Non-Gaussian Linearization: An Efficient Tool to Analyze Nonlinear MDOF-Systems

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ABSTRACT

Nuclear structures, such as contaiments, primary piping, etc. may respond nonlinearly when exposed to high intensity loading, such as earthquakes. Since the latter reveal distinct random properties, methods are required by which nonlinear MDOF-systems under stochastic excitation can be treated. So far in probabilistic nonlinear dynamics, only SDOF-systems can be analysed such that the results can be applied for a credible reliability analysis. The analysis of the stochastic response of nonlinear MDOF-systems, however, is still confined to biased estimates of its first two statistical moments. The method introduced in this paper can treat this class of problems in an efficient manner. By utilizing and expanding the method of equivalent linearization, the complete probability information of the response can be determined in every point of the structure. This implies that not only the parameters (mean and variance characterizing only a Gaussian response), but the correct *type* of distribution - which is the only basis for a realistic reliability analysis - is obtained. The extention of the conventional equivalent linearization technique is based on nonlinear transformations - with quite general properties - and on the Fokker-Planck equation by which non-Gaussian properties of the response can be evaluated.

INTRODUCTION

It is a well known fact that the characterization of the performance of structures or structural systems under severe loading requires the utilization of most realistic and accurate load as well as structural, i.e. mechanical modeling. For the case of ultimate load failure analysis - which is generally of primary interest in required safety evaluations - this implies a very good knowledge and hence description of the inelastic range of the response. It is also known, that in most cases future loading conditions, which a structure might experience during its design life, can not be predicted with absolute certainty. Severe loading conditions, such as earthquake loads - which might lead to structural collapse - for example, are associated with considerable uncertainties. In other words, even if the record of of a severe earthquake at a particular site is known, the characteristics of the next event might be quite different. Hence, a realistic and accurate analysis has to take into account these uncertainties in quantitative terms. This implies that the uncertainties of earthquake events have to be described by random processes. The effect of earthquakes on structures is generally of dynamic nature. To solve this class of problems, i.e. to determine the inelastic dynamic response of structures under dynamic stochastic excitation, various methods are available. They may be categorized into the following mayor types: (1) Methods based on the Fokker-Planck equation, (2) equivalent linearization methods, (3) perturbation methods,

(4) series representations and (5) simulation methods. Their merits, accuracy and range of applicability are summarized and compared in review papers e.g.(Roberts, 1981; Spanos,1981; Crandall & Zhu, 1983). Among these approaches, those based on equivalent linearization have the widest range of applicability to practical MDOF-systems (Wen, 1988). All other procedures - except the costly simulation method - are confined to a very small number of degrees of freedom (≤2) and, hence, are of limited practical interest.
Equivalent linearization replaces the set of coupled nonlinear differential equations by a "statistically equivalent" linear set for which well developed and efficient procedures exist to determine the statistics of the response. The linearized system is determined such that the error between the true nonlinear and linearized system is minimized in a mean square sense. Investigations, however showed (see e.g. Pradlwarter & Schuëller,1987, Wen,1988) that errors in the prediction of the variance, i.e. the second statistical moment, of the response may be as high as 25 %. In this paper it is shown why the presently applied procedures of equivalent linearization - which in fact are limited to the prediction of the variance of the response - are prone to errors. In addition, the method is improved by minimizing these errors to a negligible quantity. Moreover, the procedure of equivalent linearization is expanded such that the probability distribution of the response is derived as well. Only this information lays the ground for a credible reliability analysis of structures of large size under stochastic dynamic loading conditions.
The improvement and extension of the method of equivalent linearization is accomplished in the following steps. First, the conventional procedure assumes that the response follows a Gaussian distribution. This, of course can not be true generally for a nonlinear system, even if the excitation is Gaussian, i.e. normally distributed. In the present procedure, the realistic shape of the distribution of the response is determined by utilizing the diffusion, i.e. Fokker-Planck equation. Since knowing the accurate variance of the response depends on knowing the accurate type of the distribution of the response, it is quite clear that the problem can be solved only numerically in an iterative way by applying a nonlinear transformation.
The practical applicability of the procedure is demonstrated by determing the structural response information of the containment structure of the HDR-containment (see Steinhilber & Ludwig,1983) under earthquake loading, where the structural response is described by a set of $N = 21$ coupled first order differential equations. The procedure is derived such that it is applicable to any other system irrespectivly of the type of nonlinearity or number of degrees of freedom. A respective generally applicable software has been developed.

NON-GAUSSIAN STATISTICAL EQUIVALENT LINEARIZATION

The response of a nonlinear system can generally be represented by a state vector following a set of first order differential equations

$$\frac{d\mathbf{y}(t)}{dt} = \mathbf{g}(\mathbf{y}(t))+\mathbf{b}(t)+\mathbf{w}(t) \tag{1}$$

where $\mathbf{g}(\mathbf{y}(t))$ is a nonlinear vector function. The state vector of the response is denoted by $\mathbf{y}(t)$ and the excitation is split into a deterministic part $\mathbf{b}(t)$ and stochastic part $\mathbf{w}(t)$. The random excitation is assumed to consist of white noise of zero mean, i.e.

$$E\{\mathbf{w}(t_1)\mathbf{w}^T(t_2)\} = \mathbf{I}(t_1)\cdot \delta(t_1-t_2) \tag{2}$$

where $\delta(t)$ denotes Dirac's impulse function and $\mathbf{I}(t)$ the intensity matrix.
Unfortunately, there is presently no generally applicable method available to determine completely the stochastic response of the state vector $\mathbf{y}(t)$, i.e. the joint probability function $f_{\mathbf{Y}(t)}(\mathbf{y})$. Conversely to nonlinear systems, however, exist efficient methods to evaluate the stochastic response of linear systems subjected to white noise excitation. Its particularity is a jointly normal distributed response where the first two moments suffices to characterize completely the stochastic response.

These procedures become applicable also for nonlinear systems if eq.(1) is converted to a set of linear differential equations

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{C}(t)\cdot \mathbf{x}(t)+\mathbf{h}(t)+\mathbf{w}(t) \tag{3}$$

where $\mathbf{C}(t)$ and $\mathbf{h}(t)$ are determined such that

$$E\{\mathbf{y}(t)\} = E\{\mathbf{x}(t)\} \tag{4}$$

$$E\{\mathbf{y}(t)\mathbf{y}^T(t)\} = E\{\mathbf{x}(t)\mathbf{x}^T(t)\} \tag{5}$$

It has been proofed (Kozin,1988; Pradlwarter,1989) that the above relations are satisfied *exactly* for *any* nonlinear system if

$$\mathbf{C}(t)=[E\{\mathbf{g}(\mathbf{y}(t))\mathbf{y}^T(t)\}-E\{\mathbf{g}(\mathbf{y}(t))\}\cdot E\{\mathbf{y}(t)\}^T]\cdot \mathbf{S}^{-1}(t) \tag{6}$$

$$\mathbf{h}(t)=\mathbf{b}(t)+E\{\mathbf{g}(\mathbf{y}(t))\}-\mathbf{C}(t)\cdot E\{\mathbf{y}(t)\} \tag{7}$$

where $\mathbf{S}(t)$ denotes the covariance matrix

$$\mathbf{S}(t)=E\{\mathbf{y}(t)\mathbf{y}^T(t)\}-E\{\mathbf{y}(t)\}\cdot E\{\mathbf{y}(t)\}^T \tag{8}$$

For the evaluation of $\mathbf{C}(t)$ and $\mathbf{h}(t)$, the joint distribution $f_{\mathbf{Y}(t)}(\mathbf{y})$ would be required. Clearly, the joint distribution is not available, since if it were known, there would be no need to linearize. Instead the true statistics being applied, the conventional practice is to assume the nonlinear response normally distributed, i.e. $f_{\mathbf{Y}(t)}(\mathbf{y}) = f_{\mathbf{X}(t)}(\mathbf{y})$. Although it is well known that $f_{\mathbf{Y}(t)}(\mathbf{y})$ is generally non-Gaussian, this simplification has been useful since 30 years in many practical applications to obtain an estimate of the first two moments of the nonlinear response. However, due to the unjustified assumption, a bias is introduced leading to considerable errors in case $f_{\mathbf{Y}(t)}(\mathbf{y})$ deviates far from a jointly normal PDF. Moreover, approximate first two moments of the joint distribution are generally not sufficient for a credible reliability analysis where the tails of $f_{\mathbf{Y}(t)}(\mathbf{y})$ might differ considerably from a jointly normal distribution.For an essential progress toward a credible reliability analysis, it is therefore important to take non-Gaussian properties of the response into account. Instead of using the identity relation $f_{\mathbf{Y}(t)}(\mathbf{y}) = f_{\mathbf{X}(t)}(\mathbf{y})$, suitable nonlinear transformation

$$\mathbf{y}(\mathbf{x})=F_{\mathbf{Y}}^{-1}(F_{\mathbf{X}}(\mathbf{x})) \tag{9}$$

between the nonlinear state vector $\mathbf{y}(t)$ and the linearized (Gaussian) state vector $\mathbf{x}(t)$ are considered. F denotes in eq.(9) the cumulative distribution function and F^{-1} its inverse. Note, that these nonlinear relations must satisfy eq.(4) and (5). Hence,

$$E\{y_i(\mathbf{x})\}=E\{x_i\} \tag{10}$$

$$E\{y_i(\mathbf{x})y_j(\mathbf{x})\}=E\{x_i x_j\} \tag{11}$$

Generally,each component $y_i(\mathbf{x})$, $1\le i\le N$, is function of all components of the state vector $\mathbf{x}(t)$. This leads for MDOF-systems to quite involved relations, whose evaluations might become untractable. A practicable simplification is the construction of non-Gaussian joint distributions by its marginals $f_{Y_i}(y_i)$ and the covariance matrix $\mathbf{S}$. For the establishment of suitable marginal distributions, the special case $y_i(\mathbf{x})=y_i(x_i)$ suffices. In order to meet the constraints in eq.(11), modified correlation coefficient ρ'_{ij} are evaluated from the following integral equation (Nataf, 1962; Liu & Der Kiureghian, 1986),

$$E\{x_i x_j\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y_i(x_i)y_j(x_j)f_{X_iX_j}(x_i,x_j;\rho'_{ij})dx_i dx_j \tag{12}$$

where $f_{X_iX_j}(x_i,x_j;\rho'_{ij})$ denotes a two dimensional normal PDF with correlation coefficient ρ'_{ij}. The above integral equation leads to Nataf's simplified approximate joint distribution of the state vector $\mathbf{y}$,

$$f_{\mathbf{Y}}(\mathbf{y})=f_{\mathbf{X}}(\mathbf{x};\ \mathbf{R}')\prod_{k=1}^{N}\left(\frac{\partial y_k}{\partial x_k}\right)^{-1} \tag{13}$$

where the modified correlation matrix $\mathbf{R}'$ contains all correlation coefficients ρ'_{ij} determined by eq.(12) and $f_{\mathbf{X}}(\mathbf{x})$ denotes a jointly normal PDF. Utilizing the above representation for the non-Gaussian joint distribution, all expectations required in eq.(6) or (7) can be evaluated and subsequently the statistically linearized system of eq.(3), e.g.:

$$E\{\mathbf{g}(\mathbf{y}(t))\mathbf{y}^T(t)\} = \int_{-\infty}^{+\infty} \mathbf{g}(\mathbf{y}(\mathbf{x}(t)))\mathbf{y}^T(\mathbf{x}(t))f_{\mathbf{X}}(\mathbf{x};\mathbf{R}')d\mathbf{x} \tag{14}$$

The accuracy of the above outlined procedure depends mainly on two ingredients. First, on the appropriateness of the representation of the joint distribution by marginal distributions in eq.(13). Secondly, on the accuracy of the utilized marginal distributions, defined by normalized nonlinear relations $\hat{y}_i(\hat{x}_i)$, in which $\hat{x}_i$ and $\hat{y}_i$ are defined as follows:

$$\hat{x}_i = \frac{x_i - E\{x_i\}}{\sigma_{x_i}} \quad \text{and} \quad \hat{y}_i = \frac{y_i - E\{x_i\}}{\sigma_{x_i}} \tag{16}$$

The evaluation of $\hat{y}_i(\hat{x}_i)$ is shown in the following section.

GOVERNING EQUATION OF NONLINEAR TRANSFORMATIONS

The nonlinear transformations $y_i(x_i)$, $1 \le i \le N$, are evaluated by utilzing the Fokker-Planck equation (Kolmogorov forward equation). The Fokker-Planck equation (FPE) for the linearized system of order N in eq.(3) reads

$$FPE_L(\mathbf{x}) = \frac{\partial f_{\mathbf{X}}(\mathbf{x})}{\partial t} + \sum_{m=1}^{N} \frac{\partial}{\partial x_m}[\dot{x}_m f_{\mathbf{X}}(\mathbf{x})] - \frac{1}{2}\sum_{m=1}^{N}\sum_{n=1}^{N} I_{mn}(t)\frac{\partial^2 f_{\mathbf{X}}(\mathbf{x})}{\partial x_m \partial x_n} = 0 \tag{17}$$

and analogous for the nonlinear system in eq.(1):

$$FPE_N(\mathbf{y}) = \frac{\partial f_{\mathbf{Y}}(\mathbf{y})}{\partial t} + \sum_{m=1}^{N} \frac{\partial}{\partial y_m}[\dot{y}_m f_{\mathbf{Y}}(\mathbf{y})] - \frac{1}{2}\sum_{m=1}^{N}\sum_{n=1}^{N} I_{mn}(t)\frac{\partial^2 f_{\mathbf{Y}}(\mathbf{y})}{\partial y_m \partial y_n} = 0 \tag{18}$$

Since the response of the state vector $\mathbf{x}(t)$ is jointly normal, its joint distribution can be represented as

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^N/\det(\mathbf{A})}} \exp\{-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_{ij}x_i x_j\} \tag{19}$$

in which

$$[\alpha_{ij}] = \mathbf{A} = \mathbf{S}^{-1} \tag{20}$$

A representation of the joint distribution of the state vector $\mathbf{y}(t)$ can be established by utilizing eq.(13):

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{\exp\{-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha'_{ij}x_i(y_i)x_j(y_j)\}}{\sqrt{(2\pi)^N/\det(\mathbf{A}')}} \prod_{k=1}^{N}\left(\frac{\partial y_k(x_k)}{\partial x_k}\right)^{-1} \tag{21}$$

where

$$[\alpha'_{ij}] = \mathbf{A}' = [\rho'_{ij}\sigma_{xi}\sigma_{xj}]^{-1} \tag{22}$$

Also the derivative $\dot{x}_m$ and $\dot{y}_m$ of eq.(17) and (18), respectively, can be further specified as:

$$\dot{x}_m = h_m + \sum_{k=1}^{N} c_{mk}\, x_k \tag{23}$$

$$\dot{y}_m = b_m + g_m(\mathbf{y}) \tag{24}$$

The above relations allow the derivation of the derivatives

$$\frac{\partial f_{\mathbf{X}}(\mathbf{x})}{\partial x_i};\quad \frac{\partial^2 f_{\mathbf{X}}(\mathbf{x})}{\partial x_i \partial x_j};\quad \frac{\partial f_{\mathbf{Y}}(\mathbf{y})}{\partial y_i};\quad \frac{\partial^2 f_{\mathbf{Y}}(\mathbf{y})}{\partial y_i \partial y_j};\qquad 1 \le i,j \le N$$

given in the Appendix (A.1)÷(A.5) and subsequently the Fokker-Planck equations $FPE_L(\mathbf{x})$ and $FPE_N(\mathbf{y})$ to be found in eq.(A.6) and eq.(A.7). Note, that in case $\mathbf{g}(\mathbf{y}) = \mathbf{C}\cdot\mathbf{y}$ is linear the right hand side of eq.(A.6) and eq.(A.7) coincide, where $y_k(x_k) = x_k$, $\partial y_k/\partial x_k = 1$ and higher derivatives vanish.

The Fokker-Planck equation for the nonlinear system $FPE_N(\mathbf{y}(\mathbf{x})) = 0$ in eq. (A.7) defines basically a partial differential equation in the whole N-dimensional space for the required transformations $y_i(x_i)$, $1 \le i \le N$. The $FPE_N(\mathbf{y}(\mathbf{x}))$ is derived under the assumption that eq.(4) and (5) are satisfied exactly. This is true only in case the exact PDF $f_{\mathbf{Y}(t)}(\mathbf{y})$ has been found. It is justified, however, to neglect errors in eq.(4) and (5) by use of eq.(13) after all nonlinear transformations $y_i(x_i)$ have been evaluated. Moreover, the coefficients α_{ij}', defined in eq.(22) and (12), depend on the nonlinear transformations $y_i(x_i)$ and $y_j(x_j)$. These dependencies are taken into account in a iterative manner, where $f_{\mathbf{X}}(\mathbf{x})$ and all coefficients α_{ij}' are calculated from privious transformations $y_i(x_i)$, $1 \le i \le N$, but held fixed while improved updated transformations $y_i(x_i)$ are evaluated.
In order to avoid complications due to the derivate $\partial f_{\mathbf{x}}(\mathbf{x})/\partial t \neq 0$ in case of nonstationarity, appearing in eq.(A.6) and (A.7), the potential (scalar) $F(\mathbf{y}(\mathbf{x}))$ is introduced

$$F(\mathbf{y}(\mathbf{x})) = \frac{FPE_N(\mathbf{y}(\mathbf{x}))}{f_{\mathbf{Y}}(\mathbf{y}(\mathbf{x}))} - \frac{FPE_L(\mathbf{x})}{f_{\mathbf{X}}(\mathbf{x})} \cong 0 \tag{25}$$

which should be approximately zero. Only in case eq.(13) is an exactly valid representation of the joint distribution $f_{\mathbf{Y}(t)}(\mathbf{y})$, $F(\mathbf{y}(\mathbf{x})) = 0$ follows. The scalar function $F(\mathbf{y}(\mathbf{x}))$ defines all relations $y_i(x_i)$, $1 \le i \le N$, in terms of a partial nonlinear differential equation of third order and will be called "governing equation of $\mathbf{y}(\mathbf{x})$", since it provides the basic relation for the determination of $\mathbf{y}(\mathbf{x})$. The somewhat lengthy expression, valid for an arbitrary nonlinear system of order N, is given in Appendix A.3., eq.(A.8).

METHOD OF SOLUTION

Since the governing equation of $\mathbf{y}(\mathbf{x})$ in eq.(25) respectively eq.(A.8) is only approximately zero, $\mathbf{y}(\mathbf{x})$ is evaluated such that the scalar

$$\Pi = \int_{-\infty}^{+\infty} F^2(\mathbf{y}(\mathbf{x}))\, f_{\mathbf{x}}(\mathbf{x})\; w(\mathbf{x})\, d\mathbf{x} \rightarrow \text{Minimum!} \tag{26}$$

is minimized, where $f_{\mathbf{X}}(\mathbf{x})$ denotes the joint PDF of the state vector $\mathbf{x}(t)$ and $w(\mathbf{x})$ is a weighting function. The conditions of eq.(4) and (5) play in the present case a similar role as boundary conditions do by solving a partial differential equation.

A solution of eq.(26) is found iteratively by constraint minimizations of

$$\Pi_r = \int_{-\infty}^{+\infty} E\{F^2(\mathbf{y}(\mathbf{x}|x_r))\}\; f_{X_r}(x_r)\; w_r(x_r)\; dx_r \rightarrow \text{Minimum} \tag{27}$$

for $1 \le r \le N$, where $y_i(x_i)$ for $i \neq r$, is held fixed when Π_r is minimized and $\mathbf{x}|x_r$ denotes the state vector $\mathbf{x}$ conditioned on x_r. The associated constraints are

$$c_{rj} = E\{x_r^j\} - \int_{-\infty}^{+\infty} y_r^j(x_r)\; f_{X_r}(x_r)\; dx_r = 0; \quad 1 \le j \le 2 \tag{28}$$

Possible solutions $y_r(x_r)$ respectively $\hat{y}_r(\hat{x}_r)$ in eq.(26) are represented by a discrete series

$$\hat{y}_{r,k} = \hat{y}_r(\hat{x}_{r,k}) = \hat{y}_r(k\cdot\Delta_r) \tag{29}$$

where Δ_r is a constant increment of the order 0.05 ÷ 0.4. Function values $\hat{y}_r(\hat{x}_r)$ in between are found by interpolation. All derivatives appearing in eq.(A.8) are represented by central differences, given in the Appendix A.4, eq.(A.9)÷(A.12). Hence, the sets $\hat{y}_{r,k}$, $K_r \le k \le K_r$, $1 \le r \le N$, are the variables to be found by the minimization procedure.

For the purpose of a numerical evaluation, the scalar functions in eq.(27) and eq.(28) are discretized. Introducing,

$$u_{r,k} = \int_{(2k-1)\Delta_r/2}^{(2k+1)\Delta_r/2} e^{-\frac{\xi^2}{2}} d\xi \quad \text{and} \quad w_{r,k} = w_r(E\{x_r\}+k\Delta_r\sigma_{x_r}) \tag{30}$$

the constraints in eq.(28) read:

$$c_{rj} \doteq \sum_{k=-K_r}^{+K_r} (\hat{y}^j_{r,k} - (k\Delta_r)^j)\, u_{r,k} = 0; \quad 1 \le j \le 2 \tag{31}$$

The expectations $E\{F^2(\mathbf{y}(\mathbf{x}|x_r))\}$ at discrete points $x_{r,k} = E\{x_r\}+k\Delta_r\delta_{x_r}$ are approximated by a finite set of Monte Carlo simulation of sample size J,

$$E\{F^2(\mathbf{y}(\mathbf{x}|x_{r,k}))\} \doteq \frac{1}{J} \sum_{j=1}^{J} {}^jF_r^2\,(k) \tag{32}$$

where

$${}^jF_r(k) = F(\mathbf{y}({}^j\mathbf{x}|x_{r,k}=E\{x_r\}+k\Delta_r\delta_{x_r})) \tag{33}$$

in which ${}^j\mathbf{x}$ denotes the j-th simulated vector following the conditioned PDF

$$f_{X|x_{r,k}}(\mathbf{x}) = \frac{f_{\mathbf{X}}(x)}{f_{X_r}(x_{r,k})} \tag{34}$$

Thus, the scalar Π_r in eq.(27) can be approximated by the following expression

$$\Pi_r = \sum_{K=-K_r}^{+K_r} u_{r,k}\, w_{r,k} \frac{1}{J} \sum_{j=1}^{J} {}^jF_r^2(k) \quad \rightarrow \text{ Minimum} \tag{35}$$

A solution of the constraint minimization, defined by eq.(35) and eq.(31), is found by utilizing the method of Lagrange multipliers,

$$p_{r,k} = \frac{\partial \Pi r}{\partial \hat{y}_{r,k}} - \lambda_{r1} \frac{\partial c_{r1}}{\partial \hat{y}_{r,k}} - \lambda_{r2} \frac{\partial c_{r2}}{\partial \hat{y}_{r,k}} = 0; \; -K_r \le k \le K_r \tag{36a}$$

$$c_{rj} = 0; \quad 1 \le j \le 2 \tag{36b}$$

denoted by λ_{r1} and λ_{r2}. The above relations define a vector function

$$\mathbf{q}_r(\mathbf{z}_r) = 0 \tag{37}$$

as a necessary condition for Π_r being a minimum, where the $2K_r+3$ dimensional vectors $\mathbf{q}_r$ and $\mathbf{z}_r$ are defined as follow:

$$\mathbf{q}_r^T = \left[p_{r,-K_r}, p_{r,1-K_r}, \ldots, p_{r,0}, \ldots, p_{r,K_{r-1}}, p_{r,K_r}, c_{r1}, c_{r2} \right]^T \tag{38}$$

$$\mathbf{z}_r^T = \left[\hat{y}_{r,-K_r}, \hat{y}_{r,1-K_r}, \ldots, \hat{y}_{r,0}, \ldots, \hat{y}_{r,K_r-1}, \hat{y}_{r,K_r}, \lambda_{r1}, \lambda_{r2} \right]^T \tag{39}$$

The solution vector $\mathbf{z}_r$ is evaluated iteratively by Newton's method including "line search"

$$\mathbf{z}_r^{(j+1)} = z_r^{(j)} - \kappa^{(j)} \left[\frac{\partial \mathbf{q}_r(\mathbf{z}_r^{(j)})}{\partial \mathbf{z}_r} \right]^{-1} \cdot \mathbf{q}_r(\mathbf{z}_r^{(j)}) \tag{40}$$

where $\kappa^{(j)}$, $0.05 \le \kappa^{(j)} \le 1$, is varried systematically within the j-th iteration to find the smallest solution of $||\mathbf{q}_r(\mathbf{z}_r^{(j)})||$. For the efficiency of Newton's procedure, it is quite important to evaluate the vector $\mathbf{q}_r(\mathbf{z}_r)$ and the Hessenberg matrix $[\partial\mathbf{q}_r(\mathbf{z}_r^{(j)})/\partial\mathbf{z}_r]$ in closed form without resorting to numerical differentiation of Π_r in eq.(35). For each discrete function ${}^jF_r(k)$ (see eq.(33)), the differentials

$$\frac{\partial({}^jF_r(k))}{\partial\hat{y}_{r,k+m}} = {}^jF_r(k;k+m) \neq 0 \quad \text{for } -2 \leq m \leq 2 \tag{41}$$

$$\frac{\partial^2({}^jF_r(k))}{\partial\hat{y}_{r,k+m}\partial\hat{y}_{r,k+n}} = {}^jF_r(k;k+m,k+n) \neq 0 \quad \text{for } -2 \leq m,n \leq 2 \tag{42}$$

can be computed in closed form. The required derivatives of Π_r read

$$\frac{\partial\Pi_r}{\partial\hat{y}_{r,k}} = \sum_{i=\max(-2,-K_r-k)}^{\min(2,K_r-k)} u_{r,k+i}\, w_{r,k+i}\, \frac{2}{J} \sum_{j=1}^{J} {}^jF_r(k+i)\cdot{}^jF_r(k+i,k) \tag{43}$$

$$\frac{\partial^2\Pi_r}{\partial\hat{y}_{r,m}\partial\hat{y}_{r,n}} = \sum_{i=\max(-2,-K_r-m,n-m-2)}^{\min(2,K_r-m,m-n+2)} u_{r,m+i}w_{r,m+i}\cdot\frac{2}{J}\sum_{j=1}^{J} {}^jF(m+i;n)\cdot$$
$$\cdot\, {}^jF(m+1;m) + {}^jF(m+i)\cdot{}^jF(m+i;m,n) \tag{44}$$

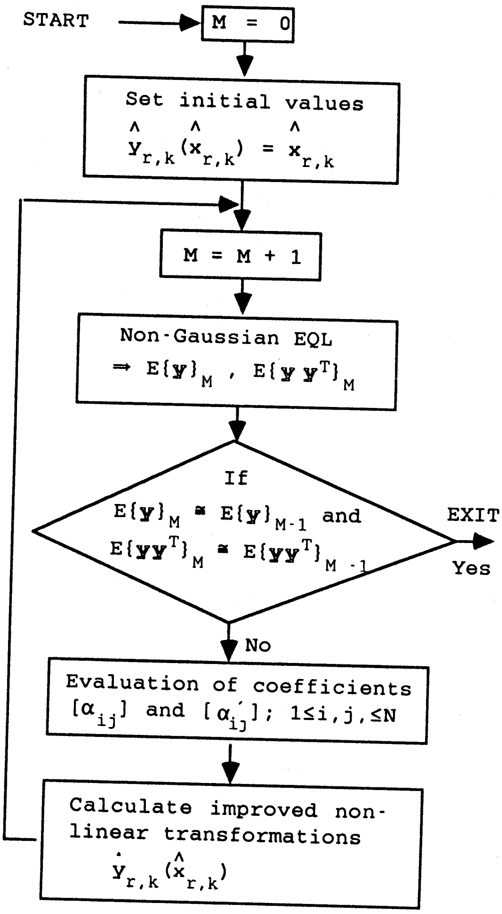

Fig.1: Flow chart of Non-Gaussian Linearization

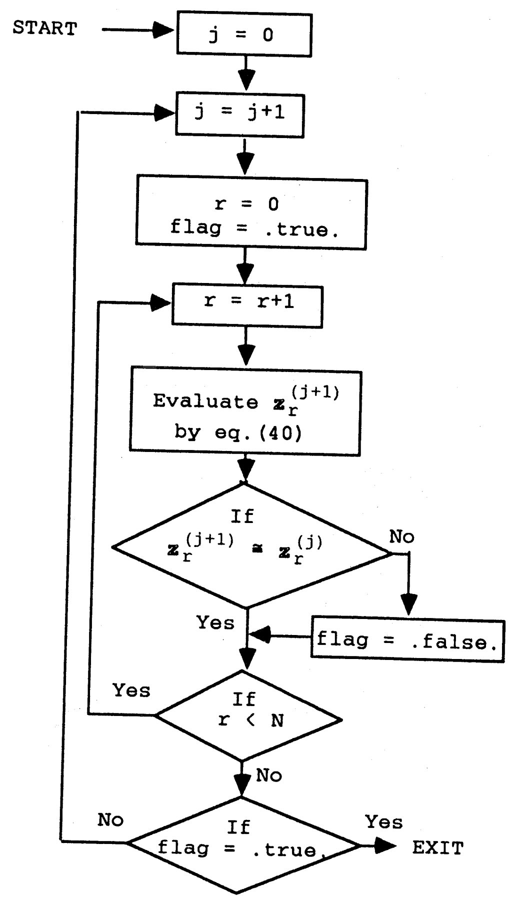

Fig.2: Flow chart for the Evaluation of nonlinear transformations

The method of non-Gaussian EQL described previously and the iterative procedure of eq.(40) require as starting vectors approximate solutions of the nonlinear transformations $y_r(x_r), 1\leq r\leq N$, described by the parameters $\hat{y}_{r,k}, -K_r\leq k\leq K_r$. Such approximate solutions might be established by other procedures, based on experience and physical considerations of limit cycles of the nonlinear restoring force (Pradlwarter, 1988) or utilization of Monte Carlo simulation and numerical integration over a short time. Close approximations help to reduce the numerical effort, however, they are not required to assure a solution. In case no apriori informations on the shape of marginal PDF's $f_{Y_i}(y_i)$ are available, it is suggested to assume in the very first step the marginals to be normally distributed, i.e. $\hat{y}_{r,k}=k\cdot\Delta_r$.

Finally, a flow chart of the overall procedure called "Non-Gaussian linearization" is shown in Fig.1. It combines non-Gaussian equivalent linearization (NG-EQL) with a procedure to evaluate non-Gaussian properties of the state vector $\mathbf{y}(t)$. Note, that the accuracy of the procedure for non-Gaussian EQL and of the nonlinear transformations depend on each other. These dependencies are taken into account by an iteration, denoted by the subscript M. The flow chart for successive evaluation of improved nonlinear transformation is shown in Fig.2. Note, that MDOF-systems of order N can be treated by the suggested procedure, where all required transformation $y_r(x_r)$, $1\le r\le N$, are evaluated separately. Since the results of $y_i(x_i)$ and $y_j(x_j), i\neq j$ influence each other, all transformations (index r, $1\le r\le N$) are improved within one iteration (index j) as simplified version for evaluating all transformations simultaneously.

NUMERICAL EXAMPLE

The effect of strong earthquakes on the outer shell containment of the power plant in Fig.3 is analysed. Due to underreinforcement of the outer shell a critical cross section of the outer shell is found at a height of -5.8m. Failure occurs in case the tensile stress exceeds the strength of concrete in the critical section which is under normal conditions under compression due to the dead load. The structural behaviour of the outer shell can be represented sufficiently well by a beam model where the flow of forces between internal and external structure and the behaviour of the soil requires particular attention. The model as utilized is shown in Fig.4. The structural parameters are found by static condensation of more detailed FE-models.

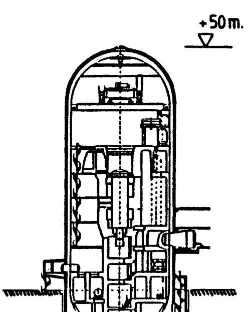

Fig.3: Containment structure

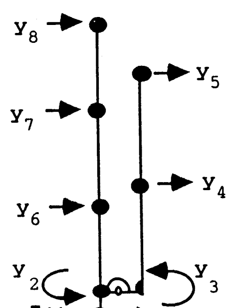

Fig.4: Structural model

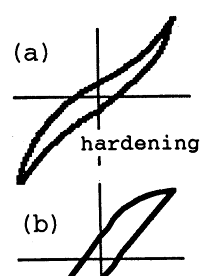

Fig.5: Nonlinear Hysteresis

The concrete structure reacts essentially linear when subjected to strong earthquake motion ($a_{max} < \sim 4$ m/sec^2). However, nonlinear (hysteretic) restoring forces P_1, P_2, P_3 due to the adjacent soil associated with the displacement y_1 and rocking y_2 and y_3 of the foundation must be expected. Due to the embeddment (see Fig.3), a hardening hysteretic type behaviour is found for horizontal motion, shown in Fig.5(a). For the rocking motion, however, a softening type hysteretic moment (see Fig.5(b)) is expected reflecting the soil behaviour. The hysteretic restoring forces $P_i(t)$, $1<i\leq3$, can be represented as superposition of a linear part ${}^LP_i(t)$ and nonlinear hysteretic part ${}^NP_i(t)$,

$$P_i(t) = {}^LP_i(t) + {}^NP_i(t) \; ; \qquad 1\leq i\leq3 \tag{45}$$

$${}^LP_i(t) = -K_{ii}y_i(t) + \sum_{j=1}^{NN} K_{ij}y_j(t) + C_{ij}\dot{y}_j(t) \tag{46}$$

where K_{ij} and C_{ij} are constant stiffness and damping coefficients, respectively, and NN=8 is the degree of freedom different from the order N=21 of the nonlinear system. The nonlinear hysteretic part of the restoring force ${}^NP_i(t)$ can be described uniquely by the following differential equations,

$${}^N\dot{P}_i(t) = K_{ii}\dot{y}_i \cdot h_i(\dot{y}_i, {}^NP_i(t), P_i^*) \tag{47}$$

where the function $h_i(\dot{y}_i, {}^NP(t), P_i^*)$ is shown in Fig.6a for i=1 (hardening type) and in Fig.6b for $2\leq i\leq3$ (softening type).

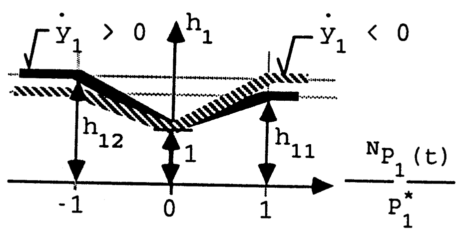

Fig.6a: $h_1(\dot{y}_1, {}^NP_1(t), P_1^*)$

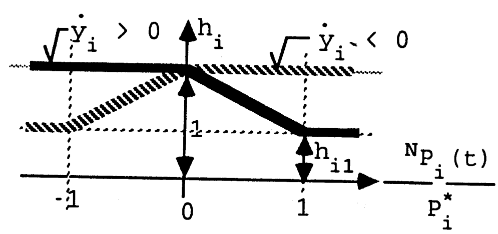

Fig.6b: $h_i(\dot{y}_i, {}^NP_i(t), P_i^*)$

The following parameters have been utilized to define the hysteric loops: h_{11} = 1.8, h_{12} = 2.0, P_1^*= 25.6 MN; h_{21} = 0.5, P_2^*= 391.4 MNm; h_{31} = 0.6, P_3^*= 441.3 MNm. The stochastic ground motion $a_g(t)$ is represented by filtered white noise leading to a Kanai-Tajimi spectrum (ω_g=28.9 rad/sec, ζ_g=0.18) where the white noise intensity S_n = 0.00565 m^2/sec^2 corresponds to σ_{ag} = 1.20 m/sec^2 standard deviation of ground acceleration equivalent to a maximum of a_{max} = 3.4 m/sec^2 within a duration T = 3.2 sec.

All equations defining the stochastic response, i.e. the equation of motion of the 8-DOF-system, the filter equation and the nonlinear equation (47) of the hysteretic nonlinear restoring forces, are converted into a set of coupled first order differential equation of order N = 21, where the state vector $\mathbf{y}$ comprises the components:

$$\mathbf{y}^T = \lfloor y_1, y_2, \ldots, y_7, y_8, \dot{y}_1, \dot{y}_2, \ldots, \dot{y}_7, \dot{y}_8, P_1, P_2, P_3 \rfloor^T \tag{45}$$

Using the procedure described in this paper, the second moments as well as the distributions of the stochastic response have been calculated. It is found from Fig.7a ÷ Fig.7c that the probability density functions (PDF's) of the nonlinear response quantities y_1 and y_2 deviate only in the tails considerably from a normal distribution. However, the behaviour of the tails is of paramount importance for a reliability analysis, since it describes the extreme values of the stochastic response. Moreover, the present small deviations from a normal distribution should not be generalized. Quite significant deviations has been found for strongly nonlinear systems. The present structure, however, reacts essentially linearly due to the counterbalancing effects of hardening and softening hysteretic behaviour. A comparison with a linear response analysis (constant tangential initial stiffness) yields the following standard deviations: σ_{P_1} = 4.71 (9.29) MN, σ_{P_2} = 112.7 (274.2) MNm, σ_{P_3} = 129.7 (322.6) MNm, where the values for the linear case are given in brackets. Hence, the stochastic response is reduced to less than a half due to the hysteretic nonlinear behaviour (damping). Since the ratios σ_{Pi}/P_i^* are considerable smaller than one, an almost Gaussian distributed response has been found.

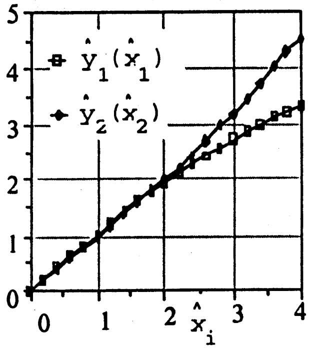

Fig.7a: Nonlinear transf. $y_1(x_1)$ and $y_2(x_2)$

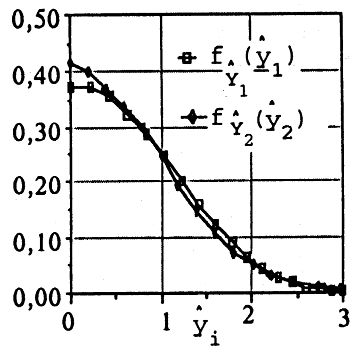

Fig.7b: PDF's of y_1 and y_2 linear scale

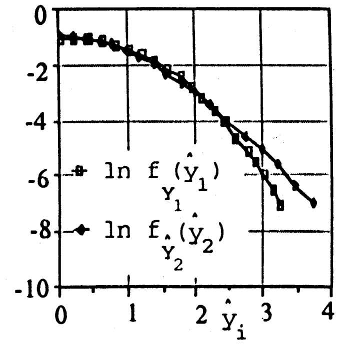

Fig.7c: PDF's of y_1 and y_2 log scale

CONCLUSION

Based on the results presented herein, the following conclusion are drawn:

(1) The conventional method of equivalent linearization has been extended to take into account most realistically non-Gaussian properties of the nonlinear response by means of nonlinear transformations.
(2) Distributions of the nonlinear response defined by nonlinear transformations are obtained by utilizing both non-Gaussian equivalent linearization and the Fokker-Planck equation of the nonlinear system.
(3) Quite accurate estimates for the first two moments of the nonlinear response are obtained by non-Gaussian equivalent linearization, since systematic errors of the first two statistical moments can be attributed to assuming inaccurate shapes of the joint distribution only.
(4) The procedure called "non-Gaussian linearization" does not involve any restricting assumption. Hence, it is generally applicable - with high accuracy - to any nonlinear MDOF-system.
(5) On the basis of the presented stochastic nonlinear response analysis, it is now possible to obtain realistic reliability estimates for nonlinear MDOF-systems under stochastic dynamic excitation. For this class of problems - which is for practical applications of paramount importance - no soltutions were available so far.

ACKNOWLEDGEMENT

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APPENDIX

A.1: Derivatives of joint distributions

$$\frac{\partial f_{\mathbf{X}}(\mathbf{x})}{\partial x_i} = -f_{\mathbf{X}}(\mathbf{x})\cdot\sum_{k=1}^{N} \alpha_{ik}\, x_k \tag{A.1}$$

$$\frac{\partial^2 f_{\mathbf{X}}(\mathbf{x})}{\partial x_i \partial x_j} = -f_{\mathbf{X}}(\mathbf{x})\cdot[\alpha_{ij} - \left(\sum_{k=1}^{N} \alpha_{ik}x_k\right)\left(\sum_{m=1}^{N} \alpha_{jm}x_m\right)] \tag{A.2}$$

$$\frac{\partial f_{\mathbf{Y}}(\mathbf{y})}{\partial y_i} = -f_{\mathbf{Y}}(\mathbf{y})\cdot[\frac{\partial^2 y_i}{\partial x_i}\left(\frac{\partial y_i}{\partial x_i}\right)^{-2} + \left(\frac{\partial y_i}{\partial x_i}\right)^{-1}\sum_{k=1}^{N} \alpha'_{ik}x_k] \tag{A.3}$$

$$\frac{\partial^2 f_{\mathbf{Y}}(\mathbf{y})}{\partial y_i \partial y_j} = -f_{\mathbf{Y}}(\mathbf{y})\cdot\Big\{\left(\frac{\partial y_i}{\partial x_i}\right)^{-1}\left(\frac{\partial y_j}{\partial x_j}\right)^{-1}\alpha'_{ij} - [\frac{\partial^2 y_i}{\partial x_i^2}\left(\frac{\partial y_i}{\partial x_i}\right)^{-2} +$$

$$+\left(\frac{\partial y_i}{\partial x_i}\right)^{-1}\sum_{k=1}^{N}\alpha'_{ik}x_k]\cdot[\frac{\partial^2 y_j}{\partial x_j^2}\left(\frac{\partial y_j}{\partial x_j}\right)^{-2} + \left(\frac{\partial y_j}{\partial x_j}\right)^{-1}\sum_{m=1}^{N}\alpha'_{jm}x_m]\Big\} \quad \text{for } i \neq j \tag{A.4}$$

$$\frac{\partial^2 f_{\mathbf{Y}}(\mathbf{y})}{\partial y_i^2} = -f_{\mathbf{Y}}(\mathbf{y})\cdot\Big\{\frac{\partial^3 y_k}{\partial x_k^3}\left(\frac{\partial y_k}{\partial x_k}\right)^{-3} - 3\left(\frac{\partial y_k^2}{\partial x_k^2}\right)^{-2}\left(\frac{\partial y_k}{\partial x_k}\right)^{-4} -$$

$$-3\,\frac{\partial^2 y_k}{\partial x_k^2}\left(\frac{\partial y_k}{\partial x_k}\right)^{-3}\left(\sum_{i=1}^{N}\alpha'_{ki}x_i\right) + \left(\frac{\partial y_k}{\partial x_k}\right)^{-2}\cdot[\alpha'_{kk} - \left(\sum_{j=1}^{N}\alpha'_{kj}x_j\right)^2]\Big\} \tag{A.5}$$

A.2: Fokker-Plank equation for linear and nonlinear system

$$FPE_L(\mathbf{x}) = f_{\mathbf{X}}(\mathbf{x})\cdot\Big\{\frac{\partial f_{\mathbf{X}}(\mathbf{x})}{\partial t}\cdot\frac{1}{f_{\mathbf{X}}(\mathbf{x})} + \sum_{m=1}^{N}\Big\{c_{mm} - \left(h_m + \sum_{k=1}^{N} c_{mk}x_k\right)$$

$$\cdot\left(\sum_{j=1}^{N} c_{mj}x_j\right) + \frac{1}{2}\sum_{n=1}^{N} I_{mn}(t)\cdot\left[\alpha_{mn} - \left(\sum_{k=1}^{N} c_{mk}x_k\right)\left(\sum_{j=1}^{N}\alpha_{nj}x_j\right)\right]\Big\}\Big\} \tag{A.6}$$

$$FPE_N(\mathbf{y}) = f_Y(\mathbf{y})\cdot\Big\{\frac{\partial f_X(\mathbf{x})}{\partial t}\cdot\frac{1}{f_X(\mathbf{x})} + \sum_{m=1}^{N}\Big\{\frac{\partial g_m(\mathbf{y}(\mathbf{x}))}{\partial y_m} - \left(b_m + g_m(\mathbf{y}(\mathbf{x}))\right)$$

$$\cdot\Big[\frac{\partial^2 y_m}{\partial x_m}\Big(\frac{\partial y_m}{\partial x_m}\Big)^{-2}+\Big(\frac{\partial y_m}{\partial x_m}\Big)^{-1}\Big(\sum_{j=1}^{N}\alpha'_{mj}x_j\Big)\Big]+\frac{1}{2}\sum_{n=1}^{N}I_{mn}(t)\cdot\Big\{\Big[\alpha'_{mn}-\Big(\sum_{i=1}^{N}\alpha'_{mi}x_i\Big)$$

$$\Big(\sum_{j=1}^{N}\alpha'_{nj}x_j\Big)\Big]\cdot\Big(\frac{\partial y_m}{\partial x_m}\Big)^{-1}\Big(\frac{\partial y_n}{\partial x_n}\Big)^{-1}+\delta_{mn}\cdot\Big[-\frac{\partial^3 y_m}{\partial x_m^3}\Big(\frac{\partial y_m}{\partial x_m}\Big)^{-3}+3\Big(\frac{\partial^2 y_m}{\partial x_m^2}\Big)^2\Big(\frac{\partial y_m}{\partial x_m}\Big)^{-4}$$

$$+3\frac{\partial^2 ym}{\partial x^2 m}\cdot\Big(\frac{\partial y_m}{\partial x_m}\Big)^{-3}\Big(\sum_{k=1}^{N}\alpha'_{mk}x_k\Big)\Big]+(1-\delta_{mn})\cdot\Big[\frac{\partial^2 y_m}{\partial x_m^2}\Big(\frac{\partial y_m}{\partial x_m}\Big)^{-2}\cdot\Big(\frac{\partial^2 y_n}{\partial x_n^2}\Big(\frac{\partial y_n}{\partial x_n}\Big)^{-2}$$

$$+\Big(\frac{\partial y_n}{\partial x_n}\Big)^{-1}\sum_{k=1}^{N}\alpha'_{nk}x_k\Big)+\frac{\partial^2 y_n}{\partial x_m^2}\Big(\frac{\partial^2 y_n}{\partial x_n^2}\Big)^{-2}\Big(\frac{\partial y_m}{\partial x_m}\Big)^{-1}\sum_{k=1}^{N}\alpha'_{mk}x_k\Big]\Big\}\Big\}\Big\} \qquad \text{(A.7)}$$

A.3: Governing equation of $\mathbf{y}(\mathbf{x})$

$$F(\mathbf{y}(\mathbf{x})) = \frac{FPE_N(\mathbf{y}(\mathbf{x}))}{f_{\mathbf{Y}}(\mathbf{y}(\mathbf{x}))} - \frac{FPE_L(\mathbf{x})}{f_{\mathbf{X}}(\mathbf{x})}$$

$$=\sum_{m=1}^{N}\frac{\partial g_m(\mathbf{y}(\mathbf{x}))}{\partial y_m}-c_{mm}+\Big(h_m+\sum_{k=1}^{N}c_{mk}x_k\Big)\Big(\sum_{j=1}^{N}c_{mj}x_j\Big)-(b_m+g_m(\mathbf{y}(\mathbf{x})))\cdot$$

$$\cdot\Big[\frac{\partial^2 y_m}{\partial x_m}\Big(\frac{\partial y_m}{\partial x_m}\Big)^{-2}+\Big(\frac{\partial y_m}{\partial x_m}\Big)^{-1}\Big(\sum_{j=1}^{N}\alpha'_{mj}x_j\Big)\Big]+\frac{1}{2}\sum_{n=1}^{N}I_{mn}(t)\cdot\Big\{\Big[\alpha'_{mm}-\Big(\sum_{i=1}^{N}\alpha'_{mi}x_i\Big)$$

$$\Big(\sum_{j=1}^{N}\alpha'_{nj}x_j\Big)\Big]\cdot\Big(\frac{\partial y_m}{\partial x_m}\Big)^{-1}\Big(\frac{\partial y_n}{\partial x_n}\Big)^{-1}-\Big[\alpha_{mm}-\Big(\sum_{i=1}^{N}\alpha_{mi}x_i\Big)\Big(\sum_{j=1}^{N}\alpha_{nj}x_j\Big)\Big]+\delta_{mn}\cdot$$

$$\cdot\Big[-\frac{\partial^3 y_m}{\partial x_m^3}\Big(\frac{\partial y_m}{\partial x_m}\Big)^{-3}+3\Big(\frac{\partial^2 y_m}{\partial x_m}\Big)^2\Big(\frac{\partial y_m}{\partial x_m^2}\Big)^{-4}+3\frac{\partial^2 y_m}{\partial x_m^2}\Big(\frac{\partial y_m}{\partial x_m}\Big)^{-3}\Big(\sum_{k=1}^{N}\alpha'_{mk}x_K\Big)\Big]+$$

$$+(1-\delta_{mn})\cdot\Big[\frac{\partial^2 y_m}{\partial x_m^2}\Big(\frac{\partial y_m}{\partial x_m}\Big)^{-2}\cdot\Big(\frac{\partial^2 y_n}{\partial x_n^2}\Big(\frac{\partial y_n}{\partial x_n}\Big)^{-2}+\Big(\frac{\partial y_n}{\partial x_n}\Big)^{-1}\Big(\sum_{k=1}^{N}\alpha'_{nk}x_K\Big)\Big)+$$

$$+\frac{\partial^2 y_n}{\partial x_n^2}\Big(\frac{\partial^2 y_n}{\partial x_n^2}\Big)^{-2}\Big(\frac{\partial y_m}{\partial x_m}\Big)^{-1}\Big(\sum_{k=1}^{N}\alpha'_{mk}x_K\Big)\Big]\Big\}\Big\}\Big\} \qquad \text{(A.8)}$$

A.4. Derivatives

$$\frac{\partial y_j(x_{j,k})}{\partial x_j} = \frac{1}{2\Delta_j}(\hat{y}_{j,k+1}-\hat{y}_{j,k-1}) \qquad \text{(A.9)}$$

$$\frac{\partial^2 y_j(x_{j,k})}{\partial x_j^2} = \frac{1}{\sigma x_j \Delta_j^2}(y_{j,k+1}-2y_{j,k}+y_{j,k-1}) \qquad \text{(A.10)}$$

$$\frac{\partial^3 y_j(x_{j,k})}{\partial x_j^3} = \frac{1}{\sigma_{xj}^2\Delta_j^3}(y_{j,k+2}-2y_{j,k+1}+2y_{j,k-1}-y_{j,k-2}) \qquad \text{(A.11)}$$

$$\frac{\partial g_m(\mathbf{y}(\mathbf{x}))}{\partial y_m} = \frac{1}{2\Delta_m\sigma_{x_m}}(g_m(\mathbf{y}(\mathbf{x}+\mathbf{e}_m\Delta_m\sigma_{x_m}))-g_m(\mathbf{y}(\mathbf{x}-\mathbf{e}_m\Delta_m\sigma_{x_m})))\cdot\Big(\frac{\partial y_m}{\partial x_m}\Big)^{-1} \qquad \text{(A.12)}$$

where $\mathbf{e}_m$ is a unit coordinate vector of the m-th component.