

## DIRECT FLEXIBILITY FINITE ELEMENT ELASTOPLASTIC ANALYSIS

R.H. GALLAGHER,

*Department of Structural Engineering, Cornell University, Ithaca, New York, U.S.A.*

A.K. DHALLA,

*Foster-Wheeler Corp., Livingston, New Jersey, U.S.A.*

### ABSTRACT

The direct flexibility method, a finite element procedure for complementary energy solutions, is formulated for the elastoplastic plane stress analysis. Flexibility relationships are derived for a rectangular element, using an Airy stress function representation which corresponds to the displacement field representation for an inter-element-compatible rectangle in bending. This model enables satisfaction of equilibrium conditions in both the local and global sense, exclusive of boundary conditions. The constitutive relationships of incremental plasticity theory, with the Prandtl-Reuss flow rule, are utilized in conjunction with an initial stress approach in application to elastoplastic analysis. Illustrative examples are performed and the results compared with alternative solutions.

### 1. INTRODUCTION

The duality of matrix formulations of the displacement (or stiffness) and force (or flexibility) procedures for finite element structural analysis is well known (Langefors [1], Argyris and Kelsey [2]). Practical applications of the displacement method, however, are rarely cast in the form dual to the traditional force method, but are rather performed by application of the seemingly more efficient "direct stiffness" concept (Martin [3]). The direct stiffness concept enables formation of the system equations from the element equations by simple addition of element stiffness coefficients possessing identical subscripts.

Heretofore, a "direct flexibility" approach to matrix force analysis has not been available and dependence has been placed upon a format which features redundant, self-equilibrating, force systems as basic unknowns. Redundant force systems have represented an impediment to the computationally efficient utilization of matrix force procedures in finite element analysis. The emergence of procedures for the automatic selection of redundants (Denke [4], Robinson [5]) has enhanced the appeal of matrix force analysis to only a limited extent, since a substantial computational expense is accrued in the redundant selection process.

A motivation for matrix force analysis is the accomplishment of a valid minimum complementary solution, enabling the establishment of an upper bound value for strain energy. This has been achieved (Fraeijs de Veubeke [6]) through inversion of directly-derived element flexibilities to the form of element stiffnesses and the subsequent use of the displacement method in performance of the analysis of the complete system, avoiding the difficulties attendant upon matrix (redundant) force analysis. This approach suffers from difficulties in the construction of a kinematically-stable system stiffness model.

The need for and advantages of a method of "direct flexibility" analysis, analogous (or dual) to direct stiffness analysis and capable of producing valid minimum complementary energy solutions, is well recognized. Firstly, the many large scale general purpose computer programs (Gallagher [7]), based upon direct stiffness concepts, are rendered amenable to utilization for minimum complementary energy analyses. Secondly, many of the already formulated types of elements, established in direct stiffness analysis, are transferable to the direct flexibility method. Thirdly, complementary energy formulations incorporate the direct, as-measured format of the material constitutive law, i.e., the expression for strain in terms of stress, which is advantageous in elastoplastic analysis where the coefficients are explicitly functions of stress.

Recently, considerable progress has been made in the development of the method of direct flexibility analysis. The papers by Fraeijs de Veubeke [8], Sander [9], Morley [10, 11], Elias [12], Watwood and Hartz [13], Anderheggen [14] and Charlwood [15] are noteworthy for their development of concepts and general procedures in linear analysis. Contributions to elastoplastic analysis have been made by Schmit and Rybicki [16], Belytschko and Hodge [17] and Belytschko [18].

The present paper differs from the aforementioned in the form of element representation, the detailed approach to construction of boundary constraint equations--with associated implications in the solution procedure for the equations of the complete system--and in the elastoplastic analysis procedure. The element representation consists of a rectangle in plane stress whose stress field is approximated by means of a 16-term expression for the Airy stress function. Force boundary conditions yield constraints upon the complementary energy function to be minimized and here these constraint equations are established by integration of the applied stress field on the boundary. Finally, elastoplastic analysis is treated via an approach which parallels the "initial stress" approach of Zienkiewicz, et al [19] for potential energy.

## 2. COMPLEMENTARY ENERGY FOR PLANE STRESS

The basic constitutive relationships for plane stress, including initial strains  $\{\epsilon^i\}$ , are

$$\{\sigma\} = [D]\{\epsilon\} + [D]\{\epsilon^i\} \quad (1)$$

where  $\{\sigma\} = [\sigma_x \sigma_y \tau_{xy}]^T$  is the vector of stress components (the superscript T designates the transpose of a matrix),  $\{\epsilon\} = [\epsilon_x \epsilon_y \gamma_{xy}]^T$  is the vector of strain components, and [D] is the matrix of material stiffness coefficients. The complementary energy of a structure ( $\pi_c$ ) is defined as

$$\pi_c = \frac{1}{2} \int_V [\sigma] [D]^{-1} \{\sigma\} dV + \int_V [\sigma] \{\epsilon^i\} dV - \int_{S_u} \bar{u} \cdot T dS \quad (2)$$

$\bar{u}$  defines prescribed displacements, T is the surface traction in the direction of  $\bar{u}$ ,  $S_u$  is the surface upon which  $\bar{u}$  is prescribed, and V is the volume of the structure.

We now express the stress vector  $\{\sigma\}$  in terms of the Airy stress function  $\phi$  where, by the usual definition,  $\sigma_x = \phi_{yy}$ , etc., and the subscripts on  $\phi$  denote differentiation with respect to the indicated variables. Thus

$$[\sigma] = [\phi_{yy} \quad \phi_{xx} - \phi_{xy}] \quad (3)$$

and, by substitution of (3) into (2) and with  $\{\sigma^i\} = [D]^{-1} \{\epsilon^i\}$  defined as the initial stress vector

$$\pi_c = \frac{1}{2} \int_V [\phi_{yy} \quad \phi_{xx} - \phi_{xy}] [D]^{-1} [\phi_{yy} \quad \phi_{xx} - \phi_{xy}]^T dV + \int_V [\phi_{yy} \quad \phi_{xx} - \phi_{xy}] [D]^{-1} \{\sigma^i\} dV - \int_{S_u} \bar{u} \cdot T dS \quad (4)$$

$$\text{or} \quad \pi_c = U_c - W \quad (4a)$$

where  $U_c$ , the complementary strain energy, is composed of the first two integrals on the r.h.s. of eq. (4) and  $W = \int_{S_u} \bar{u} \cdot T dS$ . We will exclude from this paper, however, the possibility of non-zero prescribed displacements so that  $W = 0$  and  $\pi_c = U_c$ . Also, with respect to force boundary conditions, we consider only distributed edge loadings. These limitations are introduced merely to permit development of the present concepts in the simplest form within the allotted space and do not represent inherent limitations of the approach.

### 3. ELEMENT EQUATIONS

The element studied in this paper is the constant-thickness rectangle of dimension a x b (Figure 1). The necessary interelement equilibrium conditions, requiring continuity of both  $\phi$  and  $\frac{\partial \phi}{\partial n}$  on each boundary, are met by the bivariate Hermite polynomial interpolation of fourth order. We have, for the desired 16-term function

$$\phi = [N] \{\phi^e\} \quad (5)$$

where

$$[N] = [N_1(x)N_1(y) \quad \dots \quad N_{x1}(x)N_{y2}(y)]_{16 \times 1} \quad (6a)$$

$$\{\phi^e\} = [\phi_{11} \quad \dots \quad \phi_{xy12}]_{16 \times 1}^T \quad (6b)$$

The shape functions  $N_1(x) \dots N_{x1}(x)$ , etc., are presented in Figure 1.

$N_{1}(y) \dots N_{y2}(y)$  are similarly defined by replacing  $a$  with  $b$  and  $\zeta = \frac{x}{a}$  by  $\eta = \frac{y}{b}$ . We shall refer to the stress function parameters  $\{\phi^e\}$  as d.o.f. (degrees of freedom) in the following.

Establishing now the stress vector by appropriate differentiation of eq. (5), we have, symbolically

$$\{\sigma\} = \begin{Bmatrix} \phi_{yy} \\ \phi_{xx} \\ -\phi_{xy} \end{Bmatrix} = \begin{bmatrix} [N_{yy}] \\ [N_{xx}] \\ -[N_{xy}] \end{bmatrix} \{\bar{\phi}^e\} = [C]\{\bar{\phi}^e\} \quad (7)$$

By substitution of eq. (7) into eq. (4), the element complementary strain energy ( $U_c^e$ ) becomes

$$U_c^e = \frac{[\bar{\phi}^e]}{2} [f^e]\{\bar{\phi}^e\} + [\bar{\phi}^e]\{G_i^e\} \quad (8)$$

where

$$[f^e] = \left[ \int_A [C]^T [D]^{-1} [C] t dA \right] \quad (9)$$

$$\{G_i^e\} = \left\{ \int_A [C]^T [D]^{-1} \{\sigma^i\} t dA \right\} \quad (10)$$

$[f^e]$  denotes the element flexibility matrix and  $\{G_i^e\}$  is the initial stress matrix.

Because of the analogy between the homogeneous forms of the equilibrium governing differential equation for plate flexure (in terms of transverse displacement,  $w$ ) and the compatibility differential equation for plate stretching (in terms of the Airy stress function  $\phi$ ) (Fraeijs de Veubeke and Zienkiewicz [20]), an existing stiffness formulation for plate flexure (Bogner, et al [21]) gives the flexibility coefficients defined by eq. (9). Due account must be taken of the differences in the constants and signs between terms of the material compliance ( $[D]^{-1}$ ) and stiffness  $[D]$  matrices.

We next examine the applied loads on the element boundary, which lead to constraint conditions on the d.o.f. Consider an element with an edge parallel to the  $x$ -axis and acted upon by distributed stresses  $\sigma_y(x)$  (Figure 2). At any point along the edge:

$$\frac{\partial^2 \phi}{\partial x^2} = \sigma_y(x) \quad (11)$$

Integrating this expression twice and evaluating the constants of integration in terms of  $\phi$  and  $\frac{\partial \phi}{\partial x}$  at the end points of the segment (i.e.,  $\phi_i, \phi_j, \phi_{xi}, \phi_{xj}$ ) there results

$$-\phi_{xi} + \phi_{xj} = \int_0^a \sigma_y(x) dx \quad (12)$$

and

$$-\phi_i + \phi_j - \phi_{xi} a = \int_0^a \int_0^x \sigma_y(x) dx \quad (13)$$

For the shearing stress, by definition,  $\tau_{xy}(x) = -\phi_{xy}$  and since the twist derivative is a d.o.f., direct evaluation at the end points of the edge gives the two conditions  $-\phi_{xy_i} = \tau_{xy_i}$  and  $-\phi_{xy_j} = \tau_{xy_j}$ . A third condition on the shear stress is needed to account for its quadratic variation. By integration of  $\tau_{xy}(x) = -\phi_{xy}$  along the edge i-j and with evaluation of  $\phi_y$  at i and j

$$\phi_{y_i} - \phi_{y_j} = \int_0^a \tau_{xy}(x) dx \quad (14)$$

Thus, five constraint equations (12, 13, 14 and the conditions on  $\tau_{xy_i}$  and  $\tau_{xy_j}$ ) emerge from conditions along a single edge. At exterior corners, of course, conditions on the shear stress are common to the adjacent edges so that in a representation of a rectangular plate with a single element there will be the necessary total of sixteen constraint conditions (four per edge). These constraint equations are sufficient to describe the force boundary conditions exactly when the edge direct stresses vary linearly and the shear stress varies quadratically. More complicated boundary conditions are approximated in an integral sense.

It is useful to distinguish between the constraints which assign a value to a single d.o.f. ( $\tau_{xy_i}$  and  $\tau_{xy_j}$ ) and those which are of a form of an equation connecting various d.o.f. (eqs. 12-14). The former type of constraint is imposed directly on  $U_c^e$  and requires no further representation. The equations of the latter type of constraint cannot be imposed directly on the element complementary energy but must be treated specially in the system analysis. They are written, collectively for a single element, in the form

$$[\Gamma^e]\{\bar{\phi}^e\} - \{B^e\} = 0 \quad (15)$$

where  $[\Gamma^e]$  and  $\{B^e\}$  consist of the coefficients resulting from integration of eqs. 12-14.

#### 4. SYSTEM EQUATIONS

The operations to be performed in establishment of equations describing the behavior of the complete system are: (a) the synthesis of the system equations from the element equations, including constraint equations, and (b) the formation of a solvable final representation which accounts for constraints.

The synthesis of the system equations begins with the determination of the complementary strain energy ( $U_c$ ), using as a basis the component element complementary energies ( $U_c^e$ ). The requirements to be met in interconnection of elements to retain a valid minimum complementary energy formulation are those of continuity of the normal and shear stresses across the element boundaries. As noted above, use of the present bivariate Hermitian polynomial interpolation function for  $\phi$  meets such requirements. It is possible to satisfy a condition of continuity of direct stress parallel to the element edges by use of a sixth-order function (Rybicki and Schmit [16]) but this

goes beyond the requirements for a valid minimum complementary energy solution. Thus, the required continuity is achieved when, at each joint in the system, the d.o.f.  $\{\bar{\Phi}^e\}$  of the elements meeting at each joint are equated to each other.

It follows, from the well-known concepts of direct stiffness analysis (Martin [3]), that the system flexibility matrix is formed by simple addition of all element flexibility coefficients with like subscripts. Thus

$$U_c = \frac{[\bar{\Phi}]}{2} [f]\{\bar{\Phi}\} \quad (16)$$

where  $[\bar{\Phi}] = [\phi_j \phi_{x_j} \phi_{y_j} \phi_{xy_j}]$  lists all joint d.o.f. in the system and  $[f]$  is the system flexibility matrix, established as

$$f_{ij} = \sum_{e=1}^{e=q} f_{ij}^e \quad (17)$$

for the  $q$  elements at the joint  $i$ .

From eq. (8), the contribution of the initial strains to the element complementary strain energy is described by the product of the row and column vectors,  $[\bar{\Phi}^e]$  and  $\{G_i^e\}$ , respectively. Combination of these for all elements of the system, in the same manner as for initial force terms in minimum potential energy analysis yields the system product  $[\bar{\Phi}]\{G_i\}$ . The system complementary energy is therefore

$$\pi_c = U_c = \frac{[\bar{\Phi}]}{2} [f]\{\bar{\Phi}\} + [\bar{\Phi}]\{G_i\} \quad (18)$$

For a system with  $n$  d.o.f. the matrix  $[f]$  is  $n \times n$  and  $[\bar{\Phi}]$  and  $\{G_i\}$  are  $1 \times n$  and  $n \times 1$ , respectively.

Similarly, the element boundary conditions (eq. 15) assemble to form the  $m$  system constraint equations

$$[\Gamma]\{\bar{\Phi}\} - \{B\} = 0 \quad (19)$$

Before the condition of stationary complementary energy can be invoked account must be taken of the constraint equations. Alternative procedures include: (a) use of the constraint equations to solve for certain d.o.f. in terms of others, removing the former from the functional (Greene [22]), special operations devised by Morley [10] in which the size of the original system is preserved, and (c) the classical method of Lagrange multipliers, which expands the size of the system of equations to be solved.

For conciseness, we discuss here the method of Lagrange multipliers. We define one such multiplier,  $\lambda_i$ , for each of the  $m$  constraint equations (eq. 19), form the product of the vector of Lagrange multipliers,  $[\lambda] = [\lambda_1 \dots \lambda_i \dots \lambda_m]$  and the constraint equations, and add the result to the original functional, i.e., (with  $[\lambda][\Gamma]\{\bar{\Phi}\} = [\bar{\Phi}][\Gamma]^T\{\lambda\}$ )

$$\bar{\pi}_c = \frac{[\bar{\Phi}]}{2} [f]\{\bar{\Phi}\} + \{\bar{\Phi}\}\{G_i\} + [\bar{\Phi}]\{H\} - [\lambda]\{B\} + [\bar{\Phi}][\Gamma]^T\{\lambda\} + C_1 \quad (20)$$

where now  $\bar{\pi}_c$  is termed the augmented functional.



and

$$d\bar{\epsilon}_p = \left[ \frac{2}{9} (d\epsilon_x^p - d\epsilon_y^p)^2 + (d\epsilon_x^p)^2 + (d\epsilon_y^p)^2 + \frac{1}{3} (d\gamma_{xy}^p)^2 \right]^{1/2} \quad (26)$$

Having obtained the incremental stress-strain relationship in the plastic range, the total strain components are now written as the sum of elastic and plastic strains

$$\{d\epsilon\} = \{[D]^{-1} \{d\sigma\} + [D_p]^{-1} \{d\sigma\}\} \quad (27)$$

or, inversely

$$\{d\sigma\} = \left[ [D]^{-1} + [D_p]^{-1} \right]^{-1} \{d\epsilon\} \quad (28)$$

The procedure for tracing the behavior of the system through the non-linear regime is the "initial stress" approach wherein the flow rule (eq. 28) and the system compatibility equation (eq. 21) are satisfied in an iterative manner in a given load increment. The procedure is as follows (see Figure 3):

- (1) An elastic analysis for an arbitrary load level is performed and, by scaling, the load level at which yield first occurs in an element is established. Designating the onset of first yielding by the subscript "0", we have the corresponding load  $\{P_0\}$  and stresses  $\{\sigma_0\}$ .
- (2) The load level is incremented by an amount  $\{\Delta P\}$  and the associated stress  $\{\Delta\sigma'\}_1$  and strain changes  $\{\Delta\epsilon'\}_1$  are computed via elastic analysis. The subscript "1" outside the bracket denotes the first iterative step.
- (3) Calculate the change in stress  $\{\Delta\sigma\}_1$  due to the change in strain ( $\{\Delta\epsilon'\}_1$ ) in accordance with the flow rule (eq. 28). It is important here to employ in the computation a value of H' which corresponds to the slope of the effective stress-effective strain diagram between the interval end points. An estimate based upon the approximate stress change in the load increment is generally satisfactory.
- (4) Since the increase in plastic strain produced the stress change  $\{\Delta\sigma\}_1$  (Step 3) rather than  $\{\Delta\sigma'\}_1$  (Step 2), a corrective analysis for the discrepancy must be performed. Compute  $\{\Delta\sigma''\}_1 = \{\Delta\sigma'\}_1 - \{\Delta\sigma\}_1$  and employ this in the formation of the system initial stress vector  $\{G_1\}$  (eq. 10).
- (5) Compute and store the current stress and strain from

$$\{\sigma\} = \{\sigma_0\} + \{\Delta\sigma\}_1 \quad (29)$$

$$\{\epsilon\} = \{\epsilon_0\} + \{\Delta\epsilon'\}_1 \quad (30)$$

Eq. (29) also defines a new yield surface for a strain hardening material and the corresponding effective stress is the new yield stress for the next iteration.

- (6) Using the  $\{G_1\}$  calculated in Step 4, perform a system analysis which results in new stresses  $\{\Delta\sigma'\}_2$ . Employ these in an elastic determination of  $\{\Delta\epsilon'\}_2$ .

(7) Repeat Steps 3-6 until the stresses  $\{\Delta\sigma\}_n$  in the nth cycle are negligible.

Once convergence is reached for a given load increment a new increment is defined and Steps (2)-(7) are repeated. The process of load incrementation is continued until either the total desired load level is reached or the strains are excessive for small increases in load, i.e., "collapse."

If an element which is elastic at the start of a load increment becomes plastic during the application of the fictitious initial stress, its contribution to the inelastic terms should be taken into account in the succeeding iterative steps. If unloading takes place in any part of the structure during the incrementation of load, the subsequent action is elastic and requires no special treatment. Also, if the material is elastic-perfectly plastic, then the stress does not increase beyond  $\{\sigma\}_0$ . Thus,  $\{\Delta\sigma\}_n$  in the above discussion is zero for all values of the plastic strain and the initial stress is immediately determined as  $\{\Delta\sigma\}_n$ .

## 6. ILLUSTRATIVE EXAMPLES

Two numerical examples, one elastic and one elastoplastic, are presented. An elastic analysis evaluation is desirable since the element presented in this paper has not previously been employed in elastic complementary energy analysis.

For the elastic analysis consider a rectangular plate (Fig. 4) subjected to parabolically distributed stresses  $\sigma_y$  along two opposite faces. Symmetry about two axes permits study of a quadrant of the plate. Idealizations based upon 1, 4, and 16 elements are employed in 3 separate analyses; Table 1 summarizes the numbers of constraint equations and solution parameters for each analysis.

Results, in the form of plots of the stress distributions on various sections are shown in Figs. 5 and 6. Comparisons are given in Table 1 with a "classical" solution (Timoshenko and Goodier [25] which uses a 3-term series representation) for the stress at the center of the plate. From the above-mentioned figures and Table 1 it can be observed that the stresses approach the Ref. [25] solution as the grid is consistently refined.

In order to check the feasibility of the present method in the plastic range the problem of a uniformly loaded, simply supported beam was analyzed. The problem data and element idealization are shown in Figure 7. The material is elastic-perfectly plastic. This same problem was solved by Anand, et al [26], who employed the displacement approach, 272 constant strain triangular elements, and the Tresca yield condition.

The solution in the elastic range is summarized in Table 2. The 16-element stress function solution is of superior accuracy to the 272-element displacement-based solution in the prediction of stresses, as might be expected of an equilibrium model. In the plastic range the chosen load incre-

ment was  $0.1 P_e$ , where  $P_e = 23.8$  kips is the total load on the half span to cause initial yielding. The fictitious "initial stresses" were evaluated at each corner of the rectangular plastified element and the average value was then redistributed. The  $\sigma_x$  and  $\sigma_y$  stresses in the plastic range at a distance  $L/8$  from the midspan of the beam are reproduced from Ref. [26], together with the complementary energy results at two loads  $P = 1.4 P_e$  and  $P = 1.6 P_e$  are shown in Figure 8. The stress distributions give good correlation. The elastic plastic boundary at a given load increment is also given in Ref. [26]; these results could not be duplicated here due to the coarse gridwork. To obtain meaningful results at higher load levels by the present method it would be necessary to employ a finer gridwork.

#### 7. CONCLUDING REMARKS

A finite element method of elastoplastic plane stress analysis, founded in complementary energy procedures and based upon the use of stress function fields, has been presented. The limited results obtained thus far, which are not described here in entirety, indicate that the many necessary extensions in the direction of element representation and inclusion of alternative inelastic constitutive relationships would be profitable. Especially attractive is the possibility of employing existing, displacement-based, finite element computer programs for the present approach; only limited modifications to such programs should be required.

REFERENCES

- [1] Langefors, B., "Analysis of Elastic Structures by Matrix Transformation with Special Regard to Semi-monocoque Structures", *Journal of the Aeronautical Sci.*, V. 19, No. 7, July, 1962, pp. 451-458.
- [2] Argyris, J. H. and Kelsey, S., "Energy Theorems and Structural Analysis", Butterworths Publishers, London, 1960.
- [3] Martin, H. C., "Introduction to Matrix Methods of Structural Analysis", McGraw-Hill Book Co., New York, 1965.
- [4] Denke, P. H., "Engineering Aspects and Mathematical Formulation of the Problem of a Computerized Aircraft Structural Analysis System", SAE Trans. Paper 650066, 1965.
- [5] Robinson, J. R. and Haggemacher, G. R., "Optimization of Redundancy Selection in the Finite Element Force Method", *AIAA Journal*, V. 8, No. 8, Aug. 1970.
- [6] Fraeijs de Veubeke, B., "Upper and Lower Bounds in Matrix Structural Analysis", *AGARDograph*, 72, Pergamon Press, 1964.
- [7] Gallagher, R. H., "Large-Scale Computer Programs for Structural Analysis", *ASME Annual Meeting*, Dec. 1970.
- [8] Fraeijs de Veubeke, B., "Displacement and Equilibrium Models in the Finite Element Method", Chapter 9 in *Stress Analysis 1965*, O. Zienkiewicz and G. Holister, Ed., John Wiley and Sons, Inc., New York.
- [9] Sander, G., "Application of the Dual Analysis Principle", *Proc. of the IUTAM Symposium on High Speed Computing of Elastic Structures*, Liege, Belgium, 1970.
- [10] Morley, L. S. D., "The Triangular Equilibrium Element in the Solution of Plate Bending Problems", *The Aero. Quarterly*, May 1968.
- [11] Morley, L. S. D., "A Triangular Equilibrium Element with Linearly Varying Bending Moments for Plate Bending Problems", *Jl. of the Royal Aeronautical Society*, Vol. 71, Oct. 1967.
- [12] Elias, Z. M., "Duality in Finite Element Methods", *Jn. of Eng. Mech. Div. Proc. ASCE*, Vol. EM4, Aug. 1968.
- [13] Watwood, V. B. and Hartz, B. J., "An Equilibrium Stress Field Model for Finite Element Solutions of Two Dimensional Elastostatic Problems", *Int. J. Solid Structures*, Vol. 4, 1968.
- [14] Anderheggen, E., "Finite Element Plate Bending Equilibrium Analysis", *Proc. ASCE, Journal of the EM Div.*, V. 95, No. EM4, Aug. 1969, pp. 841-857.
- [15] Charlwood, R. G., "Dual Formulations of Linear Elasticity Using Finite Elements", *Computer Aided Engineering*, G.L.M. Gladwell (Ed.), Univ. of Waterloo, 1971.
- [16] Rybicki, E. F. and Schmit, L. A., "An Incremental Complimentary Energy Method of Non-Linear Stress Analysis", *AIAA Journal*, V. 119, Jan. 1969.
- [17] Belytschko, T. and Hodge, P. G., "Plane Stress Limit Analysis by Finite Elements", *Proc. ASCE, Journal of the Struct. Div.*, V. 96, No. EM6, Dec. 1970, pp. 931-944.
- [18] Belytschko, T., "Error Bounds in Elastic-Plastic Finite Element Analysis of Plates", *Computer Aided Engineering*, G. L. M. Gladwell (Ed.), Univ. of Waterloo, 1971.

- [19] Zienkiewicz, O., Valliappan, S. and King, I., "Elastic-Plastic Solutions of Engineering Problems, Initial Stress, Finite Element Approach", International Journal for Numerical Methods in Engineering, Vol. 1, No. 1, Jan-March 1969.
- [20] Fraeijns de Veubeke, B. and Zienkiewicz, O. C., "Strain-Energy Bounds in Finite Element Analysis by Slab-Analogy", JI. of Strain Analysis, Vol. 2, No. 4, 1967.
- [21] Bogner, F. K., Fox, R. L. and Schmit, L. A., "The Generation of Inter-element, Compatible Stiffness and Mass Matrices by the Use of Interpolation Formulas", Proc. of Conf. on Matrix Methods in Struc. Mech., Dayton, O., AFFDL-TR 66-80, Oct. 1965.
- [22] Greene, B. E., "Application of Generalized Constraints in the Stiffness Method of Structural Analysis", AIAA Journal, V. 4, No. 9, Sept. 1966, pp. 1531-1537.
- [23] Marcal, P. V. and King, I. P., "Elastic-Plastic Analysis of Two Dimensional Stress Systems by the Finite Element Method", International Journal of Mech. Science, Vol. 9, pp. 143-155, 1967.
- [24] Mendelson, A., "Plasticity; Theory and Application", MacMillan, N.Y., 1968.
- [25] Timoshenko, S. and Goodier, J. N., "Theory of Elasticity", McGraw-Hill Book Co., Inc., Third Edition, 1970.
- [26] Anand, S. C., Lee, S. L., and Rossow, E. C., "Finite Element Analysis of Elastic-Plastic Plane Stress Problems Based Upon Tresca Yield Criterion", Ingenieur-Archiv, Vol. 39, 1970, pp. 73-86.

Table 1

Square Plate Subjected to Parabolic Loading Along Two Opposite Faces

No. of Elements	Degrees of Freedom n	Boundary Constraint Eqs. m	$\sigma_y$ at Center of plate (F.E. analysis)	$\sigma_y$ at Center of pl. ("Exact", Ref. 25)
1	16	12	0.912 $\sigma_{20}$	0.862 $\sigma_{20}$
4	36	24	0.887 $\sigma_{20}$	0.862 $\sigma_{20}$
16	100	48	0.867 $\sigma_{20}$	0.862 $\sigma_{20}$

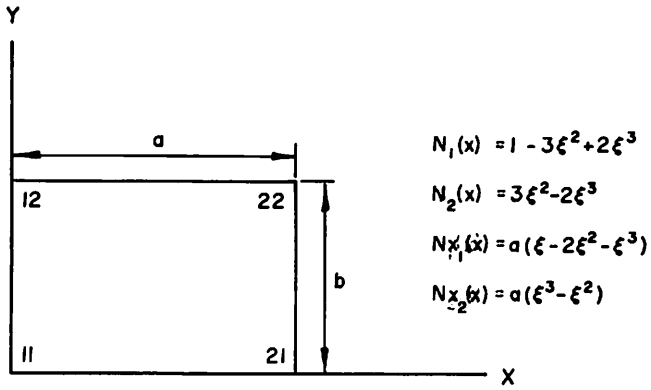
Table 2

Elastic Stresses at Quarter Span at Load  $P_e = 23.811$  kips

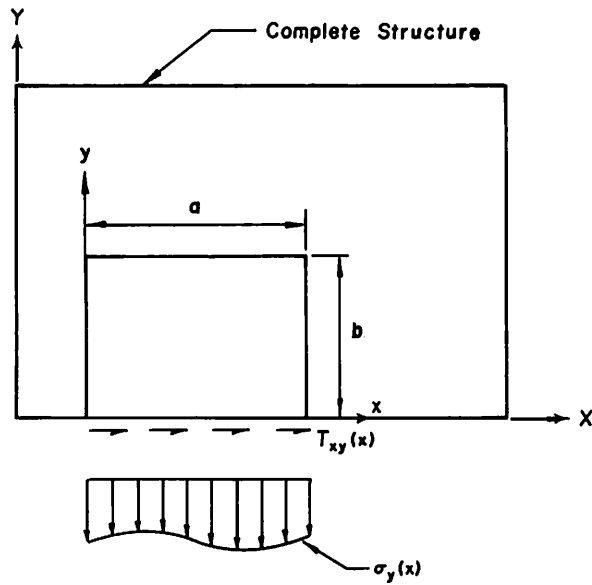
Dist. from top of beam	$\sigma_x$ (ksi.)			$\sigma_y$ (ksi.)			$\tau_{xy}$ (ksi.)		
	Finite Elements		Exact Eq. (33) Ref. 25	Finite Elements		Exact Eq. (d) P. 41 Ref. 25	Finite Elements		Exact Eq. (d) P. 41 Ref. 25
	P.E. <sup>a</sup>	C.E. <sup>b</sup>		P.E. <sup>a</sup>	C.E. <sup>b</sup>		P.E. <sup>a</sup>	C.E. <sup>b</sup>	
0	-24.76	-27.23	-27.40	-2.61	-2.98	-2.98	0.93	0.00	0.00
d/4	-12.81	-13.08	-13.15	-2.52	-2.51	-2.52	3.21	3.35	3.34
d/2	0.03	-0.04	0.00	-1.53	-1.49	-1.49	4.32	4.46	4.46
3d/4	12.81	13.00	13.15	-0.50	-0.47	-0.46	3.26	3.35	3.34
d	24.70	27.23	-27.40	-0.21	0.00	0.00	0.95	0.00	0.00

<sup>a</sup>Potential Energy solution [26] with 272 constant strain triangles.

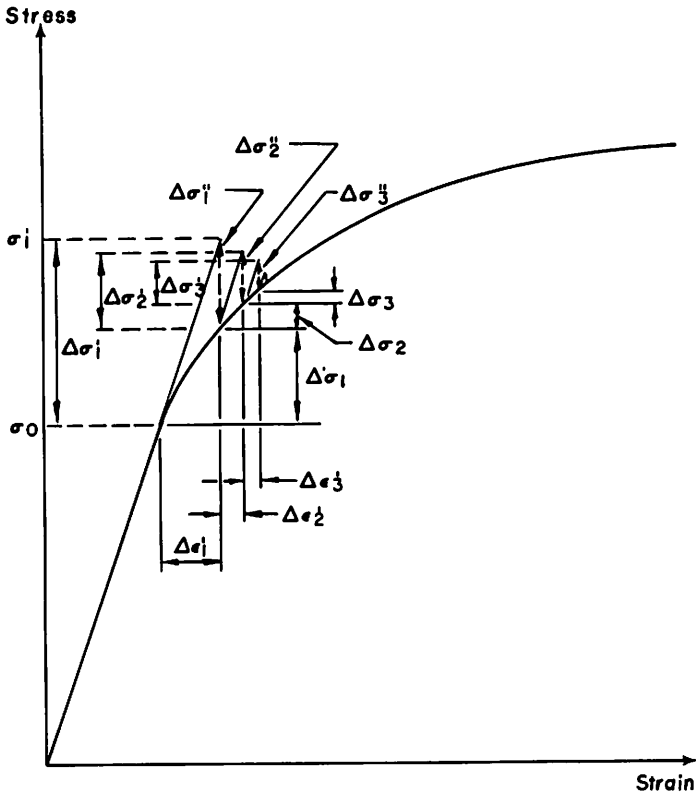
<sup>b</sup>(Present) Complementary Energy solution with 16 rectangle elements.



1. RECTANGULAR ELEMENT

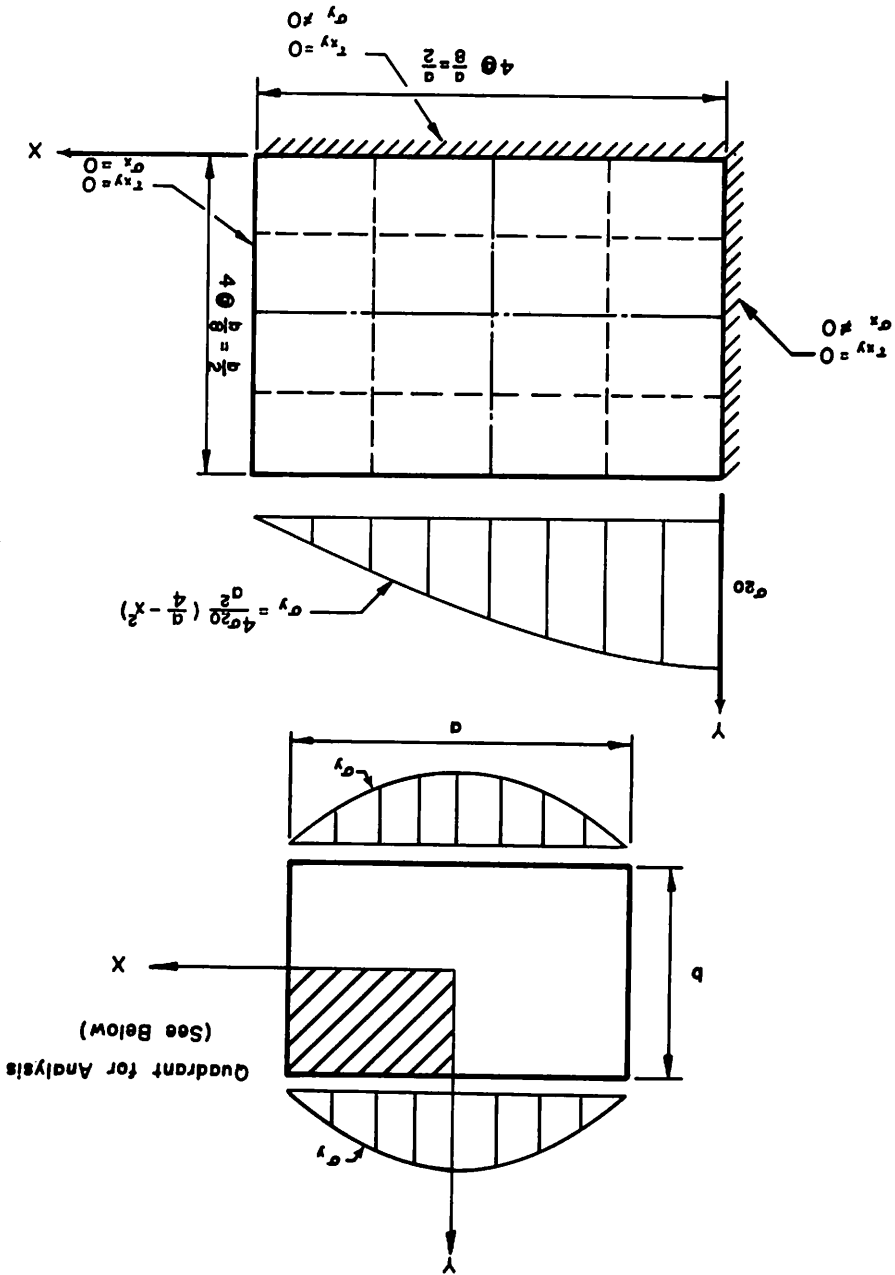


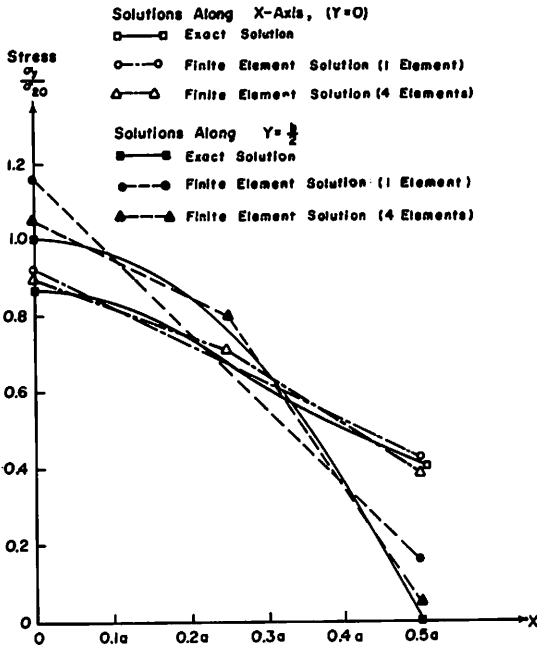
2. DISTRIBUTED EDGE STRESSES



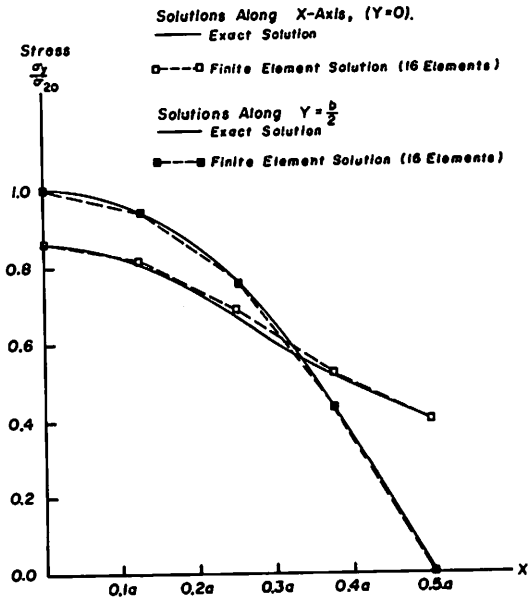
3. INCREMENTAL-ITERATIVE (INITIAL STRESS) PROCEDURE FOR ELASTOPLASTIC ANALYSIS

4. RECTANGULAR PLATE UNDER PARABOLICALLY-VARYING EDGE STRESS

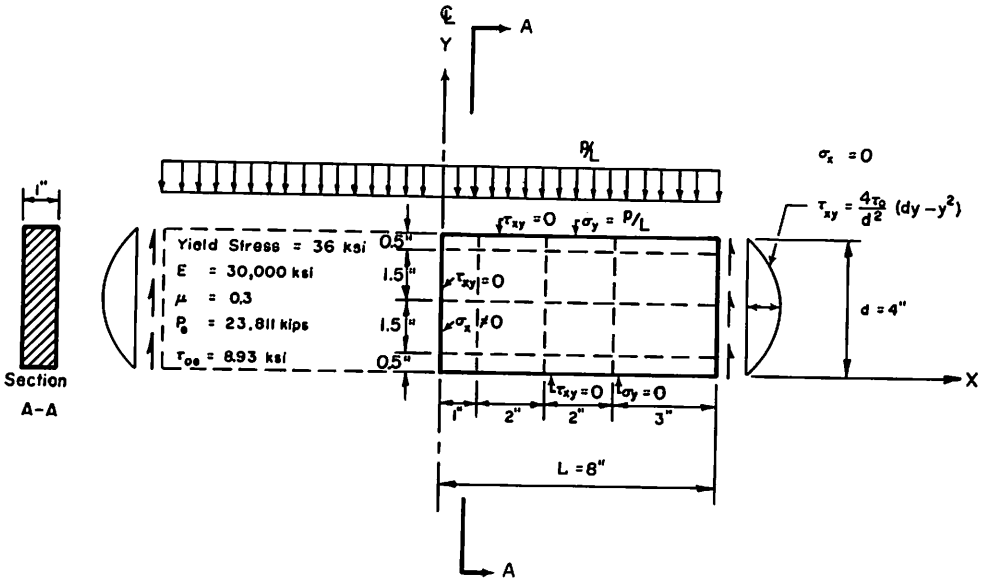




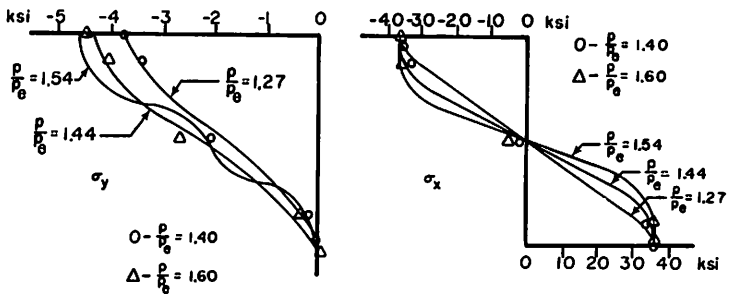
5. RECTANGULAR PLATE - ELASTIC SOLUTION - 1 & 4 ELEMENTS



6. RECTANGULAR PLATE - ELASTIC SOLUTION - 16 ELEMENTS



7. SIMPLY-SUPPORTED BEAM. ANALYSIS DATA



8. SIMPLY-SUPPORTED BEAM. ELASTOPLASTIC ANALYSIS RESULTS

**Q** D. HARTIG, Germany

Your first example showed a plate with two edges free and two edges clamped, in two points of this example the solution is singular and in my experience these singularities are indicated by discontinuous solutions for the stresses. You showed smooth stress curves.

**A** R. H. GALLAGHER, U. S. A.

The figure intended to represent the conditions along an axis of symmetry, not a fixed edge (this is described in the text). The point raised is an interesting observation, however, of the character of this approach for the situation you describe.

**C** Y. R. RASHID, U. S. A.

With reference to the creep formulation, the problem of inverting the creep compliance to obtain the relaxation modulus is quite a significant part of the solution process and can amount to several hours of computed time for a practical problem. I will be quite interested to learn of future success in this area and I believe it will be a definite contribution.

**Q** K. WILLAM, Germany

Prof. Gallagher, you indicated the implementation of force boundary conditions for the direct equilibrium approach using Lagrangian constraint equations. This procedure destroys the positive definiteness of the resulting set of equations and complicates very much the usage of standard finite element software packages such as ASKA, NASTRAND, STRUDEL. . . Do you think this procedure will prove useful in large scale programs ?

**A** R. H. GALLAGHER, U. S. A.

We employed the frontal solution of B. Irons, published in the International Journal of Numerical Methods in Engineering, Jan. 1970, for equation-solving. This program, which has gained wide utilization, because of the publication of the detailed listing, incorporates the treatment of constraints by use of Lagrange multipliers. If this provision is not in the equation-solver used, and if the presence of zeros on the main-diagonal presents no difficulty, then one may employ the concept of "constraint elements" described by the author in the Proceedings of the ONR Symposium on Numerical Methods in Structural Engineering, University of Illinois, Sept. 1971. We did not encounter difficulties in the conditioning of the equations due to this source.

A. JEZERNIK, U. K.

**Q**

I would like to ask you two questions: Did you 1) compare the computing time and central memory required when solving different classes of problems and 2) investigate the suitability of different finite element approaches when solving different problems.

R. H. GALLAGHER, U. S. A.

**A**

1) and 2) No comparison studies of computer time or of a wider range of approaches in finite element plasticity analysis were attempted.