ASYMPTOTICALLY EFFICIENT ESTIMATOR FOR THE INDEX OF A STABLE DISTRIBUTION

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Summary. Based on the sample characteristic function, a method of estimating the index parameter of a stable distribution is considered. Weak convergence of the empirical characteristic function (process) is incorporated in the choice of an asymptotically optimal estimator and in the study of its asymptotic properties.

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1. Introduction.

Although, the recent years have witnessed some developments in the area of inference concerning parameters of the *stable laws*, the efforts have hardly matched their importance in many areas of applications including astronomy, business and economics. The attempts have been mostly disconcerted, in the sense that often these lack generality and apply to specific situations only. The main reason for such a state of affairs lies in the fact that while the probability density function (p.d.f.) of a stable distribution function (d.f.) always exists, it may not always be expressible in a closed form. However, the characteristic function (c.f.) $\phi(t)$ of a stable d.f. F is representible as

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$$\phi(t) = \exp\{ita-|t\delta|^{\alpha}[1+i\beta(\text{sgn}t)\omega(t,\alpha)]\} \quad , \qquad (1.1)$$

where t is real, the *index* (characteristic exponent) α satisties $0<\alpha\leq 2$, $\delta\geq 0$, $|\beta|\leq 1$, $-\infty<a<\infty$, and $\omega(t,\alpha) = (2/\pi)\log|t|$ or $\tan(\pi\alpha/2)$ according as $\alpha=1$ or $\neq 1$. Press (1972) exploited this canonical representation of $\phi(t)$ to suggest suitable estimators for the parameters α,β,δ and a; in particular, for F symmetric, these estimators were shown to have asymptotically a (multi-) normal distribution. His estimators are combinations of estimators of $\phi(t)$ at four fixed (but arbitrary) t points, and hence, the asymptotic dispersion matrix (as well as the efficiency) may depend on the choice of these t-points. In a followup, Press (1975) has given a chronological accorent of the efforts in this area, while, for the positive stable laws, some alternative (and efficient) methods have been considered by Brockwell and Brown (1979, 1981). For a general stable law, de Haan and Resnick (1980) proposed an estimator of the index parameter α, based on the order statistics. Unfortunately, their estimator is not generally asymptotically efficient.

The object of the present investigation is to have a deeper look into the method of Press (1972) and to locate the optimal t points (leading to asymptotically efficient estimators of the parameters). In this context, the weak convergence of the sample characteristic function, studied earlier by Feuerverger and Mureika (1977) and Csörgö (1981), had been incorporated to provide valid asymptotic solutions under minimal regularity conditions. An algorithm has also been provided. In fact, for the sake of simplicity and for the purpose of getting a straight-forward insight into the issues involved, we have considered here the case of a stable d.f. F symmetric about the origin (so that $a=\beta=0$) with $\delta=1$. The problem of simultaneous estimation of α,δ in such a (symmetric) case can be treated similarly at the expense of a bit more algebra, while the general case of α,β,δ and a requires a more complicated algorithm (but no new tools).

2. Empirical Characteristic Function Based Estimator of α

We are concerned with the estimation of the index parameter $\alpha (0<\alpha\leq 2)$ of a symmetric stable law for which in (1.1) $\delta=1$ (and $a=\beta=0$). Thus, (1.1) reduces to

$$\phi(t) = \exp(-|t|^{\alpha}) \ , \quad \text{for all} \quad t \in (-\infty,\infty) \quad . \tag{2.1}$$

Let $X_1,\ldots,X_n$ be a set of n independent and identically distributed random variables (i.i.d.r.v.) with a d.f. F with c.f. ϕ, given by (2.1). The *empirical characteristic function* (e.c.f.) is defined as

$$\phi_n(t) = n^{-1}\Sigma_{j=1}^{n} \exp\{itX_j\} \ , \quad t \in (-\infty,\infty) \tag{2.2}$$

Note that by (2.1),

$$\alpha = (\log|t|)^{-1}\log(-\log\phi(t)) \ , \quad \forall \text{ real } t \quad . \tag{2.3}$$

As such, following Press's (1972) moment-estimation method, for an arbitrary t, we consider the estimator

$$\hat{\alpha}_n = \hat{\alpha}_n(t) = (\log|t|)^{-1}\log(-\log U_n(t)) \ , \tag{2.4}$$

where

$$U_n(t) = \mathrm{Re}(\phi_n(t)) = \text{real part of } \phi_n(t). \tag{2.5}$$

Note that (2.4) is properly defined for all real t, excepting $t=0$ and ± 1. Since $\hat{\alpha}_n(-t) = \hat{\alpha}_n(t)$, $\forall$ real t, in the sequel, we shall consider only positive values of $t(\neq 1)$.

Now, by Theorem 2.1 of Feuerverger and Mureika (1977), for every $T(<\infty)$,

$$\sup_{|t|\leq T}|\phi_n(t)-\phi(t)| \to 0 \text{ almost surely, as } n\to\infty \ , \tag{2.6}$$

so that the real part of $\phi_n(t)$ $(=U_n(t))$ converges a.s. to $\phi(t)$, and hence, by (2.3) and (2.4), for every fixed $T(<\infty)$

$$\sup\{|\hat{\alpha}_n(t)-\alpha| : \ 0<t(\neq 1)\leq T\} \to 0 \text{ a.s. , as } n\to\infty \quad . \tag{2.7}$$

Having established this (strong) consistency of $\hat{\alpha}_n(t)$, one would naturally like to have an optimal choice of t. In this respect, we proceed to study first the asymptotic behaviour of $\{\sqrt{n}(\hat{\alpha}_n(t)-\alpha);0\le t\le T\}$ and incorporate the same in our desired solution. Let then

$$Y_n(t) = n^{\frac{1}{2}}\{U_n(t)-\phi(t)\} \ , \quad 0\le t\le T \quad . \tag{2.8}$$

Note that as F is a stable distribution and, in (2.1), $\alpha>0$,

$$\int_{-\infty}^{\infty} |x|^p dF(x) < \infty \ , \quad \forall\ 0\le p<\alpha \quad , \tag{2.9}$$

so that by Theorem 3.1 of Feuerverger and Mureika (1977) [and an improved and generalized version of this theorem by Csörgö (1981)], as $n\to\infty$, for every fixed $T<\infty$,

$$Y_n = \{Y_n(t),\ 0\le t\le T\} \xrightarrow[\mathcal{D}]{} Y = \{Y(t),\ 0\le t\le T\} \tag{2.10}$$

where Y is Gaussian with 0 drift and

$$EY(s)Y(t) = \tfrac{1}{2}[\phi(s+t)+\phi(s-t)] - \phi(s)\phi(t) \quad , \tag{2.11}$$

for every $(s,t) \in [0,T]^2$. Further, (2.10) ensures that

$$\sup\{|Y_n(t)|:\ 0\le t\le T\} = O_p(1) \quad . \tag{2.12}$$

Therefore, writing $U_n(t) = \phi(t)+n^{-\frac{1}{2}}Y_n(t)$ and using (2.3), (2.4) and (2.12), we obtain that for every fixed $T<\infty$,

$$-\log U_n(t) = -\log\phi(t)[1-n^{-\frac{1}{2}}Y_n(t)\{-\phi(t)\log\phi(t)\}^{-1} +$$

$$O(n^{-1}Y_n^2(t)\{-\phi^2(t)\log\phi(t)\}^{-1}] \ , \quad \forall\ t \in (0,T)/\{1\} \quad , \tag{2.13}$$

so that by some routine steps,

$$n^{\frac{1}{2}}(\hat{\alpha}_n(t)-\alpha) = -e^{t^\alpha} Y_n(t)(t^\alpha \log t^\alpha)^{-1}\alpha + O(n^{-\frac{1}{2}}Y_n^2(t)e^{2t^\alpha}(t^\alpha \log t^\alpha)^{-1}) \ . \tag{2.14}$$

As such, by (2.11), (2.12) and (2.14), for every fixed $t(\neq 0 \text{ or } 1)$,

$$n^{\frac{1}{2}}(\hat{\alpha}_n(t)-\alpha) \sim N(0, \sigma^2(t)) \quad , \tag{2.15}$$

where

$$\sigma^2(t) = \tfrac{1}{2}\alpha^2 e^{2t^\alpha}(-t^\alpha \log t^\alpha)^{-2}(1-2e^{-2t^\alpha}+e^{-2^\alpha t^\alpha})$$

$$= \tfrac{1}{2}\alpha^2 h_\alpha(t^*); \; t^* = t^\alpha \;, \quad t\geq 0 \quad , \tag{2.16}$$

and

$$h_\alpha(s) = e^{2s}(-s\log s)^{-2}(1-2e^{-2s}+e^{-2^\alpha s}) \;, \quad s\geq 0 \quad . \tag{2.17}$$

Note that for every $\alpha \in (0,2]$, $h_\alpha(s) \to +\infty$ as $s\downarrow 0$ or to 1 or to ∞. Further, we shall show in the Appendix that for every fixed $\alpha(\in(0,2])$, there exists a unique $s_0 \in (0,e^{-2})$, such that

$$\inf_{0<s<\infty} h_\alpha(s) = h_\alpha(s_0) \quad , \tag{2.18}$$

where $s_0(=s_{0\alpha})$ may depend on α. Let then $t_{0\alpha}$ be the solution of the equation $t_{0\alpha}^\alpha = s_{0\alpha}$. Then, from (2.15) through (2.18), we conclude that within the class $\{\hat{\alpha}_n(t),\ 0<t(\neq 1)<\infty\}$ of estimators of α, an asymptotically optimal estimator is $\hat{\alpha}_n(t_{0\alpha})$. Unfortunately, $s_{0\alpha}$ as well as $t_{0\alpha}$ may generally depend on the unknown parameter α (as $h_\alpha(s)$ does so), and hence, we may not be in a position to compute $\hat{\alpha}_n(t_{0\alpha})$. For this reason, in the next section, we proceed to construct an iterative procedure yielding an alternative asymptotically efficient esitmator.

3. An Asymptotically Optimal Iterative Estimator.

Basically, we consider an iterative method of estimating $t_{0\alpha}$ and incorporate this in the formulation of the estimator $\hat{\alpha}_n$. The suggested procedure depends implicitly on the weak convergence result in (2.6)-(2.7). Note that by virtue of (2.10)-(2.11), for every $\varepsilon>0$ and $\eta>0$, there exist a $\delta>0$ and an $n_0(=n_0(\varepsilon,\eta))$, such that for every $n\geq n_0$,

$$P\{\sup_{t:\ |t-t_{0\alpha}|\le\delta} \left|Y_n(t)-Y_n(t_{0\alpha})\right|>\varepsilon\} < \eta \quad . \tag{3.1}$$

Hence, using (2.16) and (3.1), we conclude that for every $\varepsilon'>0$ and $\eta'>0$ (and fixed $\alpha\in(0,2]$), there exist a $\delta>0$ and an $n_0'(=n_0'(\alpha,\varepsilon',\eta'))$, such that for every $n\ge n_0'$,

$$P\{\sup_{t:\ |t-t_{0\alpha}|\le\delta} n^{\frac{1}{2}}|\hat{\alpha}_n(t)-\hat{\alpha}_n(t_{0\alpha})|>\varepsilon'\} < \eta' \tag{3.2}$$

As a result, to find an asymptotically optimal estimator (within the class $\{\hat{\alpha}_n(t),t>0\}$, it suffices to consider a (stochastic) sequence $\{\hat{t}_{nm},m\ge0\}$ of t-points, such that $\hat{t}_{nm}$ stochastically converges to $t_{0\alpha}$ as m increases and $n\to\infty$, and then to use $\hat{\alpha}_n(\hat{t}_{nm})$, for some appropriate m, as the desired estimator. Towards this objective, we consider the following iterative procedures.

As will be seen in the Appendix, $s_{0\alpha}$ is always $\le e^{-2}$, for all α: $0<\alpha\le2$. As such, at the initial stage, we choose an arbitrary $\hat{t}_{n0}\in(0,1)$, preferably close to ½, compute $\hat{\alpha}_{n_0}=\hat{\alpha}_n(\hat{t}_{n0})$ and $\hat{s}_{0\alpha}^{(0)}=s_0(\hat{\alpha}_{n0})$, by (2.4) and (2.18), respectively. In the next step, we consider

$$\hat{t}_{n1}:\ \hat{t}_{n1}^{\hat{\alpha}_{n0}}=\hat{s}_{0\alpha}^{(0)}\ ,\quad \hat{\alpha}_{n1}=\hat{\alpha}_n(\hat{t}_{n1})\ \text{ and }\ \hat{s}_{0\alpha}^{(1)}=s_0(\hat{\alpha}_{n1}) \quad . \tag{3.3}$$

In this way, at the m-th step, we define

$$\hat{t}_{nm}:\ \hat{t}_{nm}^{\hat{\alpha}_{n(m-1)}}=\hat{s}_{0\alpha}^{(m-1)}\ ,\quad \hat{\alpha}_{nm}=\hat{\alpha}_n(\hat{t}_{nm})\ \text{ and }\ \hat{s}_{0\alpha}^{(m)}=s_0(\hat{\alpha}_{nm}) \quad , \tag{3.4}$$

for every $m\ge1$. The *stopping number* $M(=M_{n\varepsilon})$, corresponding to some preassigned $\varepsilon>0$, is defined by

$$M=\min\{m\ge2:\ \sqrt{n}|\hat{\alpha}_{nm}-\hat{\alpha}_{n(m-1)}|\le\varepsilon\} \quad . \tag{3.5}$$

In the Appendix, we shall show that $s_{0\alpha}$ is a continuous function of α: $0<\alpha\le2$, and hence, by (2.6), (2.7) and the definition of $s_{0\alpha}$, $\hat{t}_{nm}\to t_{0\alpha}$ a.s., as $n\to\infty$ (for every $m\ge1$), and hence, $M(=M_n)$ is a.s. finite. Actually, a few iteration will lead to the desired estimator

$$\hat{\alpha}_{nM} = \hat{\alpha}_n(\hat{t}_{nM}) \quad . \tag{3.6}$$

Since $\hat{t}_{nM}$ stochastically converges to t_0, by virtue of (3.2), (3.5), (3.6) and (2.15)-(2.17), we conclude that

$$n^{\frac{1}{2}}(\hat{\alpha}_{nM}-\alpha) \sim N(0,\tfrac{1}{2}\alpha^2 h_\alpha(s_{0\alpha})) \quad , \tag{3.7}$$

which, by (2.18), reveals the desired asymptotic optimality property.

It may be noted that if, as in Press (1972), we consider an arbitrary t and $\hat{\alpha}_n(t)$, then by (2.15) and (3.7), the asymptotic relative efficiency (A.R.E.) of $\hat{\alpha}_n(t)$ with respect to $\hat{\alpha}_{nM}$ is given by

$$\begin{aligned} e(t,t_{0\alpha}) &= h_\alpha(s_{0\alpha})/h_\alpha(t^\alpha) \\ &= \{h_\alpha(t^\alpha)\}^{-1}\{\inf_s h_\alpha(s)\} \\ &\leq 1, \ \forall t \quad , \end{aligned} \tag{3.8}$$

where the equality sign holds when $t=t_{0\alpha}$. Actually, (3.8) may be quite below 1 depending on $t, t_{0\alpha}$ and α. This explains the utility of the iterative procedure considered here. We may term $\hat{\alpha}_{nM}$ an *adaptive estimator* of α also, since $\hat{t}_{nM}$ is adapted from the data set.

4. Appendix.

The function $h_\alpha(x)$ in (2.17) plays a fundamental role in the procedure described in Section 3. We define $s_0=s_{0\alpha}$ as in (2.18). Then, we have the following.

Lemma 4.1. *For every (fixed)* α: $0<\alpha\leq 2$,

$$0 < s_{0\alpha} < e^{-2} \quad . \tag{4.1}$$

Proof. Consider first the case of $h_\alpha(x)$, $0<x<1$. Note that for every $\alpha(0<\alpha\leq 2)$, $e^{2x}(1-2e^{-2x}+e^{-x2^\alpha})$ is nondecreasing in $x(\geq 0)$, while $(-x\log x)^{-2}$ is a convex function $(0 \le x \le 1)$ with values at 0 and 1 equal to $+\infty$ and a unique minimum at $x=e^{-1}$. Hence, by (2.17), $h_\alpha(x)$ is nondecreasing on $[e^{-1},1]$, so that for $0 \le x \le 1$, the minimum value of $h_\alpha(x)$ occurs at some point below e^{-1}. Note that

$$
\begin{aligned}
h_\alpha(e^{-1}) &= e^2(e^{2/e}+e^{\gamma/e}-2) \\
&\le e^2(e^{2/e}+e^{1/e}-2) \quad (\text{as } -2 \le \gamma \le 1) \\
&= 11.74 \ , \quad \text{for all } \alpha:\ 0<\alpha\le 2 \quad . \qquad (4.2)
\end{aligned}
$$

Next, observe that for every $x \ge 1$,

$$
\begin{aligned}
h_\alpha(x) &= (x\log x)^{-2} e^{2x}(1-2e^{-2x}+e^{-2^\alpha x}) \\
&\ge (x\log x)^{-2} e^{2x}(1-2e^{-2x}+e^{-4x}) \quad (\forall\ 0<\alpha\le 2) \\
&= (x\log x)^{-2} e^{2x}(1-e^{-2x})^2 \\
&= \{(e^x-e^{-x})/(x\log x)\}^2 = g(x) \ , \quad \text{say} \quad . \qquad (4.3)
\end{aligned}
$$

Also, note that

$$
\tfrac{1}{2}\frac{d}{dx}(\log g(x)) = \frac{e^x+e^{-x}}{e^x-e^{-x}} - \frac{1}{x} - \frac{1}{x\log x} \quad , \qquad (4.4)
$$

where the first term on the right hand side of (4.4) is $\downarrow$ in x and converges to 1 as $x\to\infty$, while the other two go to 0 as $x\to\infty$. As such, if x_0 be defined by $x_0^{-1} + (x_0\log x_0)^{-1} = 1$ $(x_0 \sim 2.35)$, then (4.4) is positive for all $x \ge x_0$, so that $g'(x)$ is strictly positive for every $x \ge x_0$. Hence, the minimum of $g(x)$ on $(1,\infty)$ does not occur at any $x \ge x_0$. Let us then consider the interval $(1,x_0)$ and denote x_1 by $x_1\log x_1 = e^{-1}$, $x_2=1.5$, $x_3=1.75$, $x_4=2.00$ and $x_5=x_0$. Then, for $1 \le x \le x_1$, $g(x) \ge e^2(e^1-e^{-1})^2 = e^4(1-e^{-2})^2 \ge 40 \ge h_\alpha(e^{-1})$. Also, for $x_1 \le x \le x_2$, $g(x) \ge (x_2\log x_2)^{-2} \times (e^{x_1}-e^{-x_1})^2 \ge 20 \ge h_\alpha(e^{-1})$, and a similar case holds for $x_2 \le x \le x_3$ as well as $x_3 \le x \le x_4$ and $x_4 \le x \le x_0$. Hence, $g(x) \ge h_\alpha(e^{-1})$, for every $x \ge 1$. Thus, by (4.3), $\inf\{h_\alpha(x):\ x \ge 1\} \ge h_\alpha(e^{-1})$, $\forall\ \alpha \in (0,2]$. Therefore, the global minima of $h_\alpha(x)$ $(0<x<\infty)$ occurs at a value of $x \leq e^{-1}$.

Note that by definition, for every α: $0<\alpha\leq 2$,

$$h_\alpha(x) = \Sigma_{r=1}^{\infty}\{(2^r+\gamma^r)/r!\}x^{r-2}(-\log x)^{-2}$$

$$= \Sigma_{r=1}^{\infty}c_r x^{r-2}(-\log x)^{-2}\ ,\quad 0<x\leq e^{-1}\ , \tag{4.5}$$

where the c_r are all non-negative. Thus, for every $x>0$,

$$\frac{d}{dx}h_\alpha(x) = \Sigma_{r=1}^{\infty}c_r\{(r-2)+2(-\log x)^{-1}\}x^{r-3}(-\log x)^{-2}\ , \tag{4.6}$$

so that for every x: $2 \geq -\log x$ (i.e., $x\geq e^{-2}$), $(r-2) + 2(-\log x)^{-1} \geq r-1 \geq 0$, $\forall\ r\geq 1$, and $(d/dx)h_\alpha(x) > 0$. Therefore, the minima of $h_\alpha(x)$ does not occur within the interval (e^{-2},e^{-1}). Hence, the minima of $h_\alpha(x)$, no matter whether unique or not, occur at a point lying in the interval $(0,e^{-2})$ Q.E.D.

Note that though $h_\alpha(x)$ is the product of two convex functions, it is not itself convex everywhere. Hence, (4.2), (4.3) and (4.6) provide a convenient tool for proving (4.1).

It may be remarked that for $\alpha=2$, $h_2(x)$ is equal to $\Sigma_{r=1}^{\infty}\{2^{2r+1}/(2r)!\}x^{2(r-1)}(-\log x)^{-2}$, $x\geq 0$, and hence, is $\uparrow$ in $x \in (0,1)$, so that $s_{02}=0$. On the other hand, for $0<\alpha<2$, we like to show that $0<s_{0\alpha}<e^{-2}$ and is unique. Towards this, we have the following.

Lemma 4.2. *For every (fixed)* α: $0<\alpha<2$, $h_\alpha(x)$ *has a unique minima on* $(0,e^{-2})$.

Proof. Note that by (4.6) and the non-negativity of $x^{-2}(-\log x)^{-3}$ (on $(0,e^{-2})$), $(d/dx)h_\alpha(x)$ and $g_\alpha(x) = x^2(-\log x)^3(d/dx)h_\alpha(x)$ both have the same sign. Thus, it suffices to show that $g_\alpha(x)$ has a unique root in $(0,e^{-2})$. Note that by (4.6)

$$g_\alpha(x) = \Sigma_{r=1}^{\infty}c_r\{(r-2)(-\log x)+2\}x^{r-1}$$

$$= c_1(2+\log x)+2c_2x+\Sigma_{r=3}^{\infty}c_r x^{r-1}((r-2)(-\log x)+2)\ . \tag{4.7}$$

Note that $g_\alpha(0) = -\infty$, $g_\alpha(e^{-2}) > 0$, and hence, it suffices to show that $(d/dx)g_\alpha(x) > 0$, $\forall\ x \in (0,e^{-2})$. Towards this, we have by (4.7),

$$\frac{d}{dx} g_\alpha(x) = x^{-1}c_1 + 2c_2 + \Sigma_{r=3}^{\infty} c_r\{(r-1)x^{r-2}((r-2)(-\log x)+2)+x^{r-1}(r-2)(-\frac{1}{x})\}$$

$$= x^{-1}c_1 + 2c_2 + \Sigma_{r=3}^{\infty} c_r\{x^{r-2}[(r-1)(r-2)(-\log x)+2(r-1)-(r-2)]\}$$

$$= x^{-1}c_1 + 2c_2 + \Sigma_{r=3}^{\infty} c_r x^{r-2}\{(r-1)(r-2)(-\log x)+r\} > 0 ,$$

$$\text{for every } x \in (0,1) \quad . \tag{4.8}$$

Hence, $g_\alpha(x)$ has a unique root $(s_{0\alpha})$ in $(0,e^{-2})$. Q.E.D.

Note that the c_r are polynomial functions of $\gamma(=2-2^\alpha)$, and hence, are continuous and differentiable functions of α. Hence, by (4.7) and (4.8), we conclude that

$$s_{0\alpha} \text{ is a continuous function of } \alpha:\ 0<\alpha\leq 2 \quad . \tag{4.9}$$

We conclude this section with the remark that for $\alpha=2$, $s_{02}=0$ and for $\alpha\to 2$, $s_{0\alpha}\downarrow 0$. Hence, looking at (2.4), (2.14), (3.3) and (3.4), we may argue that for $\alpha=2$ or very, very close to 2, the iterative procedure may not work out well. However, in such a case, if we let

$$\hat{s}_{0\alpha}^{(m)} = \max\{n^{-\lambda}, s_0(\hat{\alpha}_{nm})\} \ , \quad m\geq 1 \quad , \tag{4.10}$$

where $\lambda(0<\lambda<\frac{1}{4})$ is some prefixed positive number, then the procedure works out -- though we may not have the asymptotically optimal $\hat{\alpha}_{nM}$.

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