An equilibrium segmental finite element for plane elasticity problems

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1 INTRODUCTION

The displacements of structural elements due to environmental changes and or due to externally applied loads are made up of two major components. The first component is due to the strain free-rigid body modes and the second is due to straining of the elements and the final displacements are obtained by adding the corresponding two components for each of the various fields. This fact is inherent in any analytical solution for such problems. For in such an analysis the displacements are usually obtained by integrating the relevant strain-displacement equations. If the equations for the various components of strains in terms of the displacements are put to zero the resulting solutions for the various components of the displacements are due to strain free-rigid body modes. In mathematical terms these are referred to as the complementary functions of the solutions to the differential equations. The complete solution is obtained by adding the particular integrals to the complementary functions and hence the particular integrals are the displacements corresponding to straining of the element.

The application of such an approach to the development of displacement fields for finite elements was introduced by Sabir et al. It was first applied to curved members such as arches (1971) and (1975), and to two dimensional analysis of shells (1972), (1982), (1983) and (1985). The resulting displacement fields satisfy the exact representation of the components corresponding to rigid body modes of displacements, since exact solutions, for these cases, to the differential equations relating strains to displacements can be found. The other component is developed by assuming polynomial expressions for the various strains and hence are approximate within the context of the finite element method of analysis and this approach is referred to as the strain based approach. It is to be noted that unlike the usual finite element approach in which the displacement fields are assumed and also each field is independent since the constants appearing in the displacement fields are different, the strain based approach leads to linking of the constants appearing in the various displacement fields due to rigid body modes of displacements and also due to straining. Analytical solutions also exhibit such linking. Furthermore both components satisfy the compatibility equations separately, for sometimes the assumed strains are linked to achieve this. The strain based approach was developed to remove some shortcomings which were highlighted when finite elements based on assumed independent polynomial displacements were tested by applying them to the analysis of curved members and shells.

Meanwhile Sabir et al, (1983), (1984), (1985) and (1986) have shown that the strain based approach is not confined to curved structures but can be used in the development of displacement fields for flat elements in both Cartesian and polar coordinate systems.

A basic rectangular inplane element having only the two essential external degrees of freedom at each of the four corner nodes is first developed in Cartesian coordinates, Sabir (1983). A method of coordinate transformation from Cartesian to polar coordinates was used to obtain a sector inplane element equivalent to that of the basic rectangular element, Sabir and Salhi (1986). This sector element was applied to the solution of rotationally symmetric and unsymmetric deformations of circular plates to show its satisfactory performance. In the present paper, a sector inplane element having the only essential degrees of freedom, satisfying the strain free-rigid body and compatibility requirements and based on strain assumptions satisfying the equilibrium equations is developed. The performance of this new equilibrium sector element is compared with a previous sector element to show the degree of improvement of convergence of the results.

2 DERIVATION OF THE DISPLACEMENT FUNCTIONS AND STIFFNESS MATRIX

The equilibrium equations in polar coordinates are more complex than in Cartesian coordinates and thus it is difficult to assume functions for the radial, circumferential and shearing strains which satisfy the equilibrium equations in polar coordinates. We therefore proceed by obtaining the displacement fields for an equilibrium element in Cartesian coordinates and then use the method of coordinate transformation referred to earlier to obtain the displacement fields for the equilibrium sector element in polar coordinates.

Consider the sector element shown in Figure 1a: the three components of strains at any point p in the Cartesian coordinate system x and y are given by

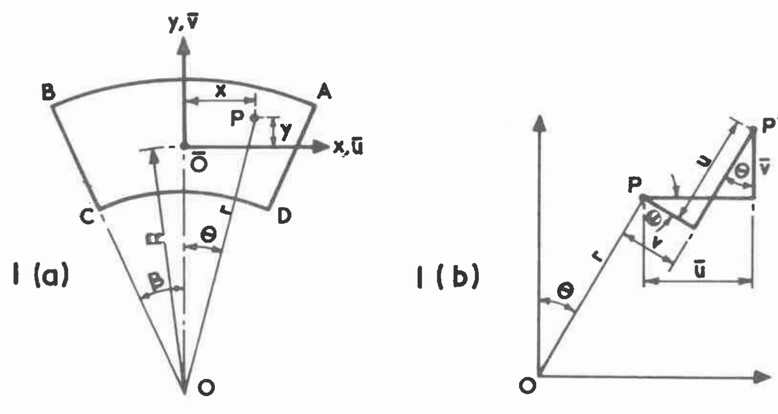

Figure 1. Coordinate systems and displacements for the sector element

$$
(1) \qquad \begin{aligned} \varepsilon_x &= \partial\bar{u}/\partial x \\ \varepsilon_y &= \partial\bar{v}/\partial y \\ \gamma_{xy} &= \partial\bar{u}/\partial y + \partial\bar{v}/\partial y \end{aligned}
$$

where ε_x and ε_y are the direct strains, γ_{xy} is the shearing strain, and $\bar{u}$ and $\bar{v}$ are the translational displacements in the x and y directions, respectively.

If these three strains are equal to zero, the resulting differential equations from (1) can be integrated to obtain

$$
(2) \qquad \begin{aligned} \bar{u} &= a_1 - a_3 y \\ \bar{v} &= a_2 + a_3 x \end{aligned}
$$

Equations (2) represent the displacement fields for the sector element in terms of its three rigid body displacements components a_1, a_2 and a_3. If the sector element is to have four corner modes at A, B, C and D (Figure 1a) and each node is to have two degrees of freedom, the element displacement functions should contain eight independent constants. Having used three for the representation of the rigid body components, the remaining five are to be apportioned among the three strains. The usual way of doing this is to assume

$$
(3) \qquad \begin{aligned} \varepsilon_x &= a_4 + a_5 y \\ \varepsilon_y &= a_6 + a_7 x \\ \gamma_{xy} &= a_8 \end{aligned}
$$

If the element is required to satisfy the equilibrium equations which are

$$
(4) \qquad \begin{aligned} \frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} &= 0 \\ \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{xy}}{\partial x} &= 0 \end{aligned}
$$

where σ_x and σ_y are the direct stresses and τ_{xy} is the shearing stress, equations (4) can be written in terms of the strains ε_x, ε_y and γ_{xy} from Hooke's Law and if equations (3) are then substituted into the resulting equilibrium equations we find that these equations can not be satisfied unless the strains are linked in some way. One possible way of this linking is given by

$$
(5) \qquad \begin{aligned} \varepsilon_x &= a_4 + a_5 y - \nu a_7 x \\ \varepsilon_y &= a_6 + a_7 x - \nu a_5 y \\ \gamma_{xy} &= a_8 \end{aligned}
$$

when equations (5) and (1) are equated and integrated, we obtain

$$
(6) \qquad \begin{aligned} \bar{u} &= a_4 x + a_5 xy + a_8 y/2 - a_7 (y^2 + \nu x^2)/2 \\ \bar{v} &= a_6 y + a_7 xy - a_5 (x^2 + \nu y^2)/2 + a_8\, x/2 \end{aligned}
$$

where ν is Poisson's ratio.

The final displacement functions will be given by adding equations (2) and (6) to obtain

$$\bar{u} = a_1 - a_3 y + a_4 x + a_5 xy + a_8 y/2 - a_7(y^2 + \nu x^2)/2 \tag{7}$$
$$\bar{v} = a_2 + a_3 x + a_6 y + a_7 xy - a_5(x^2 + \nu y^2)/2 + a_8 x/2$$

To convert the above in terms of a polar coordinates system we use, from Figure 1b,

$$x = r\sin\Theta \tag{8}$$
$$y = r\cos\Theta - R$$

where R is the radius of curvature of the central circumferential line of the element and the polar coordinates r and Θ are as shown in Figure 1a. Furthermore, if any point p is displaced to p' with components $\bar{u}$ and $\bar{v}$ in the x and y directions, the displacement components in the r and Θ directions u and v will be given (see Figure 1b) by

$$u = \bar{u}\sin\Theta + \bar{v}\cos\Theta \tag{9}$$
$$v = \bar{u}\cos\Theta - \bar{v}\sin\Theta$$

The final displacement functions will be obtained by substituting equations (7) into (9) and substituting the resulting equations for x and y from (8), hence

$$\begin{aligned} u = {} & a_1\sin\Theta + a_2\cos\Theta + a_3 R\sin\Theta + a_4 r\sin^2\Theta \\ & + a_5\left|r\sin\Theta(r\cos\Theta - 2R\sin\Theta) - \nu\cos\Theta(r^2\cos^2\Theta - 2rR\cos\Theta + R^2)\right|/2 \\ & + a_6\cos\Theta(r\cos\Theta - R) + a_7\sin\Theta(r^2\cos^2\Theta - \nu r^2\sin^2\Theta - R^2)/2 \\ & + a_8\sin\Theta(r\cos\Theta - R/2) \end{aligned} \tag{10}$$

$$\begin{aligned} v = {} & a_1\cos\Theta - a_2\sin\Theta + a_3(r\cos\Theta - R) + a_4 r\sin\Theta\cos\Theta \\ & + a_5 r\left|\sin\Theta\cos\Theta(r - R + \nu r\cos\Theta/2 - \nu R) + \sin\Theta(r\sin^2\Theta + \nu R^2)/2\right| \\ & + a_6\sin\Theta(R - r\cos\Theta) + a_7\left|\cos\Theta(-r^2\sin^2\Theta - R^2/2 - r^2\cos^2\Theta/2 - \nu r^2\sin^2\Theta/2) + rR\right| \\ & + a_8(r\cos^2\Theta - r\sin^2\Theta - R\cos\Theta)/2 \end{aligned}$$

The strains in polar coordinates can be calculated from equations (10) and the following strain displacement equations

$$\varepsilon_r = \partial u/\partial r \tag{11}$$
$$\varepsilon_\Theta = u/r + \frac{1}{r}\,\partial u/\partial\Theta$$
$$\gamma_{r\Theta} = \frac{1}{r}\,\partial u/\partial\Theta + \partial v/\partial r - v/r$$

Having obtained the strains and the displacement fields, the usual finite element method is used to obtain the stiffness matrix for this new equilibrium sector element.

3 PROBLEM CONSIDERED

The performance of the new sector equilibrium element is tested by applying it to the problem of semi-circular annulus plate shown in Figure 2. The plate is subjected to two equilibrating shearing forces applied in the radial direction at the free ends. To test the element in a comprehensive manner a considerable range of the aspect ratios of the annulus expressed as the ratio of the outer to the inner radius (b/a) are considered. (The inner radius was kept constant and equal to 20 and b was varied accordingly). Young's modulus E and Poisson's ratio ν are taken to be 200kNmm^{-2} and 0.3.

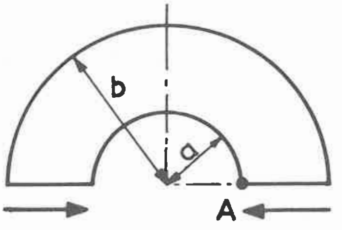

Figure 2. Annulus subjected to radial shear

Due to lack of space only a small sample of results are given below. Table 1 shows a comparison of the results for the radial deflection at A for the case of b/a = 3.0. It is seen that especially for the smaller meshes of (2x2) and (4x4) the new element gives better results than those obtained from a previous sector element Sabir and Salhi (1985). Similar results were obtained for b/a = 2.0 and 1.3 and are shown in Tables 2 and 3. Corner stresses and mid element stresses are also obtained. A sample of these results is given in Table 4 where a comparison is shown between the results obtained from the new element and the analytical values given by Timoshenko and Goodier (1951). All the results were obtained by the use of the NODAL solution Sabir (1976).

Table 1 Radial deflection at A in mm b/a = 3.0

Mesh	Present element	Previous element
2x2	0.042	0.037
4x4	0.048	0.046
6x6	0.050	0.049
8x8	0.051	0.050

Table 2 Radial deflection at A in mm b/a = 2.0

Mesh	Present element	Previous element
2x2	0.155	0.144
4x4	0.165	0.162
6x6	0.167	0.165
8x8	0.168	0.167

Table 3 Radial deflection at A in mm b/a = 1.3

Mesh	Present element	Previous element
2x2	2.487	2.347
4x4	2.683	2.673
6x6	2.686	2.684
8x8	2.684	2.684

Table 4 Shearing stress near free end 10x10 mesh a=20 b=40(kNmm^{-2})

Radius	Present element	Analytical
21.0	0.028	0.028
25.0	0.071	0.075
29.0	0.070	0.074
33.0	0.051	0.053
37.0	0.021	0.024

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