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NUMERICAL IMPLEMENTATION OF A TRANSVERSE-ISOTROPIC INELASTIC, WORK-HARDENING CONSTITUTIVE MODEL

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SUMMARY

During the past few decades the dramatic growth of computer technology has been paralleled by an increasing degree of complexity in material constitutive modeling. This paper documents the numerical implementation of one of these models, specifically a transverse-isotropic, inelastic, work-hardening constitutive model which is developed elsewhere by the author.

A brief overview of the mathematical formulation of the model is presented to facilitate the understanding of its numerical implementation. The model is based on incremental flow theories for materials which have time- and temperature-independent properties and which are capable of undergoing small plastic as well as small elastic strain at each loading increment. In addition, the model is written in terms of "pseudo" stress invariants so that the incremental anisotropic stress-strain relationship can be readily incorporated into existing finite-difference or finite-element computer codes. The isotropic version of the model is retrieved without any changes in the mathematical formulation or in the numerical implementation (algorithm) of the model.

Various methods exist for incorporating inelastic constitutive models into computer programs. The method presented in this paper is appropriate for both finite-difference and finite-element codes, and is applicable for solving static as wall as dynamic problems. This method expresses the material constitutive properties as a matrix of coefficients, C (generalized tangent moduli), which relates incremental stresses to incremental strains. It possesses desirable convergence properties. In either finite-difference or finite-element applications the input quantities are the initial stress components, σ_{ij}^0, obtained at the end of the previous strain increment, and the new strain increments, $\Delta\varepsilon_{ij}$. The output quantities are the new values of the stress components σ_{ij}.

In the first step of the algorithm a set of elastic trial stresses, $\sigma^E{}_{ij}$ are computed by means of the incremental Hooke's law. These stresses are then tested with respect to the failure criteria. If they do not violate the failure criteria, the behavior of the material is truly elastic and these stresses are the correct stresses for the given strain increments. On the other hand, if the elastic trial stresses violate the failure criteria, they are corrected through an iteration scheme using an appropriate flow rule. In order to reduce the inaccuracies inherent in computing inelastic stress states by this iteration scheme, the algorithm allows the strain increment to be split into N equal increments, each of which is processed through the iteration scheme. Since the number of iterations and the value of N increase the cost of numerical computation, an automatic control feature has been introduced such that "splitting" only occurs when it is absolutely necessary.

Expanded version: Nucl. Eng. & Design 46 (1978) 263–272

1. Introduction

During the past few decades the dramatic growth of computer technology and the development of new methods of numerical analysis have been paralleled by an increasing degree of complexity in material constitutive modeling. Stress-strain relationships for a number of materials, such as wood, crystals, rolled or drawn metals, soil, rock, reinforced concrete, fiber-filled plastics, etc., are often found to be anisotropic and inelastic even when the magnitudes of the strains involved are small. Here, the term anisotropic refers to the dependence of the moduli or strengths of the material upon direction of loading.

In order to rationally solve complex load-deformation problems involving such materials, one must resort to the theory of continuum mechanics and have available an appropriate constitutive relation. The author [1] has previously developed a transverse-isotropic, inelastic, work-hardening constitutive model for use in numerical solutions to such problems. The present paper summarizes the mathematical formulation of this model and documents its numerical implementation.

2. Summary of Model Formulation

2.1 General

From a general point of view, the model falls within the framework of the classical incremental theory of plasticity (Hill [2]) for materials which have time- and temperature-independent properties and which are capable of undergoing small elastic as well as small plastic strains during each loading increment, i.e., the total strain increment, $d\varepsilon_{ij}$, is assumed to be the sum of the elastic (or recoverable) strain increment, $d\varepsilon^E_{ij}$, and the plastic (or permanent) strain increment, $d\varepsilon^P_{ij}$.

$$d\varepsilon_{ij} = d\varepsilon^E_{ij} + d\varepsilon^P_{ij} \tag{1}$$

2.2 Fundamental Definitions of "Pseudo Stress Invariants"

The numerical implementation of a transverse-isotropic model can be greatly simplified if strain can be separated into hydrostatic and deviatoric components. This can be done, as will be shown later in this section, by defining "pseudo stress invariants" as follows (Baladi [1], Sandler and DiMaggio [3]):

$$\emptyset_1 = \sigma_{11} + \alpha(\sigma_{22} + \sigma_{33}) \tag{2}$$

$$\emptyset_2 = \frac{\beta}{6}\,[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{11} - \sigma_{33})^2] + \frac{(\sigma_{22} - \sigma_{33})^2}{6} + \gamma(\sigma_{12}{}^2 + \sigma_{13}{}^2) + \frac{\beta + 2}{3}\,\sigma_{23}{}^2 \tag{3}$$

where σ_{ij} are components of the stress tensor and α, β, and γ are dimensionless material properties (constants). The axis of material symmetry, in this definition, is assumed to be normal to the $\sigma_{22} - \sigma_{33}$ plane (the plane of isotropy). In reality, eqs. (2) and (3) represent a simplified version of the complete transverse-isotropic elastic, transverse-isotropic plastic model described by Baladi [1]. The complete model contains two descriptions of $\emptyset_1$ and $\emptyset_2$, i.e., one for the elastic portion of the model ($\emptyset^E_1$, $\emptyset^E_2$) and one for the plastic portion ($\emptyset^P_1$, $\emptyset^P_2$). These descriptions require two sets of the parameters α, β and γ, namely, α^E, β^E, and γ^E, and α^P, β^P, and γ^P. For the present discussion, the model has been simplified such that $\alpha^E = \alpha^P = \alpha$, etc. Note that when $\alpha = \beta = \gamma = 1$, $\emptyset_1$ becomes J_1 , the familiar first invariant of the stress tensor and $\emptyset_2$ becomes J_2' , the familiar second invariant of the stress deviator tensor.

2.3 The Plastic Potential Functions

The mathematical basis of the incremental theory of plasticity was established by Drucker [4] when he introduced the concept of material stability, which has the following implications:

(a) Yield surface (loading function) should be convex in stress space.

(b) Yield surface and plastic potential should coincide (this is to guarantee an associated flow rule).

(c) Work "softening" should not occur.

These three conditions can be summarized mathematically by the following inequality

$$d\sigma_{ij}\ d\varepsilon_{ij}^P \geqslant 0 \tag{4}$$

The above conditions allow considerable flexibility in choosing a loading function, f , for the model which serves as both a yield surface and a plastic potential. For the present model, the loading function consists of two parts: an ultimate failure envelope which effectively serves to limit the maximum shear strength of the material and a strain-hardening surface, Figure 1. The failure envelope portion of the loading function is denoted by

$$f = G(\emptyset_1 , \sqrt{\emptyset_2}) = \sqrt{\emptyset_2} - f(\emptyset_1) = 0 \text{ for } \emptyset_1 < L(\kappa) \tag{5}$$

and the strain-hardening surface by

$$f = H(\emptyset_1 , \sqrt{\emptyset_2} , \kappa) = \sqrt{\emptyset_2} - F(\emptyset_1 , \kappa) = 0 \text{ for } X(\kappa) \geqslant \emptyset_1 \geqslant L(\kappa) \tag{6}$$

Note in Figure 1 that $L(\kappa)$ and $X(\kappa)$ define the intersection of the hardening surface with the failure envelope, $G(\emptyset_1 , \emptyset_2)$ and the $\emptyset_1$ axis, respectively, and κ is, in general, a function of the history of plastic volumetric strain, ε_{kk}^P . κ is chosen so that

$$L(\kappa) = \begin{cases} \kappa & \text{if } \kappa > 0 \\ 0 & \text{if } \kappa \leqslant 0 \end{cases} \tag{7}$$

The hardening parameter κ can be one of the following functions

$$\kappa = g_1\ (\varepsilon_{kk}^P) \tag{8}$$

$$\kappa = g_2\ [(\varepsilon_{kk}^P)_{max}] \tag{9}$$

The use of eq. (8) permits the hardening surface to move back toward the origin (Figure 1) when a point on the failure envelope, G , is reached, thus controlling the dilatancy of the material. The use of eq. (9), on the other hand, only permits the hardening surface to move away from the origin, thus ensuring dilatancy while still permitting hysteresis in a hydrostatic load-unload cycle.

The plastic loading criteria for the function f are given by

$$df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} \begin{cases} > 0 \text{ loading} \\ = 0 \text{ neutral loading} \\ < 0 \text{ unloading} \end{cases} \tag{10}$$

Plastic strains will occur only when df is positive and $f = 0$. During unloading or neutral loading, as well as for $f < 0$, the material will behave elastically. The prescription that neutral loading produces no plastic strain is called the continuity condition. Its satisfaction leads to coincidence of elastic and plastic constitutive laws during neutral loading (Drucker [4] and Handelman [5]).

2.4 Plastic Strain Increment Tensor

Based on eq. (4), the plastic strain increments in the model are defined through the associated plastic flow rule

$$d\varepsilon_{ij}^{P} = \begin{cases} d\lambda \dfrac{\partial f}{\partial \sigma_{ij}} & \text{if} \quad f = 0 \\ 0 & \text{if} \quad f < 0 \end{cases} \tag{11}$$

where f is defined by either eq. (5) or (6) and $d\lambda$ is a positive function of proportionality which is nonzero only when plastic deformation occurs.

The plastic stress-strain relation can be expressed in terms of the hydrostatic and deviatoric components of strain through the chain rule of differentiation applied to the right-hand side of eq. (11), thus

$$d\varepsilon_{kk}^{P} = d\lambda(1 + 2\alpha) \frac{\partial f}{\partial \phi_1} \tag{12}$$

and

$$d\bar{e}_{ij}^{P} = \frac{d\lambda}{2\sqrt{\phi_2}} \frac{\partial f}{\partial \sqrt{\phi_2}} n_{ij} \tag{13}$$

in which

$$n_{ij} = \frac{\partial \phi_2}{\partial \sigma_{ij}} \text{ and } n_{ii} = 0 \tag{14}$$

where $d\varepsilon_{kk}^{P}$ is the increment of plastic volumetric strain and $d\bar{e}_{ij}^{P}$ is the "pseudo plastic strain deviation increment tensor." The plastic strain deviation increment tensor, de_{ij}^{P} , is related to $d\bar{e}_{ij}^{P}$ through the following relation

$$de_{ij}^{P} = d\bar{e}_{ij}^{P} - \left(\frac{\delta_{ij}}{3} - \frac{A_{ij}}{1 + 2\alpha}\right) d\varepsilon_{kk}^{P} \tag{15}$$

in which δ_{ij} is the Kronecker delta function and

$$[A_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} \quad (16)$$

Hence, the plastic strain increment tensor becomes

$$d\varepsilon_{ij}^{P} = d\lambda\left(\frac{\partial f}{\partial \phi_1} A_{ij} + \frac{1}{2\sqrt{\phi_2}} \frac{\partial f}{\partial\sqrt{\phi_2}} \eta_{ij}\right) \quad (17)$$

In order to use eqs. (12) through (15) or eq. (17), the proportionality factor $d\lambda$ must be determined. This can be accomplished in a straightforward manner; however, since the resulting expression for $d\lambda$ contains elastic moduli, the determination of the expression is deferred to Section 2.7.

2.5 The Elastic Potential Function

Within the yield surface, f , the material behavior is transverse-isotropic elastic. The complementary energy function for this material is chosen to be

$$d\Omega = \frac{d(\phi_1)^2}{18B} + \frac{d\phi_2}{2S} \quad (18)$$

where B and S are material response functions that will be defined subsequently. The elastic strain increment tensor, $d\varepsilon_{ij}^{E}$, becomes

$$d\varepsilon_{ij}^{E} = d\left(\frac{\partial\Omega}{\partial\sigma_{ij}}\right) \quad (19)$$

It can be seen from eqs. (2), (3), (18) and (19) that the elastic behavior is governed by the three elastic constants (α, β and γ) and by the two response functions B and S which may take the following form

$$B = B(\phi_1 , \kappa) \quad (20)$$

$$S = S(\sqrt{\phi_2} , \kappa) \quad (21)$$

Note that the definition of eq. (18) and the form of eqs. (20) and (21) dictate that the model is path independent during purely elastic deformation (Prager [6] and Sandler, DiMaggio, and Baladi [7]).

Certain limitations must be placed on the constants α , β , and γ of eqs. (2) and (3) and the response functions, B and S . These limitations are discussed in detail by Lekhnitskii [8] for a linear elastic transverse isotropic material. In the present model these limitations imply:

$$\frac{\alpha^2}{3B} + \frac{\beta + 1}{2S} > 0 \quad (22)$$

$$\frac{1}{3B} + \frac{\beta}{S} > 0 \quad (23)$$

$$\frac{\gamma}{S} > 0 \tag{24}$$

$$\frac{\beta + 2}{S} > 0 \tag{25}$$

$$\frac{2\alpha^2}{3B} + \frac{\beta}{2S} \geqslant 0 \tag{26}$$

$$\left(\frac{2\alpha^2}{3B} + \frac{\beta}{2S}\right)\left(\frac{1}{3B} + \frac{\beta}{S}\right) - 2\left(\frac{\alpha}{3B} - \frac{\beta}{2S}\right)^2 \geqslant 0 \tag{27}$$

2.6 Elastic Strain Increment Tensor

The elastic strain increment tensor, $d\varepsilon^E_{ij}$, can be obtained from eq. (19), with the help of eqs. (14) and (16); thus

$$d\varepsilon^E_{ij} = \frac{1}{9B}\, d\phi_1\, A_{ij} + \frac{1}{2S}\, d\eta_{ij} \tag{28}$$

The hydrostatic and deviatoric components of strain can be easily obtained from eq. (28) as

$$d\varepsilon^E_{kk} = \frac{1 + 2\alpha}{9B}\, d\phi_1 \tag{29}$$

$$d\bar{e}^E_{ij} = \frac{1}{2S}\, d\eta_{ij} \tag{30}$$

where $d\varepsilon^E_{kk}$ is the increment of elastic volumetric strain and $d\bar{e}^E_{ij}$ is the "pseudo elastic strain deviation increment tensor." The elastic strain deviation increment tensor, de^E_{ij}, is related to $d\bar{e}^E_{ij}$ through the following relation

$$de^E_{ij} = d\bar{e}^E_{ij} - \left(\frac{\delta_{ij}}{3} - \frac{A_{ij}}{1 + 2\alpha}\right) d\varepsilon^E_{kk} \tag{31}$$

Note that when $\alpha = \beta = \gamma = 1$, the above model reduces to the isotropic models described by Sandler, DiMaggio, and Baladi [7] and Baladi [9] and the response functions B and S become the isotropic bulk and shear modulus response functions, respectively.

2.7 Flow Rule Proportionality Factor

We return now to the determination of the proportionality factor $d\lambda$. Applying the chain rule of differentiation to the right-hand side of eq. (10) and using eqs. (1), (12), (13), (14), (15), (17), (29), (30), and (31) leads to

$$d\lambda = \frac{\left[\frac{9B}{(1 + 2\alpha)}\frac{\partial f}{\partial \phi_1} - \frac{S}{\beta\sqrt{\phi_2}}\left(\frac{1 - \alpha}{1 + 2\alpha}\right)\frac{\partial f}{\partial \sqrt{\phi_2}}\eta_{11}\right] d\varepsilon_{kk} + \frac{S}{\sqrt{\phi_2}}\frac{\partial \phi_2}{\partial \eta_{ij}}\frac{\partial f}{\partial \sqrt{\phi_2}}\, de_{ij}}{9B\left(\frac{\partial f}{\partial \phi_1}\right)^2 + S\left(\frac{\partial f}{\partial \sqrt{\phi_2}}\right)^2 - (1 + 2\alpha)\frac{\partial f}{\partial \phi_1}\frac{\partial f}{\partial \kappa}} \tag{32}$$

2.8 Total Incremental Strain-Stress Relations

The hydrostatic and deviatoric components of the total strain increment tensor can be obtained by adding eqs. (12) and (29) to obtain $d\varepsilon_{kk}$, and eqs. (15) and (31) to obtain de_{ij}; thus,

$$d\varepsilon_{kk} = \frac{1 + 2\alpha}{9B} d\phi_1 + d\lambda(1 + 2\alpha) \frac{\partial f}{\partial \phi_1} \tag{33}$$

$$de_{ij} = \frac{1}{2S} d\eta_{ij} + d\lambda \frac{\eta_{ij}}{2\sqrt{\phi_2}} \frac{\partial f}{\partial \sqrt{\phi_2}} - (\frac{\delta_{ij}}{3} - \frac{A_{ij}}{1 + 2\alpha})(1 + 2\alpha)(\frac{d\phi_1}{9B} + d\lambda \frac{\partial f}{\partial \phi_1}) \tag{34}$$

Hence, the total strain increment tensor becomes

$$d\varepsilon_{ij} = \frac{1}{9B} d\phi_1 A_{ij} + \frac{1}{2S} d\eta_{ij} + d\lambda(\frac{\partial f}{\partial \phi_1} A_{ij} + \frac{1}{2\sqrt{\phi_2}} \frac{\partial f}{\partial \sqrt{\phi_2}} \eta_{ij}) \tag{35}$$

Similarly, the "pseudo hydrostatic and deviatoric components of the stress increment tensor" can be written as

$$d\phi_1 = \frac{9B}{1 + 2\alpha} d\varepsilon_{kk} - 9B\, d\lambda \frac{\partial f}{\partial \phi_1} \tag{36}$$

$$d\eta_{ij} = 2S\, de_{ij} + 2S(\frac{\delta_{ij}}{3} - \frac{A_{ij}}{1 + 2\alpha})\, d\varepsilon_{kk} - \frac{S\, d\lambda}{\sqrt{\phi_2}} \frac{\partial f}{\partial \sqrt{\phi_2}} \eta_{ij} \tag{37}$$

Equations (33) and (34), or eqs. (36) and (37), are the general constitutive equations for a transverse-isotropic, inelastic, work-hardening constitutive model in which the axis of material symmetry is normal to the $\sigma_{22} - \sigma_{33}$ plane. To use these equations it is only necessary to specify the functional forms of B , S , f , and κ , and, of course, to determine experimentally the values of the coefficients in these functions as well as the values of α , β , and γ .

3. Numerical Implementation of the Model

Various methods exist for incorporating the model into computer programs (e.g., Sandler and Rubin [10]). The method presented in this section is appropriate for both finite difference and finite element codes, and is applicable for solving static as well as dynamic problems. This method makes use of eqs. (36) and (37), i.e., the input quantities are the stress components, σ_{ij}^{n} , and the hardening parameter κ^{n} obtained at the end of the nth loading increment, and the components of the new strain increments, $d\varepsilon_{ij}^{n+1}$, which are obtained from the field equations. The output quantities are the new values of the stress components σ_{ij}^{n+1} .

In the first step of the numerical algorithm a set of elastic trial stresses, σ_{ij}^{E} , is computed by means of eqs. (36) and (37) in which $d\lambda = 0$, i.e.,

$$\phi_1^E = \phi_1^n + \frac{9B}{1 + 2\alpha} d\varepsilon_{kk}^{n+1} \tag{38}$$

$$\eta_{ij}^E = \eta_{ij}^n + 2S\, de_{ij}^{n+1} + 2S(\frac{\delta_{ij}}{3} - \frac{A_{ij}}{1 + 2\alpha})\, d\varepsilon_{kk}^{n+1} \tag{39}$$

where

$$B = B(\phi_1^E , \kappa^n) \tag{40}$$

and

$$S = S(\sqrt{\phi_2^E} , \kappa^n) \tag{41}$$

These trial stresses are then tested, first with respect to the failure envelope, $G(\phi_1^E , \sqrt{\phi_2^E})$, and second with respect to the hardening surface, $H(\phi_1^E , \sqrt{\phi_2^E} , \kappa^n)$. If these trial stresses do not violate either loading function, the behavior of the material is truly elastic, the hardening parameter κ^n remains unchanged, and the final stresses at the end of the (n + 1) loading increment are

$$\phi_1^{n+1} = \phi_1^E \tag{42}$$

$$\eta_{ij}^{n+1} = \eta_{ij}^E \tag{43}$$

Note that B and S given in eqs. (40) and (41) represent the values of the elastic moduli at the end of the nth loading increment, i.e., they are assumed to be constant during the (n + 1) loading increment. The effect of this assumption on the accuracy of the computations is negligible if the given strain increments, $d\epsilon_{ij}^{n+1}$, are relatively small. For relatively large strain increments, on the other hand, accuracy will suffer accordingly. This problem can be overcome, however, by splitting the given strain increment into m equal increments and sequentially going through the numerical algorithm m times, using updated values for B and S each time.

If the failure envelope, $G(\phi_1^E , \sqrt{\phi_2^E})$, is violated by the elastic trial stresses, i.e.,

$$\sqrt{\phi_2^E} \geq \min \left\{ f(\phi_1^E) , F[L(\kappa^n) , \kappa^n] \right\} \tag{44}$$

$$\phi_1^E \leq L(\kappa^n) \tag{45}$$

and

$$G(\phi_1^E , \sqrt{\phi_2^E}) \geq 0 \tag{46}$$

then the trial stresses have to be corrected such that the final stresses at the end of (n + 1) loading increment satisfy the following relation

$$G(\phi_1^{n+1} , \sqrt{\phi_2^{n+1}}) = 0 \tag{47}$$

The method for correcting these stresses is illustrated in Figure 2. Mathematically, the statement that the final stress point must be on the failure surface is given by eq. (10); thus

$$dG = \frac{\partial G}{\partial \sigma_{ij}} d\sigma_{ij} = 0 \tag{48}$$

For small strain increments, eq. (48) can be approximated numerically by

$$dG = G(\phi_1^E , \sqrt{\phi_2^E}) - G(\phi_1^n , \sqrt{\phi_2^n}) \tag{49}$$

but since

$$G(\emptyset_1^n , \sqrt{\emptyset_2^n}) = 0 \tag{50}$$

we have

$$dG = G(\emptyset_1^E , \sqrt{\emptyset_2^E}) \tag{51}$$

which, when substituted into eq. (48), leads to

$$G(\emptyset_1^E , \sqrt{\emptyset_2^E}) = \frac{\partial G}{\partial \emptyset_1} d\emptyset_1 + \frac{1}{2\sqrt{\emptyset_2}} \frac{\partial G}{\partial \sqrt{\emptyset_2}} d\emptyset_2 \tag{52}$$

Then, substituting eq. (32) into eq. (52) results in, after some simplification and elaboration,

$$d\lambda = \frac{G(\emptyset_1^E , \sqrt{\emptyset_2^E})}{9B(\frac{\partial G}{\partial \emptyset_1})^2 + S} \tag{53}$$

Moreover, substitution of eq. (53) into eq. (12) leads to

$$d\varepsilon_{kk}^P = (1 + 2\alpha)\left[\frac{G(\emptyset_1^E , \sqrt{\emptyset_2^E})}{9B(\frac{\partial G}{\partial \emptyset_1})^2 + S}\right] \frac{\partial G}{\partial \emptyset_1} \tag{54}$$

Hence, the final value of $\emptyset_1$ at the end of the (n + 1) loading increment is

$$\emptyset_1^{n+1} = \emptyset_1^E - \frac{9B}{1 + 2\alpha} d\varepsilon_{kk}^P \tag{55}$$

Our work is not yet done, however, for the hardening parameter κ^n, which is a function of ε_{kk}^P, must also be updated, i.e., the hardening surface must intersect the failure envelope (thereby creating a corner) at this new value of $\emptyset_1^{n+1}$. This can be done by testing $\emptyset_1^{n+1}$ against $L(\kappa^n)$. If $\emptyset_1^{n+1} > L(\kappa^n)$, "corner coding" is triggered.

The treatment of "corner coding" depends on the functional form of κ. If eq. (8) is used, i.e., if the hardening surface is permitted to move back toward the origin, then eqs. (7) and (55) give

$$\emptyset_1^E - \frac{9B}{1 + 2\alpha} d\varepsilon_{kk}^P = \emptyset_1^{n+1} = L(\kappa^{n+1}) = \kappa^{n+1} = \kappa^n + \left.\frac{\partial \kappa}{\partial \varepsilon_{kk}^P}\right|_{\kappa^n} d\varepsilon_{kk}^P \tag{56}$$

Eliminating $d\varepsilon_{kk}^P$ from eq. (56) and solving for $\emptyset_1^{n+1} = \kappa^{n+1}$ leads to the final corner value of κ, i.e.,

$$\kappa^{n+1} = \emptyset_1^{n+1} = \frac{\left(\left.\frac{\partial \kappa}{\partial \varepsilon_{kk}^P}\right|_{\kappa^n} \emptyset_1^E + \frac{9B}{1 + 2\alpha} \kappa^n\right)}{\left(\left.\frac{\partial \kappa}{\partial \varepsilon_{kk}^P}\right|_{\kappa^n} + \frac{9B}{1 + 2\alpha}\right)} \tag{57}$$

If eq. (9) is used as the functional form for κ, i.e., the hardening surface is not permitted to reverse itself, then κ^{n+1} is updated by simply setting

$$L(\kappa^{n+1}) = \emptyset_1^{n+1} \tag{58}$$

Having now determined $\emptyset^{n+1}$ and κ^{n+1}, the final values of the "pseudo deviator stresses" η_{ij}^{n+1} can be easily determined (see Figure 2) from

$$\eta_{ij}^{n+1} = \frac{f(\emptyset_1^{n+1})}{f(\emptyset_1^E)} \eta_{ij} \tag{59}$$

This completes the correction process for stress states violating the failure envelope. We now proceed to examine the case where the failure envelope is not violated, but the hardening surface is.

The elastic trial stresses are tested against the hardening surface, i.e., if

$$\sqrt{\emptyset_2^E} > F(\emptyset_1^E , \kappa^n) \tag{60}$$

and

$$X(\kappa^n) > \emptyset_1^E \geq L(\kappa^n) \tag{61}$$

or

$$\emptyset_1^E > X(\kappa^n) \tag{62}$$

then the hardening surface computations are performed.

Since both the plastic potential and the elastic potential are implicit functions of their arguments, an iterative scheme such as a Newton-Raphson technique is used. First, a value of $d\kappa$ is assumed from which the corresponding values of $L(\kappa)$, $X(\kappa)$, and $d\varepsilon_{kk}^P$ are computed. Second, a trial value of $\emptyset_1$, i.e., $\emptyset_1^t$, corresponding to $\kappa^t = \kappa^n + d\kappa$ is computed in a manner similar to eq. (55)

$$\emptyset_1^t = \emptyset_1^E - \frac{9B}{1 + 2\alpha} d\varepsilon_{kk}^P \tag{63}$$

Third, the trial values of η_{ij} are computed using eqs. (12), (13), (15), (32), and (37) as

$$\eta_{ij}^t = \frac{\eta_{ij}^E + 2S\left(\frac{\delta_{ij}}{3} - \frac{A_{ij}}{1 + 2\alpha}\right) d\varepsilon_{kk}^P}{1 + \cfrac{S\ d\varepsilon_{kk}^P}{(1 + 2\alpha)\ F(\emptyset_1^t , \kappa^t)\ \cfrac{\partial H}{\partial \emptyset_1^t}}} \tag{64}$$

Finally, $\emptyset_2^t$ is computed. If

$$H(\emptyset_1^t , \sqrt{\emptyset_2^t} , \kappa^t) \leq r \tag{65}$$

where r is a very small input number (of the order of 10^{-6}, with dimensions of stress) which controls the level of accuracy in the iterative scheme, then the computation of the hardening surface is completed and the final "corrected" stresses are

$$\phi_1^{n+1} = \phi_1^t \tag{66}$$

and

$$\eta_{ij}^{n+1} = \eta_{ij}^t \tag{67}$$

If, on the other hand, eq. (65) is not satisfied within the specified value of r, then a new value of $d\kappa$ is obtained (by the Newton-Raphson technique, operating on eq. (65) and the previous value of $d\kappa$) and the procedure is repeated until eq. (65) is satisfied.

4. Application of the Model

A particular form of the above model has been coded by the author [1] and incorporated into a finite difference computer code called "TLAYER." This code has been used for blast-induced ground shock calculations. For brevity's sake, typical results of such calculations cannot be included herein; however, the interested reader is referred to reference [1].

5. Acknowledgment

The work reported herein was sponsored by the Defense Nuclear Agency (DNA) under Nuclear Weapons Effects Subtask SB209, "Propagation of Ground Shock Through Soils and Rock."

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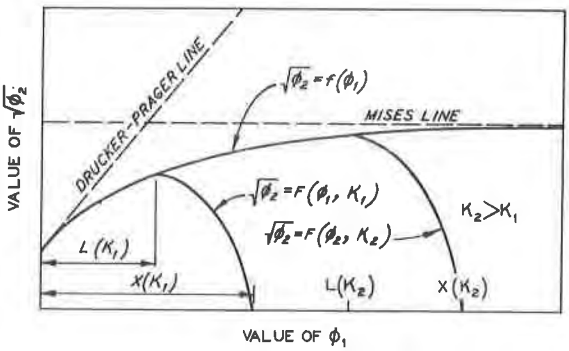

Figure 1. Typical yield surface in transverse-isotropic, inelastic, work-hardening model.

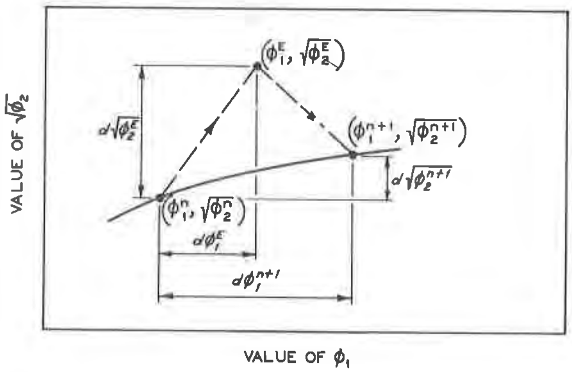

Figure 2. Method of correcting a trial stress state to the final stress state on the failure envelope.