AN IMPROVED MODAL SYNTHESIS METHOD FOR THE TRANSIENT RESPONSE OF PIPING SYSTEM

J.W. LEONARD

Department of Civil Engineering,
Illinois Institute of Technology, Chicago, Illinois 60616, U.S.A.

SUMMARY

In this work is explored the feasibility of employing the modal-acceleration method as an improved modal-synthesis method for the dynamic analysis of piping systems. In the modal-acceleration method, the equivalent static responses to transients loadings are combined with the partial sum of superposed modes, each mode being weighted by a participating factor for the transient loading under consideration. Thus the modal-acceleration method can be considered as a refinement on the classical method of modal superposition.

The two basic techniques in current use to predict the forced dynamic response of structural systems are 1) the direct time integration of the equations of motion, and 2) the modal-superposition method. The modal-superposition method has the relative advantage of using a single set of basic information describing the dynamic characteristics of the structural system, i.e., the natural modes and frequencies, to generate with slight additional computation the solutions to a multiplicity of different forcing functions for the same structure.

There have been two compounding disadvantages to the successful application of the modal-superposition method computationally. First, for complicated forcing functions, or for forcing functions which tend to excite the higher modes of vibration (as in the case of fluid transients in piping systems), it is often necessary to include the effects of numerous natural modes in the superposition in order to obtain sufficient accuracy. Secondly, there are no current methods of efficiently obtaining uniform accuracy of modal shapes for all the desired modes for a given analysis. The compounding effect of these two disadvantages is that it is the additional higher modes of vibration desired for additional accuracy in the modal superposition that are the most inaccurate. Even if accurate frequencies are calculated for the higher modes, the corresponding mode shapes are usually inaccurate.

One method of alleviating these difficulties is to modify the modal-superposition method so as to require a lesser number of mode shapes to represent the forced response for a given set of transient loadings. There would be additional benefit on a computational-cost basis in that, since less modes are required, the expense of calculating the higher modes could be eliminated. One method which requires less modes in the dynamic analysis is the modal-acceleration method.

In this work the fundamentals of the modal-superposition method are briefly reviewed for a damped multidegree-of-freedom system. Then the modification by inclusion of equivalent static displacement shapes is detailed and used to develop the modal-acceleration method for damped multidegree-of-freedom systems. The potential application of this method in the numerical calculation of forced responses of computer idealizations of multidegree-of-freedom structural systems is explored.

1. Introduction

The two basic techniques in current use to predict the forced dynamic response of structural systems are 1) the direct time integration of the equations of motion, and 2) the modal-superposition method. The modal-superposition method has the relative advantage of using a single set of basic information describing the dynamic characteristics of the structural system, i.e., the natural modes and frequencies, to generate with slight additional computation the solutions to a multiplicity of different forcing functions for the same structure.

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One potential method of alleviating these difficulties is to modify the modal superposition method so as to require a lesser number of mode shapes to represent the forced response for a given set of transient loadings. There would be additional benefit on a computational-cost basis in that, since less modes are required, the expense of calculating the higher modes could be eliminated. One method which requires less modes in the dynamic analysis is the modal-acceleration method (1).

In the succeeding sections the fundamentals of the modal-superposition method are reviewed for a damped multidegree-of-freedom system. Then the modification by inclusion of equivalent static displacement shapes is detailed and used to develop the modal-acceleration method for damped multidegree-of-freedom systems. The potential application of this method in the numerical calculation of forced responses of computer idealizations of multidegree-of-freedom structural systems will be explored.

2. Review of Modal Superposition

Let us assume that we are given the matrix equations describing the forced response of a structure in the form

$$[M]\{\ddot{X}\} + [C]\{\dot{X}\} + [K]\{X\} = \{P(t)\} \qquad (1)$$

where $\{X\}$ = unknown time-dependent displacement

$\{P(t)\}$ = forcing function, t = time

$[M]$ = mass matrix

$[K]$ = stiffness matrix

$[C]$ = $\alpha M + \beta K$ = damping matrix, assumed proportional to the mass and/or stiffness matrix

and a dot (.) denotes a time derivative.

As a prelude to the modal-superposition method, it is necessary to collect the natural modes of undamped motion $\{Y(t)\}$ from solutions of the equation

$$[M]\{\ddot{Y}\} + [K]\{Y\} = 0 \qquad (2)$$

The solution has the form

$$\{Y(t)\} = \{Z\} e^{ipt} \tag{3}$$

where $\{Z\}$ is the unknown mode shape and p the unknown frequency of vibration. Upon substitution of Eq. (3) into Eq. (2), we obtain

$$([K] - p^2[M])\{Z\} = 0 \tag{4}$$

The nontrivial solutions to Eq. (4) occur for a discrete number of frequencies p_j, j = 1, N, N = total number of degrees of freedom, which are solutions to

$$\det([K] - p_j^2 [M]) = 0 \tag{5}$$

Let us assume that the determinant equation, Eq. (5), has been solved for L of the N frequencies--p_j, j = 1, L. Then, the normalized mode shapes $\{Z_j\}$, j = 1, L can be determined, except for a normalizing constant, from

$$[K]\{Z_j\} - p_j^2 [M]\{Z_j\} = 0 \tag{6}$$

Once the natural frequencies and mode shapes have been determined, the superposition method proceeds as follows: Since $\{Z_j\}$, j = 1, N, constitute a set of linearly independent bases for the configuration space of $\{X\}$ in Eq. (1), we can assume a solution to Eq. (1) in the form

$$\{X\} = \sum_{j=1}^{N} F_j(t)\{Z_j\} \tag{7}$$

where $F_j(t)$ are the N unknown scalar time functions called modal participation factors. Substitution of Eq. (7) into Eq. (1) yields

$$\sum_{j=1}^{N} (\ddot{F}_j [M]\{Z_j\} + \dot{F}_j [C]\{Z_j\} + F_j [K]\{Z_j\}) = \{P\} \tag{8}$$

Equation (6) and the fact that $[C] = \alpha M + \beta K$ can be used to reduce Eq. (8) to the form

$$\sum_{j=1}^{N} (\ddot{F}_j + (\alpha + \beta p_j^2)\dot{F}_j + p_j^2 F_j)[M]\{Z_j\} = \{P\} \tag{9}$$

which can be further reduced by use of the orthogonality property of the natural modes, i.e.

$$\{Z_m\}^T[M]\{Z_j\} = \begin{pmatrix} 0 & \text{if } m \neq j \\ B_m & \text{if } m = j \end{pmatrix} \tag{10}$$

where B_m is a scalar normalization constant for the m^{th} mode. Hence, premultiplication of Eq. (9) by $\{Z_m\}^T$ yields

$$\ddot{F}_m + (\alpha + \beta p_m^2)\dot{F}_m + p_m^2 F_m = R_m(t) \tag{11}$$

where

$$R_m(t) = \{Z_m\}^T\{P(t)\}/B_m \tag{12}$$

Equations (11) and (12) constitute an N-tuple of scalar initial-value problems for the participation factors $F_m(t)$, m = 1, N. These equation can be solved numerically for as many $F_m(t)$ as desired, say L, and then can be used to approximate the exact solution given by Eq. (7) as

$$\{X\} = \sum_{m=1}^{L} F_m(t)\{Z_m\} \tag{13}$$

where $L \leq N$.

Theoretically, the more modes $\{Z_m\}$ included, the more accurate will be the approximation given by Eq. (13). Computationally, however, the higher modes contain more inaccuracies than the lower modes because of the numerical calculations required to obtain the shapes from Eq. (6). Hence, a method which will give comparable accuracy for $\{X\}$ with the inclusion of a lesser number of mode shapes is highly desirable.

3. Modal-Acceleration Method

Consider the static problem equivalent to Eq. (1)

$$[K]\{\overline{X}\} = \{\overline{P}\} \tag{14}$$

where the barred quantities denote static equivalents at time $t = t_1$, and assume again a solution in terms of the mode shapes

$$\{X\} = \sum_{j=11}^{N} F_j \{Z_j\} \tag{15}$$

Then Eq. (14) can be cast with the aid of Eq. (6) into the form

$$\sum_{j=11}^{N} \overline{F}_j \, p_j^2 \, [M]\{Z_j\} = \{\overline{P}\} \tag{16}$$

Again using the orthogonality properties of $\{Z_j\}$ as given by Eq. (10), we obtain

$$p_m^2 \, \overline{F}_m = \overline{R}_m \tag{17}$$

where

$$\overline{R}_m = \{Z_m\}^T \{\overline{P}\} / B_m \tag{18}$$

For every instant $t = t_1$, Eq. (11) can be written as

$$p_m^2 F_m(t_1) = R_m(t_1) - \ddot{F}_m(t_1) - (\alpha + \beta \, p_m^2) \, \dot{F}_m(t_1) \tag{19}$$

or--using Eqs. (17) and (18)--as

$$p_m^2 \overline{F}_m(t_1) = p_m^2 \overline{F}_m - \ddot{F}_m(t_1) - (\alpha + \beta \, p_m^2) \, \dot{F}_m(t_1) \tag{20}$$

since $\overline{R}_m = R_m(t_1)$ is the static equivalent at $\{\overline{P}\} = \{P(t_1)\}$.

Then, the solution to the equations of motion, Eq. (1), as given by Eq. (7) can be written as

$$\begin{aligned} \{X(t_1)\} &= \sum_{m=1}^{N} F_m(t_1) \{Z_m\} = \sum^{N} \overline{F}_m \{Z_m\} - \\ &- \sum_{m=1}^{N} (\ddot{F}_m(t_1) + (\alpha + \beta \, p_m^2) \, \dot{F}_m(t_1))/p_m^2 \{Z_m\} \\ &= \{\overline{X}\} - \sum_{m=1}^{N} \frac{1}{p_m^2} (\ddot{F}_m(t_1) + (\alpha + \beta \, p_m^2) \, \dot{F}_m(t_1)) \{Z_m\} \end{aligned} \tag{21}$$

If, now, we were to consider an approximate solution with a truncated sum of modal participation factors as in Eq. (13), the solution would have the form

$$\{X\} = \{\overline{X}\} + \sum_{m=1} F_m \{Z_m\} - \sum_{m=1} \frac{R_m}{p_m^2} \{Z_m\} \tag{22}$$

where Eq. (11) has been used to condense terms.

Equation (22) constitutes a more accurate approximation than does Eq. (13) if $L < N$. This can be seen by comparison of the two solutions as follows: substituting Eqs. (15) and (17) into Eq. (22) yields

$$\{X\} = \sum_{m=1}^{L} F_m\{Z_m\} + \sum_{m=L+1}^{N} \frac{R_m}{p_m^2}\{Z_m\} \qquad (23)$$

Equations (13), here rewritten as

$$\{X\} = \sum_{m=1}^{L} F_m\{Z_m\} \qquad (24)$$

is a truncated version of Eq. (23), i.e., Eq. (23) includes the static portion of the higher mode contributions ($m > L$), along with the truncated approximate sum.

Thus the modal acceleration method given by Eq. (22) is a more accurate solution than the classical modal-superposition method if an equal number of mode shapes are included. Comparable accuracy to the unmodified superpositon method can be obtained from the modal-acceleration method with the inclusion of a lesser number of mode shapes. Furthermore, since the calculated lower modes are more accurate than the higher modes, less precision is lost with the modal-acceleration method, especially since the statical contribution of the higher modes are calculated from a statical solution rather than from a free-vibration analysis of the higher modes.

4. Demonstration Problems

Two example problems are considered here in order to provide illustrations of the potential of the modal acceleration method. The first example, a three-story shear frame, is intended to provide comparisons between the modal acceleration and classical superposition methods for a small problem in which exact modal information is available. The second example, an axially and transversely loaded beam, is intended to provide comparisons for a slightly larger problem (formulated by finite element techniques) in which disparate ranges in modal frequencies exist.

In Fig. 1 is shown the three-story shear frame considered in Ref. (2). The masses are assumed lumped at the story levels and each floor is infinitely rigid. The masses and spring constants are assumed as follows:

$$M_2 = M_3 = M_4 = 15 \text{ lb. sec}^2/\text{in}$$

$$K_1 = 10 \text{ kpi} \quad K_2 = 9 \text{ kpi} \quad K_3 = 8 \text{ kpi}$$

Based on this information, the natural frequencies and mode shapes are determinable as

$$p_1 = 11.14 \text{ rdn/sec} \qquad X_1 = \{1,\ 1.905,\ 2.483\}$$

$$p_2 = 30.07 \text{ rdn/sec} \qquad X_2 = \{1,\ 0.604,\ -\ 0.867\}$$

$$p_3 = 43.64 \text{ rdn/sec} \qquad X_3 = \{1,\ -1.063,\ 0.413\}$$

The mode shapes are shown in Fig. 1.

The forced vibration example considered here is that of a step function loading applied to each floor of the frame:

$$F_2 = -1000 \text{ lbs},\ F_3 = -1000 \text{ lbs},\ F_y = +1000 \text{ lbs}, \quad t>0.$$

This loading, as shown in Fig. 2, will tend to excite the two higher modes of vibration. The resulting static equivalent displacements (shown in Fig. 2) are

X_2 = -0.1 in. X_3 = -0.1 in. X_4 = +0.025 in.

Two forced response analyses for this loading were performed. One analysis was based on the classic mode superposition approach and the other analysis employed the modal acceleration method. In Table 1 those analyses are compared. The quantity tabulated as a function of time is the horizontal motion of the top story of the frame. Results are presented in the form of percentages of deviation from the exact displacements for one and two modes of vibration included in each of the analyses. It can be seen that when only one mode is included, the use of the mode acceleration method led to improved results for eight out of the ten data points tabulated. When two modes were included, the use of the mode acceleration method led to improved results for seven out of the ten data points. A comparison of the average per cent error is of some interest; with only one mode included the average per cent error without mode acceleration was 262.9%; with mode acceleration -123.2%. With two modes included the average percent error without mode acceleration was 7.6%, with mode acceleration -4.9%.

The second example considered was that of a cantilevered beam subjected to step function loads in the transverse and axial directions at the free end of the beam. See Fig. 3. The beam was modelled using finite element theory. A standard, well-documented computer program, SAP4 (3), was used to perform both the static and eigenvalue analyses for the beam. Four equal length beam elements (100" each) were assumed, and the distributed mass was taken as lumped at the node points. The cross sectional area was taken as 10 in.^{2}, the inertia as 100 in.^{4}, and the Young's modulus as 30 x 10^6 psi. For this example, two distinct clusters of natural frequencies were found. One cluster corresponded to the flexural modes of vibration for the discrete model. The other cluster occurred at a higher frequency range and corresponds to the axial modes of vibration for the discrete model.

The forced vibration example considered was that of simultaneous sudden application of an axial and a transverse load at the free end of the cantilever. The transverse load had a magnitude of 100.0 lbs., the axial load a magnitude of 100,000.0 lbs. As expected, the primary response was a transverse vibration of the beam with a period of vibration in excess of 16.4 secs. The secondary response was an axial vibration with a shorter period of less than 0.3 secs. Results for the lateral vibration of the free end are given in Table 2. The axial vibration response of the free end is listed in Table 3.

In each table, the response with modal acceleration included is compared to the results without modal acceleration included. For the transverse response given in Table 2, it can be seen that in either case the inclusion of only one mode yielded a close approximation to the response as calculated with all eight modes included. This is because the shape of the first mode corresponded closely to the forced response shape. The results obtained with the inclusion of the modal acceleration method were, in general, closer to the exact answers. The axial vibration in Table 3 depicts the fact that at least five modes need to be included in the superposition. In that case, the modal acceleration method leads to significantly improved results. If only one mode were included in the superposition, the modal acceleration method would reproduce the static part equivalent to the mean axial response, whereas the response for only one mode included without acceleration indicates no motion in the axial direction.

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TABLE 1. COMPARISON FOR SHEAR FRAME

		One Mode Included		Two Modes Included	
time (sec)	exact U_4 (in)	no accel. (% error)	with accel. (% error)	no accel. (% error)	with accel. (% error)
0.1	+0.1234	-123.7*	61.1	3.2	1.1
0.2	-0.0760	10.5	9.2	7.0	3.7
0.3	+0.0404	-355.5*	-164.6*	1.0	5.2
0.4	-0.0524	24.9	-122.5*	4.7	-0.1
0.5	+0.1262	-110.1*	- 48.9	-4.7	-2.7
.6	+0.0204	-120.2*	257.7	-7.3	5.0
.7	+0.0703	-170.0*	60.3	-1.5	2.1
.8	-0.0526	85.7	- 61.1	11.0	-6.2
.9	+0.0087	-1188.7*	-305.9*	-34.4	-5.6
1.0	+0.0135	-430.2*	140.4	-1.3	17.3

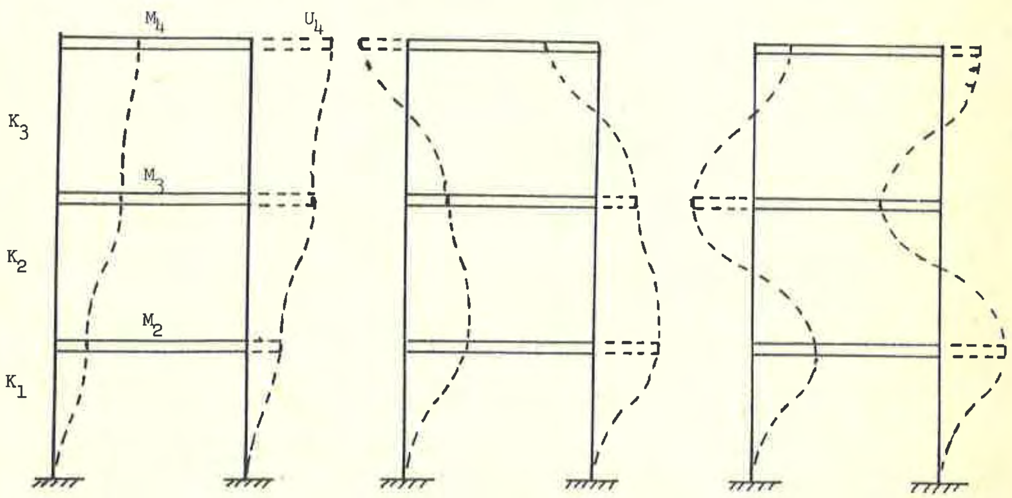
*Denotes opposite sign from exact answer

TABLE 2. COMPARISON OF TRANSVERSE BEAM DEFLECTION

time	Transverse Deflection of Free End (in.)		
	exact (8 modes)	with accel. (1 mode)	no accel. (1 mode)
0.5	0.022	0.027	0.012
1.0	0.070	0.063	0.047
1.5	0.135	0.120	0.104
2.0	0.202	0.197	0.182
2.5	0.283	0.292	0.277

TABLE 3. COMPARISON OF AXIAL BEAM DEFLECTION

time (sec)	Axial Deflection of Free End (in.)				
		one mode included		five modes included	
	8 (modes)	no accel.	with accel.	no accel.	with accel.
0.2	0.166	0.0	0.133	0.156	0.180
0.4	0.190	0.0	0.133	0.180	0.203
0.6	0.024	0.0	0.133	0.036	0.027
0.8	0.160	0.0	0.133	0.129	0.153
1.0	0.230	0.0	0.133	0.199	0.222
1.2	0.057	0.0	0.133	0.014	0.038
1.4	0.136	0.0	0.133	0.102	0.125
1.6	0.250	0.0	0.133	0.212	0.236
1.8	0.061	0.0	0.133	0.031	0.055
2.0	0.100	0.0	0.133	0.074	0.098
2.2	0.246	0.0	0.133	0.218	0.242
2.4	0.070	0.0	0.133	0.052	0.076

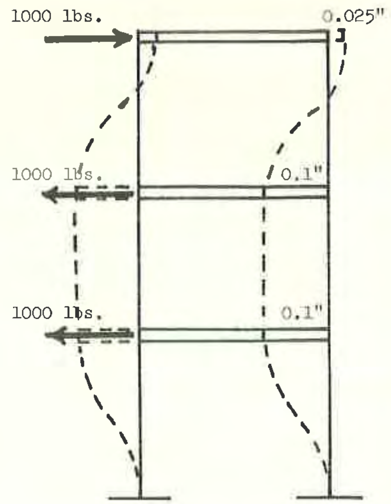


Mode 1
T = 0.56 sec.

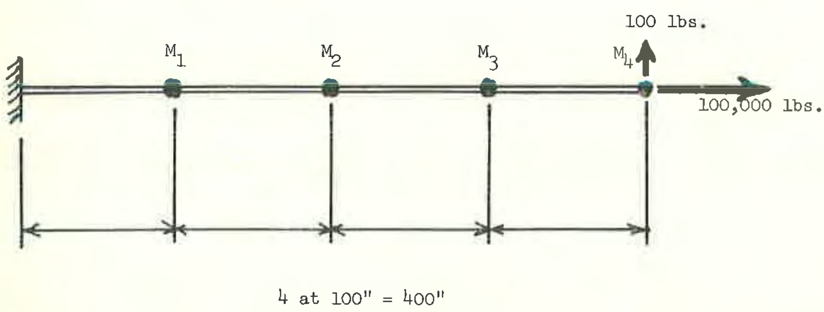
Mode 2
T = 0.21 sec.

Mode 3
T = 0.14 sec.

1 Mode Shapes of Shear Frame



2 Static Deflection of Shear Frame



3 Cantilevered Beam