Analytical Solution of Dynamic Motions of Rectangular Foundations

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Abstract

The dynamic stresses and the displacement amplitude of the oscillation of a rigid rectangular foundation resting upon a semi-infinite isotropic elastic medium were studied. The solutions of the equations of motion are expressed in terms of volumetric dilation and rotation about Z-axis. The singularity arising from the dual integral equation were then eliminated through the assumed boundary conditions. Finally, the stress and displacement expressions could then be reduced to non-dimensional forms for design purposes.

Introduction

The analysis of oscillations of rigid bodies of finite mass resting upon a semi-infinite isotropic elastic medium is of great interest for dynamic analysis of structural foundations. In process of design, a designer must know the dynamic stress to be developed and the displacement amplitude of the medium and foundation system due to dynamic actions of the rigid body or the medium. A review of literature reveals that the complete analytical solutions of the equations of motion of the rectangular foundations have not been obtained due to difficulties involved in formulating the boundary conditions. Unknown boundary stress and the singularities involved in the derivation of the boundary stress the major obstacles in formulating an analytical solution.

An attempt has been made here to provide the complete solutions of the equations of motion in terms of volumertric dilation, ε , and rotation about Z-axis, ω_z, by introducing the boundary conditions as suggested by Awojobi and Grootenhuis (1). The solutions thus obtained eliminate the singularity arising from the dual integral equations developed for the boundary conditions. Details of the solution is given in an unpublished paper (2).

Method of Solution

The governing equations of motion can be expressed as:

$$C_1^2 \text{ grad div } S - C_2^2 \text{ curl curl } S = \frac{d^2 S}{dt^2} \qquad \text{.......................................(1)}$$

where S = displacement vector

C_1 = velocity of waves of dilation

C_2 = velocity of waves of distortion

In a cartesian rectangular coordinate system, as shown in Fig. 1, Eq. (1) can be written as:

$$C_1^2 \frac{\partial^2 \varepsilon}{\partial x^2} - 2C_2^2\left(\frac{\partial \omega_z}{\partial y} - \frac{\partial \omega_y}{\partial z}\right) = \frac{\partial^2 u}{\partial t^2} \qquad \text{(2a)}$$

$$C_1^2 \frac{\partial^2 \varepsilon}{\partial y^2} - 2C_2^2\left(\frac{\partial \omega_x}{\partial z} - \frac{\partial \omega_z}{\partial x}\right) = \frac{\partial^2 v}{\partial t^2} \qquad \text{(2b)}$$

$$C_1^2 \frac{\partial^2 \varepsilon}{\partial z^2} - 2C_2^2\left(\frac{\partial \omega_y}{\partial x} - \frac{\partial \omega_x}{\partial y}\right) = \frac{\partial^2 w}{\partial t^2} \qquad \text{(2c)}$$

Considering the case of vertical translation, i.e., $\omega = 0$, and all derivatives with respect to z will be zero. Thus,

$$C_1^2 \frac{\partial^2 \varepsilon}{\partial x} - 2C_2^2 \frac{\partial \omega_z}{\partial y} = \frac{\partial^2 u}{\partial t^2} \qquad \text{(3a)}$$

$$C_1^2 \frac{\partial^2 \varepsilon}{\partial y^2} + 2C_2^2 \frac{\partial \omega_z}{\partial x} = \frac{\partial^2 v}{\partial t^2} \qquad \text{(3b)}$$

Now, differentiating Eq. (3a) with respect to x and Eq. (3b) with respect to y, then adding the resulting equations, one can obtain the following equations:

$$\nabla^2 \varepsilon + \Omega_1^2 \varepsilon = 0 \qquad \text{(4a)}$$

$$\nabla^2 \omega_z + \Omega_2^2 \omega_z = 0 \qquad \text{(4b)}$$

in which $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $\Omega_1^2 = \frac{\Omega^2}{C_1^2}$, and $\Omega_2^2 = \frac{\Omega^2}{C_2^2}$. Here, Ω is the frequency of vibration of the rigid body.

Let us assume that the solution of Eq. (4a) be the form of

$$\varepsilon = X(x)Y(y) \qquad \text{(5)}$$

thus, one can obtain the solution as:

$$\varepsilon = \exp(-\lambda_1 y)\left[A \cos p_1 x + B \sin p_1 x\right] \qquad \text{(6)}$$

Similarily, the solution of Eq. (4b) may be written as:

$$\omega_z = \exp(-\lambda_2 y)\left[C \cos p_2 x + D \sin p_2 x\right] \qquad \text{(7)}$$

where A, B, C, and D are to be determined by means of β, p_1, p_2, λ_1 and λ_2 Moreover, examining Eqs. (6) and (7), we see both ε and ω_z are of periodic nature; consequently, it is reasonable that one assumes $p_1 = m$ and $p_2 = n$, where m = 0, 1, 2, 3 ..... and n = 0, 1, 2, 3...., whence

$$\lambda_1^2 = m^2 - \Omega_1^2 \qquad \text{(8a)}$$

$$\lambda_2^2 = n^2 - \Omega_2^2 \qquad \text{(8b)}$$

As such, σ_y and τ_{xy} can be obtained as follows:

$$\sigma_y = -\frac{G}{\Omega_2^2}\left\{\left(\beta^4\lambda_1^2 - \beta^2\frac{\lambda}{G}p_1^2\right)\exp(-\lambda_1 y)\left[A\cos p_1 x + B\sin p_1 x\right] - 4p_2\lambda_2\exp(-\lambda_2 y)\left[D\cos p_2 x - C\sin p_2 x\right]\right\} \qquad \text{(9)}$$

$$\tau_{xy} = \frac{2G}{\Omega_2^2}\left\{\left(\lambda_2^2 + p_2^2\right)\exp(-\lambda_2 y)\left[C\cos p_2 x + D\sin p_2 x\right] + \beta^2\lambda_1\exp(-\lambda_1 y)p_1\left[B\cos p_1 x - A\sin p_1 x\right]\right\} \qquad \text{(10)}$$

The boundary conditions of stresses can be expressed as follows:

$\sigma_y = \sigma_x$ at x = 0, y = 0

$\sigma_y = 0$ at $x = b\ell$, y = 0, where $\ell > 1$

$\tau_{xy} = 0$ at x = 0, y = 0

$\tau_{xy} = 0$ at $x = b\ell$, y = 0, where $\ell > 1$

By means of the above boundary conditions, one can rewrite Eqs. (6) and (7), respectively, as:

$$\varepsilon = \exp(-\lambda_1 y)\left[\frac{a}{M}\cos p_1 x + \frac{b}{M}\sin p_1 x\right]\sigma(x) \qquad (11)$$

$$\omega_z = \exp(-\lambda_2 y)\left[\frac{c}{M}\cos p_2 x + \frac{d}{M}\sin p_2 x\right]\sigma(x) \qquad (12)$$

where a, b, c, d, and M are known constants, while $\sigma(x)$ is the surface stress distribution in the region $0 \leqslant x \leqslant b$, $0 < -x < -b$.

The equations for stresses and displacements have been obtained from the stress-strain and the strain-displacement relationships. The resulting equations for displacements u and v are found to be as follows:

$$u = -\left\{\frac{1}{\Omega_1^2}\exp(-\lambda_1 y)\sigma(x)\left[\left(\frac{a}{m}\frac{x}{b^2-x^2} + p_1\frac{b}{m}\right)\cos p_1 x + \left(\frac{b}{m}\frac{x}{b^2-x^2} - p_1\frac{a}{m}\right)\sin p_1 x\right] + \frac{2\lambda_2^2}{\Omega_2^2}\exp(-\lambda_2 y)\sigma(x)\left[\frac{c}{M}\cos p_2 x + \frac{d}{M}\sin p_2 x\right]\right\} \qquad (13)$$

and

$$v = -\sigma(x)\left\{-\frac{\lambda_1}{\Omega_1^2}\exp(-\lambda_1 y)\left(\frac{a}{m}\cos p_1 x + \frac{b}{m}\sin p_1 x\right) + \frac{2}{\Omega_2^2}\exp(-\lambda_2 y)\left[\left(\frac{c}{m}\frac{x}{b^2-x^2} + \frac{d}{m}p_2\right)\cos p_2 x + \left(\frac{d}{m}\frac{x}{b^2-x^2} - \frac{c}{m}p_2\right)\sin p_2 x\right]\right\} \qquad (14)$$

The stress and displacement expressions could then be reduced to non-dimensional forms. In determining the mass or inertia of the medium to be added to the rigid body, one can use the equations suggested by Awojobi and Grootenhuis in (1). For vertical translation:

$$\Omega^2 = \frac{2\pi G(1-\beta_1^2)}{m+m_c} \qquad (15)$$

and for rocking mode:

$$\Omega^2 = \frac{\pi G b^2(1-\beta_1^2)}{J + J_c + \frac{Wh}{\Omega^2}} \qquad (16)$$

where W is the weight of the rigid body per unit length and h is the distance of the center of gravity from the axis of rocking.

References

(1) Awojobi, A. O., and P. Grootenhuis, "Vibration of Rigid Bodies on Semi-infinite Elastic Media," Proc. Royal Soc., A, Vol. 287, pp 27 - 63, 1965, London.

(2) Ray, A. K., and P. C. Chan, "An Analytical Solution of The Equations of Motion of Rectangular Foundations," (Unpublished Notes), 1984.

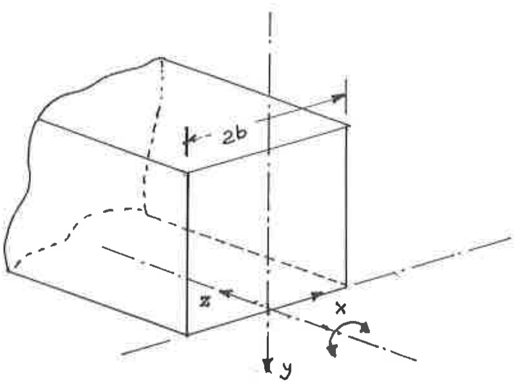

Figure 1. Coordinate axes Chosen for rectangular rigid body