

ON THE ASYMPTOTIC NORMALITY OF ROBUST REGRESSION ESTIMATES

by

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Abstract

Huber's (1973) proof of asymptotic normality of robust regression estimates is modified to include the estimates used in practice, which have unknown scale and only piecewise smooth defining functions ψ .

Key Words and Phrases: Robustness, M-estimates, regression, linear model, asymptotic theory.

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1. Introduction

We consider the general linear model

$$(1.1) \quad y_i = \tau_i + z_i/\sigma_0 \quad (i = 1, \dots, n),$$

where

$$(1.2) \quad \tau_i = \sum_{j=1}^p c_{ij} \beta_j^{(0)}.$$

Here the $\beta_j^{(0)}$ are unknown parameters, the c_{ij} are known constants, the z_i are independent and identically distributed with common distribution function F (symmetric about zero), and σ_0 is a constant to be chosen later. Huber (1973) proposed estimating β by solving the system of equations

$$(1.3) \quad \sum_i \psi(\sigma(y_i - t_i(\beta)))c_{ij} = 0 \quad (j = 1, \dots, p)$$

$$(1.4) \quad \sum_i \{\psi^2(\sigma(y_i - t_i(\beta))) - \xi\} = 0,$$

where

$$t_i(\beta) = \sum_{j=1}^p c_{ij} \beta_j$$
$$\xi = E_{\Phi} \psi^2(z),$$

(the expectation being taken under the standard normal distribution) and ψ is a monotone nondecreasing function. Let $C = \{c_{ij}\}$, $\Gamma = C(C^T C)^{-1} C^T$ be the projection matrix, and ϵ be the maximum diagonal element of Γ .

Huber (1973) considers σ fixed and known ($\sigma \equiv 1$) so that one only need solve (1.3). He proves that if $\epsilon p^2 \rightarrow 0$ as $n \rightarrow \infty$ (which implies $p^3/n \rightarrow 0$) and if ψ is bounded with two continuous bounded derivatives, then all estimates of the form $\hat{\alpha} = \sum_{j=1}^p a_j \hat{\beta}_j$ ($\sum_j a_j^2 = 1$) are asymptotically normal. It is

routine to extend his results to the system (1.3), (1.4) (see below). The smoothness conditions on ψ are not satisfied for three of the most commonly used functions, namely

Huber's function

$$\begin{aligned}\psi(x) &= x && |x| < c \\ &= c \operatorname{sign}(x) && |x| \geq c\end{aligned}$$

Hampel's function

$$\begin{aligned}\psi(x) &= -\psi(-x) = x && 0 < x < a \\ &= a && a \leq x < b \\ &= a \left(\frac{c-x}{c-b} \right) && b \leq x < c \\ &= 0 && x \geq c\end{aligned}$$

Andrew's function

$$\begin{aligned}\psi(x) &= -\psi(-x) = \operatorname{sine}(x/c) && 0 < x < \pi c \\ &= 0 && x \geq \pi c.\end{aligned}$$

In this note we weaken Huber's conditions to include the common ψ -functions, at the cost of slightly strengthened conditions on the rate of growth of p , the dimension of the problem. The results have been applied by Carroll and Ruppert (1979) to the problem of testing for heteroscedasticity (Bickel (1978)).

As in Huber's proof, we assume $C'C = I$. Because of the invariance of the problem, we can take $\beta_j^{(0)} = 0$ ($j = 1, \dots, p$) and $\sigma_0 = 1$, primarily to simplify notation.

2. Notation, Assumptions and Main Results

Let a be an arbitrary $(p \times 1)$ vector for which $\sum a_j^2 = \|a\|^2 = 1$. Define $s_i = \sum_{j=1}^p c_{ij} a_j$. Note that $\|t(\beta)\|^2 = \|\beta\|^2$. Define for $j = 1, \dots, p$,

$$\begin{aligned}\Phi_j(\beta, \sigma) &= - \sum_{i=1}^n \psi(\sigma(y_i - t_i(\beta))) c_{ij} / E\psi'(y_1) \\ \Psi_j(\beta, \sigma) &= \beta_j - \sum_{i=1}^n \psi(y_i) c_{ij} / E\psi'(y_1),\end{aligned}$$

while

$$\begin{aligned}\Phi_{p+1}(\beta, \sigma) &= -n^{-1/2} \sum_{i=1}^n \{\psi^2(\sigma(y_i - t_i(\beta))) - \xi\} / 2 E y_1 \psi(y_1) \psi'(y_1) \\ \Psi_{p+1}(\beta, \sigma) &= -n^{-1/2} \left((\sigma-1) + \sum_{i=1}^n \{\psi^2(y_i) - \xi\} / 2 E y_1 \psi(y_1) \psi'(y_1) \right).\end{aligned}$$

Our estimates are solutions to $\Phi_j(\hat{\beta}, \hat{\sigma}) = 0$ ($j = 1, \dots, p+1$), and we hope to approximate $(\hat{\beta}, \hat{\sigma})$ by $(\tilde{\beta}, \tilde{\sigma})$ which solves $\Psi_j(\tilde{\beta}, \tilde{\sigma}) = 0$ ($j = 1, \dots, p+1$).

We make the following assumptions.

(2.1) ψ is odd, bounded, and constant outside a finite interval.

(2.2) ψ is Lipschitz of order one and has two continuous bounded derivatives except at a finite number of points, which we take without loss as $\pm c$.

(2.3) F is symmetric about zero.

(2.4) F is Lipschitz in neighborhoods of $\pm c$.

(2.5) $E \psi'(y_1) \neq 0$, $E y_1 \psi(y_1) \psi'(y_1) \neq 0$.

Theorem 1. If (2.1) - (2.5) hold and, in addition, there is a sequence $a_n \rightarrow 0$ such that

$$(2.6) \quad (\epsilon p/a_n^2) \rightarrow 0, \quad (\epsilon n a_n) \rightarrow 0,$$

then there is a sequence of solutions to (1.3) - (1.4) such that

$$(2.7) \quad ||(\hat{\beta}, (np)^{1/2} (\hat{\sigma} - 1))|| = o_p(p).$$

If in addition

$$(2.8) \quad (\epsilon p^2/a_n^2) \rightarrow 0, \quad (\epsilon n p a_n) \rightarrow 0,$$

then

$$(2.9) \quad ||(\hat{\beta}, (np)^{1/2} (\hat{\sigma} - 1)) - (\tilde{\beta}, (np)^{1/2} (\hat{\sigma} - 1))|| \xrightarrow{p} 0.$$

Thus, (2.1) - (2.5) and (2.8) imply that all estimates of the form

$$\hat{\alpha} = \sum_{j=1}^p \alpha_j \hat{\beta}_j \quad (||\alpha||^2 = 1) \text{ are asymptotically normal.}$$

Remark: The result (2.7) is the starting point in constructing robust tests for heteroscedasticity (Bickel (1978), Carroll and Ruppert (1979)); it implies the crucial assumption T of Bickel's Theorem 3.1. For balanced designs ($\epsilon = p/n$), assumption (2.6) is satisfied by choosing $a_n = p^{-(1+\gamma)}$ for small $\gamma > 0$, which then requires $p^{4+2\gamma}/n \rightarrow 0$, as compared to Huber's condition $p^2/n \rightarrow 0$ when ψ has two derivatives.

3. Proofs

Proposition 1. Suppose that (2.1) - (2.5) hold, that ψ has two bounded continuous derivatives, and that $\epsilon p \rightarrow 0$. Then on the set

$$||(\beta, (pn)^{1/2} (\sigma - 1))||^2 \leq Kp,$$

the following hold:

$$(3.1) \quad ||\phi(\beta, \sigma) - \Psi(\beta, \sigma)|| = o_p((\epsilon p^2)^{\frac{1}{2}}),$$

$$(3.2) \quad ||\phi(\beta, \sigma) - (\beta, (np)^{\frac{1}{2}}(\sigma - 1))|| = o_p(p^{\frac{1}{2}} + (\epsilon p^2)^{\frac{1}{2}}).$$

Proof of Proposition 1. By a Taylor series expansion,

$$(3.3) \quad \left| \begin{aligned} & \sum_{j=1}^p a_j (\phi_j(\beta, \sigma) - \psi_j(\beta, \sigma)) + (\sigma - 1) \sum_{i=1}^n s_i y_i \psi'(y_i) / E \psi'(y_1) \\ & - \sigma \sum_{i=1}^n s_i t_i(\beta) (\psi'(y_i) - E \psi'(y_1)) / E \psi'(y_1) \end{aligned} \right| \\ = |A_n| \\ \leq |\sigma - 1| \left| \sum_{i=1}^n s_i t_i(\beta) y_i \psi''((1 + \eta_{2i})y_i) / E \psi'(y_1) \right| \\ + \sigma^2 \left| \sum_{i=1}^n s_i t_i^2(\beta) \psi''(\sigma y_i + \eta_{1i}) / E \psi'(y_1) \right|.$$

On the set $||\beta||^2 \leq Kp$, $|\sigma - 1| \leq \frac{1}{2}$, $||a|| = 1$, Huber shows that the second term on the r.h.s. of (3.3) is $o(\epsilon^{\frac{1}{2}} ||\beta||^2)$. Since $\psi'' = 0$ outside a finite set and $|\eta_{2i}| > \frac{1}{4}$, the first term on the r.h.s. of (3.3) is bounded by

$$M |\sigma - 1| (\sum s_i^2)^{\frac{1}{2}} (\sum t_i^2(\beta))^{\frac{1}{2}} = o(||\beta|| |\sigma - 1|).$$

Thus,

$$|A_n| = o(\epsilon^{\frac{1}{2}} ||\beta||^2 + |\sigma - 1| ||\beta||).$$

We next consider A_n , in particular its last two terms. Since $\sum s_i^2 = 1$ and $E y_1 \psi'(y_1) = 0$,

$$(3.4) \quad (\sigma - 1) \sum_{i=1}^n s_i y_i \psi'(y_i) / E \psi'(y_1) = o_p(|\sigma - 1|).$$

Huber shows that

$$(3.5) \quad \sum_{i=1}^n s_i t_i(\beta) (\psi'(y_i) - E \psi'(y_1)) / E \psi'(y_1) = o_p(\varepsilon^{1/2} \|\beta\|^2).$$

Thus, uniformly on the set $\|\beta\|^2 \leq Kp$, $|\sigma - 1| \leq Kn^{-1/2}$, $\|a\| = 1$ we have from (3.3) - (3.5)

$$(3.6) \quad \left| \sum_{j=1}^p a_j (\Phi_j(\beta, \sigma) - \Psi_j(\beta, \sigma)) \right| = o_p(\varepsilon^{1/2} p + pn^{-1/2}).$$

Another Taylor series expansion shows that

$$(3.7) \quad |B_n| = |\Phi_{p+1}(\beta, \sigma) - \Psi_{p+1}(\beta, \sigma) + C_{n1} + C_{n2}| \\ \leq M\{|\sigma - 1| \|\beta\| + pn^{-1/2} + n^{-1/2} |\sigma - 1|^2\},$$

where

$$- \{2 E y_1 \psi(y_1) \psi'(y_1)\} C_{n1} = 2n^{-1/2} (\sigma - 1) \sum_{i=1}^n (y_i \psi(y_i) \psi'(y_i) - E y_1 \psi(y_1) \psi'(y_1))$$

and

$$\{2 E y_1 \psi(y_1) \psi'(y_1)\} C_{n2} = 2 \sigma n^{-1/2} \sum_{i=1}^n t_i(\beta) \psi(y_i) \psi'(y_i).$$

Since $C_{n1} = o_p(|\sigma - 1|)$ and $C_{n2} = o_p((p/n)^{1/2})$, we obtain

$$|\Phi_{p+1}(\beta, \sigma) - \Psi_{p+1}(\beta, \sigma)| = o_p(pn^{-1/2}).$$

Since $(\varepsilon p^2)^{1/2} \geq (p^3/n)^{1/2} \geq pn^{-1/2}$, we thus obtain (3.1). Equation (3.2) follows from (3.20) of Huber (1973).

Proof of Theorem 1. The proof of Proposition 1 makes it clear that we need to obtain bounds for $|A_n|$ and $|B_n|$ uniformly on the set $\|\beta\| \leq Kp$,

$|\sigma - 1| \leq Kn^{-1/2}$ and $\|a\| = 1$. Rewrite

$$E \psi'(y_1) |A_n| = \left| \sum_{i=1}^n H(i, \beta, \sigma) \left\{ \begin{array}{l} s_i (\psi(\sigma(y_i - t_i(\beta))) - \psi(y_i)) \\ + (\sigma - 1) s_i y_i \psi'(y_i) \\ - \sigma s_i t_i(\beta) (\psi'(y_i) - E \psi'(y_1)) \end{array} \right\} \right|.$$

Let I be the indicator function and let $a_n \rightarrow 0$, $n^{1/2} a_n \rightarrow \infty$. Then

$$|A_n| = \left| \sum_{i=1}^n H(i, \beta, \sigma) \left\{ \begin{array}{l} I(-c + a_n \leq y_i \leq c - a_n) \\ + I(y_i \geq c + a_n) + I(y_i \leq -c - a_n) \\ + I(-c - a_n < y_i < -c + a_n) + I(c - a_n < y_i < c + a_n) \end{array} \right\} \right| \\ = |A_{n1} + A_{n2} + A_{n3} + A_{n4} + A_{n5}|.$$

For notational purposes, define $d_i = \sigma(y_i - t_i(\beta))$. Then

$$A_{n1} = \sum_{i=1}^n H(i, \beta, \sigma) I(-c + a_n < y_i < c - a_n) \\ \times \{I(-c < d_i < c) + I(|d_i| > c)\} = A_{n1}^{(1)} + A_{n1}^{(2)}.$$

By Proposition 1, $|A_{n1}^{(1)}| = o(\epsilon^{1/2} p)$. Since ψ is Lipschitz and constant outside a finite interval,

$$|A_{n1}^{(2)}| \leq M \sum_{i=1}^n |s_i| \{|\sigma - 1| + |t_i(\beta)|\} I(-c + a_n < y_i < c - a_n \quad |d_i| > c).$$

However, since $|\sigma - 1| \leq Kn^{-1/2}$ and $n^{1/2} a_n \rightarrow 0$,

$$|A_{n1}^{(2)}| \leq M \epsilon^{1/2} \sum_{i=1}^n \{|\sigma - 1| + |t_i(\beta)|\} I\{|t_i(\beta)| > a_n/2\}.$$

Now

$$(3.8) \quad \sum_{i=1}^n I\{|t_i(\beta)| > a_n/2\} \leq 4p/a_n^2$$

$$(3.9) \quad \sum_{i=1}^n |t_i(\beta)| I\{|t_i(\beta)| > a_n/2\} \leq 4p/a_n,$$

so that (3.8) and (3.9) imply

$$|A_{n1}^{(2)}| \leq 4M \varepsilon^{1/2} \{|\sigma - 1| p/a_n^2 + p/a_n\}.$$

However, $|\sigma - 1|/a_n \leq K(na_n^2)^{-1/2}$ so that

$$(3.10) \quad |A_{n1}| = O(\varepsilon^{1/2} p/a_n (1 + (na_n^2)^{-1/2})).$$

The same bound (3.10) holds for $|A_{n2}|$ and $|A_{n3}|$. Noting once again that ψ is Lipschitz and constant outside a finite interval,

$$\begin{aligned} |A_{n5}| &\leq \sum_{i=1}^n |s_i| \{|\sigma - 1| + |t_i(\beta)|\} I\{|y_i - c| < a_n\} \\ &\leq M\varepsilon^{1/2} \{|\sigma - 1| G_n + p^{1/2} G_n^{1/2}\}, \end{aligned}$$

where

$$(3.11) \quad G_n = \sum_{i=1}^n I\{|y_i - c| < a_n\}.$$

From Lemma 1 of Carroll (1978), provided $na_n \geq \log n$,

$$G_n = O_p(na_n).$$

This gives

$$(3.12) \quad |A_{n5}| = O_p((\varepsilon n p a_n)^{1/2}).$$

The bound (3.12) also holds for $|A_{n4}|$. Thus,

$$(3.13) \quad |A_n| = O_p(\varepsilon^{1/2} p/a_n (1 + (na_n^2)^{-1/2}) + (\varepsilon n p a_n)^{1/2}).$$

The same bound holds for $|B_n|$. Thus,

$$\begin{aligned}
 (3.14) \quad ||\Phi(\beta, \sigma) - \Psi(\beta, \sigma)|| &= O_p(\varepsilon^{\frac{1}{2}} p(1 + (na_n^2)^{-\frac{1}{2}})/a_n + (\varepsilon npa_n)^{\frac{1}{2}}) \\
 &= O_p(r(p, n)) .
 \end{aligned}$$

Equation (3.14) is the generalization of Huber's (3.18). As he shows, for sufficiently large K , on the set $||\beta||^2 \leq Kp$, $|\sigma - 1| \leq Kn^{-\frac{1}{2}}$,

$$(3.15) \quad ||\Phi(\beta, \sigma) - (\beta, (np)^{\frac{1}{2}} (\sigma - 1))|| \leq r(p, n) + \frac{1}{2}(Kp)^{\frac{1}{2}} .$$

Thus, if

$$(3.16) \quad r(p, n)/p^{\frac{1}{2}} \rightarrow 0 ,$$

Brouwer's fixed point theorem enables us to conclude that Φ has a zero inside the ball $||(\beta, (np)^{\frac{1}{2}} (\sigma - 1))|| < Kp$. Equation (3.16) is true if (2.6) is true. The rest of the proof parallels that of Huber. \square

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