavage parameter G_p, based on energetic considerations, is defined to characterise crack initiation in a brittle elastoplastic medium, and discussed in the context of the famous « Rice's paradox ». We start from Francfort-Marigo elastic fracture theory, based on a minimum energy principle, which generalises Griffith's theory. In this theory, the energy depends on the displacement field as well as the geometrical crack surface. The cleavage parameter is defined and determined as the value of the discrete gradient of an incremental elastoplastic potential. For that purpose, it is necessary to compute two solutions : the solution of the problem with the initial crack, and the solution of the problem after a finite crack propagation « Δl » taking into account the local unloading due to this propagation.

This parameter G_p is applied to the analysis of a centred cracked plate submitted to a loading first increasing (to open the crack) and then decreasing (to close the crack). The constitutive law is elastoplastic with an isotropic hardening. A very fine mesh is used in the area of the crack tip, and a parametric analysis is realised to study the effect of this mesh and of the Δl propagation. Therefore an important result is obtained : the G_p parameter is neither mesh dependent, nor Δl dependent. During the unloading, G_p decreases very quickly from a maximum value to zero, and we can affirm then that no propagation is possible, although the value of the J-integral is non zero.

The G_p parameter can be related to the G^{Δ} parameter proposed by Kfouri & Miller in 1976. As suggested by Rice, the value of this parameter seems to vanish to zero with the length Δl of the finite crack propagation. It means that, for an elastoplastic media, there would be no energy available for crack propagation. We will show that we would have obtained the same result for the G_p parameter with a standard mesh, but also that, if the mesh is sufficiently refined, the final result is not zero. Therefore, this G_p parameter can be used as a cleavage fracture parameter.

INTRODUCTION

The J-integral based theories are not appropriate when applied to the analysis of cracked structures submitted to unloading after loading, and more generally, in the case of non proportional loading. Briefly said, they are only valid in the context of Deformation Theory of Plasticity, but not in the context of Incremental Theory of Plasticity. Moreover, when appropriate, some constraint effects can be observed on the fracture toughness, the low values on specimen corresponding to high values of constraint. This situation needs to develop other approaches.

A first possibility is to develop the global approach. If we consider the J-parameter as the coefficient of the first term in the asymptotic development of the stress field, it is possible to introduce a second parameter, as the coefficient of the second term, and to develop an approach based on these two parameters (approach J/T, J/Q). Some triaxiality effects can then be explained, but the behaviour of the material is still limited to the Deformation Theory of Plasticity, which is very restricted. If we consider J as an energetic parameter (energy release rate), it is possible to extend it to Incremental Theory of Plasticity. Many parameters called « path-independent integral » have been proposed (T* of Atluri, Jch of Kishimoto-Aoki-Sakata, Jm of McClintock etc.) but no consensus has been obtained. The physical interpretation of these parameters is not clear, and there are numerical difficulties to calculate them, see [1].

A second possibility is to develop the local approach of fracture, taking into account the local mechanism of fracture in the microstructure of the material. The limits of the global approach can then be overcome. Different models can be used according to the fracture mechanism : Rice and Tracey, Weibull, Gurson, Rousselier etc. This approach is very much developed at the present time and seems promising, but it needs a careful identification of the model parameters, and there are some numerical difficulties related to the dependence of the results with respect to the mesh size especially near the crack tip.

In this paper we reconsider the « Energetic Approach » of elastoplastic fracture mechanics, starting from Francfort and Marigo elastic fracture theory [2], based on a minimum energy principle, which generalises Griffith's theory. This theory can be extended to the case of brittle fracture in an elastoplastic medium, and an energy release rate, noted G_p, can be defined. We will apply, on a simple case, this parameter to the analysis of the problem of unloading, where the J approach is not valid. Then this G_p parameter will be discussed in the context of the « Paradox of Rice » which leads to the conclusion that the G_p value must tend to zero with the crack propagation.

THE ENERGETIC APPROACH OF ELASTOPLASTIC FRACTURE MECHANICS

Francfort and Marigo Elastic Fracture Theory

A widely used approach in fracture mechanics to predict whether a crack propagates or not in an elastic medium relies on Griffith's criterion, see [2]. This approach is limited to progressive and continuous crack growth and cannot be used to predict crack initiation in a non-craked medium and discontinuous propagation. This is the reason why Francfort and Marigo have proposed a new elastic fracture theory [3], where these two phenomena can be predicted. One of its essential features consists in a discretisation of the load history : the true evolution of the structure during a load increment is taken into account only through the state of the structure (displacement field u and cracks positions S) at the beginning and the end of the considered load increment. More precisely, they consider the total energy E of the structure depending of u and S, defined as follows :

$$\mathrm{E}(u,S) = \mathrm{E}_{\mathrm{D}}(u) + G_c \ \mathrm{Area}(S) \tag{1}$$

where E_{D} is the (elastic) strain energy and G_c the toughness. Considering only the case of « prescribed displacement » loading, they postulate that the displacement field u at the end of the current load increment and the newly cracked surface ΔS during the increment realise a minimum of E. If S^{-} represent the cracks positions at the time t, the displacement field u and the cracks positions $S = S^{-} \cup \Delta S$ at the end of the current load increment realise :

$$\underset{\substack{\Delta S \\ u \in CA(S^{-} \cup \Delta S)}}{\mathrm{Min}} \quad \mathrm{E}\left(u,\, S^{-} \cup \Delta S\right) \tag{2}$$

where $CA(S^{-} \cup \Delta S)$ represents the displacements cinematically admissible with the loading and the cracks.

In case of progressive crack propagation, the Griffith criterium and the theory of Francfort and Marigo are equivalent [4]. The Griffith criterion can be retrieved by restricting ΔS (Δl in 2D) to a sufficiently small propagation dS (dl in 2D), and we can define an energy release rate G_{el} as the opposite of the variation of the potential energy $\mathrm{W_e}$ with respect to dS :

$$\mathrm{G}_{\mathrm{el}}(dS) \underset{\text{déf}}{=} -\frac{\mathrm{W_e}(dS) - \mathrm{W_e}(\varnothing)}{\mathrm{Area}(dS)} < G_c \tag{3}$$

Extension to a Brittle Elastoplastic Medium

The minimisation principle can be linked to incremental formulations with global internal variables as soon as the first term (elastic energy) along with the kinematic admissibility conditions are identified as the free Helmhotz' energy F and the second term (the energy dissipated when the crack propagates) is identified as the dissipation potential D, see [5].

Exploiting this link, we extend Francfort and Marigo theory in the case of an elastoplastic material by introducing the energetic contribution due to plasticity into the free Helmhotz' energy and the dissipation potential, assuming that fracture mechanisms and plasticity are independent [4]. If we call α the internal variables for plasticity, $\mathrm{E_{bl}}$ the hardening energy, $\mathrm{D_{pl}}$ the plastic dissipation potential, and $\mathrm{I}_{CA(S)}$ the indicator function of $CA(S)$, we define :

$$\begin{cases} \mathrm{F}(u,\alpha,S) = \mathrm{E}_{\mathrm{D}}(u,\alpha) + \mathrm{E}_{\mathrm{bl}}(\alpha) + \mathrm{I}_{CA(S)}(u) \\ \mathrm{D}(\Delta\alpha,\Delta S) = \mathrm{D}_{\mathrm{pl}}(\Delta\alpha) + G_c \mathrm{Area}(\Delta S) \end{cases} \tag{4}$$

Using the same process than the one used in the elastic case, we define an incremental elastoplastic potential W as :

$$\mathrm{W}(\Delta S) = \underset{(\Delta u,\Delta\alpha)}{\mathrm{Min}} \left[\mathrm{F}(u^{-} + \Delta u, \alpha^{-} + \Delta\alpha, S^{-} \cup \Delta S) + \mathrm{D}_{\mathrm{pl}}(\Delta\alpha) - \mathrm{W}_{\mathrm{ext}} \cdot \Delta u \right] \tag{5}$$

and an energy rate G_p for a sufficiently small propagation dS, as :

$$\mathrm{G}_p(dS) = -\frac{\mathrm{W}(dS) - \mathrm{W}(\varnothing)}{\mathrm{Area}(dS)} < G_c \tag{6}$$

Numerical Aspects

The solution of the problem can be computed by using the finite element method. The Δl propagation corresponds to one element behind the crack tip. The finite element calculation of G_p needs to release this element, for each step of the problem resolution, and to obtain once again the convergence on the equilibrium. Unilateral conditions have to be taken into account on the crack lips in order to avoid interpenetrating. The numerical convergence must be very severe to obtain a very accurate solution and to be able to calculate the very small variation of the energy W during the crack propagation Δl.

APPLICATION TO THE PROBLEM OF UNLOADING ON A SIMPLE CASE

Definition of the Problem

Let us consider a Centred Cracked Plate submitted to a loading in mode I that is first increasing and then decreasing to zero. The material is elastoplastic with a Von Mises criterion and a linear isotropic hardening. The data related to geometry, loading and material are presented on **Fig. 1**. Due to symmetries only a quarter of the structure is represented, and the plane strain hypothesis is assumed.

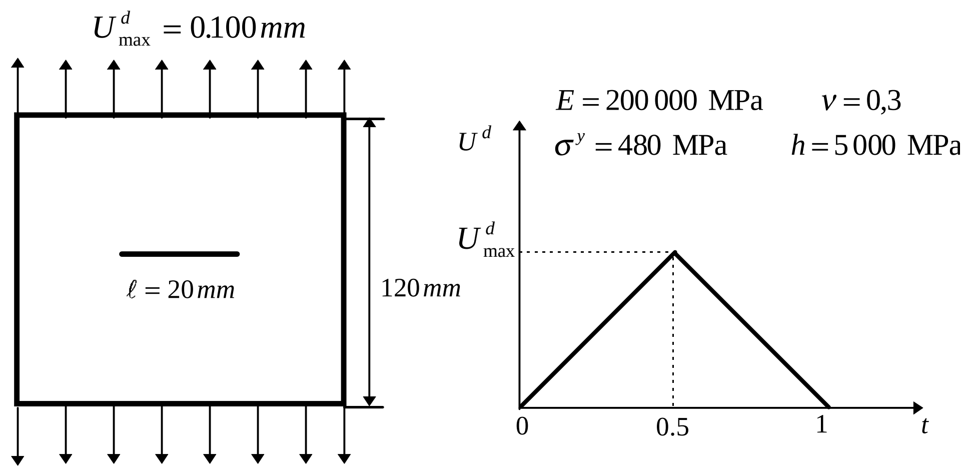

Fig. 1 - Geometry, loading and material data

The mesh is presented on **Fig. 2**, a zoom of the mesh in the crack tip area is presented on **Fig. 3**, and a zoom of the mesh near the crack tip is presented on **Fig. 4**. We can see that the mesh is very refined. More than 10000 nodes are used and the size of the smallest element near the crack tip is about one micron. Different values of the size « Δl » of the element located at the crack tip and corresponding to different meshes are investigated as follows :

- M1 : mesh for Δl = 0.0500 mm, M2 : mesh for Δl = 0.0125 mm, M3 : mesh for Δl = 0.0062 mm,
- M4 : mesh for Δl = 0.0025 mm, M5 : mesh for Δl = 0.0012 mm

We consider 4 values for the maximum loading after which the loading decreases :

- Umax = 0.016 mm, Umax = 0.030 mm, Umax = 0.050 mm, Umax = 0.100 mm,

For each loading and for each mesh we make the computation in 20 steps (10 steps when the loading increases and 10 steps when the loading decreases). After that and for each step, one element is released to obtain the propagation of the crack.

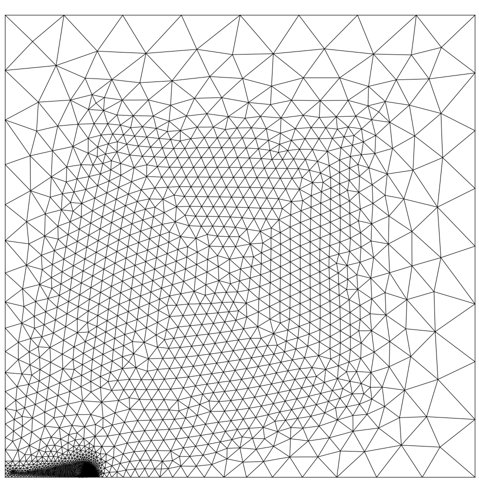

Fig. 2 - Mesh of a quarter of the structure

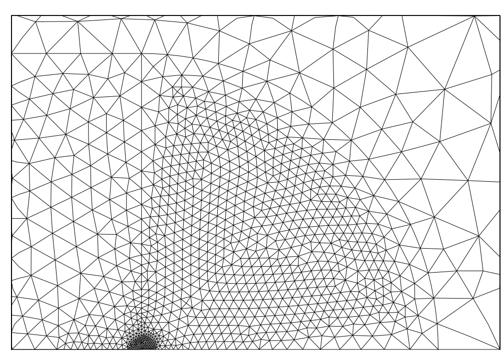

Fig. 3 - Zoom of the crack tip area

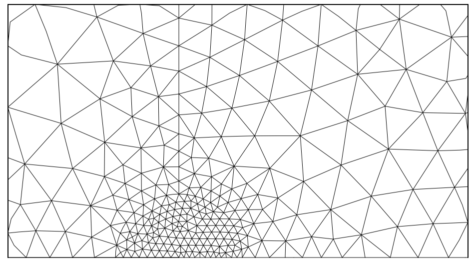

Fig. 4 - Zoom of the mesh near the crack tip

Results Obtained

On **Fig. 5** the variation of G_p as a function of the loading is presented for the 3 meshes M3, M4 and M5 and for different maximum values of the loading. We can see that the G_p value steadily increases while the loading is increasing, and afterwards this value falls down suddenly to zero while the loading is decreasing. Then the G_p value stabilises at this zero value corresponding to the « closure » of the crack tip, when unilateral conditions are taken into account. The values obtained for the 3 meshes seem to converge to a non zero value.

On **Fig. 6** we can see that the curve obtained with the mesh M5 is very close to the curve extrapolated with all the results in order to obtain the result corresponding to Δl equal 0. So in this case, we can conclude that the G_p value is not zero and that we can use G_p as a fracture parameter.

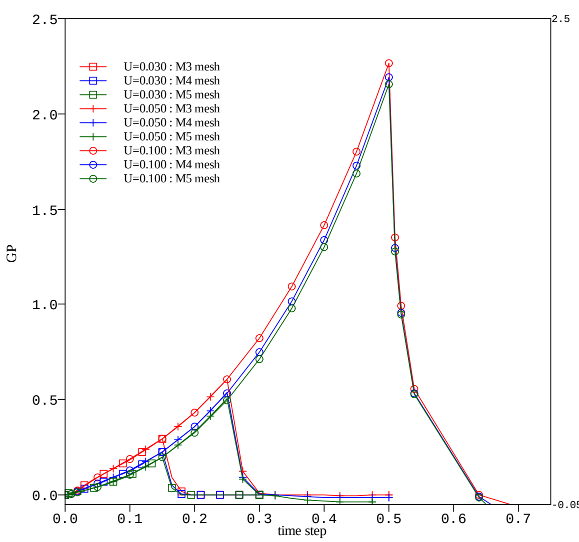

Fig. 5 - Results for different Δl : Umax = 0.100 mm

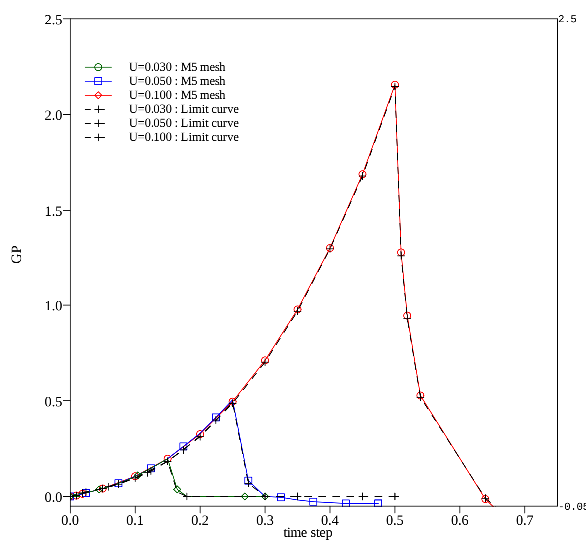

Fig. 6 - Convergence of the results

As we know, as soon as the loading decreases, it becomes non proportional, and the J-approach cannot be applied. So the variation of the G_p parameter during unloading is very interesting and it is possible to take advantage of that. Indeed, $G_p = 0$ means that there is no energy available in the structure to make the crack propagate. So the conclusion is that in such a case the initiation of the crack is impossible.

It has to be pointed out that this result can be obtained only if unilateral contact conditions are prescribed at the crack tip, and more precisely on the side of the finite element corresponding to the crack propagation. These conditions are triggered as soon as the crack tip area become in compression, and without any contact on the crack lips. Indeed, the crack remains open during the unloading, due to the plastic deformation accumulated during the loading. The change of sign of the opening stresses behind the crack tip corresponds precisely to the zero value of the parameter G_p.

Furthermore, we have a problem of numerical accuracy in the case of unloading after the higher maximum value : Umax = 0.100 mm. Indeed, we can notice some negative values of G_p, which is impossible. The reason for this problem is that it is difficult to obtain convergence with, at the same moment, unilateral conditions and very high deformations near the crack tip. This point needs to be improved.

RECONSIDERATION OF THE « PARADOX OF RICE »

Background

The G_p parameter represents the energy available in the structure to obtain a *dl* propagation, divided by *dl*. In 1966, Rice considered a continuously growing crack in an elastoplastic material where the flow strength saturates to a finite value at large strain, and demonstrated that this value must be zero, see [6]. This is the « Paradox of Rice ». This result cannot be applied to a stationary crack or to a non bounded flow strength material, but in 1976, Kfouri and Miller [7], considered this case and they found a result in agreement with the Paradox of Rice. They proposed a parameter called G^{Δ} defined as $\Delta W/\Delta l$, ΔW being the work of the opening forces released during a small amount Δl of the crack, and found : G^{Δ} equal 0 when Δl goes to zero. Although the G_p definition is more general, it can be proved, in the case of a 2D elastoplastic media and for a sufficiently small propagation Δl, that G_p is equal to G^{Δ}. In 1977, Rice used the result $G^{\Delta} = 0$ to generalise his paradox [8].

In 1980, Nguyen presented the analysis of the problem of plane crack extension in dissipative continuum within the classical thermodynamic description [9]. He introduced the thermodynamic crack extension force F (equivalent here to the G_p parameter) and pointed out the hypothesis leading to the result F = 0. In 1987, Bui and Dang Van [10], showed that the Paradox of Rice is valid for a large class of elastoplastic materials, including hardening materials which show a power law relation.

It seems that all these works lead to the conclusion that $G_p = 0$ in the large context of incremental theory of plasticity and that it is now a general and well accepted result. The work of Kfouri and Miller is often mentioned in synthetic papers on elastoplastic fracture mechanics [1], [11], as a definite proof of this result. Nevertheless, if we reconsider it 25 years after, it is not so clear, in particular because some numerical aspects of the modelisation used by Kfouri and Miller are insufficient.

Discussion

Observation : if we return to the curves presented on the **Fig. 5**, we observe that the results converge to a non zero value, and this observation is confirmed on the **Fig. 6** giving the extrapolated curve. Now if we look very carefully at the beginning of the curves on the **Fig. 5**, that is to say when the loading is small, it seems that there is a certain dependency of the results with respect to the mesh or to the Δl value. To make it clear, let us make a zoom on this region, and let us consider the case corresponding to the smallest value of the maximum loading applied : U = 0.016 mm. Then, we obtain the result presented on **Fig. 7**, which amplifies significantly the effect detected on the **Fig. 5**. The curves are obviously mesh or Δl dependent, and then we could conclude : the G_p parameter seems to tend to zero, in agreement with the Paradox of Rice.

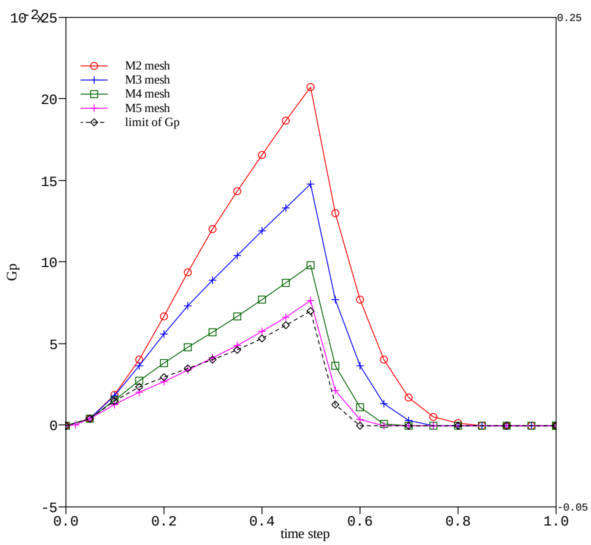

Fig. 7 - Results for different Δl : Umax = 0.016 mm

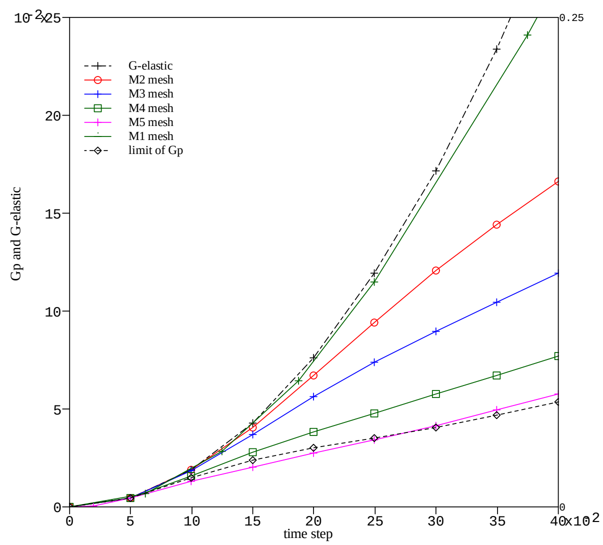

Fig. 8 - Zoom of Fig. 7

Explanation : an important parameter has to be defined in order to understand these results : Np = Rp/ Δl (Rp being the radius of the plastic zone). Obviously, the value of this parameter must be sufficiently high, and because the radius Rp is increasing from zero with the loading, it will be impossible to obtain a precise result on G_p if the loading is very low. That means that the value of Δl (even very low but fixed), will always be too high compared to the loading or to the radius Rp. We can verify this point on **Fig. 8** (zoom of **Fig. 7** for very low loading) where the different curves « Numerical G_p » are distributed, with respect to Δl, between the low curve « limit of G_p » (low, but not zero) and the high curve « G-elastic (or J)».

These curves are located in a very large area and we can conclude that the result obtained for a very low loading cannot be precise. We will always have an apparent dependence of the result with respect to Δl.

Moreover, it is possible to exhibit a relation between « Np » and the accuracy of the results noted « Eps ». For that purpose, Rp is calculated from an analytical formula corresponding to the case of an infinite medium and a perfect plastic material. The accuracy Eps is obtained from the relative difference between the calculated value and the extrapolated value of G_p. Then, we obtain the following table giving an idea of the mesh refinement corresponding to a given value of Eps :

Np	5	20	50	120
Eps	50%	20%	10%	5%

We can see that it is necessary to refine the mesh in the plastic zone with a lot of elements (more than 50 !) if we want to obtain accurate results. So if the mesh is not refined enough, or the value of Δl not small enough, the results will always show a Δl dependency. It will be always the case for very low loading. Of course if the loading is higher, this phenomenon disappears and a precise result can be obtained.

Validation

Some aspects of the methodology proposed need a validation. The first aspect is related to the propagation of the crack in one step and on a length Δl. It is of course necessary to verify if the results depend on the mesh or not, or more precisely if they depend on the number of elements released, for a given Δl. On **Fig. 9**, the variation of G_p as a function of the loading is presented for Umax = 0.016 mm and for the M2 mesh (corresponding to Δl = 0.0125 mm). In this case, only one element is released. Two other results are presented on **Fig. 9**, for the M2b and the M2c meshes. These two meshes correspond to the same length of propagation (Δl = 0.0125 mm), but the meshes are more refined, 2 elements are released for the M2b mesh, and 4 elements are released for the M2c mesh. We can see that the 3 curves are very close from each other. Therefore we have no effect of the mesh (or the number of elements released) for a given Δl.

Another aspect concerns the calculation of the G_p parameter as the difference between the total energy of the structure before the propagation and the total energy of the structure after propagation. As this difference is very small in comparison with the value of the energy it is necessary to verify if the result is significant. So, we suppose that we are in the context of the Deformation Theory of Plasticity (we assume a non linear elastic constitutive law for the material), and we compare the values of G-theta, the energy release rate calculated by the Theta method [12], - that is to say J-integral - on 4 different crowns area, with the values of the G_p parameter, calculated for the 4 values of Δl. On the **Fig. 10**, we can see that the 8 curves are very close from each other. That means that our G_p calculation is very precise.

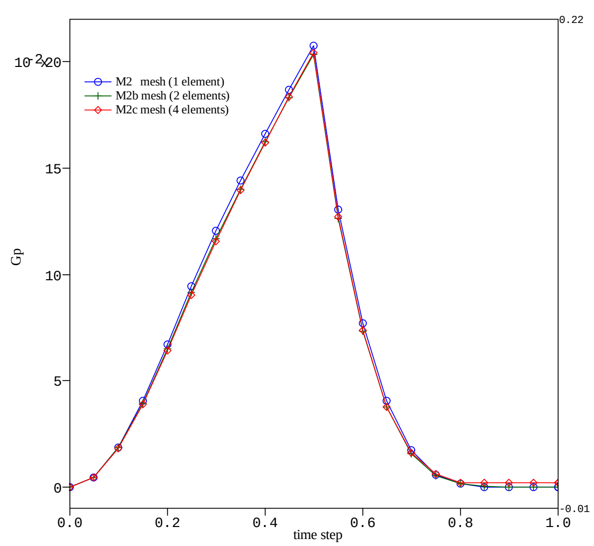

Fig. 9 - Mesh dependence (number of elements released)

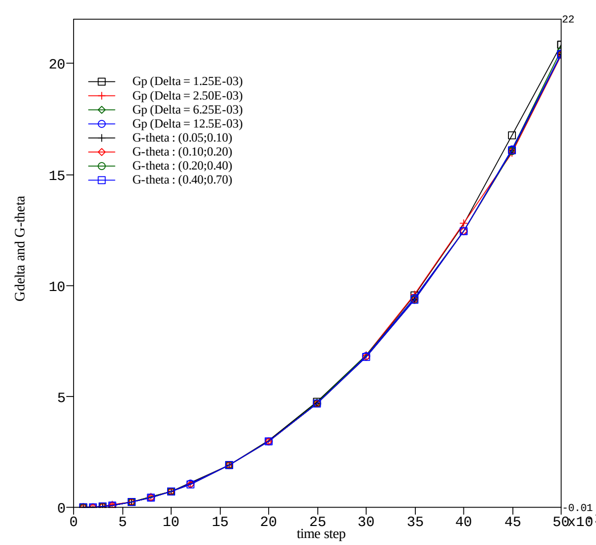

Fig. 10 - Gp compared to G-theta (J-integral)

Dependence of the Results on the Hardening Coefficient

We have obtained the following result considering an elastoplastic material with linear hardening : G_p is not zero, in contradiction with the Paradox of Rice. A clear explanation has been presented and we can affirm that the Paradox of Rice is not valid in this case. Does this result depend on the hardening ? In fact, the Paradox of Rice concerns only a continuously growing crack in an elastoplastic material where the flow strength saturates to a finite value at large strain. What would be the conclusion for a stationary crack and a bounded flow strength material, or more simply a perfect plastic material ? To answer this question, two different cases of hardening are considered : one case with a low value of the hardening coefficient (h = 100.), and one case with a zero value of the hardening coefficient (h = 0.) corresponding to the case of perfect plasticity.

On **Fig. 11** the variation of G_p as a function of the loading is presented for the 4 meshes M2, M3, M4 and M5, for the case Umax = 0.100 (we do not consider here the decreasing of the loading), and for the small value of the hardening coefficient. Now, we can see that the G_p values seem to tend to zero with Δl, except at the end of the loading. This is due probably to the non zero value of the hardening coefficient. Indeed we can verify on the **Fig. 12**, that when this coefficient is zero, the G_p values are clearly going to zero with Δl. Therefore, in this case the Paradox of Rice is verified.

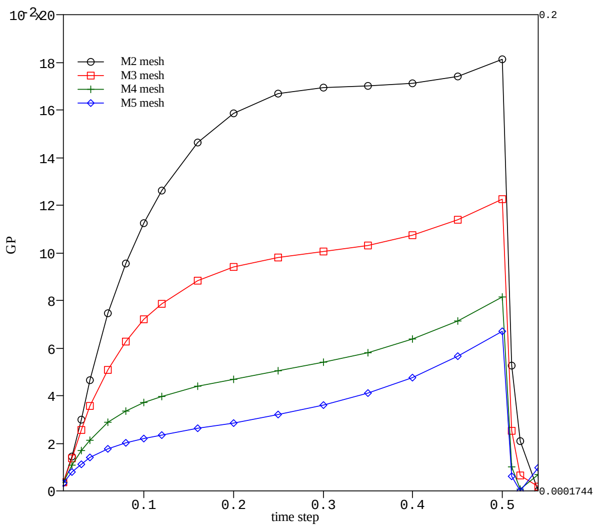

FIG. 11 - Small hardening coefficient (H = 100.)

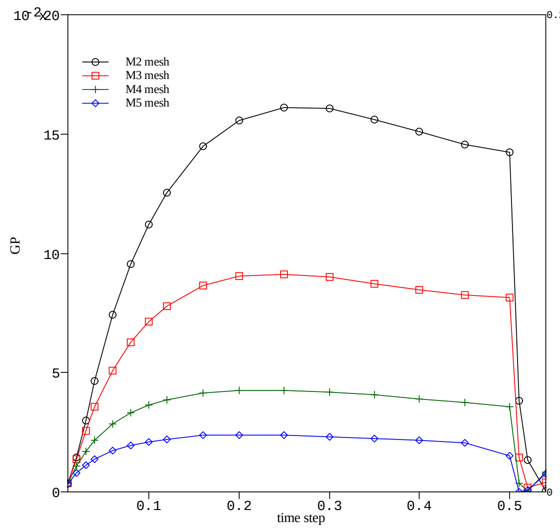

FIG. 12 - Perfect plasticity (H = 0.)

CONCLUSION

As the Griffith's approach is limited to progressive and continuous crack growth and cannot be used to predict crack initiation in a non-craked medium and discontinuous propagation, Francfort and Marigo have proposed a new elastic fracture theory, which generalises the Griffith approach, and where these two phenomena can be predicted. We have extended this theory to brittle fracture in an elastoplastic medium, and we have defined an energy release rate noted Gp as the opposite of the variation of the (incremental) potential energy W of the structure. This parameter G_p has been applied to the analysis of a centred cracked plate submitted to a loading first increasing and then decreasing. We have obtained the following results :

- the G_p value steadily increases while the loading is increasing, and afterwards this value falls down suddenly to zero while the loading is decreasing, then the G_p value stabilises at this zero value corresponding to the « closure » of the crack tip, when unilateral conditions are taken into account,
- the curve obtained with the M5 mesh (the more refined) is very close to the curve extrapolated with all the results in order to obtain the result corresponding to Δl (crack propagation) equal to zero, that is to say : Gp converge to a finite value,
- when the loading decreases, $G_p = 0$ means that there is no energy available in the structure to make the crack propagate, and in such a case the initiation of the crack is impossible.

So, we have conclude that, in the case of a stationary crack and for an elastoplastic material with linear hardening, the G_p parameter is not identically zero and that we can use it as a fracture parameter. This conclusion has been discussed in the context of the Paradox of Rice. Indeed, as the G_p parameter can be related to the G^{Δ} parameter proposed by Kfouri & Miller in 1976, as suggested by Rice, the value of this parameter should vanish to zero with the length Δl of the finite crack propagation. Nevertheless the analysis of the results obtained with very fine meshes reveals that :

- if we consider the results obtained for very low values of the applied loading, these results are necessary mesh dependent, or Δl dependent, and it seems that these values tend to zero when Δl goes to zero,
- this apparent dependence could be explained if we consider the parameter Rp/ Δl corresponding to the mesh refinement in the plastic zone area, which must be sufficiently high : Δl must be about 1% of the plastic zone size.

The conclusion is that in the case of a stationary crack in an elastoplastic material with linear hardening, the G_p parameter is not zero, and the Paradox of Rice is not applicable. The result obtained by Kfouri and Miller can be explained by an insufficient refinement of the mesh in the plastic zone.

In order to validate the methodology proposed we have shown that the results obtained do not depend on the number of elements released for a given crack propagation Δl, and that the methodology used to calculate G_p is accurate. Indeed, assuming the deformation theory of plasticity hypothesis, we have compared the values of G-theta calculated on different crown areas, to the values of G_p calculated for different values of Δl, and we have found that the results are very close.

Then, we have considered the case of a stationary crack and in a perfect plastic material and we have verified that the G_p parameter tends to zero with the crack propagation Δl. Therefore, in such a situation, the Paradox of Rice is verified.

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