ABSTRACT

WANG, JU. On the Cohomology of Poisson Pseudoalgebras. (Under the direction of Bojko Bakalov).

As generalizations of Lie conformal superalgebras (abbreviated LCA) in higher dimensions, Lie superpseudoalgebras (abbreviated LPA) are Lie superalgebras in a specific pseudotensor category $\mathscr{M}^*(H)$, where H is a cocommutative Hopf algebra. By definition, $\mathscr{M}^*(H)$ consists of a collection of left H-modules being the objects, together with vector superspaces of "polylinear" maps between them, such that the composition of these maps is associative, the action of symmetric groups on these maps is equivariant, and there exists a unity map that is compatible with the composition. In particular, for a LPA structure on a specific vector superspace V, one can only consider the "subcategory" $P(V)$ of $\mathscr{M}^*(H)$ with V being the only object. This "subcategory" $P(V)$ is called an operad (a pseudotensor category with only one object). It governs the LPA structures on V because of the following construction: we can associate to $P(V)$ a universal Lie superalgebra $W(P(V))$ such that there is a one-to-one correspondence between the LPA structures on V and the odd elements $X \in W_1(P(\Pi V))$ satisfying $[X, X] = 0$, where ΠV is the same space as V but with reverse parity. In fact, this operadic approach also works for Lie conformal superalgebras and Poisson vertex superalgebras (abbreviated PVA): the $\mathfrak{Chom}$ operad governs the LCA structures and the classical operad P^{cl} governs the PVA structures.

Enlightened by this approach, we construct a new operad P_H^{cl}, called the generalized classical operad, that governs the Poisson H-superpseudoalgebra (abbreviated PPA) structures. The proof that P_H^{cl} is an operad is based on the $\circ$-products formulation, which is an equivalent way to describe the axioms of an operad. Given an LPA V, one can take the symmetric algebra $S(V)$ and extend the pseudobracket on V by the Leibniz rules to make $S(V)$ a PPA. With this construction, we get important examples of PPA analogous to the PVA setting: the free superboson PPA, the free superfermion PPA, the affine PPA and the W_M PPA. In the case when $H = \mathbb{F}[\partial_1, \ldots, \partial_N]$, the LPA structure is indeed a LCA in dimension N, the corresponding PPA can be thought of as a PVA in dimension N.

For the second half of this thesis, we study the reduced variational PPA cohomology $H_{PP}(P, M)$. After some discussions on the general theory and the basic cohomology $\widetilde{H}_{PP}(P, M)$, we generalize the energy operators to the reduced and basic variational PPA

cohomology for $H=\mathbb{F}[\partial_1,\dots,\partial_N]$. By Cartan's formula, the energy operators are trivial on the basic cohomology. Using this observation and the spectral sequence associated to a free resolution of $C_{PP}(P,M)$, which is obtained from the Koszul complex of the H-module $\widetilde{C}^n_{PP}(P,M)$ for $n\geq 1$, we prove the second main result that under certain assumptions on the PPA P and the P-module M, the energy operators N_1 and N_2 have only eigenvalues 0 and 1 on the the reduced variational PPA cohomology $H_{PP}(P,M)$ when $H=\mathbb{F}[\partial_1,\partial_2]$. This theorem gives strong constraints on the cohomology in high dimensions for some PPA. As an application of this theorem, we show that when $H=\mathbb{F}[\partial_1,\partial_2]$, the only nontrivial cohomology of the free superfermion $F_{\mathfrak{h}}$ with coefficients in the adjoint module is the 0-th cohomology, which is 1-dimensional.

On the Cohomology of Poisson Pseudoalgebras

by
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DEDICATION

To my parents.

BIOGRAPHY

The author was born in Meizhou, Guangdong Province, China in 1990. He received his B.S. degree in Pure and Applied Mathematics from Sun Yat-sen University in Guangzhou in 2014. After that, he began his graduate study in University of Washington in Seattle and received an M.S. degree in Statistics in 2016. He then moved to Raleigh and started his doctoral program in mathematics in NC State University. Under the direction of Dr. Bojko Bakalov, his research mainly focuses on the cohomology theory of Lie and Poisson pseudoalgebras.

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INTRODUCTION

The study of Poisson algebras has a long history and is closely related to Hamiltonian systems. By construction, a Poisson algebra has both an associative algebra structure and a Lie algebra structure, such that two of them are intertwined by the Leibniz rules. A good example of a Poisson algebra to have in mind is the set of smooth functions $C^\infty(M)$ on a symplectic manifold (M,ω). The associative product of $f,g \in C^\infty(M)$ is just the usual point-wise product of two functions, while the Lie bracket (often called the Poisson bracket) is given through the closed non-degenerate 2-form ω:

$$\{f,g\} = \omega(X_f, X_g), \tag{1.0.1}$$

where X_f is the Hamiltonian vector field of f (for details see [20]). Although a Poisson algebra need not be commutative, we will focus on commutative ones (which show up in important examples like the one above) if not otherwise specified. Often times, when given a Lie algebra, one way to construct a commutative Poisson algebra is to take the symmetric algebra of the Lie algebra and extend the Lie bracket by the Leibniz rules. For the super case one only needs to add signs properly to the algebra structures in the discussion above.

There is a similar relation between Poisson vertex algebras and Lie conformal algebras. Conformal algebras, first introduced by Victor G. Kac [17], provide an axiomatic description of the operator product expansion of chiral fields in conformal field theory. Meanwhile, they appear naturally as formal distribution Lie algebras and are a useful tool to study infinite-dimensional Lie algebras. Specifically, a Lie conformal algebra is a left $\mathbb{F}[\partial]$-module A with a λ-bracket $[\cdot_\lambda\cdot]: A\otimes A \to \mathbb{F}[\lambda]\otimes A$ that satisfies the sesquilinearity, skew-symmetry and the Jacobi identity (see Definition 2.3.4 in Chapter 2). Since their introduction, conformal algebras have been intensively studied by different authors. For example, the finite simple and semisimple Lie conformal algebras are classified in [12]. The conformal modules and their extensions are studied in [10; 11]. The cohomology theory of conformal algebras with coefficients in an arbitrary module is developed in [6]. Combining both an associate algebra structure and a Lie conformal algebra structure with the Leibniz rules, Poisson vertex algebras are studied a lot in recent years: for instance the application of Poisson vertex algebras on the integrability of Hamiltonian partial differential equations [7]; the cohomology theory of Poisson vertex algebras [4; 5; 14], etc.

Motivated by the work on Poisson vertex algebras and with a hope to generalize their applications to Hamiltonian PDEs, the main goal of this thesis is to generalize Poisson vertex algebras to "higher dimensions", to what we call Poisson (super)pseudoalgebras, and to study the energy operators on the cohomology of Poisson (super)pseudoalgebras. An indispensable ingredient of a Poisson pseudoalgebra is the Lie pseudoalgebra, which is the generalization of Lie conformal algebras in higher dimensions. In a Lie pseudoalgebra, one substitutes the $\mathbb{F}[\partial]$-module in a Lie conformal algebra with a cocommutative Hopf algebra H-module L, and replaces the λ-bracket with the pseudobracket $[*]: L\otimes L \to (H\otimes H)\otimes_H L$. The detailed axioms of the pseudobracket can be found in Definition 2.3.3. The theory of finite pseudoalgebras is developed in [1]. In particular, all finite simple and semisimple Lie pseudoalgebras for H being the universal enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{d}$ are classified in [1]. In a more general perspective, a Lie H-pseudoalgebra can be viewed as a Lie algebra in the pseudotensor category $\mathscr{M}^*(H)$ (cf. Example 2.2.3 and Definition 2.3.3). Furthermore, there is a unifying operadic perspective, which states that these algebra structures (Lie superalgebras, Lie conformal superalgebras, vertex superalgebras and Poisson vertex superalgebras, we consider super cases here) on a vector superspace V, correspond to some specific elements in the universal Lie superalgebra of some specific operads [3; 14; 25]. An operad can be thought of as a pseudotensor category with only one object. Explicitly, a linear, unital symmetric superoperad P, abbreviated as an operad, is a collection of vector superspaces $P(n)$ (of morphisms), together with a composition of elements in $P(n)$'s and a right action of symmetric group S_n on $P(n)$, such that the composition is associative, the right action of the symmetric group is equivariant and there exists a unity that is compatible with the composition (cf. Subsection 3.1.1). The universal Lie superalgebra $W(P)$ associated to P is $W(P) = \bigoplus_{n\geq -1} W_n(P)$ where:

$$W_n(P) = \{\, f \in P(n+1) \mid f^{\sigma} = f, \text{ for any } \sigma \in S_{n+1} \,\}, \tag{1.0.2}$$

with the bracket:

$$[f, g] = f \square g - (-1)^{p(f)p(g)} g \square f. \tag{1.0.3}$$

The square product $\square$ is defined as:

$$f \square g = \sum_{\sigma \in S_{m+1,n}} (f \circ_1 g)^{\sigma^{-1}} \in W_{n+m}. \tag{1.0.4}$$

The (m, n)-shuffles $S_{m,n}$ is the subset of of S_{m+n}:

$$S_{m,n} = \{\, \sigma \in S_{m+n} \mid \sigma(1) < \cdots < \sigma(m),\ \sigma(m+1) < \cdots < \sigma(m+n) \,\}. \tag{1.0.5}$$

Let V be a vector superspace with parity p, the only object in the operad P. Consider a Lie conformal superalgebra (abbreviated LCA later) structure on V, in which case we have an even derivation ∂ on V. We can choose the operad $\mathfrak{Chom}(V)$ to be the family of vector supserspaces:

$$\mathfrak{Chom}(V)(n) = \mathrm{Hom}_{\mathbb{F}[\partial]^{\otimes n}}(V^{\otimes n}, \mathbb{F}[\lambda_1, \ldots, \lambda_n] \otimes_{\mathbb{F}[\partial]} V), \tag{1.0.6}$$

together with the compositions given by (3.3.3) and the right action of S_n by permuting the tensor factors in $V^{\otimes n}$ and λ's simultaneously. Then there is a one-to-one correspondence between the LCA structures on V and the odd elements $X \in W_1(\mathfrak{Chom}(\Pi V))$ such that $X \square X = 0$, where ΠV is the vector superspace V but with reverse parity $\bar{p} = 1 - p$. Explicitly, the correspondence is given by

$$[a_\lambda b] = (-1)^{p(a)} X_{\lambda, -\lambda-\partial}(a \otimes b), \tag{1.0.7}$$

for any $a, b \in V$. We say the $\mathfrak{Chom}(\Pi V)$ operad governs the LCA structures on V. Essentially the same operad governs the Lie superpseudoalgebra (abbreviated LPA later) structures on V if we replace $\mathbb{F}[\partial]$ with H and $\mathbb{F}[\lambda_1,\dots,\lambda_n]$ with $H^{\otimes n}$ in (1.0.6). Similarly for the Poisson vertex superalgebra structures (abbreviated PVA later) on V, one can choose what we call the classical operad $P^{cl}(V)$, which consists of a family of vector superspaces $P^{cl}(V)(n)$ of linear maps:

$$G(n)\times V^{\otimes n} \to V[\lambda_1,\dots,\lambda_n]/\langle \partial+\lambda_1+\cdots+\lambda_n\rangle, \tag{1.0.8}$$

where $G(n)$ denotes all n-graphs without tadpoles. Any linear map $f\in P^{cl}(V)(n)$ should also satisfy two cycle conditions and two sesquilinearity conditions (see Subsection 3.5.1 for details). The composition of these maps is given by (3.5.7) and the action of symmetric groups is permuting the graph $\Gamma\in G(n)$ (by definition permuting the vertices of the graph), the tensor factors in $V^{\otimes n}$ and λ's simultaneously. Then the one-to-one correspondence between the PVA structures on V and the odd elements $X\in W_1(P^{cl}(\Pi V))$ such that $X\square X=0$ is given explicitly by, for any $a,b\in V$,

$$ab=(-1)^{p(a)}X^{\bullet\!\!\rightarrow\!\!\bullet}(a\otimes b),\quad [a_\lambda b]=(-1)^{p(a)}X^{\bullet\ \ \bullet}_{\lambda,-\lambda-\partial}(a\otimes b). \tag{1.0.9}$$

One sees that essentially X evaluated on the disconnected 2-graph gives the λ-bracket and on the connected 2-graph gives the usual product. It is worth to note that a similar correspondence also appears for the vertex superalgebra structures on V. The operad governing such structures on V is called the chiral operad $P^{ch}(V)$ [3]. The classical operad $P^{cl}(V)$ is closely related to the associated graded operad $\operatorname{gr}P^{ch}(V)$ of $P^{cl}(V)$. It is shown in [3] that if V has an increasing $\mathbb{Z}_+$-filtration by $\mathbb{F}[\partial]$-submodules, there exist an injective homomorphism ϕ from $\operatorname{gr}P^{ch}(V)$ to $P^{cl}(\operatorname{gr}V)$. Moreover, if this filtration on V is induced by a $\mathbb{Z}_+$-grading of $\mathbb{F}[\partial]$-modules, then ϕ above is an isomorphism [2].

Another construction from the universal Lie superalgebra $W(P)$ is: given an odd element $X\in W_1(P(\Pi V))$ such that $X\square X=0$, one has a cohomology complex $(C=\bigoplus_{j\geq 0}C^j,\operatorname{ad}X)$, where $C^j=W_{j-1}(P(\Pi V))$. For the $\mathfrak{Chom}(\Pi V)$ operad, this complex is just the cohomology complex of the LCA V with λ-bracket given by $X_{\lambda,-\lambda-\partial}$ in (1.0.7), with coefficients in the adjoint module. For the classical operad $P^{cl}(V)$, we get what is called the classical Poisson cohomology, which is shown to be isomorphic to the variational Poisson cohomology, provided that as a differential algebra, V is a finitely generated algebra of differential polynomials [4].

In this thesis, the first new result is the construction of a new operad called the generalized classical operad and denoted as $P^{cl}_H(V)$ (where H is a cocommutative Hopf algebra that is purely even). This operad governs the Poisson superpseudoalgebra (abbreviated PPA later) structures on the vector superspace V. In particular, H need not be commutative here. This new operad tells us exactly what the Leibniz rules should be for a PPA (see 4.2.1), which one has no clue only from the PVA case. The one-to-one correspondence part is similar to the PVA case once the operad $P^{cl}_H(V)$ is set up, while it is a bit more technical to construct the operad $P^{cl}_H(V)$ itself. For this purpose, pick the family of vector superspaces $P^{cl}_H(V)(n)$ to be all linear maps

$$Y^\Gamma : V^{\otimes n} \to H^{\otimes s} \otimes_H V,$$

for each $\Gamma \in G(n)$, where s denote the number of connected components of Γ. We require that Y^Γ also satisfy two cycle conditions:

$$Y^\Gamma = 0 \quad \text{if } \Gamma \text{ has a cycle,} \tag{1.0.10}$$

and if C is a cycle in Γ, then

$$\sum_{e\in C} Y^{\Gamma\backslash e} = 0, \tag{1.0.11}$$

where $\Gamma\backslash e$ denotes the graph obtained by deleting edge e from Γ. Another condition that Y should also satisfy is the componentwise H-linearity. For any connected component $\Gamma_k \subset \Gamma$, suppose there are n_k vertices in Γ_k. The componentwise H-linearity can be written as:

$$Y^\Gamma(h_k \cdot (v_1 \otimes \cdots \otimes v_n)) = h_k \cdot Y^\Gamma(v_1 \otimes \cdots \otimes v_n). \tag{1.0.12}$$

Here $h_k \cdot (v_1 \otimes \cdots \otimes v_n)$ means h_k acts by coproduct on v_i's where i belongs to the vertex set of Γ_k. That is

$$h_k \cdot (v_1 \otimes \cdots \otimes v_n) = (1 \otimes \cdots \otimes h_{k(1)} \otimes \cdots \otimes h_{k(n_k)} \otimes \cdots \otimes 1)(v_1 \otimes \cdots \otimes v_n). \tag{1.0.13}$$

For the right hand side of (1.0.12), h_k acts only on the tensor factor H that corresponds to the connected component Γ_k in $H^{\otimes s} \otimes_H V$.

Next, we define the action of symmetric groups and the composition, while the unity is just the identity map $\mathrm{id}_V \in P_H^{cl}(V)(1)$. The symmetric group S_n acts on $P_H^{cl}(V)(n)$ by permuting the graph Γ, the tensor factors in $V^{\otimes n}$ and the tensor factors in $H^{\otimes s}$. For $\Gamma \in G(n)$ and $\sigma \in S_n$, $\sigma(\Gamma)$ is the graph obtained by permuting the vertices of Γ by σ. Note that this will also induce a permutation $\widetilde{\sigma} \in S_s$ of the connected components of Γ. We let $\widetilde{\sigma}$ act on $H^{\otimes s}$. Explicitly, the action can be written as:

$$(Y^{\sigma})^{\Gamma}(v_1 \otimes \cdots \otimes v_n) = (\widetilde{\sigma} \otimes_H 1)\big(Y^{\sigma(\Gamma)}(\sigma(v_1 \otimes \cdots \otimes v_n))\big). \tag{1.0.14}$$

The composition is defined in terms of $\circ$-products, which is an equivalent way to giving the composition directly provided that the associativity axiom holds. In general, $Y \circ_k X$ is defined as inserting X into the k-th input of Y (see (3.1.11)). We start with the $\circ_1$-product for $X \in P_H^{cl}(V)(m)$, $Y \in P_H^{cl}(V)(n)$. Given a graph $\Gamma \in G(m+n-1)$, let $\Delta_0^{m1\ldots1}(\Gamma) \in G(n)$ denote the graph obtained by clasping the first m vertices into a single vertex in Γ, and let $\Delta_1^{m1\ldots1}(\Gamma) \in G(m)$ denote the graph formed by the first m vertices and edges of Γ (cf. the cocomposition maps in (3.4.1)). Suppose that

$$X^{\Delta_1^{m1\ldots1}(\Gamma)}(v_1 \otimes \cdots \otimes v_m) = \sum_i (f_{i1} \otimes \cdots \otimes f_{is}) \otimes_H x_i(v_1 \otimes \cdots \otimes v_m), \tag{1.0.15}$$

$$Y^{\Delta_0^{m1\ldots1}(\Gamma)}(w_1 \otimes \cdots \otimes w_n) = \sum_j (g_{j1} \otimes \cdots \otimes g_{jt}) \otimes_H y_j(w_1 \otimes \cdots \otimes w_n), \tag{1.0.16}$$

where $x_i : V^{\otimes m} \to V$ and $y_j : V^{\otimes n} \to V$ are linear maps, and s, t are the number of connected components of $\Delta_1^{m1\ldots1}(\Gamma)$ and $\Delta_0^{m1\ldots1}(\Gamma)$, respectively. Then the $\circ_1$-product is given by

$$\begin{aligned}
&(Y \circ_1 X)^{\Gamma}(v_1 \otimes \cdots \otimes v_{m+n-1}) \\
=&(Y(X,1,\ldots,1))^{\Gamma}(v_1 \otimes \cdots \otimes v_{m+n-1}) \\
=&\sum_{i,j} \big((f_{i1(1)} \otimes \cdots \otimes f_{is(1)} \otimes 1 \otimes \cdots \otimes 1)(\Delta^{(s-1)}(g_{j1}) \otimes g_{j2} \otimes \cdots \otimes g_{jt})\big) \otimes_H y_j\big(x_i(v_1 \otimes \cdots \\
&\otimes v_m) \otimes ((f_{i1(-2)} \otimes \cdots \otimes f_{is(-2)}) \cdot (v_{m+1} \otimes \cdots \otimes v_{m+n-1}))\big).
\end{aligned} \tag{1.0.17}$$

For detailed explanations of (1.0.17), see Subsection 4.1.1. It is worth mentioning here that the transformation $(\mathrm{id} \otimes S)\Delta(f_{ik}) = f_{ik(1)} \otimes f_{ik(-2)}$ plays an important role in the formula; moreover, $f_{ik(-2)}$ act disjointly on $v_{m+1} \otimes \cdots \otimes v_{m+n-1}$ for different k. It is shown in Chapter 4.1 that $Y \circ_1 X$ is indeed well defined and it is an element of $P_H^{cl}(V)(m+n-1)$. One can generalize the $\circ_1$-product to the $\circ_k$-product. The formula is similar and is given by (4.1.23). The additional step there compared to the $\circ_1$-product is to keep track of the correct connected component when combining the "coefficients" in $H^{\otimes(s+t-1)}$. Now that we have all $\circ$-products, the composition of $X_1,\ldots,X_n$ and Y is defined to be a sequence of $\circ$-products:

$$Y(X_1,\ldots,X_n)=(\cdots((Y\circ_1 X_1)\circ_{M_1+1} X_2)\cdots)\circ_{M_{n-1}+1} X_n, \tag{1.0.18}$$

where M_k is given by (3.1.2). With all ingredients ready, we proved that $P_H^{cl}(V)$ given above is indeed an operad, which is done in Theorem 4.1.1. We verified the three axioms of being an operad in terms of $\circ$-products. After this new operad $P_H^{cl}(V)$ is established, we gave the definition of a PPA. A PPA V is a commutative associative H-differential superalgebra with a Lie superpseudobracket that satisfies the left Leibniz rule: for any $a,b,c\in V$,

$$[a*bc]=[a*b]c+(-1)^{p(b)p(c)}[a*c]b, \tag{1.0.19}$$

where $[a*b]c$ is defined as

$$[a*b]c=\sum_i (f_i\otimes g_{i(1)})\otimes_H v_i(a\otimes b)(g_{i(-2)}c), \tag{1.0.20}$$

if $[a*b]=\sum_i(f_i\otimes g_i)\otimes_H v_i(a\otimes b)$ for some $f_i,g_i\in H$ and linear maps $v_i: V\otimes V\to V$. It is not hard to see that $[a*b]c$ is well defined. The right Leibniz rule below follows from the left Leibniz rule:

$$[ab*c]=a[b*c]+(-1)^{p(a)p(b)}b[a*c], \tag{1.0.21}$$

where

$$a[b*c]=\sum_j (d_{j(1)}\otimes l_j)\otimes_H (d_{j(-2)}a)w_j(b\otimes c), \tag{1.0.22}$$

if $[b*c]=\sum_j(d_j\otimes l_j)\otimes_H w_j(b\otimes c)$. As in the PVA case, there exists a one-to-one correspondence between the PPA structures on V with odd elements $X\in W_1(P_H^{cl}(\Pi V))$ such that $X\square X=0$. Explicitly, the correspondence is given by: for any such X and $a,b\in V$,

$$ab=(-1)^{p(a)}X^{\bullet\!\rightarrow\!\bullet}(a\otimes b),\quad [a*b]=(-1)^{p(a)}X^{\bullet\;\;\bullet}(a\otimes b). \tag{1.0.23}$$

After proving this theorem, some important examples of PPAs were given explicitly. When

H is commutative, we get direct analogues of PVAs in higher dimensions. In this case we have $H=\mathbb{F}[\partial_1,\ldots,\partial_N]$ in place of $\mathbb{F}[\partial]$ and the λ-bracket now becomes the vector version $\vec{\lambda}$-bracket. Other examples, including the free superboson PPA, the free superfermion PPA, the affine PPA and the W_M PPA, are obtained by taking the symmetric algebras of some corresponding LPAs. For these examples, we assume H to be the universal enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{d}$.

After the construction of PPAs, we moved on to study the variational cohomology of a PPA. We considered the cochain complex $C_{PP}(P,M)=\bigoplus_{n\geq 0} C^n_{PP}(P,M)$ where P is a PPA and M is a P-module. An n-cochain $\gamma\in C^n_{PP}(P,M)$ is a linear map from $P^{\otimes n}$ to $H^{\otimes n}\otimes_H M$ that satisfies the H-multilinearity, the symmetry conditions and the Leibniz rule (cf. Section 6.1). The differential is given by (6.1.9). The cohomology of this complex is what we call the reduced variational PPA cohomology of P (abbreviated the reduced cohomology of P). If one replaces all $\otimes_H$ with $\otimes$ in the cochain complex and the differential map above, the resulting complex and its cohomology will be called the basic complex and the basic cohomology, respectively. The relation of these two types of cohomology, together with the information on the basic cohomology will help us compute the reduced cohomology. This idea has been utilized already in [5; 6; 7; 13]. For this purpose, we studied the energy operators on the two types of cohomology above when H is commutative, specifically $H=\mathbb{F}[\partial_1,\ldots,\partial_N]$. The energy operator N_i is defined through a Virasoro element L_i with respect to ∂_i. A Virasoro element L_i is an element of P such that (in $\vec{\lambda}$-notation):

$$[L_{i\,\vec{\lambda}}L_i]=(\partial_i+2\lambda_i)L_i, \tag{1.0.24}$$

and

$$L_{i(0)}:=[L_{i\,\vec{\lambda}}\cdot]\Big|_{\vec{\lambda}=0}=\partial_i, \tag{1.0.25}$$

$$L_{i(t_i)}:=\frac{\partial}{\partial\lambda_i}[L_{i\,\vec{\lambda}}\cdot]\Big|_{\vec{\lambda}=0}\in \mathrm{End}(P)\quad \text{is diagonalizable.} \tag{1.0.26}$$

A P-module M is called conformal with respect to L_i if (1.0.25) and (1.0.26) hold with the $\vec{\lambda}$-bracket replaced with the $\vec{\lambda}$-action. One can extend the t_i-mode $L^M_{i\ (t_i)}=\frac{\partial}{\partial\lambda_i}(L_{i\,\vec{\lambda}}\cdot)\Big|_{\vec{\lambda}=0}$ to an energy operator N_i on $M[\vec{\lambda}_1,\ldots\vec{\lambda}_n]$ by letting

$$N_i:=L^M_{i\ (t_i)}+\sum_{k=1}^{n}\lambda_{ki}\frac{\partial}{\partial\lambda_{ki}}. \tag{1.0.27}$$

Then the energy operator N_i on the reduced complex $C_{PP}(P,M)$ is defined as:

$$
\begin{aligned}
(N_i\gamma)_{\vec\lambda_1,\dots,\vec\lambda_n}(a_1\otimes\cdots\otimes a_n) = (N_i+n)(\gamma_{\vec\lambda_1,\dots,\vec\lambda_n}(a_1\otimes\cdots\otimes a_n)) \\
-\sum_{k=1}^{n}\gamma_{\vec\lambda_1,\dots,\vec\lambda_n}(a_1\otimes\cdots\otimes L_{i(t_i)}a_k\otimes\cdots\otimes a_n),
\end{aligned}
\tag{1.0.28}
$$

for $\gamma\in C^n_{PP}(P,M)$. This induces an endomorphism, still denoted as N_i, on the reduced cohomology since it commutes with differential map. One can also define the energy operators $\widetilde{N}_i$ on the basic complex $\widetilde{C}_{PP}(P,M)$ via the Lie derivative by L_i:

$$
\widetilde{N}_i = \frac{\partial}{\partial\lambda_i}\big((L_i)_{\vec\lambda}\,\cdot\big)\Big|_{\vec\lambda=0}. \tag{1.0.29}
$$

It turned out that $\widetilde{N}_i$ is given by the same formula (1.0.28) on the basic complex as N_i on the reduced complex. By Cartan's formula, we have the important result that $\widetilde{N}_i$ is trivial on the basic cohomology, or equivalently, the image of a cocycle under $\widetilde{N}_i$ is a coboundary.

To relate the reduced and basic cohomology, we explored the spectral sequence associated to the Koszul complex built from the basic cochain complex $\widetilde{C}^n_{PP}(P,M)$. By construction, regrading $\widetilde{C}^n_{PP}(P,M)$ $(n\geq 1)$ as an H-module with ∂_i acting by (6.1.26), the Koszul complex is:

$$
0\to\widetilde{C}^n_{PP}(P,M)\xrightarrow{\phi_N}\widetilde{C}^n_{PP}(P,M)^{\binom{N}{N-1}}\xrightarrow{\phi_{N-1}}\cdots\xrightarrow{\phi_2}\widetilde{C}^n_{PP}(P,M)^{\binom{N}{1}}\xrightarrow{\phi_1}\widetilde{C}^n_{PP}(P,M)\to 0, \tag{1.0.30}
$$

with the differential ϕ_p given by (6.1.48). Moreover, $\partial_1,\dots,\partial_N$ is a $\widetilde{C}^n_{PP}(P,M)$-regular sequence (for $n\geq 1$) and $\widetilde{C}^n_{PP}(P,M)/\big(\partial_1\widetilde{C}^n_{PP}(P,M)+\cdots+\partial_N\widetilde{C}^n_{PP}(P,M)\big)\cong C^n_{PP}(P,M)$ (for $n\geq 0$) provided that P, as a differential superalgebra, is a superalgebra of differential polynomials in even or odd variables (cf. Proposition 6.1.2). Hence, by Corollary 6.1.1, one has the following exact sequence:

$$
0\to\widetilde{C}^n_{PP}(P,M)\xrightarrow{\phi_N}\widetilde{C}^n_{PP}(P,M)^{\binom{N}{N-1}}\xrightarrow{\phi_{N-1}}\cdots\xrightarrow{\phi_2}\widetilde{C}^n_{PP}(P,M)^{\binom{N}{1}}\xrightarrow{\phi_1}\widetilde{C}^n_{PP}(P,M)\xrightarrow{\epsilon}C^n_{PP}(P,M)\to 0, \tag{1.0.31}
$$

where $\epsilon:\widetilde{C}^n_{PP}(P,M)\to C^n_{PP}(P,M)$ is the quotient map. Considering this exact sequence for all $n\geq 1$, one indeed has a double complex, to which we can associate a spectral sequence. This spectral sequence collapses at the $(N+2)$-th page (cf. Lemma 5.3.1). We have shown that there exists a medium exact sequence on the N-th page of this spectral sequence (Theorem

5.3.3). This is true for any spectral sequence associated to a double complex arising from an exact sequence, which can be thought of as a generalization of the well-known fact that a short exact sequence induces a long exact sequence on cohomology (which is the $N=1$ case). We have applied these results to (1.0.31) for the case $H=\mathbb{F}[\partial_1,\partial_2]$. Under this assumption, the spectral sequence collapses at the 4-th page and one has the following exact sequence on the 2-nd page of the spectral sequence (see the proof of Theorem 6.2.1):

$$0\to H(H^{n+1}(\widetilde{C})\oplus H^{n+1}(\widetilde{C}))\xrightarrow{d_2^{1,n+1}} H^n(C)/\mathrm{im}^n\epsilon_*\xrightarrow{\theta_n}\ker^{n+2}(\phi_2)_*\xrightarrow{d_2^{0,n+2}} H(H^{n+1}(\widetilde{C}))\to 0. \tag{1.0.32}$$

Furthermore, we chose polynomials of N_i $(i=1,2)$ for each (p,q)-term on the 0-th page of the spectral sequence. By doing so we obtained an endomorphism of the double complex that commutes with the differentials of the double complex in both directions. This endomorphism induces an endomorphism on each page of the spectral sequence (Proposition 6.2.5). In particular, the induced endomorphism commutes with the differential d_r on the r-th page for all r. It is also true that this endomorphism commutes with the connecting homomorphism θ_n in (1.0.32) above.

With there results discussed above, as the second main result of this thesis, we proved that the energy operators N_1 and N_2 have only eigenvalues 0 and 1 on the reduced cohomology (see Theorem 6.2.1). This theorem imposes strong constraints on the cohomology in high dimensions for some PPAs. As an application of this theorem, we computed the reduced cohomology of the free superfermion PPA $F_{\mathfrak{h}}$ with coefficients in the adjoint module when $H=\mathbb{F}[\partial_1,\partial_2]$. The only nontrivial cohomology group turns out to be at degree 0 (Theorem 6.3.1):

$$\dim(H^n_{PP}(F_{\mathfrak{h}},F_{\mathfrak{h}}))=\delta_{n0},\quad n\geq 0. \tag{1.0.33}$$

The thesis is organized in the following way.

In Chater 2, we reviewed the following basic materials of Lie pseudoalgebras: the definition, basic properties and two examples of Hopf algebras, the Fourier transform, the definition of pseudotensor categories, the definition of Lie pseudoalgebras and examples of Lie pseudoalgebras.

In Chapter 3, we reviewed the following material related to operads: the definition of an operad and the $\circ$-product formulation, the construction of the universal Lie superalgebra of an operad, the $\mathfrak{Chom}$ operad that governs the LCA structures and the classical operad P^{cl} that governs the PVA structures.

In Chapter 4, we constructed the new operad P_H^{cl}, which is called the generalized classical operad. We proved our first main result: P_H^{cl} so defined is indeed an operad. Then we defined the PPA structure and showed that P_H^{cl} governs the PPA structures. We gave some important examples of PPAs at the end of the chapter.

In Chapter 5, we reviewed the definition of a first quadrant cohomological spectral sequence and the spectral sequence associated to a double complex. In the last section of the chapter, we applied the spectral sequence tool to an exact sequence and showed that for an exact sequence of complexes that contains K nonzero complexes, one has a medium exact sequence on the $(K-2)$-th page.

In Chapter 6, we first discussed some general theory of PPA and defined the Virasoro elements in a PPA P, which is then extended to energy operators on the cochain complex and the cohomology (both reduced and basic) for $H = \mathbb{F}[\partial_1, \dots, \partial_N]$. In particular when $H = \mathbb{F}[\partial_1, \partial_2]$, we proved the second main result of this thesis: the energy operators N_1 and N_2 only have eigenvalues 0 and 1 on the reduced cohomology. With this theorem, we computed the cohomology of the free superfermion PPA $F_{\mathfrak{h}}$ (for $H = \mathbb{F}[\partial_1, \partial_2]$) with coefficients in the adjoint module.

In Chapter 7, we listed some of the open problems that are related to the topic (but beyond the scope) of this thesis for future reference.

Unless otherwise specified, all vector spaces, linear maps and tensor products without any subscript are considered over an algebraically closed field $\mathbb{F}$ of characteristic 0. The set of all non-negative integers is denoted as $\mathbb{Z}_+$. We may sometimes suppress the word "super" and just say, for example, "Poisson pseudoalgebras" instead of "Poisson superpseudoalgebras" to avoid making the word overwhelming. It should be clear from the context if a super case, or just the even case is considered (or maybe just not important in that discussion).

CHAPTER

2

LIE PSEUDOALGEBRAS

As an important building block of Poisson pseudoalgebras, we review some basic facts of Lie pseudoalgebras needed in further discussions in this chapter. With appropriate signs added, these notions can be extended to the super case without much difficulty. For more detailed and in depth discussions of Lie pseudoalgebras, we refer the reader to [1].

2.1 Hopf algebras

The construction of Lie pseudoalgebras starts with a left module of a Hopf algebra. In this section, we review some notions of Hopf algebras that can be found in [1] and [28].

2.1.1 Definitions and properties of Hopf algebras

Definition 2.1.1 ([28])**.** *A Hopf algebra H is a bialgebra with an antipode.*

We denote the coproduct of H by $\Delta : H \to H \otimes H$, the antipode by $S : H \to H$ and the counit by ϵ. With Sweedler's notation (cf. [1; 28]), we have the following equalities which will be useful in the computations later:

$$\Delta(h) = h_{(1)} \otimes h_{(2)}, \tag{2.1.1}$$

$$(S \otimes \mathrm{id})\Delta(h) = h_{(-1)} \otimes h_{(2)}, \tag{2.1.2}$$

$$(\mathrm{id} \otimes S)\Delta(h) = h_{(1)} \otimes h_{(-2)}, \tag{2.1.3}$$

and the coassociativity of Δ:

$$(\Delta \otimes \mathrm{id})\Delta(h) = (\mathrm{id} \otimes \Delta)\Delta(h) = h_{(1)} \otimes h_{(2)} \otimes h_{(3)}. \tag{2.1.4}$$

We also have the axiom of the antipode and the counit:

$$\epsilon(h) = h_{(-1)}h_{(2)} = h_{(1)}h_{(-2)} \tag{2.1.5}$$

$$h = \epsilon(h_{(1)})h_{(2)} = h_{(1)}\epsilon(h_{(2)}) \tag{2.1.6}$$

which imply:

$$h_{(-1)}h_{(2)} \otimes h_{(3)} = 1 \otimes h = h_{(1)}h_{(-2)} \otimes h_{(3)} \tag{2.1.7}$$

$$h_{(1)} \otimes h_{(-2)}h_{(3)} = h \otimes 1 = h_{(1)} \otimes h_{(2)}h_{(-3)} \tag{2.1.8}$$

A Hopf algebra is called cocommutative if its coalgebra structure is cocommutative, that is $h_{(1)} \otimes h_{(2)} = h_{(2)} \otimes h_{(1)}$. In this thesis, we will only consider cocommutative Hopf algebras. It is worth to know a non-cocommutative (four dimensional) example would be the Sweedler's Hopf algebra (cf. [9]).

Definition 2.1.2. *An H-differential (super)algebra A is an associative (super)algebra which is also a left H-module such that the multiplication is a homomorphism of H-modules, that is: $h \cdot (xy) = (h_{(1)}x)(h_{(2)}y)$ for any $h \in H$ and $x, y \in A$.*

2.1.2 Fourier transform

Given an arbitrary Hopf algebra H, we define the Fourier transform the map $\mathscr{F} : H \otimes H \to H \otimes H$,

$$\mathscr{F}(f \otimes g) = (f \otimes 1)(S \otimes \mathrm{id})\Delta(g) = f g_{(-1)} \otimes g_{(2)}. \tag{2.1.9}$$

The inverse $\mathscr{F}^{-1}: H \otimes H \to H \otimes H$ is given as:

$$\mathscr{F}^{-1}(f \otimes g) = (f \otimes 1)\Delta(g) = f g_{(1)} \otimes g_{(2)}. \tag{2.1.10}$$

Straightforward calculation using (2.1.4) and (2.1.7) shows that:

$$f \otimes g = \mathscr{F}^{-1}(\mathscr{F}(f \otimes g)) = (f g_{(-1)} \otimes 1)\Delta(g_{(2)}). \tag{2.1.11}$$

An important consequence of (2.1.11) is: let $H \otimes H$ be a right H module with the action by coproduct, and M a left H module. Suppose $(f \otimes g) \otimes_H m \in (H \otimes H) \otimes_H M$, it can be written as:

$$(f \otimes g) \otimes_H m = (f g_{(-1)} \otimes 1) \otimes_H g_{(2)} m \tag{2.1.12}$$

We call this process straightening by making the second tensor factor equal to 1.

2.1.3 Two important examples of Hopf algebras

In this section we lay out two well-known important examples of cocommutative Hopf algebras: the group algebra and the universal enveloping algebra of a Lie algebra.

Example 2.1.1. *Given a group G, the group algebra $\mathbb{F}[G]$ is a cocommutative Hopf algebra with the coproduct: $\Delta(g) = g \otimes g$, the counit $\epsilon(g) = 1$ and the antipode $S(g) = g^{-1}$ for any $g \in G$.*

Example 2.1.2. *For a Lie algebra $\mathfrak{d}$, its universal enveloping algebra $U(\mathfrak{d})$ is a cocommutative Hopf algebra with the coproduct: $\Delta(a) = a \otimes 1 + 1 \otimes a$, the counit $\epsilon(a) = 0$ and the antipode $S(a) = -a$ for any $a \in \mathfrak{d}$. In particular, if $\mathfrak{d}$ is a one dimensional Lie algebra spanned by ∂, $U(\mathfrak{d})$ is the polynomial algebra $\mathbb{F}[\partial]$.*

Remark 2.1.1. *For an arbitrary Hopf algebra H, if $g \in H$ is such that $\Delta(g) = g \otimes g$, it is called a group-like element of H. Note by (2.1.5) and (2.1.6), $\epsilon(g) = 1$ and $g^{-1} = S(g)$. The set of all group-like elements $G(H)$ of H forms a group. In addition, all elements $h \in H$ such that $\Delta(h) = h \otimes 1 + 1 \otimes h$ are called primitive, and they form a Lie algebra $P(H)$ with the commutator bracket. The group $G(H)$ acts on $P(H)$ by inner automorphisms: $g \cdot p = g p g^{-1}$ for any $g \in G(H)$ and $p \in P(H)$.*

Remark 2.1.2. *It is a theorem of Kostant that any cocommutative Hopf algebra over an an algebraically closed field $\mathbb{F}$ of characteristic 0 is isomorphic to the smash product of the* *universal enveloping algebra $U(P(H))$ and the group algebra $\mathbb{F}[G(H)]$* [28]*. For the purpose of Lie pseudoalgebras, we can and will reduce to the case when H is a universal enveloping algebra* [1].

2.2 Pseudotensor categories

In this section, we review the definition of a pseudotensor category, which was first introduced by Lambek [19] under the name multicategories.

Definition 2.2.1 ([8])**.** *A pseudotensor category is a class of objects $\mathcal{M}$ together with vector spaces of "polylinear" maps $Lin(\{L_i\}_{i\in I}, M)$ between these objects, equipped with actions of the symmetric groups among them and composition of these maps, satisfying the following three properties:*

1. *The composition of maps is associative;*
2. *There exists a unit element $\mathbb{1}_M \in Lin(\{M\}, M)$ for any object M, such that: $\mathbb{1}_M(f) = f(\otimes_{i\in I} \mathbb{1}_{L_i}) = f$ for any $f \in Lin(\{L_i\}_{i\in I}, M)$;*
3. *The compositions are equivariant with respect to the natural action of the symmetric group.*

Example 2.2.1. *A familiar example is the vector spaces with the maps defined as the usual Hom spaces:*

$$Lin(\{L_i\}_{i\in I}, M) := Hom(\otimes_{i\in I} L_i, M),$$

where the symmetric group S_I acts by permuting the elements in $\otimes_{i\in I} L_i$ and the compositions of maps are just the usual ones. In this case we have a pseudotensor category of vector spaces.

Example 2.2.2. *Another example is to consider objects as left H-modules, where H is a cocommutative bialgebra with coproduct Δ. Still take the Hom space but over H this time:*

$$Lin(\{L_i\}_{i\in I}, M) := Hom_H(\otimes_{i\in I} L_i, M).$$

Note that H acts on $\otimes_{i\in I} L_i$ by the iterated coproduct:

$$h \cdot (g_1 \otimes \cdots \otimes g_n) = h_{(1)} g_1 \otimes \cdots \otimes h_{(n)} g_n,$$

which is well defined because of the coassociativity of the coproduct. The symmetric group action and compositions are as in the previous example. In this case, we get a pseudotensor category of left H-modules.

Example 2.2.3. *Another pseudotensor category, which allows us to introduce the notion of a Lie pseudoalgebra, can be defined as follows. Consider left H-modules as above and define the maps as:*

$$Lin(\{L_i\}_{i\in I}, M) := Hom_{H^{\otimes I}}(\boxtimes_{i\in I} L_i, H^{\otimes I} \otimes_H M).$$

Here we consider $\boxtimes_{i\in I} L_i$ as a left $H^{\otimes I}$-module by letting the jth copy H in $H^{\otimes I}$ acts on L_j. $H^{\otimes I} \otimes_H M$ is a left $H^{\otimes I}$-module by letting $a\cdot(b\otimes_H m) = (ab)\otimes_H m$ for $a, b \in H^{\otimes I}$. The action of the symmetric group is by permuting the factors in $\boxtimes_{i\in I} L_i$ and $H^{\otimes I}$ simultaneously. To write the composition of maps explicitly, let $f \in Hom_{H^{\otimes I}}(\boxtimes_{i\in I} L_i, H^{\otimes I} \otimes_H M)$ and $g_i \in Hom_{H^{\otimes J_i}}(\boxtimes_{j\in J_i} N_j, H^{\otimes J_i} \otimes_H L_i)$, and $\pi : J = \sqcup J_i \to I$ be a surjection that sends J_i to $i \in I$. Then for $\otimes_{j\in J} n_j \in \boxtimes_{j\in J} N_j$, we have

$$(f(\otimes_{i\in I} g_i))(\otimes_{j\in J} n_j) = \sum_{S,t} (\otimes_{i\in I}\ a_i^{s_i})\ (\Delta^{(\pi)}(b^t)) \otimes_H v^{S,t},$$

if

$$g_i(\otimes_{j\in J_i} n_j) = \sum_{s_i} a_i^{s_i} \otimes_H l_i^{s_i},\quad a_i^{s_i} \in H^{J_i},\ l_i^{s_i} \in L_i,$$

and

$$f(\otimes_{i\in I} l_i^{s_i}) = \sum_t b^t \otimes_H v^{S,t},\quad b^t \in H^{\otimes I},\ v^{S,t} \in M,$$

where S denotes the multiple index $\prod_{i\in I} s_i$ and $\Delta^{(\pi)}$ is the iterated coproduct determined by π. For example, if $\pi : \{1,2,3\} \to \{1,2\}$ is such that $\pi(1) = \pi(2) = 1$ and $\pi(3) = 2$, $\Delta^{(\pi)} = \Delta \otimes id$; if $\pi(1) = 1$ and $\pi(2) = \pi(3) = 2$, $\Delta^{(\pi)} = id \otimes \Delta$. The idea is to expand the ith component to match J_i part in J.

Intuitively, we may think that the composition happens on both the "coefficient" $H^{\otimes I}$ part and the "vector" M part. The composition of vector parts follows the usual composition as in earlier examples. The composition of coefficient parts is incorporating the corresponding coefficients from lower level maps into the coefficients of higher level maps after expanded by an iterated coproduct. With this composition, we get a pseudotensor category which is denoted as $\mathscr{M}^*(H)$.

Definition 2.2.2. *An operad is a pseudotensor category with only one object.*

Remark 2.2.1. *Note that above we only consider the even case. In Chapter 3, we will discuss superoperads, which are essentially the same except appropriate signs are added. Another equivalent description for superoperads is also presented there.*

2.3 Lie pseudoalgebras

2.3.1 Definition of Lie pseudoalgebras

Definition 2.3.1 ([1]). *A Lie algebra in a pseudotensor category $\mathscr{M}$ is a pair (L,β), where L is an object in $\mathscr{M}$, and $\beta \in Lin(\{L,L\},L)$ satisfies the skew-symmetry and Jacobi identity below:*

- *skew-symmetry:* $\beta = -\sigma_{12}\beta$, *where* $\sigma_{12} = (12) \in S_2$.
- *Jacobi identity:* $\beta(\beta(\cdot,\cdot),\cdot) = \beta(\cdot,\beta(\cdot,\cdot)) - \sigma_{12}\beta(\cdot,\beta(\cdot,\cdot))$, *here* $\sigma_{12} = (12)(3) \in S_3$.

Definition 2.3.2 ([1]). *A Lie H-pseudoalgebra is a Lie algebra in the pseudotensor category $\mathscr{M}^*(H)$.*

Remark 2.3.1. *We can also define an associative algebra in a pseudotensor category $\mathscr{M}$ as a pair (A,μ), where A is an object in $\mathscr{M}$ and $\mu \in Lin(\{A,A\},A)$ is a product such that $\mu(\mu(\cdot,\cdot),\cdot) = \mu(\cdot,\mu(\cdot,\cdot))$. The commutativity condition is just $\mu = \sigma_{12}\mu$ where $\sigma_{12} = (12) \in S_2$.*

For a Lie pseudoalgebra, we specify all properties encoded in **Definition 2.3.2** and have a more explicit version of the definition below:

Definition 2.3.3 ([1]). *A Lie pseudoalgebra is a pair (L,β), where L is an object in $\mathscr{M}^*(H)$, and $\beta \in Lin(\{L,L\},L)$. β is called a pseudobracket and written as $[\cdot * \cdot]$, satisfying the following properties for any $a,b,c \in L$, $f,g \in H$ and $\sigma = (12) \in S_2$:*

- *H-bilinearity:*

$$[f a * g b] = ((f \otimes g) \otimes_H 1)[a * b]; \tag{2.3.1}$$

- *Skew-symmetry:*

$$[b * a] = -(\sigma \otimes_H id)[a * b]; \tag{2.3.2}$$

- *Jacobi identity:*

$$[a*[b*c]] - ((\sigma \otimes id) \otimes_H id)[b*[a*c]] = [[a*b]*c]. \tag{2.3.3}$$

Note that $[a*[b*c]]$ and $[[a*b]*c]$ are in $H^{\otimes 3} \otimes_H L$ and should be computed using the composition in $\mathscr{M}^*(H)$.

In the next example we see that a Lie conformal algebra is a special case of a Lie pseudoalgebra. First recall the definition of a Lie conformal algebra introduced by Kac [17].

Definition 2.3.4 ([17]). *A Lie conformal algebra over a field $\mathbb{F}$ is an $\mathbb{F}[\partial]$-module A equipped with a linear map over $\mathbb{F}$, called the λ-bracket, $[\cdot_\lambda \cdot] : A \otimes A \to A[\lambda]$ subject to the following axioms:*

- *Sesquilinearity:*

$$\begin{aligned} [\partial a_\lambda b] &= -\lambda[a_\lambda b], \\ [a_\lambda \partial b] &= (\partial + \lambda)[a_\lambda b]; \end{aligned} \tag{2.3.4}$$

- *Skew-symmetry:*

$$[a_\lambda b] = -[b_{-\lambda-\partial}\, a]; \tag{2.3.5}$$

- *Jacobi identity:*

$$[a_\lambda[b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + [b_\mu[a_\lambda c]]. \tag{2.3.6}$$

To define a Lie conformal superalgebra, we only need to add appropriate signs in the axioms above in Definition 2.3.4: the sesquilinearity stays the same; multiply by $(-1)^{p(a)p(b)}$ the right-hand side of the skew-symmetry equation; and insert a factor $(-1)^{p(a)p(b)}$ before $[b_\mu[a_\lambda c]]$. Here $p(a)$ here denote the parity of a. λ and μ are regarded even. These signs follow the Koszul-Quillen rule: every time we swap two odd elements, we change the sign.

2.3.2 Examples of Lie pseudoalgebras

Below we list a few examples of Lie pseudoalgebras (cf. [1]).

Example 2.3.1. ***Lie conformal algebras****: a Lie conformal algebra is a Lie $F[\partial]$-pseudoalgebra. The pseudobracket is identified with the λ-bracket via: $\lambda = -\partial \otimes 1$. For example, the Virasoro Lie conformal algebra is defind as the $\mathbb{F}[\partial]$-module:*

$$Vir = \mathbb{F}[\partial]L, \quad [L_\lambda L] = (\partial + 2\lambda)L. \tag{2.3.7}$$

We will see later that Vir is indeed a special example of $W(\mathfrak{d})$ (cf. Example 2.3.3) after we identify the λ-backet with the pseudobracket.

Example 2.3.2. ***Current Lie pseudoalgebras****: Let H' be a Hopf subalgebra of a Hopf algebra H, and A be an H'-pseudoalgebra. A current H-pseudoalgebra Cur $^H_{H'}A$, also written as Cur A for short, is defined to be $H \otimes_{H'} A$, with the pseudobracket extended by H-bilinearity from that of an H'-pseudoalgebra, which is given as follows: for any $f \otimes_{H'} a$, $\ g \otimes_{H'} b \in H \otimes_{H'} A$,*

$$\begin{aligned}[(f \otimes_{H'} a) * (g \otimes_{H'} b)] &= ((f \otimes g) \otimes_H 1)[a * b] \\ &= \sum_i (f f_i \otimes g g_i) \otimes_H (1 \otimes_{H'} e_i)\end{aligned} \tag{2.3.8}$$

if $[a * b] = \sum_i (f_i \otimes g_i) \otimes_{H'} e_i$. One can verify that Cur A with this pseudobracket is a Lie H-pseudoalgebra. In particular, if H' is the base field $\mathbb{F}$, a Lie $\mathbb{F}$-pseudoalgebra $\mathfrak{g}$ is indeed just a usual Lie algebra, and Cur $\mathfrak{g} = H \otimes \mathfrak{g}$ with the pseudobracket:

$$[(f \otimes a) * (g \otimes b)] = (f \otimes g) \otimes_H (1 \otimes [a, b]).$$

Example 2.3.3. ***Primitive pseudoalgebra of vector fields*** *$W(\mathfrak{d})$: Let $H = U(\mathfrak{d})$ be the universal enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{d}$. Then $W(\mathfrak{d})$ is defined as $W(\mathfrak{d}) = H \otimes \mathfrak{d}$ with the pseudobracket: for $f \otimes a$, $g \otimes b \in H \otimes \mathfrak{d}$,*

$$[(f \otimes a) * (g \otimes b)] = (f \otimes g) \otimes_H (1 \otimes [a, b]) - (f \otimes g a) \otimes_H (1 \otimes b) + (f b \otimes g) \otimes_H (1 \otimes a). \tag{2.3.9}$$

One can check that $W(\mathfrak{d})$ is a Lie pseudoalgebra. In fact, for an N-dimensional Lie algebra $\mathfrak{d}$ ($N \geq 1$), the annihilation algebra of $W(\mathfrak{d})$ is $H^* \otimes_H W(\mathfrak{d}) \cong H^* \otimes \mathfrak{d}$, which is the Lie algebra W_N. Elements of W_N are continuous vector fields on the topological algebra $\mathcal{O}_N = \mathbb{F}[[t_1, \ldots, t_N]]$, the formal power series over $\mathbb{F}$ in indeterminates $t_1, \ldots, t_N$.

For Lie algebras of vector fields, we also consider some subalgebras of W_N: the vector fields S_N that annihilate the standard volume form, the vector fields H_N that annihilate the standard symplectic form, and the vector fields K_N that preserves the standard contact from up to some formal power series in $\mathscr{O}_N$. The famous theorem by E. Cartan (see [15]) concludes that any infinite-dimensional simple linearly compact Lie algebra is isomorphic to one of the topological Lie algebras W_N, S_N, H_N or K_N.

Similarly to the Lie algebra case above, one can also define subalgebras of $W(\mathfrak{d})$ whose annihilation algebras are S_N, P_N and K_N, respectively (here P_N is a non-trivial central extension of H_N). For details of the construction, we refer the reader to [1]. Here we will call these subalgebras of type $S(\mathfrak{d})$, $H(\mathfrak{d})$ and $K(\mathfrak{d})$. A Lie pseudoalgebra of vector fields is called primitive if it is one of: $W(\mathfrak{d})$, $S(\mathfrak{d})$, $H(\mathfrak{d})$ or $K(\mathfrak{d})$. The classification theorem for the finite (finitely generated as an H-module) simple Lie $U(\mathfrak{d})$-pseudoalgebras by B. Bakalov et al. is the following:

Theorem 2.3.1 ([1]). *For a finite-dimensional Lie algebra $\mathfrak{d}$, any finite simple Lie $U(\mathfrak{d})$-pseudoalgebra is isomorphic to a current pseudoalgebra over a finite-dimensional simple Lie algebra or over one of the primitive pseudoalgebras of vector fields.*

CHAPTER

3

OPERADS

In this chapter, we review some material from [3]. We recall another equivalent definition of an operad by formulating the composition in terms of circle products. This allows us to construct the universal Lie superalgebra associated to an operad. It turns out that, depending on the operad taken, there is a one-to-one correspondence between some elements in the associated universal Lie superalgebra and some specific algebra structures. For more detailed reviews about operads, readers can see [21] and [22].

3.1 Definitions of an operad

3.1.1 Axioms defining an operad

An operad is a pseudotensor category with only one object. We list the axioms explicitly below for a (linear, unital, symmetric) superoperad.

Let $P(n), n \geq 0$, be a collection of vector superspaces with parity p. Suppose $f \in P(n)$, $g_i \in P(m_i)$ for $m_1, \ldots, m_n \geq 0$ and $M_n = m_1 + \cdots + m_n$. The composition of f and g_i is the parity preserving linear map:

$$
\begin{aligned}
P(n)\otimes P(m_1)\otimes\cdots\otimes P(m_n) &\to P(M_n),\\
f\otimes g_1\otimes\cdots\otimes g_n &\to f(g_1\otimes\cdots\otimes g_n).
\end{aligned}
$$

- Associativity axiom: suppose $h_j \in P(l_j)$ for $j = 1,\ldots,M_n$. The associativity of the composition is the following equation:

$$
f((g_1\otimes\cdots\otimes g_n)(h_1\otimes\cdots\otimes h_{M_n})) = (f(g_1\otimes\cdots\otimes g_n))(h_1\otimes\cdots\otimes h_{M_n}). \tag{3.1.1}
$$

  In the left-hand side of (3.1.1), for $k=0,\ldots,n$, let

$$
M_k = \sum_{j=1}^{k} m_k, \tag{3.1.2}
$$

  then

$$
\begin{aligned}
&(g_1\otimes\cdots\otimes g_n)(h_1\otimes\cdots\otimes h_{M_n})\\
&= \pm g_1(h_1\otimes\cdots\otimes h_{M_1})\otimes\cdots\otimes g_n(h_{M_{n-1}+1}\otimes\cdots\otimes h_{M_n}),
\end{aligned} \tag{3.1.3}
$$

  where the sign

$$
\pm = (-1)^{\sum_{i<j,M_{i-1}<k\le M_i} p(g_j)p(h_k)} \tag{3.1.4}
$$

  follows the Koszul-Quillen rule mentioned earlier.

- Unity axiom: there exist a unit $1\in P(1)$ such that:

$$
f(1\otimes\cdots\otimes 1) = 1(f) = f, \quad \text{for any } f\in P(n). \tag{3.1.5}
$$

- Equivariance axiom: there is a right action of the symmetric group S_n on $P(n)$, $n\ge 1$:

$$
\begin{aligned}
P(n)\times S_n &\to P(n),\\
f\cdot\sigma &\to f^{\sigma}.
\end{aligned}
$$

The equivalariance axiom holds if: for any $\tau_1 \in S_{m_1}, \ldots, \tau_n \in S_{m_n}$,

$$f^{\sigma}(g_1^{\tau_1} \otimes \cdots \otimes g_n^{\tau_n}) = (f(\sigma(g_1 \otimes \cdots \otimes g_n)))^{\sigma(\tau_1, \ldots, \tau_n)}, \tag{3.1.6}$$

where

$$(\sigma(\tau_1, \ldots, \tau_n))(v_1 \otimes \cdots \otimes v_{M_n}) = \sigma(\tau_1(v_1 \otimes \cdots \otimes v_{M_1}) \otimes \cdots \otimes \tau_n(v_{M_{n-1}+1} \otimes \cdots \otimes v_{M_n})) \tag{3.1.7}$$

and

$$\sigma(g_1 \otimes \cdots \otimes g_n) = \epsilon_g(\sigma)(g_{\sigma^{-1}(1)} \otimes \cdots \otimes g_{\sigma^{-1}(n)}). \tag{3.1.8}$$

Here the sign factor $\epsilon_g(\sigma)$ is defined again by the Koszul-Quillen rule:

$$\epsilon_g(\sigma) = \prod_{i<j \mid \sigma(i)>\sigma(j)} (-1)^{p(g_i)p(g_j)}. \tag{3.1.9}$$

A direct calculation shows that:

$$\epsilon_g(\sigma\tau) = \epsilon_{\tau(g)}(\sigma)\epsilon_g(\tau). \tag{3.1.10}$$

Definition 3.1.1. *A superoperad P is a collection of vector superspaces $P(n)$, $n \geq 0$, with parity p such that the associativity, unity and equivariance axioms above hold.*

We will use the term operad in place of superoperad for simplicity in the following discussion. Next, we describe the notion of an operad in terms of circle products.

3.1.2 $\circ$-product formulation

For an operad P, one can define the $\circ_i$-product as the insertion at the ith position. To be precise, for any $Y \in P(n)$ and $X \in P(m)$ (In order to be consistent with the notations used in Section 4.1, we use X, Y, Z in place of f, g, h in this subsection),

$$\begin{aligned} \circ_i : P(n) \times P(m) &\rightarrow P(n+m-1), \\ Y \circ_i X &= Y(1 \otimes \cdots \otimes 1 \otimes X \otimes 1 \otimes \cdots \otimes 1), \end{aligned} \tag{3.1.11}$$

where X is inserted at the i-th position in the tensor product above. By the associativity axiom, we have for any $Y \in P(n)$, $X_k \in P(m_k)$, $k = 1, \ldots, n$:

$$Y(X_1 \otimes \cdots \otimes X_n) = (\cdots((Y \circ_1 X_1) \circ_{M_1+1} X_2) \cdots) \circ_{M_{n-1}+1} X_n, \tag{3.1.12}$$

assuming all $\circ_i$-products are known. In fact, the associativity axiom is equivalent to the following identities: for any $X \in P(m)$, $Y \in P(n)$ and $Z \in P(l)$:

$$(Z \circ_i Y) \circ_j X = \begin{cases} (-1)^{p(Y)p(X)}(Z \circ_j X) \circ_{i+m-1} Y\,, & \text{if } 1 \le j < i, \\ Z \circ_i (Y \circ_{j-i+1} X)\,, & \text{if } i \le j < i+n, \\ (-1)^{p(Y)p(X)}(Z \circ_{j-n+1} X) \circ_i Y\,, & \text{if } i+n \le j < n+l. \end{cases} \tag{3.1.13}$$

Moreover, the unity axiom (3.1.5) is equivalent to:

$$1 \circ_1 Y = Y \circ_i 1 = Y, \text{ for any } i = 1, \ldots n. \tag{3.1.14}$$

And the equivariance axiom (3.1.6) is equivalent to:

$$Y^\sigma \circ_i X^\tau = (Y \circ_{\sigma(i)} X)^{\sigma \circ_i \tau}. \tag{3.1.15}$$

Note that the $\circ_i$-products of permutations are defined similarly as above (cf. (3.1.7)): for $\sigma \in S_n$ and $\tau \in S_m$:

$$\sigma \circ_i \tau = \sigma(1, \ldots, 1, \tau, 1, \ldots, 1) \in S_{m+n-1}, \tag{3.1.16}$$

where τ is inserted at the i-th position. One can check if these axioms hold by checking (3.1.13), (3.1.14) and (3.1.15). In particular, for (3.1.13), the third identity is equivalent to the first one after flipping the equality; this further simplifies the verification process. Another observation is that the $\circ_1$-product is associative, which is obvious from (3.1.13).

3.2 Universal Lie superalgebra associated to an operad

Before introducing some examples of operads, we review the construction of the universal Lie superalgebra associated to an operad.

Suppose P is an operad. For $n \ge -1$, Define W_n to be the set of all permutation invariant elements in $P(n+1)$, that is:

$$W_n = \{\, f \in P(n+1) \mid f^{\sigma} = f, \text{ for any } \sigma \in S_{n+1} \,\}. \tag{3.2.1}$$

Let $W(P) = \bigoplus_{n\geq -1} W_n$ be a $\mathbb{Z}$-graded vector superspace. Next, we define the $\square$-product as follows: for any $f \in W_n$ and $g \in W_m$,

$$f \square g = \sum_{\sigma \in S_{m+1,n}} (f \circ_1 g)^{\sigma^{-1}} \in W_{n+m}. \tag{3.2.2}$$

Here $S_{m,n}$ is called the subset of (m,n)-shuffles of S_{m+n}:

$$S_{m,n} = \{\, \sigma \in S_{m+n} \mid \sigma(1) < \cdots < \sigma(m),\ \sigma(m+1) < \cdots < \sigma(m+n) \,\}. \tag{3.2.3}$$

A quick example that will be useful later is for $f, g \in W_1$, their $\square$-product is by definition:

$$\begin{aligned} f \square g &= \sum_{\sigma \in S_{2,1}} (f \circ_1 g)^{\sigma^{-1}} \\ &= f \circ_1 g + (f \circ_1 g)^{(23)^{-1}} + (f \circ_1 g)^{(123)^{-1}} \\ &= f \circ_1 g + (f \circ_1 g)^{(23)} + (f \circ_1 g)^{(132)} \\ &= f \circ_1 g + f \circ_2 g + (f \circ_2 g)^{(12)}. \end{aligned} \tag{3.2.4}$$

The last equality follows from the equivariance axiom (3.1.15), $S_{2,1} = \{(1), (23), (123)\}$ and the fact that $(23) = (132)(12)$ and $(132) = (12) \circ_2 (1)$. Another example is when $f \in W_{-1}$, $S_{m+1,-1} = \emptyset$, then $f \square g = 0$ is trivial. Consider the commutator bracket of the $\square$-product:

$$[f, g] = f \square g - (-1)^{p(f)p(g)} g \square f. \tag{3.2.5}$$

Theorem 3.2.1 ([3])**.** *$W(P)$ is a $\mathbb{Z}$-graded Lie superalgebra with the bracket given by* (3.2.5)*, called the universal Lie superalgebra associated to the operad P.*

3.3 Operad governing Lie conformal superalgebras

In this section, we review the $\mathfrak{Chom}$ operad, which governs the Lie conformal superalgebra structures from Chapter 2.

3.3.1 The $\mathfrak{Chom}$ operad

Let V be a vector superspace with an even endomorphism ∂. The $\mathfrak{Chom}(V)$ operad is defined as the following collection of vector superspaces:

$$\mathfrak{Chom}(n) = \mathrm{Hom}_{\mathbb{F}[\partial]^{\otimes n}}(V^{\otimes n}, \mathbb{F}[\lambda_1, \dots, \lambda_n] \otimes_{\mathbb{F}[\partial]} V), \quad n \geq 0, \tag{3.3.1}$$

with the parity induced by that of V. In the right-hand side of (3.3.1), $V^{\otimes n}$ is an $\mathbb{F}[\partial]^{\otimes n}$-module since by assumption V is an $\mathbb{F}[\partial]$-module, the action is componentwise. Here $\lambda_1, \dots, \lambda_n$ are commuting indeterminants that are even. $\mathbb{F}[\lambda_1, \dots, \lambda_n]$ is a left $\mathbb{F}[\partial]^{\otimes n}$-module: $\partial_i = 1 \otimes \cdots \otimes \partial \otimes \cdots \otimes 1$ (∂ is at the i-th position) acts as multiplication by $-\lambda_i$. Meanwhile it is a right $\mathbb{F}[\partial]$-module: ∂ acts as multiplication by $-\lambda_1 - \cdots - \lambda_n$.

For small n, it is not hard to see that $\mathfrak{Chom}(0) = V/\partial V$ and $\mathfrak{Chom}(1) = \mathrm{End}_{\mathbb{F}[\partial]}(V)$. In general, any $f_{\lambda_1,\dots,\lambda_n} \in \mathfrak{Chom}(n)$ by definition satisfies the sesquilinearity:

$$f_{\lambda_1,\dots,\lambda_n}(v_1 \otimes \cdots \otimes \partial v_i \otimes \cdots \otimes v_n) = -\lambda_i f_{\lambda_1,\dots,\lambda_n}(v_1 \dots v_n). \tag{3.3.2}$$

Note, in particular, that any element in $\mathfrak{Chom}(2)$ can be identified with a λ-bracket that satisfies the sesquilinearity (2.3.4) in the definition of a Lie conformal superalgebra.

The composition in $\mathfrak{Chom}$ is defined as follows: let $f \in \mathfrak{Chom}(n)$, $g_i \in \mathfrak{Chom}(m_i)$, $i = 1, 2, \dots, n$. Set $M_i = \sum_{j=1}^{i} m_j$ and $\mu_i = \sum_{j=M_{i-1}+1}^{M_i} \lambda_j$ for $i = 1, \dots, n$. Then

$$\begin{aligned}
&(f(g_1 \otimes \cdots \otimes g_n))_{\lambda_1,\dots,\lambda_{M_n}}(v_1 \otimes \cdots \otimes v_{M_n}) \\
:=& f_{\mu_1,\dots,\mu_n}(((g_1)_{\lambda_1,\dots,\lambda_{M_1}} \otimes \cdots \otimes (g_n)_{\lambda_{M_{n-1}+1},\dots,\lambda_{M_n}})(v_1 \otimes \cdots \otimes v_{M_n})) \\
=& \pm f_{\mu_1,\dots,\mu_n}((g_1)_{\lambda_1,\dots,\lambda_{M_1}}(v_1 \otimes \cdots \otimes v_{M_1}) \otimes \cdots \otimes (g_n)_{\lambda_{M_{n-1}+1},\dots,\lambda_{M_n}}(v_{M_{n-1}+1} \otimes \cdots \otimes v_{M_n})),
\end{aligned} \tag{3.3.3}$$

where the sign $\pm$ is given by the Koszul-Quillen rule:

$$\pm = (-1)^{\sum_{i<j} p(g_j)(p(v_{M_{i-1}+1}) + \cdots + p(v_{M_i}))}. \tag{3.3.4}$$

The unity is the identity map on V: $\mathrm{id}_V \in \mathfrak{Chom}(1)$, which implies the unity axiom immediately.

The right action of S_n on $\mathfrak{Chom}(n)$ is defined by permuting the λ's and the tensor factors simultaneously: for any $\sigma \in S_n$,

$$(f^{\sigma})(v_1 \otimes \cdots \otimes v_n) = f_{\sigma(\lambda_1,\dots,\lambda_n)}(\sigma(v_1 \otimes \cdots \otimes v_n))$$
$$= \epsilon_v(\sigma)\, f_{\lambda_{\sigma^{-1}(1)},\dots,\lambda_{\sigma^{-1}(n)}}(v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}), \tag{3.3.5}$$

where $\epsilon_v(\sigma)$ is given by (3.1.9).

It is not difficult to check that the associativity axiom and the equivariance axiom hold with the composition and the right action of S_n defined above; see [3]. Hence, $\mathfrak{Chom}$ is an operad.

3.3.2 The $\mathfrak{Chom}$ operad governs Lie conformal superalgebras

To see that the $\mathfrak{Chom}$ operad governs Lie conformal superalgebras, consider the following construction: for a vector superspace V with parity p, and an even endomorphism ∂, let $\bar{p} = 1 - p$ denote the reverse parity of p, and ΠV denote the same space as V but with parity $\bar{p}$. ∂ is also an endomorphism of ΠV. For the operad $\mathfrak{Chom}(\Pi V)$, let $W^{\partial}(\Pi V) = W(\mathfrak{Chom}(\Pi V))$ be the universal Lie superalgebra associated to it, and consider the set of elements:

$$S = \{X \in W_1^{\partial}(\Pi V)) \mid X \text{ is odd and } X \square X = 0\}. \tag{3.3.6}$$

Theorem 3.3.1 ([14], [3])**.** *There exist a bijective correspondence $S \longleftrightarrow \Lambda$ given by:*

$$[a_\lambda b] = (-1)^{p(a)} X_{\lambda,-\lambda-\partial}(a \otimes b), \quad \text{any } a, b \in V, \tag{3.3.7}$$

where Λ is the set of all λ-brackets $[\cdot_\lambda \cdot] : V \times V \to V[\lambda]$ on the Lie conformal superalgebra V.

Proof. Pick any $X \in S$ and let λ, μ be the indeterminates. By definition, $X_{\lambda,\mu} : \Pi V \otimes \Pi V \to \mathbb{F}[\lambda,\mu] \otimes_{\mathbb{F}[\partial]} \Pi V$. Moreover,

$$X_{\lambda,\mu}(\partial a \otimes b) = -\lambda X_{\lambda,\mu}(a \otimes b),$$
$$X_{\lambda,\mu}(a \otimes \partial b) = -\mu X_{\lambda,\mu}(a \otimes b) = (\lambda + \partial) X_{\lambda,\mu}(a \otimes b), \tag{3.3.8}$$

which is equivalent to the sesquilinearity of the λ-bracket. The assumption that X is odd with respect to $\bar{p}$ indicates:

$$\bar{p}([a_\lambda b]) = \bar{p}(a) + \bar{p}(b) + 1, \tag{3.3.9}$$

which is equivalent to:

$$p([a_\lambda b]) = p(a) + p(b), \tag{3.3.10}$$

that is the λ-bracket is parity preserving (with respect to p). The skew-symmetry of the λ-bracket is equivalent to the assumption that X is symmetric, that is $X^{(12)} = X$, since:

$$\begin{aligned}
-(-1)^{p(a)p(b)}[b_{-\lambda-\partial} a] &= (-1)^{1+p(a)p(b)}[b_\mu a] \\
&= (-1)^{1+p(a)p(b)+p(b)} X_{\mu,\lambda}(b \otimes a) \\
&= (-1)^{1+p(a)p(b)+p(b)+\bar{p}(a)\bar{p}(b)} X^{(12)}_{\lambda,\mu}(a \otimes b) \\
&= (-1)^{p(a)} X_{\lambda,\mu}(a \otimes b) \\
&= [a_\lambda b].
\end{aligned} \tag{3.3.11}$$

It is left to see that the Jacobi identity is equivalent to $X \square X = 0$, which follows from the calculation below: by (3.2.4),

$$\begin{aligned}
(X \square X)_{\lambda,\mu,\nu}(a \otimes b \otimes c) =& (X \circ_1 X)_{\lambda,\mu,\nu}(a \otimes b \otimes c) + (X \circ_2 X)_{\lambda,\mu,\nu}(a \otimes b \otimes c) \\
&+ (X \circ_2 X)^{(12)}_{\lambda,\mu,\nu}(a \otimes b \otimes c) \\
=& X_{\lambda+\mu,\nu}(X_{\lambda,\mu}(a \otimes b) \otimes c) + (-1)^{\bar{p}(a)} X_{\lambda,\mu+\nu}(a \otimes X_{\mu,\nu}(b \otimes c)) \\
&+ (-1)^{\bar{p}(a)\bar{p}(b)+\bar{p}(b)} X_{\mu,\lambda+\nu}(b \otimes X_{\lambda,\nu}(a \otimes c)) \\
=& (-1)^{p(a)+p(a)+p(b)}[[a_\lambda b]_{\lambda+\mu} c] + (-1)^{1-p(a)+p(b)+p(a)}[a_\lambda [b_\mu c]] \\
&+ (-1)^{p(a)(1-p(b))+p(a)+p(b)}[b_\mu [a_\lambda c]] \\
=& (-1)^{p(b)}([[a_\lambda b]_{\lambda+\mu} c] - [a_\lambda [b_\mu c]] + (-1)^{p(a)p(b)}[b_\mu [a_\lambda c]]).
\end{aligned} \tag{3.3.12}$$

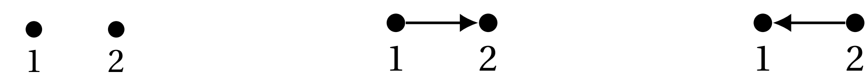
Thus $X \square X = 0$ is equivalent to the Jacobi identity. $\square$

3.4 The cooperad of n-graphs

3.4.1 n-graphs

The goal of this section is to review some basic notions about graphs from [3], which are necessary ingredients of the operad that we will introduce later. For $n \geq 1$, an n-graph Γ is a set of vertices $V(\Gamma)$ labeled from 1 to n and a collection of oriented edges $E(\Gamma)$ between them. Let $G(n)$ denote all graphs without tadpoles (edges that start and end at the same vertex), and $G_0(n)$ be the set of all acyclic n-graphs, which are graphs in $G(n)$ with no (unoriented) cycles, including multiple edges. By convention, we let $G_0(0) = G(0)$ be the set consisting of a single element (the empty graph, with no vertex).

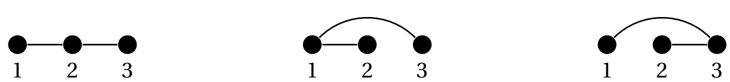
Example 3.4.1. *For n=1, the only graph in $G_0(1) = G(1)$ is a single vertex. When $n = 2$, there are 3 graphs Γ in $G_0(2)$:*



(a) $E(\Gamma) = \emptyset$ (b) $E(\Gamma) = \{1 \to 2\}$ (c) $E(\Gamma) = \{2 \to 1\}$

For $n = 3$, the graphs in $G_0(3)$ are the following, with any combination of directions on the edges:

(a) (b) (c) (d)



(e) (f) (g)

For an n-tuple of positive integers $(m_1, \ldots, m_n)$, and M_i defined as in (3.1.2), we define the cocomposition map $\Delta^{m_1 \dots m_n}$ as follows [3]:

$$\begin{aligned} \Delta^{m_1 \dots m_n} : G(M_n) &\to G(n) \times G(m_1) \times \cdots \times G(m_n), \\ \Gamma &\to \left(\Delta_0^{m_1 \dots m_n}(\Gamma), \Delta_1^{m_1 \dots m_n}(\Gamma), \dots, \Delta_n^{m_1 \dots m_n}(\Gamma)\right), \end{aligned} \tag{3.4.1}$$

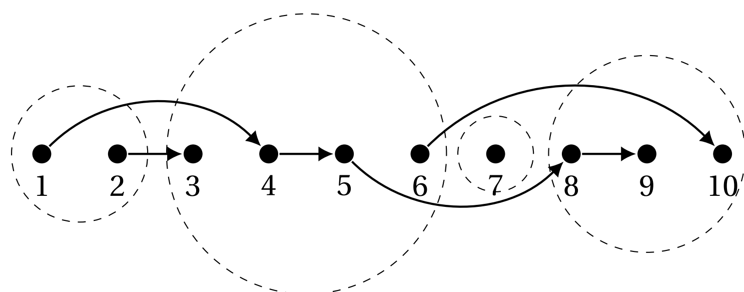
where:

1. $\Delta_0^{m_1 \dots m_n}(\Gamma)$ is the graph obtained by clasping the m_k vertices in group k into a single vertex, or in other words, for each $1 \le k \le n$, identifying all m_k vertices in Group k;
2. For each $1 \le k \le n$, $\Delta_k^{m_1 \dots m_n}(\Gamma)$ is the graph obtained by taking all m_k vertices in Group k and all edges between them.

We will call $\Delta_k^{m_1 \dots m_n}(\Gamma)$ Subgraph k of Γ.

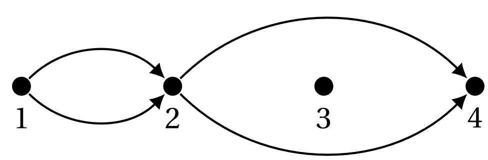
Example 3.4.2. *Consider the following graph* $\Gamma \in G(10)$ *with* $(m_1, m_2, m_3, m_4) = (2, 4, 1, 3)$*,*

$\Gamma:$



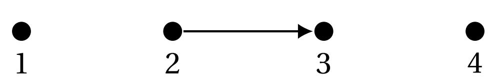
where the dashed circles denote the groupings. By definition of the cocomposition map, after relabeling the vertices starting from 1*, we have:*

$\Delta_0^{2413}(\Gamma) =$

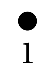


$\Delta_1^{2413}(\Gamma) =$

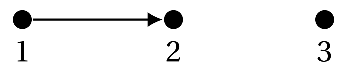
$\Delta_2^{2413}(\Gamma) =$



$\Delta_3^{2413}(\Gamma) =$



$\Delta_4^{2413}(\Gamma) =$



Another notion we will need is the external connectedness defined below:

Definition 3.4.1 ([3])**.** *For an n-tuple of positive integers $(m_1,\ldots,m_n)$, let M_i be defined as in* (3.1.2)*, and $\Gamma \in G(M_n)$. For a vertex $k \in \{1,\ldots,M_n\}$ and a group $j \in \{1,\ldots,n\}$, suppose k is in group i, that means $M_{i-1}+1 \le k \le M_i$. We say that j is externally connected to k if there exists an unoriented path (without repeating edges) between j and i in $\Delta_0^{m_1\ldots m_n}(\Gamma)$, and the edge connecting i is the image under Δ of an edge in Γ that starts or ends at k.*

The set of j's that are externally connected to k is denoted by $\mathscr{E}(k)$ as a subset of $\{1,\ldots,n\}$. Given a set of variables $x_1,\ldots,x_n$, we define:

$$X(k) = \sum_{j\in\mathscr{E}(k)} x_j\,. \tag{3.4.2}$$

Example 3.4.3. *For the graph $\Gamma \in G(10)$ from* Example 3.4.2*, we have:*

$$X(7) = X(9) = 0, \quad X(i) = x_1 + x_2 + x_4 \text{ for all other } i \in \{1,\ldots,10\}. \tag{3.4.3}$$

3.4.2 Properties of the cocomposition map of n-graphs

The properties of the cocomposition map of n-graphs listed in this section will be needed in the sequel. In fact, $(G(n),\Delta)$ defines a cooperad, which is a dual notion of an operad [21]. The notion of a cooperad is not needed later, so we focus on the properties themselves here.

Lemma 3.4.1 ([3])**.** *For any positive integers $m_1,\ldots,m_n$, there is a natural bijection:*

$$\Delta : E(\Gamma) \to E(\Delta_0^{m_1\ldots m_n}(\Gamma)) \sqcup E(\Delta_1^{m_1\ldots m_n}(\Gamma)) \sqcup \cdots \sqcup E(\Delta_n^{m_1\ldots m_n}(\Gamma)). \tag{3.4.4}$$

Proof. This is true because an edge in Γ is either contained in one of the n subgraphs, or it connects two different subgraphs; hence, its image under Δ is in $\Delta_0^{m_1\ldots m_n}(\Gamma)$. $\square$

Lemma 3.4.2 ([3])**.** *For an oriented cycle $C \in E(\Gamma)$ in an n-graph Γ, either $\Delta(C) \subset E(\Delta_i^{m_1 \dots m_n})$ for some $1 \le i \le n$ or $\Delta(C) \cap E(\Delta_0^{m_1 \dots m_n}(\Gamma))$ is an oriented cycle in $\Delta_0^{m_1 \dots m_n}(\Gamma)$.*

Proof. If C is contained in some Subgraph i, then $\Delta(C) \subset E(\Delta_i^{m_1 \dots m_n})$. Otherwise, the set of edges connecting different subgraphs in Γ, which is $\Delta(C) \cap E(\Delta_0^{m_1 \dots m_n}(\Gamma))$, gives an oriented cycle in $\Delta_0^{m_1 \dots m_n}(\Gamma)$. $\square$

For the rest of this subsection, we review the coassociativity and the coequivariance of the cocomposition map. The coassociativity is essentially the associativity of the cocomposition map.

For positive integers $m_1, \dots, m_n$, let M_i be defined as in (3.1.2), and $l_1, \dots, l_{M_n}$ be M_n positive integers. For $i \in \{0, \dots, M_n\}$ and $j \in \{1, \dots, n\}$, let

$$L_i = \sum_{n=1}^{i} l_n, \tag{3.4.5}$$

$$K_j = \sum_{m=M_{j-1}+1}^{M_j} l_m. \tag{3.4.6}$$

Proposition 3.4.1 ([3])**.** (**Coassociativity)** The cocomposition map Δ given by (3.4.1) of graphs satisfies the following coassociativity conditions: for a graph $\Gamma \in G_0(L_{M_n})$,

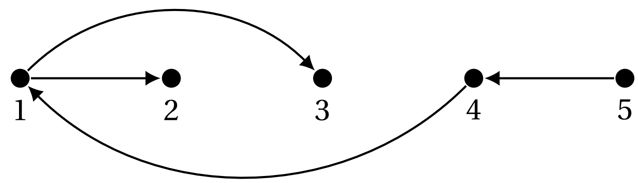
1. $\Delta_0^{m_1 \dots m_n}(\Delta_0^{l_1 \dots l_{M_n}}(\Gamma)) = \Delta_0^{K_1 \dots K_n}(\Gamma) \in G(n)$;
2. $\Delta_i^{m_1 \dots m_n}(\Delta_0^{l_1 \dots l_{M_n}}(\Gamma)) = \Delta_0^{l_{M_{i-1}+1} \dots l_{M_i}}(\Delta_i^{K_1 \dots K_n}(\Gamma)) \in G(m_i)$ for $i = 1, \dots, n$;
3. $\Delta_{M_{i-1}+j}^{l_1 \dots l_{M_n}}(\Gamma) = \Delta_j^{l_{M_{i-1}+1} \dots l_{M_i}}(\Delta_i^{K_1 \dots K_n}(\Gamma)) \in G(l_{M_{i-1}+j})$ for $i = 1, \dots, n$ and $j = 1, \dots, m_i$.

A detailed "pictorial" proof can be found in [3]. The claims become natural if the intuition behind is understood: Δ_0 simplifies the graph by clasping the subgraphs into vertices based on the groupings given and Δ_i ($i \neq 0$) just picks the i-th subgraph.

The symmetric group S_n ($n \ge 1$) acts on an n-graph $\Gamma \in G(n)$. For a permutation $\sigma \in S_n$, the action $\sigma(\Gamma)$ is defined by relabeling (permuting) Vertex i to $\sigma(i)$. This action restricts to an action on $G_0(n)$.

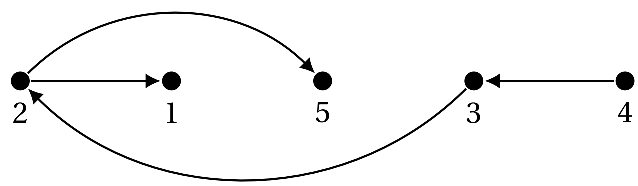
Example 3.4.4. *Suppose that $\sigma = (12)(354) \in S_5$ and $\Gamma \in G_0(5)$ is the graph below:*

$\Gamma =$



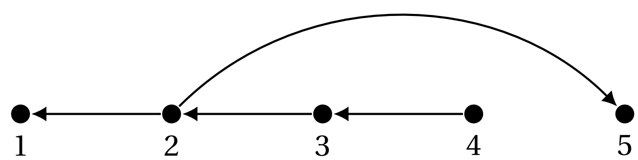
Then $\sigma(\Gamma)$ is the graph:

$\sigma(\Gamma) =$



After rearranging the vertices from 1 *to* 5*, we have:*

$\sigma(\Gamma) =$



The coequivariacne of the cocomposition map is the following:

Proposition 3.4.2 ([3])**.** (**Coequivariance**) For any positive integers $m_1, \ldots, m_n$ ($n > 0$), any permutations $\sigma \in S_n$, $\tau_i \in S_{m_i}$, $i = 1, \ldots, n$, and $\Gamma \in G_0(\sum_{i=1}^n m_i)$, we have

$$\Delta^{m_{\sigma^{-1}(1)} \ldots m_{\sigma^{-1}(n)}}\big((\sigma(\tau_1, \ldots, \tau_n))(\Gamma)\big) = \big(\sigma(\Delta_0^{m_1 \ldots m_n}(\Gamma)),\ \tau_{\sigma^{-1}(1)}(\Delta_{\sigma^{-1}(1)}^{m_1 \ldots m_n}(\Gamma)),\ \ldots,\ \tau_{\sigma^{-1}(n)}(\Delta_{\sigma^{-1}(n)}^{m_1 \ldots m_n}(\Gamma))\big), \tag{3.4.7}$$

where $\sigma(\tau_1, \ldots, \tau_n)$ is given by (3.1.7).

A "pictorial" proof is given in [3]. The intuition here is similar to the coassociativity case. One only needs to keep track of the subgraphs (and the simplified graph under Δ_0) under these permutations and the result will follow.

3.5 Operad governing Poisson vertex superalgebras

Similarly to the Lie conformal superalgebra case, one can construct an operad goverining Poisson vertex superalgebras, which will be called the classical operad and denoted as

P^{cl}. We review the definition of the classical operad and its connection to Poisson vertex superalgebras. The details can be found in [3].

3.5.1 The classical operad P^{cl}

Let V be a vector superspace with parity p and an even endomorphism ∂. The classical operad P^{cl} is defined as the collection of vector superspaces $P^{cl}(n)$ with the parity induced by the parity of V: $f \in P^{cl}(n)$ if it is a map

$$f : G(n) \times V^{\otimes n} \to V[\lambda_1, \dots, \lambda_n]/\langle \partial + \lambda_1 + \cdots + \lambda_n \rangle, \tag{3.5.1}$$

which is linear on $V^{\otimes n}$ and satisfies the following properties. Let $f^{\Gamma}_{\lambda_1,\dots,\lambda_n}(v_1 \otimes \cdots \otimes v_n)$, a polynomial in $\lambda_1, \dots, \lambda_n$, denote the image of $(\Gamma, v_1 \otimes \cdots \otimes v_n)$ under f. Then f satisfies:

- two cycle conditions:

  1. For any $\Gamma \notin G_0(n)$, that is Γ contains a cycle,

  $$f^{\Gamma} = 0. \tag{3.5.2}$$

  2. For any oriented cycle $C \subset E(\Gamma)$ of Γ,

  $$\sum_{e \in C} f^{\Gamma \backslash e} = 0. \tag{3.5.3}$$

- two sesquilinearity conditions:
  suppose Γ is the disjoint union of its s connected components, $\Gamma = \Gamma_1 \sqcup \cdots \sqcup \Gamma_s$, and for component k, the vertices in it are denoted as V_k.

  1. For any $i, j \in V_k$ for some $k \in \{1, \dots, s\}$,

  $$\frac{\partial}{\partial \lambda_i} f^{\Gamma}_{\lambda_1,\dots,\lambda_n}(v_1 \otimes \cdots \otimes v_n) = \frac{\partial}{\partial \lambda_j} f^{\Gamma}_{\lambda_1,\dots,\lambda_n}(v_1 \otimes \cdots \otimes v_n). \tag{3.5.4}$$

  2. For any $k \in \{1, \dots, s\}$,

  $$f^{\Gamma}_{\lambda_1,\dots,\lambda_n}(\partial_{\Gamma_k}(v_1 \otimes \cdots \otimes v_n)) = -\lambda_{\Gamma_k} f^{\Gamma}_{\lambda_1,\dots,\lambda_n}(v_1 \otimes \cdots \otimes v_n), \tag{3.5.5}$$

  where $\partial_{\Gamma_k} = \sum_{i \in V_k} (1 \otimes \cdots \otimes \partial \otimes \cdots \otimes 1)$ (∂ at the i-th position in the tensor product) and $\lambda_{\Gamma_k} = \sum_{i \in V_k} \lambda_i$.

Remark 3.5.1. *Applying the two cycle relations on a cycle of length* 2*, one sees that changing the orientation of a single edge in an n-graph Γ amounts to a sign change of f^Γ. The second sesquilinearity condition (3.5.5) is also true for the whole graph Γ rather than on a connected component k, that is:*

$$f^\Gamma_{\lambda_1,\dots,\lambda_n}(\partial_\Gamma(v_1\otimes\cdots\otimes v_n)) = -\lambda_\Gamma f^\Gamma_{\lambda_1,\dots,\lambda_n}(v_1\otimes\cdots\otimes v_n) = \partial f^\Gamma_{\lambda_1,\dots,\lambda_n}(v_1\otimes\cdots\otimes v_n). \tag{3.5.6}$$

The composition of maps in P^{cl} is defined as follows. For positive integers $m_1,\dots,m_n$ ($n \geq 1$), let $f \in P^{cl}(n)$ and $g_i \in P^{cl}(m_i)$ for $i = 1,\dots,n$. Set $M_i = \sum_{j=1}^i m_j$ for $i = 0,\dots,n$ and $\mu_i = \sum_{j=M_{i-1}+1}^{M_i} \lambda_j$ for $i = 1,\dots,n$. Suppose $\Gamma \in G(M_n)$; then

$$(f(g_1,\dots,g_n))^\Gamma : V^{\otimes n} \to V[\lambda_1,\dots,\lambda_{M_n}]/\langle \partial + \lambda_1 + \cdots + \lambda_{M_n}\rangle$$

is defined to be:

$$\begin{aligned}
&(f(g_1,\dots,g_n))^\Gamma_{\lambda_1,\dots,\lambda_{M_n}}(v_1\otimes\cdots\otimes v_{M_n})\\
=&f^{\Delta_0^{m_1\dots m_n}(\Gamma)}_{\mu_1,\dots,\mu_n}\Big(\Big(\Big(\big|_{x_1=\mu_1+\partial}(g_1)^{\Delta_1^{m_1\dots m_n}(\Gamma)}_{\lambda_1+X(1),\dots,\lambda_{M_1}+X(M_1)}\Big)\otimes\cdots\otimes\Big(\big|_{x_n=\mu_n+\partial}(g_n)^{\Delta_n^{m_1\dots m_n}(\Gamma)}_{\lambda_{M_{n-1}+1}+X(M_{n-1}+1),\dots,\lambda_{M_n}+X(M_n)}\Big)\Big)\\
&(v_1\otimes\cdots\otimes v_{M_n})\Big)\\
=&\pm f^{\Delta_0^{m_1\dots m_n}(\Gamma)}_{\mu_1,\dots,\mu_n}\Big(\Big(\big|_{x_1=\mu_1+\partial}(g_1)^{\Delta_1^{m_1\dots m_n}(\Gamma)}_{\lambda_1+X(1),\dots,\lambda_{M_1}+X(M_1)}(v_1\otimes\cdots\otimes v_{M_1})\Big)\otimes\dots\\
&\otimes\Big(\big|_{x_n=\mu_n+\partial}(g_n)^{\Delta_n^{m_1\dots m_n}(\Gamma)}_{\lambda_{M_{n-1}+1}+X(M_{n-1}+1),\dots,\lambda_{M_n}+X(M_n)}(v_{M_{i-1}+1}\otimes\cdots\otimes v_{M_n})\Big)\Big),
\end{aligned} \tag{3.5.7}$$

where $\pm$ is the sign defined by (3.3.4), $X(i)$ is defined by (3.4.2), and for polynomials $P(\lambda) = \sum_m p_m\lambda^m$ and $Q(\mu) = \sum_n q_n\mu^n$ with coefficients in V, we define:

$$\Big(\big|_{x=\partial}P(\lambda+y)\Big)\otimes\Big(\big|_{y=\partial}Q(\mu+x)\Big) = \sum_{m,n}((\mu+\partial)^n p_m)\otimes((\lambda+\partial)^m q_n). \tag{3.5.8}$$

Note that, in particular, for $f \in P^{cl}(n)$, $g \in P^{cl}(m)$, $n, m \geq 1$, we have $f(g,1,\dots,1) \in P^{cl}(n+m-1)$, and for any $\Gamma \in G(n+m-1)$:

$$\begin{aligned}
&(f(g,1,\dots,1))^\Gamma_{\lambda_1,\,\dots,\,\lambda_{n+m-1}}(v_1\otimes\cdots\otimes v_{n+m-1})\\
=&f^{\Delta_0^{m1\dots1}(\Gamma)}_{\sum_{i=1}^m\lambda_i,\,\lambda_{m+1},\,\dots,\,\lambda_{n+m-1}}\Big(g^{\Delta_1^{m1\dots1}(\Gamma)}_{\lambda_1+\lambda_{G_1}+\partial_{G_1},\dots,\lambda_m+\lambda_{G_m}+\partial_{G_m}}(v_1\otimes\cdots\otimes v_m)\otimes v_{m+1}\otimes\cdots\otimes v_{n+m-1}\Big),
\end{aligned} \tag{3.5.9}$$

where G_i, $i = 1,\dots,m$ is the subgraph obtained by taking the connected component of Γ that contains Vertex i but with the part in the subgraph formed by the first m vertices removed. Note that (3.5.9) is indeed $f \circ_1 g$.

The unity is the identity map on V: $1 := \mathrm{id}_V \in P^{cl}(1)$, as $G(1)$ consists of a single graph with only 1 vertex. The right action of S_n is defined as follows: for a permutation $\sigma \in S_n$ and $\Gamma \in G(n)$, we have

$$(f^{\sigma})^{\Gamma}_{\lambda_1,\dots,\lambda_n}(v_1 \otimes \cdots \otimes v_n) = f^{\sigma(\Gamma)}_{\sigma(\lambda_1,\dots,\lambda_n)}(\sigma(v_1 \otimes \cdots \otimes v_n)), \tag{3.5.10}$$

where

$$\sigma(\lambda_1, \dots, \lambda_n) = (\lambda_{\sigma^{-1}(1)}, \dots, \lambda_{\sigma^{-1}(n)}) \tag{3.5.11}$$

is just permuting $\lambda_i' s$ with σ and $\sigma(v_1 \otimes \cdots \otimes v_n)$ is given by (3.1.8).

It is shown in [3] that the composition given by (3.5.7) is well defined and P^{cl} is indeed an operad.

Theorem 3.5.1 ([3]). *The vector superspaces $P^{cl}(n)$, $n \geq 0$, together with the composition map (3.5.7) and the symmetric group action (3.5.10) form an operad.*

3.5.2 The P^{cl} operad governs Poisson vertex superalgebras

Recall the definition of a Poisson vertex superalgebra.

Definition 3.5.1. *A Poisson vertex superalgebra (abbreviated PVA) is a triple $(V, \partial, \{\cdot_\lambda \cdot\})$, where V is a commutative associative superalgebra, ∂ is an even derivation of V, and $\{\cdot_\lambda \cdot\}$ is a λ-bracket such that the following left Leibniz rule is satisfied:*

$$\{a_\lambda bc\} = \{a_\lambda b\}c + (-1)^{p(b)p(c)}\{a_\lambda c\}b. \tag{3.5.12}$$

As in the $\mathfrak{Chom}$ case in Section 3.3.2, for a vector superspace V with parity p, and an even endomorphism ∂, let $\bar{p} = 1 - p$ denote the reverse parity of p, and ΠV denote the same space as V but with parity $\bar{p}$. For the operad $P^{cl}(\Pi V)$, let $W^{cl}(\Pi V) = W(P^{cl}(\Pi V))$ be the universal Lie superalgebra associated to it, and let

$$\tilde{S} = \{X \in W_1^{cl}(\Pi V) \mid X \text{ is odd and } X \square X = 0\}. \tag{3.5.13}$$

Theorem 3.5.2 ([3])**.** *There is a bijective correspondence between $\tilde{S}$ and the Poisson vertex superalgebra structures on V, given by:*

$$ab = (-1)^{p(a)} X^{\bullet\!\to\!\bullet}(a \otimes b), \quad [a_\lambda b] = (-1)^{p(a)} X_{\overset{\bullet}{\lambda},\overset{\bullet}{-\lambda-\partial}}(a \otimes b), \tag{3.5.14}$$

for any $X \in \tilde{S}$ and $a, b \in V$.

Note that the first sesquilinearity (3.5.4) implies that $X^{\bullet\!\to\!\bullet}_{\lambda_1,\lambda_2}$ only depends on $\lambda_1 + \lambda_2 \equiv -\partial$, hence we can omit λ_1 and λ_2 and just write $X^{\bullet\!\to\!\bullet}$ and $ab \in V$. The proof is given in [3]. A similar proof is given for the Poisson superpseudoalgebra case in Chapter 4 (cf. Theorem 4.2.1).

CHAPTER

4

POISSON PSEUDOALGEBRAS

In this chapter, we state the first main result: the construction of the generalized classical operad P_H^{cl}. The composition in this operad is stated in terms of $\circ$-products, and we prove that P_H^{cl} is indeed an operad with the given composition and symmetric group action. Similar to the LCA and PVA case in Chapter 3, there exists a bijective correspondence between odd elements X in $W_{H,1}^{cl}(\Pi V)$ such that $X \Box X = 0$ and some algebra structures on V, which we call Poisson superpseudoalgebras (abbreviated PPA). At the end of the chapter, we list some examples of PPAs.

4.1 The generalized classical operad P_H^{cl}

4.1.1 Definition of P_H^{cl}

Let H be a cocommutative Hopf algebra that is purely even, and V be a vector superspace with parity p, which is also a left H-module. An element Y of $P_H^{cl}(n)$ is defined to be a collection of maps Y^Γ for any $\Gamma \in G(n)$:

$$Y^\Gamma : V^{\otimes n} \to H^{\otimes s} \otimes_H V,$$

where s is the number of connected components of $\Gamma \in G(n)$. One also imposes that Y^Γ is linear on $V^{\otimes n}$, satisfies the two cycle conditions in (3.5.2) and (3.5.3), and the following componentwise H-linearity: suppose $\Gamma = \Gamma_1 \sqcup \cdots \sqcup \Gamma_s$ where Γ_k is the k-th connected component of Γ, then

$$Y^\Gamma(h_k \cdot (v_1 \otimes \cdots \otimes v_n)) = h_k \cdot Y^\Gamma(v_1 \otimes \cdots \otimes v_n). \tag{4.1.1}$$

On the left-hand side $h_k \cdot (v_1 \otimes \cdots \otimes v_n)$ means h_k acts by coproduct on v_i's where i belongs to the vertex set $V(\Gamma_k)$ of the k-th connected component; explicitly,

$$\begin{aligned} h_k \cdot (v_1 \otimes \cdots \otimes v_n) &= \Delta^{(n_k-1)}(h_k)(v_1 \otimes \cdots \otimes v_n) \\ &= (1 \otimes \cdots \otimes h_{k(1)} \otimes \cdots \otimes h_{k(n_k)} \otimes \cdots \otimes 1)(v_1 \otimes \cdots \otimes v_n). \end{aligned} \tag{4.1.2}$$

Here $\Delta^{(n_k-1)}(h_k)$ is the iterated coproduct (cf. (2.1.4)) of h_k and is identified in $H^{\otimes n}$ by inserting 1's for vertices that are not in $V(\Gamma_k)$ as in the last line of (4.1.2), and n_k is the number of vertices in $V(\Gamma_k)$. For the right-hand side of (4.1.1), we have

$$h_k \cdot Y^\Gamma(v_1 \otimes \cdots \otimes v_n) = ((1 \otimes \cdots \otimes 1 \otimes h_k \otimes 1 \otimes \cdots \otimes 1) \otimes_H 1)\, Y^\Gamma(v_1 \otimes \cdots \otimes v_n), \tag{4.1.3}$$

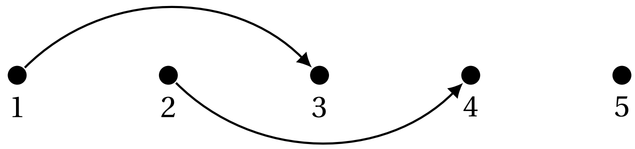
where h_k appears at the k-th position in the tensor product $1 \otimes \cdots \otimes 1 \otimes h_k \otimes 1 \otimes \cdots \otimes 1$.

Now we define the composition, unity, and the action of the symmetric groups S_n. We start with easier ones. The unity is the identity map on V: $1 := \mathrm{id}_V \in P_H^{cl}(1)$ (same as in the classical operad case). The right action of S_n is defined as follows: for a permutation $\sigma \in S_n$ and $\Gamma \in G(n)$, we let $\sigma(\Gamma)$ be the graph obtained by permuting the vertices $\{1, \ldots, n\}$ of Γ using σ, that is relabeling vertex i as $\sigma(i)$ for all $i = 1, \ldots, n$. Then σ induces a permutation $\tilde{\sigma} \in S_s$ on the connected components of Γ. Putting all information together, the action of S_n is given as:

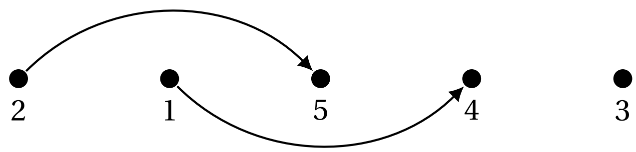
$$(Y^\sigma)^\Gamma(v_1 \otimes \cdots \otimes v_n) = (\tilde{\sigma} \otimes_H 1)\big(Y^{\sigma(\Gamma)}(\sigma(v_1 \otimes \cdots \otimes v_n))\big). \tag{4.1.4}$$

Remark 4.1.1. *If the vertices of Γ are labeled by $v_1, \ldots, v_n \in V$, then Y^σ is obtained by permuting $v_1, \ldots, v_n$ in both the domain and target of Y correspondingly.*

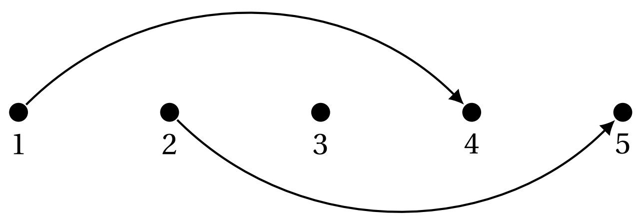
Example 4.1.1. *Consider the graph $\Gamma \in G(5)$ below:*

$\Gamma =$ 

For the permutation $\sigma = (12)(35) \in S_5$*, the graph* $\sigma(\Gamma)$ *is:*

$\sigma(\Gamma) =$ 

After rearranging the vertices from 1 *to* 5*:*

$\sigma(\Gamma) =$ 

Since there are 3 *connected components in* Γ*, the first connected component* $\{1,3\}$ *becomes the second* $\{2,5\}$ *in* $\sigma(\Gamma)$ *after rearrangement; the second connected component* $\{2,4\}$ *becomes the first* $\{1,4\}$*; and the third connected component* $\{5\}$ *stays the third* $\{3\}$*. Thus, the induced permutation of the connected components is* $\widetilde{\sigma} = (12) \in S_3$.

Next, we describe the composition of maps in terms of $\circ$-products (cf. Section 3.1.2). The $\circ_1$-product is defined as follows. Let $X \in P_H^{cl}(m)$, $Y \in P_H^{cl}(n)$, where $m, n \geq 1$. Given a graph $\Gamma \in G(m+n-1)$, suppose that

$$X^{\Delta_1^{m1\dots1}(\Gamma)}(v_1 \otimes \cdots \otimes v_m) = \sum_i (f_{i1} \otimes \cdots \otimes f_{is}) \otimes_H x_i(v_1 \otimes \cdots \otimes v_m), \tag{4.1.5}$$

and

$$Y^{\Delta_0^{m1\dots1}(\Gamma)}(w_1 \otimes \cdots \otimes w_n) = \sum_j (g_{j1} \otimes \cdots \otimes g_{jt}) \otimes_H y_j(w_1 \otimes \cdots \otimes w_n), \tag{4.1.6}$$

where $x_i : V^{\otimes m} \to V$ and $y_j : V^{\otimes n} \to V$ are linear maps, and s, t are the numbers of connected components of $\Delta_1^{m1\dots1}(\Gamma)$ and $\Delta_0^{m1\dots1}(\Gamma)$ respectively. Then

$$\begin{aligned}
&(Y \circ_1 X)^{\Gamma}(v_1 \otimes \cdots \otimes v_{m+n-1})\\
=&(Y(X,1,\dots,1))^{\Gamma}(v_1 \otimes \cdots \otimes v_{m+n-1})\\
=&\sum_{i,j}\Big((f_{i1(1)} \otimes \cdots \otimes f_{is(1)} \otimes 1 \otimes \cdots \otimes 1)(\Delta^{(s-1)}(g_{j1}) \otimes g_{j2} \otimes \cdots \otimes g_{jt})\Big) \otimes_H y_j\big(x_i(v_1 \otimes \cdots\\
&\otimes v_m) \otimes ((f_{i1(-2)} \otimes \cdots \otimes f_{is(-2)}) \cdot (v_{m+1} \otimes \cdots \otimes v_{m+n-1}))\big). \qquad (4.1.7)
\end{aligned}$$

The explanations for the notation used in (4.1.7) are as follows.

1. Recall from (2.1.3) that we have $(\mathrm{id} \otimes S)\Delta(f_{ik}) = f_{ik(1)} \otimes f_{ik(-2)}$. In the special case when no vertex in $\{m+1,\dots,m+n-1\}$ is (externally) connected to any vertex in the k-th ($k \in \{1,\dots,s\}$) connected component of $\Delta_1^{m1\dots1}(\Gamma)$, we set $f_{ik(1)} \otimes f_{ik(-2)} = f_{ik} \otimes 1$.

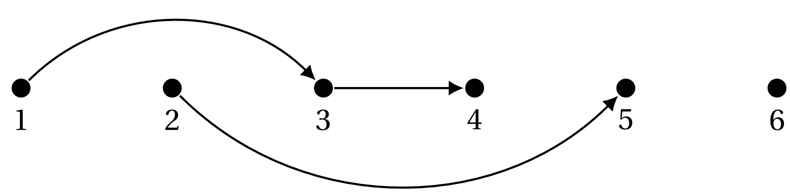
2. Explicitly, $\Delta^{(s-1)}(g_{j1}) = g_{j1(1)} \otimes \cdots \otimes g_{j1(s)}$, which is the iterated coproduct of g_{j1}. In particular, $\Delta^{(0)} = \mathrm{id}$ is the identity map when $s = 1$.

3. Identifying v_i with vertex i, we let $f_{ik(-2)}$ act on the vertices in $\{m+1,\dots,m+n-1\}$ that are (externally) connected to any vertex of the k-th connected component of $\Delta_1^{m1\dots1}(\Gamma)$ by the iterated coproduct.

To better illustrate the formula, consider the example below.

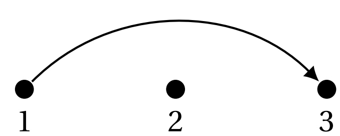
Example 4.1.2. *Let $X \in P_H^{cl}(3)$, $Y \in P_H^{cl}(4)$, and the graph $\Gamma \in G(6)$ be shown below:*

$\Gamma =$

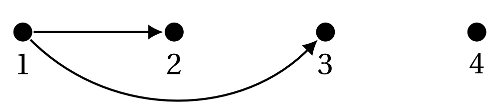


As defined in Section 3.4.1, we have:

$\Gamma_1 := \Delta_1^{3111}(\Gamma) =$



$\Gamma_0 := \Delta_0^{3111}(\Gamma) =$



The numbers of connected components in Γ_1 and Γ_0 are both 2. In addition, the vertex that is externally connected to (any vertex of) the first connected component of Γ_1 is v_4, and v_5 for the second. Suppose that

$$X^{\Gamma_1}(v_1\otimes v_2\otimes v_3)=\sum_i (f_{i1}\otimes f_{i2})\otimes_H x_i(v_1\otimes v_2\otimes v_3), \tag{4.1.8}$$

and

$$Y^{\Gamma_0}(w_1\otimes\cdots\otimes w_4)=\sum_j (g_{j1}\otimes g_{j2})\otimes_H y_j(w_1\otimes\cdots\otimes w_4). \tag{4.1.9}$$

Then

$$\begin{aligned}(Y\circ_1 X)^{\Gamma}(v_1\otimes\cdots\otimes v_6)=\sum_{i,j}&\big((f_{i1(1)}\otimes f_{i2(1)}\otimes 1)(\Delta(g_{j1})\otimes g_{j2})\big)\\ &\otimes_H y_j(x_i(v_1\otimes v_2\otimes v_3)\otimes f_{i1(-2)}v_4\otimes f_{i2(-2)}v_5\otimes v_6).\end{aligned} \tag{4.1.10}$$

Remark 4.1.2. (4.1.7) *can be extended to the case when $X\in P_H^{cl}(0)$, in which X can be regarded as a vector in V. If we define $\Delta^{(-1)}=\epsilon$ to be the counit map, then for $X\in P_H^{cl}(0)$, $Y\in P_H^{cl}(n)$, and $\Gamma\in G(n-1)$, equation* (4.1.7) *becomes:*

$$(Y\circ_1 X)^{\Gamma}(v_1\otimes\cdots\otimes v_{n-1})=\sum_j \epsilon(g_{j1})(g_{j2}\otimes\cdots\otimes g_{jt})\otimes_H y_j(X\otimes v_1\otimes\cdots\otimes v_{n-1}). \tag{4.1.11}$$

To see that (4.1.7) is well defined, we have the following lemmas.

Lemma 4.1.1. *Suppose that $\Gamma\in G_0(m+n-1)$ and $\Delta_0^{m1\dots1}(\Gamma)\in G_0(n)$. If there are s connected components in $\Delta_1^{m1\dots1}(\Gamma)$ and t connected components in $\Delta_0^{m1\dots1}(\Gamma)$, then there are $s+t-1$ connected components in Γ.*

Proof. By assumption, there are t connected components in $\Delta_0^{m1\dots1}(\Gamma)$, of which the first one corresponds to the connected components in Γ that are connected to some vertex of $\Delta_1^{m1\dots1}(\Gamma)$. The other $t-1$ connected components in $\Delta_0^{m1\dots1}(\Gamma)$ are disjoint from $\Delta_1^{m1\dots1}(\Gamma)$, hence they remain the same in Γ.

Consider the first connected component of $\Delta_0^{m1\dots1}(\Gamma)$. Note that this first connected component will split into at most s connected components in Γ, since each of its vertices is connected to some connected component of $\Delta_1^{m1\dots1}(\Gamma)$. On the other hand, we claim that the first connected component in $\Delta_0^{m1\dots1}(\Gamma)$ will split into at least s connected components in Γ. Indeed, if it is less than s, there will be a path in Γ from one of the s connected components of $\Delta_1^{m1\dots1}(\Gamma)$ to another, and this path will have to first traverse outside $\Delta_1^{m1\dots1}(\Gamma)$ and then back to another connected component of $\Delta_1^{m1\dots1}(\Gamma)$. But this contradicts with the condition that $\Delta_0^{m1\dots1}(\Gamma) \in G_0(n)$. $\square$

Lemma 4.1.2. *In* (4.1.7)*, the actions of $f_{ij(-2)}$ and $f_{ik(-2)}$ are disjoint if $j \neq k$ and $\Delta_0^{m1\dots1}(\Gamma) \in G_0(n)$.*

Proof. Note that $1 \leq j, k \leq m$. If there exits some vertex $l \in \{m+1, \dots, m+n-1\}$ that both $f_{ij(-2)}$ and $f_{ik(-2)}$ $(k \neq j)$ act on, then l is connected to both j and k, which are identified as the same vertex in $\Delta_0^{m1\dots1}(\Gamma)$. This gives a cycle in $\Delta_0^{m1\dots1}(\Gamma)$, which contradicts that $\Delta_0^{m1\dots1}(\Gamma) \in G_0(n)$. $\square$

Lemma 4.1.2 indicates that the order of the actions of $f_{ik(-2)}$'s does not matter in (4.1.7). Next we show that $Y \circ_1 X \in P_H^{cl}(m+n-1)$.

Proposition 4.1.1. *$Y \circ_1 X$, defined by* (4.1.7)*, is an element of $P_H^{cl}(m+n-1)$.*

Proof. Suppose that $\Gamma \in G(m+n-1)$ has a cycle C. If $C \subseteq \Delta_1^{m1\dots1}(\Gamma)$, then $X^{\Delta_1^{m1\dots1}(\Gamma)} = 0$ since $X \in P_H^{cl}(m)$; thus $(Y \circ_1 X)^\Gamma = 0$ by (4.1.7). Otherwise, C corresponds to a cycle $C' \in \Delta_0^{m1\dots1}(\Gamma)$ (cf. Lemma 3.4.2 and ignore the orientation). This makes $Y^{\Delta_0^{m1\dots1}(\Gamma)} = 0$ since $Y \in P_H^{cl}(n)$; thus $(Y \circ_1 X)^\Gamma = 0$ by (4.1.7). This shows that $(Y \circ_1 X)$ satisfies the first cycle condition.

For the second cycle condition, consider a cycle $C \subseteq \Delta_1^{m1\dots1}(\Gamma)$. For any edge $e \in C$, let $\Gamma \backslash e$ be the graph obtained by removing e from graph Γ. Then $\Delta_0^{m1\dots1}(\Gamma \backslash e)$ is the same for all e; hence (4.1.6) is fixed for all e. While X will be evaluated on graphs $\Delta_1^{m1\dots1}(\Gamma \backslash e) = \Delta_1^{m1\dots1}(\Gamma) \backslash e$, and removing an edge $e \in C$ from $\Delta_1^{m1\dots1}(\Gamma)$ does not change the number of connected components of $\Delta_1^{m1\dots1}(\Gamma)$, in formula (4.1.7), only the part involving f_{ik} and x_i may vary with different e. However, since the transformation $(\mathrm{id} \otimes S)\Delta$ is linear on f_{ik}, we can evaluate it on all $X^{\Delta_1^{m1\dots1}(\Gamma) \backslash e}$ together before composing with $Y^{\Delta_0^{m1\dots1}(\Gamma)}$. Since $\sum_{e \in C} X^{\Delta_1^{m1\dots1}(\Gamma) \backslash e} = 0$, we get $\sum_{e \in C} (Y \circ_1 X)^{\Gamma \backslash e} = 0$.

If the cycle C is not contained in $\Delta_1^{m1\dots1}(\Gamma)$, then $\Delta(C)$ is still a cycle in $\Delta_0^{m1\dots1}(\Gamma \backslash e)$ for any edge $e \in E(\Delta_1^{m1\dots1}(\Gamma))$. For such e, $(Y \circ_1 X)^{\Gamma \backslash e} = 0$ since $Y^{\Delta_0^{m1\dots1}(\Gamma \backslash e)} = 0$. Thus, $\sum_{e \in C} (Y \circ_1 X)^{\Gamma \backslash e} =$

$\sum_{e\in(C\cap\Delta_0^{m1\ldots 1}(\Gamma))}(Y\circ_1 X)^{\Gamma\backslash e}$. Note $\Delta_1^{m1\ldots 1}(\Gamma\backslash e)$ is the same for all $e\in(C\cap\Delta_0^{m1\ldots 1}(\Gamma))$, which implies by (4.1.7) that only the g_{jk} and y_j parts may vary with different $e\in(C\cap\Delta_0^{m1\ldots 1}(\Gamma))$. The vertices externally connected to any connected component of $\Delta_1^{m1\ldots 1}(\Gamma\backslash e)$ stay the same for all $e\in(C\cap\Delta_0^{m1\ldots 1}(\Gamma))$. Since $\Delta^{(s-1)}$ is linear and $\sum_{e\in(C\cap\Delta_0^{m1\ldots 1}(\Gamma))} Y^{\Delta_0^{m1\ldots 1}(\Gamma)\backslash e}=0$, we obtain $\sum_{e\in(C\cap\Delta_0^{m1\ldots 1}(\Gamma))}(Y\circ_1 X)^{\Gamma\backslash e}=0$. This shows that the second cycle condition is satisfied.

It is left to check the componentwise H-linearity of $Y\circ_1 X$. Pick a connected component K of $\Gamma\in G(m+n-1)$. If K is contained in the subgraph formed by vertices $\{m+1,\ldots,m+n-1\}$ and is the k-th connected component in $\Delta_0^{m1\ldots 1}(\Gamma)$ ($k>1$ in this case), we have:

$$(Y\circ_1 X)^{\Gamma}(h_K\cdot(v_1\otimes\cdots\otimes v_{m+n-1}))=(Y\circ_1 X)^{\Gamma}(v_1\otimes\cdots\otimes v_m\otimes h_K\cdot(v_{m+1}\otimes\cdots\otimes v_{m+n-1})). \tag{4.1.12}$$

The componentwise H-linearity of Y indicates:

$$\begin{aligned}Y^{\Delta_0^{m1\ldots 1}(\Gamma)}(w_1\otimes h_K\cdot(w_2\otimes\cdots\otimes w_n))&=h_K\cdot Y^{\Delta_0^{m1\ldots 1}(\Gamma)}(w_1\otimes w_2\otimes\cdots\otimes w_n)\\&=\sum_j(g_{j1}\otimes\cdots\otimes h_K g_{jk}\otimes\cdots\otimes g_{jt})\otimes_H y_j(w_1\otimes\cdots\otimes w_n),\end{aligned} \tag{4.1.13}$$

where h_K acts only on g_{jk}. Together with (4.1.5), this implies that

$$\begin{aligned}&(Y\circ_1 X)^{\Gamma}(h_K\cdot(v_1\otimes\cdots\otimes v_{m+n-1}))\\=&\sum_{i,j}\big((f_{i1(1)}\otimes\cdots\otimes f_{is(1)}\otimes 1\otimes\cdots\otimes 1)(\Delta^{(s-1)}(g_{j1})\otimes g_{j2}\otimes\cdots\otimes h_K g_{jk}\otimes\cdots\otimes g_{jt})\big)\\&\otimes_H y_j\big(x_i(v_1\otimes\cdots\otimes v_m)\otimes((f_{i1(-2)}\otimes\cdots\otimes f_{is(-2)})\cdot(v_{m+1}\otimes\cdots\otimes v_{m+n-1}))\big)\\=&h_K\cdot((Y\circ_1 X)^{\Gamma}(v_1\otimes\cdots\otimes v_{m+n-1})).\end{aligned} \tag{4.1.14}$$

Next consider the case when K intersects both subgraphs formed by the first m vertices and the last $n-1$ vertices. Suppose that K is the k-th connected component of $\Delta_1^{m1\ldots 1}(\Gamma)$ (also the k-th of Γ in this case). Let K_1 be the part of K that intersects with $\Delta_1^{m1\ldots 1}(\Gamma)$, and K_2 be the part of K that intersects with the subgraph formed by the last $n-1$ vertices and the edges among them. We have:

$$(Y\circ_1 X)^{\Gamma}(h_K\cdot(v_1\otimes\cdots\otimes v_{m+n-1}))=(Y\circ_1 X)^{\Gamma}(h_{K_1}\cdot(v_1\otimes\cdots\otimes v_m)\otimes h_{K_2}\cdot(v_{m+1}\otimes\cdots\otimes v_{m+n-1})). \tag{4.1.15}$$

By the coassociativity of the coproduct, one can choose $h_{K_1} = h_{K(1)}$ and $h_{K_2} = h_{K(2)}$ from $\Delta(h_K) = h_{K(1)} \otimes h_{K(2)}$. Both X and Y satisfy componentwise H-linearity:

$$\begin{aligned} X^{\Delta_1^{m1\ldots1}(\Gamma)}(h_{K(1)} \cdot (v_1 \otimes \cdots \otimes v_m)) &= h_{K(1)} \cdot X^{\Delta_1^{m1\ldots1}(\Gamma)}(v_1 \otimes \cdots \otimes v_m) \\ &= \sum_i (f_{i1} \otimes \cdots \otimes h_{K(1)} f_{ik} \otimes \cdots \otimes f_{is}) \otimes_H x_i(v_1 \otimes \cdots \otimes v_m) \end{aligned} \tag{4.1.16}$$

where h_{K_1} only acts on $f_{ik(1)}$, and

$$\begin{aligned} & Y^{\Delta_0^{m1\ldots1}(\Gamma)}(x_i(v_1 \otimes \cdots \otimes v_m) \otimes (h_{K(2)} \cdot (v_{m+1} \cdots \otimes v_{m+n-1}))) \\ = & \sum_j (g_{j1} \otimes \cdots \otimes g_{jt}) \otimes_H y_j(x_i(v_1 \otimes \cdots \otimes v_m) \otimes (h_{K(2)} \cdot (v_{m+1} \otimes \cdots \otimes v_{m+n-1}))). \end{aligned} \tag{4.1.17}$$

From here we obtain:

$$\begin{aligned} & (Y \circ_1 X)^\Gamma (h_K \cdot (v_1 \otimes \cdots \otimes v_{m+n-1})) \\ = & \sum_{i,j} \big((f_{i1(1)} \otimes \cdots \otimes h_{K(1)(1)} f_{ik(1)} \otimes \cdots \otimes f_{is(1)} \otimes 1 \otimes \cdots \otimes 1)(\Delta^{(s-1)}(g_{j1}) \otimes g_{j2} \otimes \cdots \otimes g_{jt}) \big) \otimes_H y_j \big(x_i(v_1 \otimes \\ & \cdots \otimes v_m) \otimes ((f_{i1(-2)} \otimes \cdots \otimes (f_{ik(-2)} h_{K(1)(-2)}) \otimes \cdots \otimes f_{is(-2)}) \cdot (h_{K(2)} \cdot (v_{m+1} \otimes \cdots \otimes v_{m+n-1}))) \big) \end{aligned} \tag{4.1.18}$$

By (2.1.8), we have:

$$h_{K(1)(1)} \otimes h_{K(1)(-2)} h_{K(2)} = h_{K(1)} \otimes h_{K(-2)} h_{K(3)} = h_K \otimes 1. \tag{4.1.19}$$

Since the actions of $f_{il(-2)}$'s are disjoint, we can replace $h_{K(1)(1)}$ with h_K, and replace $h_{K(1)(-2)}$ and $h_{K(2)}$ with 1, which makes (4.1.18) as desired:

$$(Y \circ_1 X)^\Gamma (h_K \cdot (v_1 \otimes \cdots \otimes v_{m+n-1})) = h_K \cdot ((Y \circ_1 X)^\Gamma (v_1 \otimes \cdots \otimes v_{m+n-1})). \tag{4.1.20}$$

In the special case when K is contained in $\Delta_1^{m1\ldots1}(\Gamma)$, we have $h_{K(1)(1)} = h_K$, $h_{K(1)(-2)} = 1$, and $h_{K(2)} = 1$ in (4.1.18) (cf. the first explanation of the notations below (4.1.7)). Then (4.1.20) just follows from the componentwise H-linearity of X. □

Similarly to the $\circ_1$-product, we can define $\circ_k$-products for $2 \le k \le n$ as follows. Suppose that $X \in P_H^{cl}(m)$, $Y \in P_H^{cl}(n)$, and $\Gamma \in G(m+n-1)$. Now X will be evaluated on $\Delta_k^{1\ldots1m1\ldots1}(\Gamma)$

where m shows up at the k-th position. We write:

$$X^{\Delta_k^{1\ldots1m1\ldots1}(\Gamma)}(v_1\otimes\cdots\otimes v_m)=\sum_i (f_{i1}\otimes\cdots\otimes f_{is})\otimes_H x_i(v_1\otimes\cdots\otimes v_m), \tag{4.1.21}$$

and

$$Y^{\Delta_0^{1\ldots1m1\ldots1}(\Gamma)}(w_1\otimes\cdots\otimes w_n)=\sum_j (g_{j1}\otimes\cdots\otimes g_{jt})\otimes_H y_j(w_1\otimes\cdots\otimes w_n), \tag{4.1.22}$$

where now s denotes the number of connected components of the subgraph $\Delta_k^{1\ldots1m1\ldots1}(\Gamma)$, t denotes the number of connected components of $\Delta_0^{1\ldots1m1\ldots1}(\Gamma)$, and x_i, y_j are linear maps. Suppose that the vertex in $\Delta_0^{1\ldots1m1\ldots1}(\Gamma)$ obtained by clasping the m vertices $\{k,\ldots,k+m-1\}$ in Γ is contained in the q-th connected component of $\Delta_0^{1\ldots1m1\ldots1}(\Gamma)$. Then the $\circ_k$-product is defined as follows:

$$\begin{aligned}
&(Y\circ_k X)^{\Gamma}(v_1\otimes\cdots\otimes v_{m+n-1})\\
=&(Y(1,\ldots,1,X,1,\ldots,1))^{\Gamma}(v_1\otimes\cdots\otimes v_{m+n-1})\\
=&(-1)^{p_k^X}\sum_{i,j}\rho_{0,k}^{\Gamma}\big((1\otimes\cdots\otimes1\otimes f_{i1(1)}\otimes\cdots\otimes f_{is(1)}\otimes1\otimes\cdots\otimes1)(g_{j1}\otimes g_{j2}\otimes\cdots\otimes\Delta^{(s-1)}(g_{jq})\otimes\cdots\otimes g_{jt})\big)\\
&\otimes_H y_j\big((f_{i1(-2)}\otimes\cdots\otimes f_{is(-2)})\cdot(v_1\otimes\cdots\otimes v_{k-1}\otimes x_i(v_k\otimes\cdots\otimes v_{k+m-1})\otimes v_{k+m}\otimes\cdots\otimes v_{m+n-1})\big),
\end{aligned} \tag{4.1.23}$$

where the sign $(-1)^{p_k^X}$ follows from the Koszul-Quillen rule:

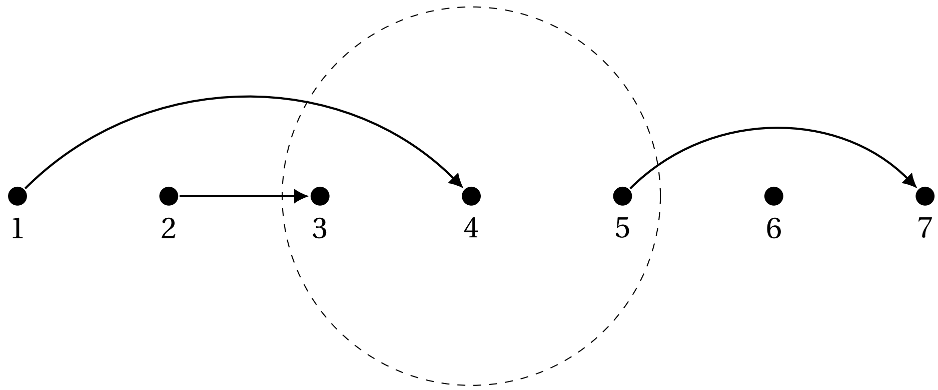
$$(-1)^{p_k^X}=(-1)^{p(X)(p(v_1)+\cdots+p(v_{k-1}))}, \tag{4.1.24}$$

and $\rho_{0,k}^{\Gamma}$ is the map identifying the connected components of $\Delta_0^{1\ldots1m1\ldots1}(\Gamma)$ and $\Delta_k^{1\ldots1m1\ldots1}(\Gamma)$ as connected components in Γ. The explanations below (4.1.7) also work here except that we focus on $\Delta_k^{1\ldots1m1\ldots1}(\Gamma)$ now instead of $\Delta_1^{m1\ldots1}(\Gamma)$. In addition, in (4.1.23), we insert $q-1$ 1's before $f_{i1(1)}$, and the $f_{il(-2)}$'s act only on the vertices $\{1,\ldots,k-1\}\cup\{k+m,\ldots,m+n-1\}$, which is consistent with the $\circ_1$-product case.

We compute $(Y\circ_3 X)^{\Gamma}$ in the following example.

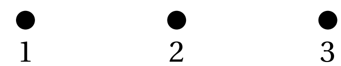
Example 4.1.3. *Let $X\in P_H^{cl}(3)$ and $Y\in P_H^{cl}(5)$, with the graph $\Gamma\in G(7)$ as shown below:*

$\Gamma =$

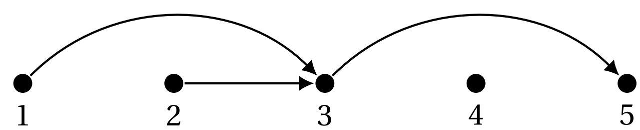


By definition of the cocomposition map, we have

$\Gamma_3 := \Delta_3^{11311}(\Gamma) =$



$\Gamma_0 := \Delta_0^{11311}(\Gamma) =$



Suppose that

$$X^{\Gamma_3}(v_1 \otimes v_2 \otimes v_3) = \sum_i (f_{i1} \otimes f_{i2} \otimes f_{i3}) \otimes_H x_i(v_1 \otimes v_2 \otimes v_3), \tag{4.1.25}$$

and

$$Y^{\Gamma_0}(w_1 \otimes \cdots \otimes w_5) = \sum_j (g_{j1} \otimes g_{j2}) \otimes_H y_j(w_1 \otimes \cdots \otimes w_5). \tag{4.1.26}$$

By (4.1.23)*, one computes:*

$$
\begin{aligned}
(Y\circ_3 X)^{\Gamma} = & (-1)^{p_3^X}\sum_{i,j}\rho_{0,3}^{\Gamma}\big((f_{i1(1)}\otimes f_{i2(1)}\otimes f_{i3(1)}\otimes 1)(g_{j1(1)}\otimes g_{j1(2)}\otimes g_{j1(3)}\otimes g_{j2})\big)\otimes_H \\
& y_j\big((f_{i1(-2)}\otimes f_{i2(-2)}\otimes f_{i3(-2)})\cdot(v_1\otimes v_2\otimes x_i(v_3\otimes v_4\otimes v_5)\otimes v_6\otimes v_7)\big) \\
= & (-1)^{p_3^X}\sum_{i,j}\big((f_{i2(1)}\otimes f_{i1(1)}\otimes f_{i3(1)}\otimes 1)(g_{j1(2)}\otimes g_{j1(1)}\otimes g_{j1(3)}\otimes g_{j2})\big)\otimes_H \\
& y_j((f_{i2(-2)}\cdot v_1)\otimes(f_{i1(-2)}\cdot v_2)\otimes x_i(v_3\otimes v_4\otimes v_5)\otimes v_6\otimes(f_{i3(-2)}\cdot v_7)). \qquad (4.1.27)
\end{aligned}
$$

In this example, $\rho_{0,3}^{\Gamma}$ *identifies the* 1*-st connected component of* Γ_3 *as the* 2*-nd connected component of* Γ*, and identifies the* 2*-nd connected component of* Γ_0 *as the* 4*-th connected component of* Γ*, etc., which explains why we get* $(f_{i2(1)}\otimes f_{i1(1)}\otimes f_{i3(1)}\otimes 1)(g_{j1(2)}\otimes g_{j1(1)}\otimes g_{j1(3)}\otimes g_{j2})$ *in* (4.1.27).

One can show the following results are also true analogously to the proofs of Lemma 4.1.1, Lemma 4.1.2 and Proposition 4.1.1. The $\circ_k$-product can be thought of as a $\circ_1$-product (possibly up to a sign change) after relabeling the vertices of Γ.

Lemma 4.1.3. *Let* $\Gamma\in G_0(m+n-1)$ *and* $\Delta_0^{1\ldots 1m1\ldots 1}(\Gamma)\in G_0(n)$*. If there are* s *connected components in* $\Delta_k^{1\ldots 1m1\ldots 1}(\Gamma)$ *and* t *connected components in* $\Delta_0^{1\ldots 1m1\ldots 1}(\Gamma)$*, then there are* $s+t-1$ *connected components in* Γ*.*

Lemma 4.1.4. *In* (4.1.23)*, the actions of* $f_{ij(-2)}$ *and* $f_{il(-2)}$ *are disjoint if* $j\neq l$ *and* $\Delta_0^{1\ldots 1m1\ldots 1}(\Gamma)\in G_0(n)$*.*

Proposition 4.1.2. $Y\circ_k X$*, defined by* (4.1.23)*, is an element of* $P_H^{cl}(m+n-1)$*.*

Now we have definitions for all $\circ_k$-products for any integer $k\geq 1$. Then, for any $f\in P_H^{cl}(n)$ and $g_l\in P_H^{cl}(m_l)$, we define the composition by (3.1.12). We will show in the following subsection that this definition makes sense by verifying the associativity axiom.

4.1.2 P_H^{cl} is an operad

In this subsection, we prove our first main theorem.

Theorem 4.1.1. *The vector superspaces* $P_H^{cl}(n)$*,* $n\geq 0$*, the actions of the symmetric groups* S_n *given by* (4.1.4)*, and the composition maps given by* (3.1.12)*, form an operad.*

To prove the theorem, we will verify (3.1.13), (3.1.14) and (3.1.15), which will be done in the following lemmas. The unity axiom is obvious, so we focus on the associativity and equivariance axioms. For the notation, let $Z \in P_H^{cl}(l)$, $Y \in P_H^{cl}(n)$, and $X \in P_H^{cl}(m)$ throughout the rest of this subsection.

Lemma 4.1.5. *If* $1 \le j < i$*, then* $(Z \circ_i Y) \circ_j X = (-1)^{p(Y)p(X)}(Z \circ_j X) \circ_{i+m-1} Y$.

Proof. By Proposition 4.1.2, both $(Z \circ_i Y) \circ_j X$ and $(Z \circ_j X) \circ_{i+m-1} Y$ are elements of $P_H^{cl}(m+n+l-2)$. Given a graph $\Gamma \in G(m+n+l-2)$, let $\Gamma_2 := \Delta_j^{1\dots 1m1\dots 1}(\Gamma) \in G(m)$ and $\overline{\Gamma}_1 := \Delta_0^{1\dots 1m1\dots 1}(\Gamma) \in G(n+l-1)$ where m appears at the j-th position; let $\Gamma_1 := \Delta_i^{1\dots 1n1\dots 1}(\overline{\Gamma}_1) \in G(n)$ and $\Gamma_0 := \Delta_0^{1\dots 1n1\dots 1}(\overline{\Gamma}_1) \in G(l)$ where n appears at the i-th position. Note that in order to compute $((Z \circ_i Y) \circ_j X)^{\Gamma}$, we need to evaluate X^{Γ_2}, Y^{Γ_1} and Z^{Γ_0}. Assume that there are r, t, s connected components in $\Gamma_0, \Gamma_1, \Gamma_2$, respectively. Suppose that

$$X^{\Gamma_2}(v_1 \otimes \cdots \otimes v_m) = \sum_{\alpha} (f_{\alpha 1} \otimes \cdots \otimes f_{\alpha s}) \otimes_H x_\alpha(v_1 \otimes \cdots \otimes v_m), \tag{4.1.28}$$

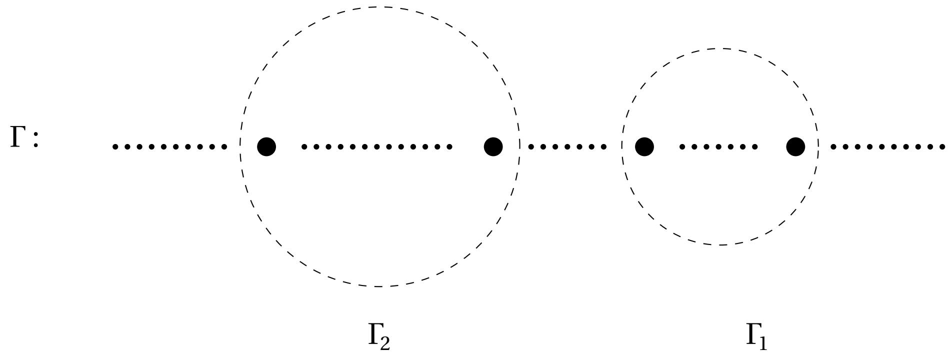
$$Y^{\Gamma_1}(w_1 \otimes \cdots \otimes w_n) = \sum_{\beta} (g_{\beta 1} \otimes \cdots \otimes g_{\beta t}) \otimes_H y_\beta(w_1 \otimes \cdots \otimes w_n), \tag{4.1.29}$$

$$Z^{\Gamma_0}(u_1 \otimes \cdots \otimes u_l) = \sum_{\gamma} (h_{\gamma 1} \otimes \cdots \otimes h_{\gamma r}) \otimes_H z_\gamma(u_1 \otimes \cdots \otimes u_l). \tag{4.1.30}$$

If Γ_1 is clasped into a vertex of the q_1-th connected component of Γ_0, then we compute by (4.1.23):

$$\begin{aligned}
&(Z \circ_i Y)^{\overline{\Gamma}_1}(v_1 \otimes \cdots \otimes v_{n+l-1}) \\
=&(-1)^{p_i^Y} \sum_{\beta,\gamma} \rho_{0,i}^{\overline{\Gamma}_1}\big((1 \otimes \cdots \otimes 1 \otimes g_{\beta 1(1)} \otimes \cdots \otimes g_{\beta t(1)} \otimes 1 \cdots \otimes 1)(h_{\gamma 1} \otimes \cdots \otimes \Delta^{(t-1)}(h_{\gamma q_1}) \otimes \cdots \otimes h_{\gamma r})\big) \otimes_H \\
&z_\gamma\big((g_{\beta 1(-2)} \otimes \cdots \otimes g_{\beta t(-2)}) \cdot (v_1 \otimes \cdots \otimes v_{i-1} \otimes y_\beta(v_i \otimes \cdots \otimes v_{i+n-1}) \otimes v_{i+n} \otimes \cdots \otimes v_{n+l-1})\big),
\end{aligned} \tag{4.1.31}$$

where p_i^Y is given by (4.1.24). To further compute $((Z \circ_i Y) \circ_j X)^{\Gamma}$, we need to consider the relations between Γ_1 and Γ_2; hence we split into the following two cases. For the first case, assume that Γ_1 and Γ_2 are disjoint after they are clasped into two vertices in Γ_0, that is there exists no path in Γ connecting any two connected components of Γ_1 and Γ_2 as shown in the graph below. The graph does not show any edges that are possible: only edges between Γ_2 and Γ_1 are not allowed.



In this case, if Γ_2 is clasped into a vertex of the $\overline{q}_2$-th connected component of $\overline{\Gamma}_1$, then $\overline{q}_2$ corresponds to some q_2-th connected component in Γ_0 that is different from the q_1-th. We have

$$
\begin{aligned}
&((Z\circ_i Y)\circ_j X)^{\Gamma}(v_1\otimes\cdots\otimes v_{m+n+l-2})\\
=&(-1)^{p^Y_{i+m-1}+p^X_j}\sum_{\alpha,\beta,\gamma}\rho^{\Gamma}_{0,j}\big((1\otimes\cdots\otimes 1\otimes f_{\alpha 1(1)}\otimes\cdots\otimes f_{\alpha s(1)}\otimes 1\cdots\otimes 1)(\Delta^{(s-1)}_{\overline{q}_2}\cdot\rho^{\overline{\Gamma}_1}_{0,i}((1\otimes\cdots\otimes 1\\
&\otimes g_{\beta 1(1)}\otimes\cdots\otimes g_{\beta t(1)}\otimes 1\cdots\otimes 1)(h_{\gamma 1}\otimes\cdots\otimes\Delta^{(t-1)}(h_{\gamma q_1})\otimes\cdots\otimes h_{\gamma r}))))\otimes_H z_\gamma\big((f_{\alpha 1(-2)}\otimes\cdots\\
&\otimes f_{\alpha s(-2)})\cdot(g_{\beta 1(-2)}\otimes\cdots\otimes g_{\beta t(-2)})\cdot(v_1\otimes\cdots\otimes v_{j-1}\otimes x_\alpha(v_j\otimes\cdots\otimes v_{j+m-1})\otimes v_{j+m}\otimes\cdots\otimes v_{i+m-2}\otimes\\
&y_\beta(v_{i+m-1}\otimes\cdots\otimes v_{i+m+n-2})\otimes v_{i+m+n-1}\otimes\cdots\otimes v_{m+n+l-2})\big).
\end{aligned}
\tag{4.1.32}
$$

Note that $g_{\beta p(1)}$ multiplies with $h_{\gamma q_1(p)}$ from $\Delta^{(t-1)}(h_{\gamma q_1})$, and $\Delta^{(s-1)}_{\overline{q}_2}$ only acts on the $\overline{q}_2$-th connected component after the permutation $\rho^{\overline{\Gamma}_1}_{0,i}$. Since Γ_1 and Γ_2 are disconnected in Γ, $\overline{q}_2$ is not any connected component corresponding to $h_{\gamma q_1(p)}$ for any p. Similarly, one sees that $f_{\alpha k(1)}$ multiplies with $h_{\gamma a(k)}$ from $\Delta^{(s-1)}(h_{\gamma a})$ (here $h_{\gamma a}$ represents the connected component in Γ_0 that becomes the $\overline{q}_2$-th connected component in $\overline{\Gamma}_1$ under $\rho^{\overline{\Gamma}_1}_{0,i}$), while $f_{\alpha k(1)}$'s and $g_{\beta p(1)}$'s do not multiply together. Eventually, $\rho^{\Gamma}_{0,j}$ permutes the connected components to the right positions in Γ. In (4.1.31), $g_{\beta p(-2)}$'s act only on vectors that are not within y_β. They do not act on vectors within x_α in (4.1.32) either; otherwise we have a path between Γ_1 and Γ_2 in Γ. Thus, $g_{\beta p(-2)}$'s only act on vectors not in x_α and y_β, which is also true for $f_{\alpha k(-2)}$'s. The last observation is that the vectors not within x_α and y_β can have at most one action: either from $g_{\beta p(-2)}$ or $f_{\alpha k(-2)}$; otherwise we have a path between Γ_1 and Γ_2 in Γ. These observations imply that we do not need to worry about the commutativity of H.

Next, we compute the right hand side $(Z\circ_j X)\circ_{i+m-1} Y$. Note that X will always be evaluated on Γ_1, Y on Γ_2, and Z on Γ_0. Let $\widetilde{\Gamma}_1=\Delta_0^{1\ldots 1n1\ldots 1}(\Gamma)$, where n appears at the $(i+m-1)$-

th position. Note that Γ_2 will be clasped into a vertex in the q_2-th connected component of Γ_0, one has

$$
\begin{aligned}
&(Z\circ_j X)^{\widetilde{\Gamma}_1}(v_1\otimes\cdots\otimes v_{m+l-1})\\
=&(-1)^{p_j^X}\sum_{\beta,\gamma}\rho_{0,j}^{\widetilde{\Gamma}_1}\big((1\otimes\cdots\otimes 1\otimes f_{\alpha 1(1)}\otimes\cdots\otimes f_{\alpha s(1)}\otimes 1\otimes\cdots\otimes 1)(h_{\gamma 1}\otimes\cdots\otimes\Delta^{(s-1)}(h_{\gamma q_2})\otimes\cdots\otimes h_{\gamma r})\big)\otimes_H\\
&z_\gamma\big((f_{\alpha 1(-2)}\otimes\cdots\otimes f_{\alpha s(-2)})\cdot(v_1\otimes\cdots\otimes v_{j-1}\otimes x_\alpha(v_j\otimes\cdots\otimes v_{j+m-1})\otimes v_{j+m}\otimes\cdots\otimes v_{m+l-1})\big).
\end{aligned}
\tag{4.1.33}
$$

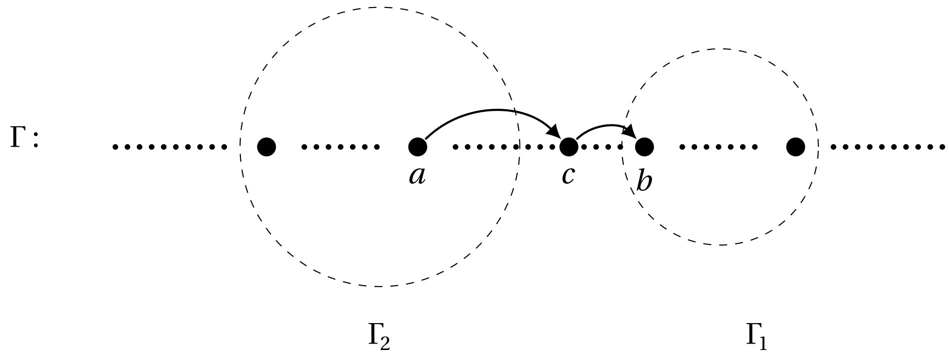
Suppose that Γ_1 is clasped into a vertex in the $\widetilde{q}_1$-th connected component of $\widetilde{\Gamma}_1$. This will correspond to the q_1-th connected component of Γ_0 as for the left-hand side. Thus

$$
\begin{aligned}
&((Z\circ_j X)\circ_{i+m-1} Y)^{\Gamma}(v_1\otimes\cdots\otimes v_{m+n+l-2})\\
=&(-1)^{p_{i+m-1}^Y+p_j^X+p(Y)p(X)}\sum_{\alpha,\beta,\gamma}\rho_{0,i+m-1}^{\Gamma}\big((1\otimes\cdots\otimes 1\otimes g_{\beta 1(1)}\otimes\cdots\otimes g_{\beta t(1)}\otimes 1\otimes\cdots\otimes 1)(\Delta_{\widetilde{q}_1}^{(t-1)}\cdot\rho_{0,j}^{\widetilde{\Gamma}_1}((1\otimes\\
&\cdots\otimes 1\otimes f_{\alpha 1(1)}\otimes\cdots\otimes f_{\alpha s(1)}\otimes 1\cdots\otimes 1)(h_{\gamma 1}\otimes\cdots\otimes\Delta^{(s-1)}(h_{\gamma q_2})\otimes\cdots\otimes h_{\gamma r})))\big)\otimes_H z_\gamma\big((f_{\alpha 1(-2)}\otimes\cdots\\
&\otimes f_{\alpha s(-2)})\cdot(g_{\beta 1(-2)}\otimes\cdots\otimes g_{\beta t(-2)})\cdot(v_1\otimes\cdots\otimes v_{j-1}\otimes x_\alpha(v_j\otimes\cdots\otimes v_{j+m-1})\otimes v_{j+m}\otimes\cdots\otimes v_{i+m-2}\\
&\otimes y_\beta(v_{i+m-1}\otimes\cdots\otimes v_{i+m+n-2})\otimes v_{i+m+n-1}\otimes\cdots\otimes v_{m+n+l-2})\big).
\end{aligned}
\tag{4.1.34}
$$

The action of $f_{\alpha k(-2)}$'s and $g_{\beta p(-2)}$'s on the vectors are the same as in (4.1.32); hence the z_γ part (the vector part) is the same as (4.1.32). For the coefficients representing each connected component, we have the following observations: $h_{\gamma b}$ (here $h_{\gamma b}$ represents the connected component in Γ_0 that becomes the $\widetilde{q}_1$-th connected component in $\widetilde{\Gamma}_1$ under $\rho_{0,j}^{\widetilde{\Gamma}_1}$) is not in $\Delta^{(s-1)}(h_{\gamma q_2})$ since Γ_1 and Γ_2 are disjoint; $\Delta^{(s-1)}(h_{\gamma q_2})$ multiplies with $f_{\alpha 1(1)}\otimes\cdots\otimes f_{\alpha s(1)}$ and then gets permuted twice, by $\rho_{0,j}^{\widetilde{\Gamma}_1}$ and $\rho_{0,i+m-1}^{\Gamma}$; while in (4.1.32), $\Delta^{(s-1)}(h_{\gamma a})$ will get permuted by $\rho_{0,j}^{\Gamma}$. These two parts will be the same, since $h_{\gamma q_2}$ and $h_{\gamma a}$ are essentially the same, both representing the connected component in Γ_0 that contains the vertex clasped from Γ_2. They are eventually identified as connected components of the same graph Γ. Similar arguments work for $\Delta^{(t-1)}(h_{\gamma q_1})$ and $\Delta^{(t-1)}(h_{\gamma b})$. For the connected components that are disjoint from Γ_1 and Γ_2, they will be the same in both cases since they are eventually identified as connected components in Γ. This shows that $(Z\circ_i Y)\circ_j X=(-1)^{p(Y)p(X)}(Z\circ_j X)\circ_{i+m-1} Y$ if Γ_1 and Γ_2 are disjoint.

The other case is when Γ_1 and Γ_2 are connected. Note that there can be only one path connecting the k_1-th connected component of Γ_1 and the k_2-th connected component of Γ_2;

otherwise we have a cycle in Γ_0. A typical example looks like the graph below, where there maybe some vertices c outside Γ_1 and Γ_2 on the path from a to b.



We can rewrite an element $(h_1 \otimes \cdots \otimes h_n) \otimes_H v$ into the form $(\widetilde{h}_1 \otimes \cdots \otimes \widetilde{h}_{i-1} \otimes 1 \otimes \widetilde{h}_{i+1} \otimes \cdots \otimes \widetilde{h}_n) \otimes_H \widetilde{v}$ for any $i \in \{1, \ldots, n\}$, where $\widetilde{h}_j$'s and $\widetilde{v}$ are unique (cf. Subsection 2.1.2). So we can assume in $f_{\alpha k_2} = 1$ in (4.1.28) and $g_{\beta k_1} = 1$ in (4.1.29). One computes $(Z \circ_i Y)^{\widetilde{\Gamma}_1}$ by (4.1.31) only letting $g_{\beta k_1(1)} = g_{\beta k_1(-2)} = 1$. For $((Z \circ_i Y) \circ_j X)^{\Gamma}$, one can still use (4.1.32) with the following changes:

1. $f_{\alpha k_2(1)} = f_{\alpha k_2(-2)} = 1$. Note that $g_{\beta k_1(1)} = 1$ and $\Delta^{(s-1)}_{\widetilde{q}_2}$ acts on the connected component corresponding to $h_{\gamma q_1(k_1)}$; hence $f_{\alpha k(1)}$'s and $g_{\beta p(1)}$'s do not multiply together.

2. Note that if a vector v_l in z_γ gets both actions from $f_{\alpha k(-2)}$ and $g_{\beta p(-2)}$, then it must be in the component that connects Γ_1 and Γ_2, and the actions are from $f_{\alpha k_2(-2)}$ and $g_{\beta k_1(-2)}$, which are trivial. Other vectors can only get one action, either from $f_{\alpha k(-2)}$ or $g_{\beta p(-2)}$; otherwise we get a cycle between Γ_1 and Γ_2 in Γ_0.

The right-hand side can be computed in a similar way. First, $(Z \circ_j Y)^{\widetilde{\Gamma}_1}$ can be obtained by letting $f_{\alpha k_2(1)} = f_{\alpha k_2(-2)} = 1$ in (4.1.33). Then $((Z \circ_j X) \circ_{i+m-1} Y)^{\Gamma}$ is computed by (4.1.34) with the following changes:

1. $g_{\beta k_1(1)} = g_{\beta k_1(-2)} = 1$. Note that $f_{\alpha k_2(1)} = 1$ and $\Delta^{(t-1)}_{\widetilde{q}_1}$ acts on the connected component corresponding to $h_{\gamma q_2(k_2)}$; hence $f_{\alpha k(1)}$'s and $g_{\beta p(1)}$'s do not multiply together.

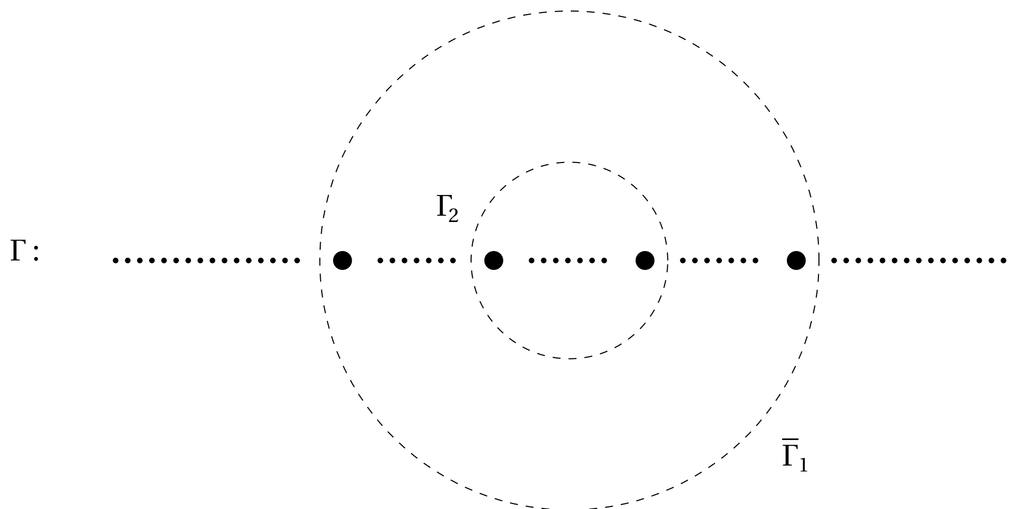
2. The second change as we compute $((Z \circ_i Y) \circ_j X)^{\Gamma}$ above also applies here.

Since on both sides we identify the connected components of Γ by either $\rho^{\Gamma}_{0,j}$ or $\rho^{\Gamma}_{0,i+m-1}$, we have $(Z \circ_i Y) \circ_j X = (-1)^{p(Y)p(X)} (Z \circ_j X) \circ_{i+m-1} Y$ when Γ_1 and Γ_2 are connected. $\square$

Remark 4.1.3. *Removing the connected component connecting Γ_1 and Γ_2 will make these two new subgraphs disjoint. Hence, the case when Γ_1 and Γ_2 are connected can be thought of as the disjoint case once $f_{\alpha k_2}$ and $g_{\beta k_1}$ are set to 1.*

Lemma 4.1.6. *If $i \le j < i+n$, then $(Z \circ_i Y) \circ_j X = Z \circ_i (Y \circ_{j-i+1} X)$.*

Proof. Consider a graph $\Gamma \in G(m+n+l-2)$. Since $i \le j < i+n$, Γ looks like:



For the notation, $\overline{\Gamma}_1 = \Delta_i^{1\ldots1(m+n-1)1\ldots1}(\Gamma) \in G(m+n-1)$, where $m+n-1$ is at the i-th position, and $\Gamma_2 = \Delta_{j-i+1}^{1\ldots1m1\ldots1}(\overline{\Gamma}_1) \in G(m)$, where m is at the $(j-i+1)$-th position. Further, let $\Gamma_0 = \Delta_0^{1\ldots1(m+n-1)1\ldots1}(\Gamma) \in G(l)$, $\Gamma_1 = \Delta_0^{1\ldots1m1\ldots1}(\overline{\Gamma}_1) \in G(n)$, and $\overline{\Gamma}_0 = \Delta_0^{1\ldots1m1\ldots1}(\Gamma) \in G(n+l-1)$ (m is at the j-th position in this last cocomposition map). Here $\overline{\Gamma}_0$ is obtained by clasping Γ_2 in Γ. As in the proof of Lemma 4.1.5, X, Y, Z will be evaluated on Γ_2, Γ_1, and Γ_0, respectively. Using the same notation on the number of connected components and assumptions (4.1.28), (4.1.29) and (4.1.30), one computes the left-hand side as follows:

$$
\begin{aligned}
&(Z \circ_i Y)^{\overline{\Gamma}_0}(v_1 \otimes \cdots \otimes v_{n+l-1}) \\
=&(-1)^{p_i^Y} \sum_{\beta,\gamma} \rho_{0,i}^{\overline{\Gamma}_0}\big((1 \otimes \cdots \otimes 1 \otimes g_{\beta 1(1)} \otimes \cdots \otimes g_{\beta t(1)} \otimes 1 \cdots \otimes 1)(h_{\gamma 1} \otimes \cdots \otimes \Delta^{(t-1)}(h_{\gamma q_1}) \otimes \cdots \otimes h_{\gamma r})\big) \otimes_H \\
&z_\gamma\big((g_{\beta 1(-2)} \otimes \cdots \otimes g_{\beta t(-2)}) \cdot (v_1 \otimes \cdots \otimes v_{i-1} \otimes y_\beta(v_i \otimes \cdots \otimes v_{i+n-1}) \otimes v_{i+n} \otimes \cdots \otimes v_{n+l-1})\big) \\
=&(-1)^{p_i^Y} \sum_{\beta,\gamma} \rho_{0,i}^{\overline{\Gamma}_0}(h_{\gamma 1} \otimes \cdots \otimes g_{\beta 1(1)} \otimes \cdots \otimes g_{\beta t(1)} \otimes \cdots \otimes h_{\gamma r}) \otimes_H z_\gamma\big((g_{\beta 1(-2)} \otimes \cdots \otimes g_{\beta t(-2)}) \cdot (v_1 \otimes \cdots \\
&\otimes v_{i-1} \otimes y_\beta(v_i \otimes \cdots \otimes v_{i+n-1}) \otimes v_{i+n} \otimes \cdots \otimes v_{n+l-1})\big),
\end{aligned}
\tag{4.1.35}
$$

where the vertex clasped from Γ_1 is contained in the q_1-th connected component of Γ_0. The last equality follows from setting $h_{\gamma q_1} = 1$. If Γ_2 is clasped into a vertex in the $\overline{q}_2$-th connected component of $\overline{\Gamma}_0$, one computes $((Z \circ_i Y) \circ_j X)^{\Gamma}$ as in (4.1.36) below. Note that in (4.1.36), since the vertex clasped from Γ_2 is in Γ_1, the $\overline{q}_2$-th connected component of $\overline{\Gamma}_0$ corresponds to some $g_{\beta l(1)}$ for some l. We assume that $g_{\beta l} = 1$; hence $\Delta^{(s-1)}_{\overline{q}_2}(g_{\beta l(1)}) = 1 \otimes \cdots \otimes 1$ and $f_{\alpha k(1)}$'s and $g_{\beta p(1)}$'s do not multiply together. In z_γ, $f_{\alpha k(-2)}$'s act only on vectors outside x_α and $g_{\beta p(-2)}$'s act only on vectors outside y_β. All vectors outside y_β can only have at most a single action either from $f_{\alpha k(-2)}$ or $g_{\beta p(-2)}$; otherwise we get a cycle in Γ_0.

$$\begin{aligned}
&((Z \circ_i Y) \circ_j X)^{\Gamma}(v_1 \otimes \cdots \otimes v_{m+n+l-2}) \\
=&(-1)^{p_i^Y + p_j^X} \sum_{\alpha,\beta,\gamma} \rho^{\Gamma}_{0,j}\Big((1 \otimes \cdots \otimes 1 \otimes f_{\alpha 1(1)} \otimes \cdots \otimes f_{\alpha s(1)} \otimes 1 \otimes \cdots \otimes 1)(\Delta^{(s-1)}_{\overline{q}_2} \cdot (\rho^{\overline{\Gamma}_0}_{0,i}(h_{\gamma 1} \otimes \cdots \otimes g_{\beta 1(1)} \otimes \\
&\cdots \otimes g_{\beta t(1)} \otimes \cdots \otimes h_{\gamma r})))\Big) \otimes_H z_\gamma\Big((f_{\alpha 1(-2)} \otimes \cdots \otimes f_{\alpha s(-2)}) \cdot (g_{\beta 1(-2)} \otimes \cdots \otimes g_{\beta t(-2)}) \cdot (v_1 \otimes \cdots \otimes v_{i-1} \otimes \\
&y_\beta(v_i \otimes \cdots \otimes v_{j-1} \otimes x_\alpha(v_j \otimes \cdots \otimes v_{j+m-1}) \otimes v_{j+m} \otimes \cdots \otimes v_{i+m+n-2}) \otimes v_{i+m+n-1} \otimes \cdots \otimes v_{m+n+l-2})\Big).
\end{aligned} \tag{4.1.36}$$

Next, we compute the right-hand side of the equation as follows. Suppose that the vertex clasped from Γ_2 is contained in the $\overline{q}_1$-th connected component of Γ_1. Then we have:

$$\begin{aligned}
&(Y \circ_{j-i+1} X)^{\overline{\Gamma}_1}(v_1 \otimes \cdots \otimes v_{m+n-1}) \\
=&(-1)^{p^X_{j-i+1}} \sum_{\beta,\gamma} \rho^{\overline{\Gamma}_1}_{0,j-i+1}\Big((1 \otimes \cdots \otimes 1 \otimes f_{\alpha 1(1)} \otimes \cdots \otimes f_{\alpha s(1)} \otimes 1 \otimes \cdots \otimes 1)(g_{\beta 1} \otimes \cdots \otimes \Delta^{(s-1)}(g_{\beta \overline{q}_1}) \otimes \cdots \otimes g_{\beta t})\Big) \\
&\otimes_H y_\beta\Big((f_{\alpha 1(-2)} \otimes \cdots \otimes f_{\alpha s(-2)}) \cdot (v_1 \otimes \cdots \otimes v_{j-i} \otimes x_\alpha(v_{j-i+1} \otimes \cdots \otimes v_{j-i+m}) \otimes v_{j-i+m+1} \otimes \cdots \otimes v_{m+n-1})\Big) \\
=&(-1)^{p^X_{j-i+1}} \sum_{\beta,\gamma} \rho^{\overline{\Gamma}_1}_{0,j-i+1}(g_{\beta 1} \otimes \cdots \otimes f_{\alpha 1(1)} \otimes \cdots \otimes f_{\alpha s(1)} \otimes \cdots \otimes g_{\beta t}) \otimes_H y_\beta\Big((f_{\alpha 1(-2)} \otimes \cdots \otimes f_{\alpha s(-2)}) \\
&\cdot (v_1 \otimes \cdots \otimes v_{j-i} \otimes x_\alpha(v_{j-i+1} \otimes \cdots \otimes v_{j-i+m}) \otimes v_{j-i+m+1} \otimes \cdots \otimes v_{m+n-1})\Big),
\end{aligned} \tag{4.1.37}$$

where the last equality follows after we set $g_{\beta \overline{q}_1} = 1$. Indeed, here $\overline{q}_1 = l$ above when computing the left-hand side, since $g_{\beta \overline{q}_1}$ corresponds to the connected component of Γ_1 that contains the vertex clasped from Γ_2. So we are imposing the same condition $g_{\beta l} = 1$ on both sides. Next, one computes $(Z \circ_i (Y \circ_{j-i+1} X))^{\Gamma}$. Suppose that $\overline{\Gamma}_1$ is clasped into a vertex contained in the q_2-th connected component of Γ_0. Then

$$
\begin{aligned}
&(Z\circ_i (Y\circ_{j-i+1} X))^{\Gamma}(v_1\otimes\cdots\otimes v_{m+n+l-2})\\
=&(-1)^{p_i^Y+p_j^X}\sum_{\alpha,\beta,\gamma}\rho^{\Gamma}_{0,j}\big((\rho^{\overline{\Gamma}_1}_{0,j-i+1}(g_{\beta 1(1)}\otimes\cdots\otimes f_{\alpha 1(1)}\otimes\cdots\otimes f_{\alpha s(1)}\otimes\cdots\otimes g_{\beta t(1)}))(h_{\gamma 1}\otimes\cdots\otimes h_{\gamma q_2(1)}\otimes\cdots\\
&\otimes h_{\gamma q_2(m+n-1)}\otimes\cdots\otimes h_{\gamma r})\big)\otimes_H z_\gamma\big((f_{\alpha 1(-2)}\otimes\cdots\otimes f_{\alpha s(-2)})\cdot(g_{\beta 1(-2)}\otimes\cdots\otimes g_{\beta t(-2)})\cdot(v_1\otimes\cdots\otimes v_{i-1}\\
&\otimes y_\beta((f_{\alpha 1(-3)}\otimes\cdots\otimes f_{\alpha s(-3)})\cdot(v_i\otimes\cdots\otimes v_{j-1}\otimes x_\alpha(v_j\otimes\cdots\otimes v_{j+m-1})\otimes v_{j+m}\otimes\cdots\otimes v_{i+m+n-2}))\\
&\otimes v_{i+m+n-1}\otimes\cdots\otimes v_{m+n+l-2})\big).
\end{aligned}
\tag{4.1.38}
$$

Note that here $q_2=q_1$ and we are imposing the same condition as in (4.1.35): $h_{\gamma q_2}=1$. Thus, $\rho^{\overline{\Gamma}_1}_{0,j-i+1}(g_{\beta 1(1)}\otimes\cdots\otimes f_{\alpha 1(1)}\otimes\cdots\otimes f_{\alpha s(1)}\otimes\cdots\otimes g_{\beta t(1)})$ will multiply with $h_{\gamma q_2(1)}\otimes\cdots\otimes h_{\gamma q_2(m+n-1)}=1\otimes\cdots\otimes 1$. In z_γ, we use the identity $f_{\alpha k(1)(1)}\otimes f_{\alpha k(1)(-2)}\otimes f_{\alpha k(-2)}=f_{\alpha k(1)}\otimes f_{\alpha k(-2)}\otimes f_{\alpha k(-3)}$. Hence, $f_{\alpha k(-2)}$'s will only act on the vectors outside y_β, and $f_{\alpha k(-3)}$'s will only act on the vectors inside y_β but out of x_α. As in (4.1.36), $f_{\alpha k(-2)}$'s and $g_{\beta p(-2)}$'s will not act on a same vector outside y_β; otherwise we get a cycle in Γ_0.

To see that (4.1.36) and (4.1.38) are equal, note that for the coefficient part, each tensor factor can only be $f_{\alpha k(1)}$, $g_{\beta p(1)}$ or $h_{\gamma q}$. No multiplication will appear since we impose the conditions $g_{\beta l}=h_{\gamma q_1}=1$. Since both (4.1.36) and (4.1.38) identify the connected components of Γ in the end, the coefficient parts are equal. For the vector part, as $\Delta(f_{\alpha k(-2)})=f_{\alpha k(-2)}\otimes f_{\alpha k(-3)}=f_{\alpha k(-3)}\otimes f_{\alpha k(-2)}$, the actions from $f_{\alpha k}$'s are the same; hence the vector parts are the same. □

Lemma 4.1.7. *For any $\sigma\in S_n$ and $\tau\in S_m$, we have $Y^\sigma\circ_i X^\tau=(Y\circ_{\sigma(i)}X)^{\sigma\circ_i\tau}$, where $\sigma\circ_i\tau$ is given by* (3.1.16).

Proof. Consider a graph $\Gamma\in G(m+n-1)$, let $X\in P^{cl}_H(m)$ and $Y\in P^{cl}_H(n)$ as usual. Let $\Gamma_0=\sigma(\Delta_0^{1\ldots 1m1\ldots 1}(\Gamma))$ and $\Gamma_i=\tau(\Delta_i^{1\ldots 1m1\ldots 1}(\Gamma))$, where m appears at the i-th position. Suppose that

$$X^{\Gamma_i}(v_1\otimes\cdots\otimes v_m)=\sum_\alpha(f_{\alpha 1}\otimes\cdots\otimes f_{\alpha s})\otimes_H x_\alpha(v_1\otimes\cdots\otimes v_m),\tag{4.1.39}$$

$$Y^{\Gamma_0}(w_1\otimes\cdots\otimes w_n)=\sum_\beta(g_{\beta 1}\otimes\cdots\otimes g_{\beta t})\otimes_H y_\beta(w_1\otimes\cdots\otimes w_n).\tag{4.1.40}$$

First, we evaluate $(X^\tau)^{\Delta_i^{1\ldots 1m1\ldots 1}(\Gamma)}(v_i\otimes\cdots\otimes v_{i+m-1})$ and $(Y^\sigma)^{\Delta_0^{1\ldots 1m1\ldots 1}(\Gamma)}(v_1\otimes\cdots\otimes v_{i-1}\otimes x_\alpha(v_i\otimes\cdots\otimes$

$v_{i+m-1})\otimes v_{i+m}\otimes\cdots\otimes v_{m+n-1})$ given (4.1.4), (4.1.39) (4.1.40):

$$\begin{aligned}
&(X^{\tau})^{\Delta_i^{1\dots1m1\dots1}(\Gamma)}(v_i\otimes\cdots\otimes v_{i+m-1})\\
=&(\widetilde{\tau}\otimes_H 1)X^{\Gamma_i}(\tau(v_i\otimes\cdots\otimes v_{i+m-1}))\\
=&\sum_{\alpha}\widetilde{\tau}(f_{\alpha1}\otimes\cdots\otimes f_{\alpha s})\otimes_H x_\alpha(v_{\tau^{-1}(i)}\otimes\cdots\otimes v_{\tau^{-1}(i+m-1)})\\
=&\sum_{\alpha}(f_{\alpha\widetilde{\tau}^{-1}(1)}\otimes\cdots\otimes f_{\alpha\widetilde{\tau}^{-1}(s)})\otimes_H x_\alpha(v_{\tau^{-1}(i)}\otimes\cdots\otimes v_{\tau^{-1}(i+m-1)}),
\end{aligned}\tag{4.1.41}$$

and

$$\begin{aligned}
&(Y^{\sigma})^{\Delta_0^{1\dots1m1\dots1}(\Gamma)}(v_1\otimes\cdots\otimes v_{i-1}\otimes x_\alpha(v_i\otimes\cdots\otimes v_{i+m-1})\otimes v_{i+m}\otimes\cdots\otimes v_{m+n-1})\\
=&(\widetilde{\sigma}\otimes_H 1)\cdot Y^{\Gamma_0}(\sigma(v_1\otimes\cdots\otimes v_{i-1}\otimes x_\alpha(v_i\otimes\cdots\otimes v_{i+m-1})\otimes v_{i+m}\otimes\cdots\otimes v_{m+n-1}))\\
=&\sum_{\beta}(g_{\beta\widetilde{\sigma}^{-1}(1)}\otimes\cdots\otimes g_{\beta\widetilde{\sigma}^{-1}(t)})\otimes_H y_\beta(v_{\sigma^{-1}(1)}\otimes\cdots\otimes v_{\sigma^{-1}(\sigma(i)-1)}\otimes x_\alpha(v_{\tau^{-1}(i)}\otimes\cdots\otimes v_{\tau^{-1}(i+m-1)})\\
&\otimes v_{\sigma^{-1}(\sigma(i)+1)}\otimes\cdots\otimes v_{\sigma^{-1}(m+n-1)}).
\end{aligned}\tag{4.1.42}$$

Then we evaluate $Y^{\sigma}\circ_i X^{\tau}$ on Γ as follows:

$$\begin{aligned}
&(Y^{\sigma}\circ_i X^{\tau})^{\Gamma}(v_1\otimes\cdots\otimes v_{m+n-1})\\
=&(Y^{\sigma}(1,\dots,1,X^{\tau},1,\dots,1))^{\Gamma}(v_1\otimes\cdots\otimes v_{m+n-1})\\
=&(-1)^{\overline{p}_i^X}\epsilon(\sigma)\epsilon(\tau)\sum_{\alpha,\beta}\rho_{0,i}^{\Gamma}\big((1\otimes\cdots\otimes1\otimes\widetilde{\tau}(f_{\alpha1(1)}\otimes\cdots\otimes f_{\alpha s(1)})\otimes1\otimes\cdots\otimes1)(g_{\beta\widetilde{\sigma}^{-1}(1)}\otimes\cdots\otimes\\
&\Delta^{(s-1)}(g_{\beta q})\otimes\cdots\otimes g_{\beta\widetilde{\sigma}^{-1}(t)})\big)\otimes_H y_\beta\big((f_{\alpha1(-2)}\otimes\cdots\otimes f_{\alpha s(-2)})\cdot(v_{\sigma^{-1}(1)}\otimes\cdots\otimes v_{\sigma^{-1}(\sigma(i)-1)}\\
&\otimes x_\alpha(v_{\tau^{-1}(i)}\otimes\cdots\otimes v_{\tau^{-1}(i+m-1)})\otimes v_{\sigma^{-1}(\sigma(i)+1)}\otimes\cdots\otimes v_{\sigma^{-1}(m+n-1)})\big),
\end{aligned}\tag{4.1.43}$$

where $g_{\beta q}$ in $\widetilde{\sigma}(g_{\beta1}\otimes\cdots\otimes g_{\beta t})$ corresponds to the connected component of Γ_0 that contains the vertex clasped from Γ_i; $\widetilde{\sigma}(q)-1$ of 1's are inserted before $\widetilde{\tau}(f_{\alpha1(1)}\otimes\cdots\otimes f_{\alpha s(1)})$ so that it will multiply with $\Delta^{(s-1)}(g_{\beta q})$; $\epsilon(\sigma)$ and $\epsilon(\tau)$ are given by (3.1.9) (the subscripts are suppressed since the tensor products these permutations act on are clear in calculation above and fixed); and

$$\overline{p}_i^X=p(X)(p(v_{\sigma^{-1}(1)})+\cdots+p(v_{\sigma^{-1}(\sigma(i)-1)})).\tag{4.1.44}$$

For the right-hand side, by Proposition 3.4.2, we will evaluate X on $\Delta_{\sigma(i)}^{1\dots1m1\dots1}((\sigma\circ_i\tau)(\Gamma))=$

Γ_i, and Y on $\Delta_0^{1\dots1m1\dots1}((\sigma\circ_i\tau)(\Gamma))=\Gamma_0$, where m appears at the $\sigma(i)$-th position. Meanwhile, one has:

$$\begin{aligned}
&(\sigma\circ_i\tau)(v_1\otimes\cdots\otimes v_{m+n-1})\\
=&\sigma(v_1\otimes\cdots\otimes v_{i-1}\otimes\tau(v_i\otimes\cdots\otimes v_{i+m-1})\otimes v_{i+m}\otimes\cdots\otimes v_{m+n-1})\\
=&\epsilon(\sigma)\epsilon(\tau)(v_{\sigma^{-1}(1)}\otimes\cdots\otimes v_{\sigma^{-1}(\sigma(i)-1)}\otimes\big(v_{\tau^{-1}(i)}\otimes\cdots\otimes v_{\tau^{-1}(i+m-1)}\big)\otimes v_{\sigma^{-1}(\sigma(i)+1)}\otimes\cdots\otimes v_{\sigma^{-1}(m+n-1)}\big).
\end{aligned}\tag{4.1.45}$$

Hence, one computes the right-hand side:

$$\begin{aligned}
&((Y\circ_{\sigma(i)}X)^{\sigma\circ_i\tau})^{\Gamma}(v_1\otimes\cdots\otimes v_{m+n-1})\\
=&(\widetilde{(\sigma\circ_i\tau)}\otimes_H 1)(Y\circ_{\sigma(i)}X)^{(\sigma\circ_i\tau)(\Gamma)}((\sigma\circ_i\tau)(v_1\otimes\cdots\otimes v_{m+n-1}))\\
=&\epsilon(\sigma)\epsilon(\tau)(\widetilde{(\sigma\circ_i\tau)}\otimes_H 1)(Y\circ_{\sigma(i)}X)^{(\sigma\circ_i\tau)(\Gamma)}(v_{\sigma^{-1}(1)}\otimes\cdots\otimes v_{\sigma^{-1}(\sigma(i)-1)}\otimes\big(v_{\tau^{-1}(i)}\otimes\cdots\otimes v_{\tau^{-1}(i+m-1)}\big)\\
&\otimes v_{\sigma^{-1}(\sigma(i)+1)}\otimes\cdots\otimes v_{\sigma^{-1}(m+n-1)}\big)\\
=&(-1)^{\overline{p}_i^X}\epsilon(\sigma)\epsilon(\tau)\sum_{\alpha,\beta}\widetilde{(\sigma\circ_i\tau)}\cdot\big(\rho_{0,\sigma(i)}^{(\sigma\circ_i\tau)(\Gamma)}((1\otimes\cdots\otimes1\otimes f_{\alpha1(1)}\otimes\cdots\otimes f_{\alpha s(1)}\otimes1\otimes\cdots\otimes1)(g_{\beta1}\otimes\cdots\\
&\otimes\Delta^{(s-1)}(g_{\beta q})\otimes\cdots\otimes g_{\beta t}))\big)\otimes_H y_\beta\big((f_{\alpha1(-2)}\otimes\cdots\otimes f_{\alpha s(-2)})\cdot(v_{\sigma^{-1}(1)}\otimes\cdots\otimes v_{\sigma^{-1}(\sigma(i)-1)}\\
&\otimes x_\alpha\big(v_{\tau^{-1}(i)}\otimes\cdots\otimes v_{\tau^{-1}(i+m-1)}\big)\otimes v_{\sigma^{-1}(\sigma(i)+1)}\otimes\cdots\otimes v_{\sigma^{-1}(m+n-1)})\big).
\end{aligned}$$

One sees immediately that the vector part is the same for both sides. For the coefficient part, without loss of generality, we assume that $g_{\beta q}=1$. Then it suffices to show that $\rho_{0,i}^{\Gamma}(\widetilde{\sigma}\circ_{\widetilde{\sigma}(q)}\widetilde{\tau})=\widetilde{(\sigma\circ_i\tau)}\rho_{0,\sigma(i)}^{(\sigma\circ_i\tau)(\Gamma)}\in S_{s+t-1}$ on $v=g_{\beta1}\otimes\cdots\otimes g_{\beta\,q-1}\otimes f_{\alpha1(1)}\otimes\cdots\otimes f_{\alpha s(1)}\otimes g_{\beta\,q+1}\otimes\cdots\otimes g_{\beta\,t}$ (here $g_{\beta\,q-1}$ has double subscript β and $q-1$ and similarly for $g_{\beta\,q+1}$). In order to prove this, observe from (4.1.4) that, for any permutation σ, the induced permutation $\widetilde{\sigma}$ identifies the k-th connected component of $\sigma(\Gamma)$ with the $\widetilde{\sigma}(k)$-th connected component of Γ. Consider the right-hand side, and pick a tensor factor in v, which represents a connected component of Γ_i or Γ_0. It is first identified by $\rho_{0,\sigma(i)}^{(\sigma\circ_i\tau)(\Gamma)}$ with a connected component of $(\sigma\circ_i\tau)(\Gamma)$; then identified by $\widetilde{(\sigma\circ_i\tau)}$ with a connected component of Γ. Now consider the left-hand side, and note that $(\widetilde{\sigma}\circ_{\widetilde{\sigma}(q)}\widetilde{\tau})$ identifies this connected component in v with a connected component in either $\Delta_0^{1\dots1m1\dots1}(\Gamma)$ or $\Delta_i^{1\dots1m1\dots1}(\Gamma)$ (m appears at the i-th position). If it is a connected component in Γ_0, it will be a connected component in $\Delta_0^{1\dots1m1\dots1}(\Gamma)$; while if it is a connected component in Γ_i, it will be a connected component in $\Delta_i^{1\dots1m1\dots1}(\Gamma)$. Then $\rho_{0,i}^{\Gamma}$ identifies this connected component with a connected component of Γ. Hence, both sides identify connected components in v with those of Γ, and they are equal. $\square$

4.2 Construction of Poisson superpseudoalgebras from P_H^{cl}

As the classical operad P^{cl} governs Poisson vertex superalgebras, the generalized classical operad P_H^{cl} governs Poisson superpseudoalgebras (abbreviated PPA) as defined below.

Definition 4.2.1. *A Poisson superpseudoalgebra V is a commutative associative H-differential superalgebra with a Lie superpseudobracket that satisfies the left Leibniz rule: for any $a, b, c \in V$,*

$$[a * bc] = [a * b]c + (-1)^{p(b)p(c)}[a * c]b, \tag{4.2.1}$$

where $[a * b]c$ is defined as

$$[a * b]c = \sum_i (f_i \otimes g_{i(1)}) \otimes_H v_i(a \otimes b)(g_{i(-2)}c), \tag{4.2.2}$$

if $[a * b] = \sum_i (f_i \otimes g_i) \otimes_H v_i(a \otimes b)$ for some $f_i, g_i \in H$ and linear maps $v_i : V \otimes V \to V$.

It is not hard to see that (4.2.2) is well defined, since if we rewrite $[a * b] = \sum_i (f_i^{'} \otimes g_i^{'}) \otimes_H (h_i v_i(a \otimes b))$, then we will have $f_i^{'} h_{i(1)} \otimes g_i^{'} h_{i(2)} = f_i \otimes g_i$ and

$$\begin{aligned}
&\sum_i (f_i^{'} h_{i(1)} \otimes g_{i(1)}^{'} h_{i(2)}) \otimes_H v_i(a \otimes b)(h_{i(-3)} g_{i(-2)}^{'} c) \\
=&\sum_i (f_i^{'} \otimes g_{i(1)}^{'}) \otimes_H h_{i(1)} \cdot (v_i(a \otimes b)(h_{i(-2)} g_{i(-2)}^{'} c)) \\
=&\sum_i (f_i^{'} \otimes g_{i(1)}^{'}) \otimes_H h_{i(1)} v_i(a \otimes b)(h_{i(2)} h_{i(-3)} g_{i(-2)}^{'} c) \\
=&\sum_i (f_i^{'} \otimes g_{i(1)}^{'}) \otimes_H h_i v_i(a \otimes b)(g_{i(-2)}^{'} c),
\end{aligned} \tag{4.2.3}$$

where we use (2.1.8) for the last equality. This shows that (4.2.2) is well defined. Similarly, one can define:

$$a[b * c] = \sum_j (d_{j(1)} \otimes l_j) \otimes_H (d_{j(-2)} a) w_j(b \otimes c), \tag{4.2.4}$$

if $[b * c] = \sum_j (d_j \otimes l_j) \otimes_H w_j(b \otimes c)$. As in (4.2.3), this is well defined too. With this definition, the right Leibniz rule can be written as:

$$[ab * c] = a[b * c] + (-1)^{p(a)p(b)} b[a * c]. \tag{4.2.5}$$

The right Leibniz rule can be obtained from the left Leibniz rule using the super-skewsymmetry of the pseudobracket. Similarly to Theorem 3.5.2, next we show that the PPA structures on V correspond to odd elements $X \in W^{cl}_{H,1}(\Pi V)$ such that $X \Box X = 0$.

Consider a vector superspace V with parity p, which is also an H-differential superalgebra where H is purely even. Let $\bar{p} = 1 - p$ denote the reverse parity of p, and ΠV denote the same space as V but with parity $\bar{p}$. For the operad $P^{cl}_H(\Pi V)$, let $W^{cl}_H(\Pi V) = W(P^{cl}(\Pi V))$ be the universal Lie superalgebra associated to it, and let

$$S_H = \{X \in W^{cl}_{H,1}(\Pi V)) \mid X \text{ is odd and } X \Box X = 0\}. \tag{4.2.6}$$

Theorem 4.2.1. *There exists a bijective correspondence between S_H and the PPA structures on V given by:*

$$a\,b = (-1)^{p(a)} X^{\bullet\!\to\!\bullet}(a \otimes b), \quad [a * b] = (-1)^{p(a)} X^{\bullet\ \ \bullet}(a \otimes b), \tag{4.2.7}$$

for any $X \in S_H$ and $a, b \in V$.

Proof. The proof here is adapted from the proof of Theorem 3.5.2 given in [3]. The ideas of the proofs are the same. For $X \in S_H$, by definition $X \in P^{cl}_H(2)$ is permutation invariant, that is $X^\sigma = X$, where $\sigma = (12) \in S_2$. This is equivalent to the commutativity of the product and the super-skewsymmetry of the pseudobracket. First, evaluate $X = X^\sigma$ on the connected graph. Since there is a unique connected component, we get

$$\begin{aligned} X^{\bullet\!\to\!\bullet}(a \otimes b) =& (X^{(12)})^{\bullet\!\to\!\bullet}(a \otimes b) = (-1)^{\bar{p}(a)\bar{p}(b)} X^{\bullet\!\leftarrow\!\bullet}(b \otimes a) \\ =& (-1)^{p(a)+p(b)+p(a)p(b)} X^{\bullet\!\to\!\bullet}(b \otimes a), \end{aligned} \tag{4.2.8}$$

which implies that

$$a\,b = (-1)^{p(a)} X^{\bullet\!\to\!\bullet}(a \otimes b) = (-1)^{p(b)+p(a)p(b)} X^{\bullet\!\to\!\bullet}(b \otimes a) = (-1)^{p(a)p(b)}\, b\,a. \tag{4.2.9}$$

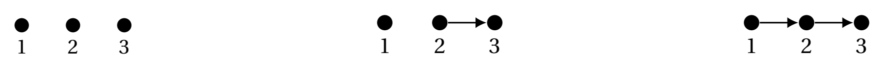
If we evaluate $X = X^\sigma$ on the disconnected graph, we get

$$X^{\bullet\ \ \bullet}(a \otimes b) = (X^{(12)})^{\bullet\ \ \bullet}(a \otimes b) = (-1)^{\bar{p}(a)\bar{p}(b)} (\sigma \otimes_H 1) X^{\bullet\ \ \bullet}(b \otimes a), \tag{4.2.10}$$

which implies that

$$[a*b] = (-1)^{p(a)} X^{\bullet\ \bullet}(a\otimes b) = -(-1)^{p(a)p(b)}(\sigma\otimes_H 1)[b*a]. \tag{4.2.11}$$

Next, we show that $X\square X = 0$ is equivalent to three properties: the associativity of the product ab, the Jacobi identity of the pseudobracket $[a*b]$ and the left Leibniz rule. We will do so by evaluating $X\square X$ on the following 3-graphs since $X\square X$ is invariant by the action of symmetric group by construction.



(a) G_1 (b) G_2 (c) G_3

Recall formula (3.2.4):

$$X\square X = X\circ_1 X + X\circ_2 X + (X\circ_2 X)^{(12)}. \tag{4.2.12}$$

Evaluating each summand above on Graph G_1, we have

$$\begin{aligned}
(X\circ_1 X)^{\bullet\ \bullet\ \bullet}(a\otimes b\otimes c) &= (-1)^{p(b)}[[a*b]*c],\\
(X\circ_2 X)^{\bullet\ \bullet\ \bullet}(a\otimes b\otimes c) &= (-1)^{1+p(b)}[a*[b*c]],\\
((X\circ_2 X)^{(12)})^{\bullet\ \bullet\ \bullet}(a\otimes b\otimes c) &= (-1)^{\bar p(a)\bar p(b)}((\sigma\otimes 1)\otimes_H 1)(X\circ_2 X)^{\bullet\ \bullet\ \bullet}(b\otimes a\otimes c)\\
&= (-1)^{p(b)+p(a)p(b)}((\sigma\otimes 1)\otimes_H 1)[b*[a*c]].
\end{aligned} \tag{4.2.13}$$

Hence $(X\square X)^{G_1} = 0$ implies the Jacobi identity:

$$[[a*b]*c] = [a*[b*c]] - (-1)^{p(a)p(b)}((\sigma\otimes 1)\otimes_H 1)[b*[a*c]]. \tag{4.2.14}$$

Next, we evaluate $X\square X$ on Graph G_2. Similarly to the previous case, we have:

$$\begin{aligned}
(X\circ_1 X)^{\bullet\ \bullet\to\bullet}(a\otimes b\otimes c) &= (-1)^{p(b)}[a*b]c,\\
(X\circ_2 X)^{\bullet\ \bullet\to\bullet}(a\otimes b\otimes c) &= (-1)^{\bar p(a)}X^{\bullet\ \bullet}(a\otimes X^{\bullet\to\bullet}(b\otimes c)) = (-1)^{1+p(b)}[a*bc],
\end{aligned} \tag{4.2.15}$$

For $(X \circ_2 X)^{(12)}$, by the equation $(X \circ_2 X)^{(12)} = (X \circ_1 X)^{(23)}$ (cf. (3.2.4)),

$$
\begin{aligned}
((X \circ_2 X)^{(12)})^{\bullet\quad \bullet\!\to\!\bullet}(a \otimes b \otimes c) =& ((X \circ_1 X)^{(23)})^{\bullet\quad \bullet\!\to\!\bullet}(a \otimes b \otimes c) \\
=& (-1)^{\bar{p}(b)\bar{p}(c)}(X \circ_1 X)^{\bullet\quad \bullet\!\leftarrow\!\bullet}(a \otimes c \otimes b) \\
=& -(-1)^{\bar{p}(b)\bar{p}(c)}(X \circ_1 X)^{\bullet\quad \bullet\!\to\!\bullet}(a \otimes c \otimes b) \\
=& (-1)^{p(b)+p(b)p(c)}[a * c]b
\end{aligned}
\tag{4.2.16}
$$

Thus $(X \Box X)^{G_2} = 0$ implies the left Leibniz rule:

$$
[a * b c] = [a * b]c + (-1)^{p(b)p(c)}[a * c]b. \tag{4.2.17}
$$

Finally, we evaluate $X \Box X$ on Graph G_3:

$$
\begin{aligned}
(X \circ_1 X)^{\bullet\!\to\!\bullet\!\to\!\bullet}(a \otimes b \otimes c) =& X^{\bullet\!\to\!\bullet}(X^{\bullet\!\to\!\bullet}(a \otimes b) \otimes c) = (-1)^{p(b)}(a b)c, \\
(X \circ_2 X)^{\bullet\!\to\!\bullet\!\to\!\bullet}(a \otimes b \otimes c) =& (-1)^{\bar{p}(a)} X^{\bullet\!\to\!\bullet}(a \otimes X^{\bullet\!\to\!\bullet}(b \otimes c)) = (-1)^{1+p(b)} a(b c), \\
((X \circ_2 X)^{(12)})^{\bullet\!\to\!\bullet\!\to\!\bullet}(a \otimes b \otimes c) =& (-1)^{\bar{p}(a)\bar{p}(b)}(X \circ_2 X)^{\bullet\!\leftarrow\!\bullet\quad\bullet}(b \otimes a \otimes c) \\
=& (-1)^{\bar{p}(b)+\bar{p}(a)\bar{p}(b)} X^{\bullet\rightleftarrows\bullet}(\cdots) \\
=& 0,
\end{aligned}
\tag{4.2.18}
$$

because of the cycle relation for X. Thus $(X \Box X)^{G_3} = 0$ is equivalent to the associativity of the product $a b$: $(a b)c = a(b c)$. $\Box$

As a remark of this proof, one can also derive the right Leibniz rule by evaluating $X \Box X$ on the graph $\bullet\!\to\!\bullet\quad\bullet$.

4.3 Examples of Poisson superpseudoalgebras

We give some examples of PPAs in this section. By definition, every PPA is a Lie superpseudoalgebra (abbreviated LPA). Conversely, given any LPA L, we construct a PPA $P(L)$ as follows. We take $P(L)$ to be the symmetric superalgebra $S(L)$ over L, which gives a supercommutative product; the action of H on $P(L)$ extends uniquely from its action on L as given in Definition 2.1.2: $h \cdot (x y) = (h_{(1)} x)(h_{(2)} y)$; the pseudobracket on $P(L)$ extends uniquely from the pseudobracket on L by the left Leibniz rule (4.2.1) and right Leibniz rule (4.2.5).

Proposition 4.3.1. *Given an LPA L, the symmetric algebra $P(L)$ defined above is a PPA, called the canonical universal PPA over L.*

In the remaining part of this section, we assume H to be the universal enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{d}$, and we let $\{\partial_1, \dots, \partial_M\}$ denote a basis of $\mathfrak{d}$, unless otherwise specified.

Example 4.3.1. *In the case of $H = \mathbb{F}[\partial]$, the notion of a PPA is equivalent to the notion of a Poisson vertex superalgebra (abbreviated PVA) from Definition 3.5.1. In particular, the underlying LPA of the PPA is a Lie conformal superalgebra (abbreviated LCA), and their relation is given in Example 2.3.1. In higher dimensions, that is if $H = \mathbb{F}[\partial_1, \partial_2, \dots, \partial_M]$, the underlying LPA of the PPA is indeed an LCA in higher dimensions* [6]*, which is a straightforward generalization of an LCA: one only needs to replace the indeterminate λ with a vector $\vec{\lambda} = (\lambda_1, \dots, \lambda_M)$ and ∂ with a vector $\vec{\partial} = (\partial_1, \dots, \partial_M)$. The pseudobracket of the PPA P can be identified with a $\vec{\lambda}$-bracket as follows:*

$$[a * b] = \sum_i (p_i(-\vec{\partial}) \otimes 1) \otimes_{\mathbb{F}[\vec{\partial}]} c_i \iff [a_{\vec{\lambda}} b] = \sum_i p_i(\vec{\lambda}) c_i. \tag{4.3.1}$$

Note that one can always write $[a * b]$ in the form in (4.3.1) using the Fourier transform (cf. Subsection 2.1.2). Furthermore, the axioms of a PPA P can now be written in terms of the $\vec{\lambda}$-bracket as follows: for any $a, b, c \in P$,

- *Sesquilinearity from H-bilinearity: for all $k = 1, \dots, M$,*

$$\begin{aligned} [\partial_k a_{\vec{\lambda}} b] &= -\lambda_k [a_{\vec{\lambda}} b], \\ [a_{\vec{\lambda}} \partial_k b] &= (\partial_k + \lambda_k)[a_{\vec{\lambda}} b]; \end{aligned} \tag{4.3.2}$$

- *Super-skewsymmetry:*

$$[a_{\vec{\lambda}} b] = -(-1)^{p(a)p(b)} [b_{-\vec{\lambda}-\vec{\partial}} a]; \tag{4.3.3}$$

- *Jacobi identity:*

$$[a_{\vec{\lambda}} [b_{\vec{\mu}} c]] = [[a_{\vec{\lambda}} b]_{\vec{\lambda}+\vec{\mu}} c] + (-1)^{p(a)p(b)} [b_{\vec{\mu}} [a_{\vec{\lambda}} c]]; \tag{4.3.4}$$

- *Leibniz rule:*

$$[a_{\vec{\lambda}} b\, c] = [a_{\vec{\lambda}} b] c + (-1)^{p(b)p(c)} [a_{\vec{\lambda}} c] b. \tag{4.3.5}$$

Example 4.3.2. (Free superboson LPA, PPA) *Let $\mathfrak{h}$ be a finite-dimensional vector superspace with parity p and a supersymmetric nondegenerate bilinear map $\beta : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{d}$, where $\mathfrak{d}$ is a finite-dimensional Lie algebra. The supersymmetry of β means that $\beta(a,b) = (-1)^{p(a)p(b)}\beta(b,a)$ for any $a, b \in \mathfrak{h}$ and $\beta(a,b) = 0$ whenever $p(a) \neq p(b)$. The free superboson LPA corresponding to $\mathfrak{h}$ is defined as the left H-module*

$$L^b_{\mathfrak{h}} = H\mathfrak{h} \oplus \mathbb{F}K, \tag{4.3.6}$$

where K satisfies $\partial K = 0$ for any $\partial \in \mathfrak{d}$ and $p(K) = \bar{0}$. The pseudobracket on $\mathfrak{h}$ is:

$$[a * b] = (\beta(a,b) \otimes 1) \otimes_H K, \quad \text{for any } a, b \in \mathfrak{h}, \quad K \text{ is central,} \tag{4.3.7}$$

which can be uniquely extended to $L^b_{\mathfrak{h}}$ by H-bilinearity. The super-skewsymmetry of the pseudobracket follows from the supersymmetry of β. The Jacobi identity is trivial since K is central. In the case when $\mathfrak{h}$ is purely even, we get the free boson LPA.

Now following the steps at the beginning of this section, we obtain the universal PPA of $L^b_{\mathfrak{h}}$ as the symmetric superalgebra

$$P(L^b_{\mathfrak{h}}) = S(H\mathfrak{h} \oplus \mathbb{F}K). \tag{4.3.8}$$

The pseudobracket on $P(L^b_{\mathfrak{h}})$ is extended uniquely from (4.3.7) by H-bilinearity and the left and right Leibniz rules. The free superboson PPA is defined as the quotient $P(L^b_{\mathfrak{h}})/(P(L^b_{\mathfrak{h}})(K-1))$, or equivalently:

$$B_{\mathfrak{h}} = S(H\mathfrak{h}), \tag{4.3.9}$$

with the pseudobracket induced from pseudobracket of $P(L^b_{\mathfrak{h}})$, given by (4.3.7) with $K = 1$ on the generators. In the case when $\mathfrak{h}$ is purely even, $B_{\mathfrak{h}}$ is called the free boson PPA. As an H-differential algebra, it is isomorphic to the H-differential algebra of polynomials in N

generators

$$B_{\mathfrak{h}} \cong \mathbb{F}[\,\partial_I u_i \mid i = 1, \dots, N], \tag{4.3.10}$$

where $\{u_1, \dots, u_N\}$ *is a basis for* $\mathfrak{h}$, $\{\partial_1, \dots, \partial_M\}$ *is a basis for* $\mathfrak{d}$, *and* ∂_I *is any ordered product of elements of* $\{\partial_1, \dots, \partial_M\}$ *in* H.

Example 4.3.3. (Free superfermion LPA, PPA) *Following the notation in Example 4.3.2 unless otherwise specified, consider the same space* $\mathfrak{h}$, *but now let* $\gamma : \mathfrak{h} \times \mathfrak{h} \to \mathbb{F}$ *to be a super-skewsymmetric nondegenerate bilinear form, that is* $\gamma(a, b) = -(-1)^{p(a)p(b)}\gamma(b, a)$ *for any* $a, b \in \mathfrak{h}$ *and* $\gamma(a, b) = 0$ *if* $p(a) \neq p(b)$. *The free superfermion LPA corresponding to* $\mathfrak{h}$ *is the left* H*-module*

$$L^f_{\mathfrak{h}} = H\mathfrak{h} \oplus \mathbb{F}K, \tag{4.3.11}$$

where K *satisfies* $\partial K = 0$ *for any* $\partial \in \mathfrak{d}$ *and* $p(K) = \bar{0}$. *The pseudobracket on* $\mathfrak{h}$ *is:*

$$[a * b] = (\gamma(a, b) \otimes 1) \otimes_H K, \quad \text{for any } a, b \in \mathfrak{h}, \quad K \text{ is central,} \tag{4.3.12}$$

which is uniquely extended to $L^f_{\mathfrak{h}}$ *by* H*-bilinearity. In the case when* $\mathfrak{h}$ *is purely even,* $L^f_{\mathfrak{h}}$ *is called the free fermion LPA. The universal PPA over* $L^f_{\mathfrak{h}}$ *is the symmetric superalgebra*

$$P(L^f_{\mathfrak{h}}) = S(H\mathfrak{h} \oplus \mathbb{F}K), \tag{4.3.13}$$

with the pseudobracket uniquely extended from (4.3.12) by the left and right Leibniz rules and H*-bilinearity. The free superfermion PPA is defined as the quotient* $P(L^f_{\mathfrak{h}})/(P(L^f_{\mathfrak{h}})(K-1))$, *or equivalently:*

$$F_{\mathfrak{h}} = S(H\mathfrak{h}) \tag{4.3.14}$$

with the pseudobracket given by (4.3.12) with $K = 1$ *on the generators. When* $\mathfrak{h}$ *is purely odd,* $F_{\mathfrak{h}}$ *is called the free fermion PPA. As an* H*-differential algebra, it is isomorphic to the* H*-differential algebra of polynomials in* N *odd generators below:*

$$F_{\mathfrak{h}} \cong \bigwedge(\partial_I u_i \mid i = 1, \dots, N), \tag{4.3.15}$$

where $\{u_1,\ldots,u_N\}$ *is a basis for* $\mathfrak{h}$, $\{\partial_1,\ldots,\partial_M\}$ *is a basis for* $\mathfrak{d}$, *and* ∂_I *is any ordered product of elements of* $\{\partial_1,\ldots,\partial_M\}$ *in* H.

Example 4.3.4. **(Affine LPA, PPA)** *Let* $\mathfrak{g}$ *be a Lie algebra with a nondegenerate invariant symmetric bilinear map* $\rho : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{d}$. *The affine LPA associated to* $\mathfrak{g}$ *is defined as the purely even left* H*-module*

$$L_{\mathfrak{g}} = H\mathfrak{g} \oplus \mathbb{F}K, \tag{4.3.16}$$

where $\partial K = 0$ *for any* $\partial \in \mathfrak{d}$. *The pseudobracket on the generators is:*

$$[a * b] = (1\otimes 1)\otimes_H [a,b] + (\rho(a,b)\otimes 1)\otimes_H K, \quad \text{for any } a,b\in\mathfrak{g}, \quad K \text{ is central.} \tag{4.3.17}$$

The universal PPA over the affine LPA $L_{\mathfrak{g}}$ *is the purely even symmetric algebra*

$$P(L_{\mathfrak{g}}) = S(H\mathfrak{g}\oplus\mathbb{F}K), \tag{4.3.18}$$

with the pseudobracket uniquely extended from (4.3.17) by the left and right Leibniz rules and the H*-bilinearity. The affine PPA at level* $k\in\mathbb{F}$ *is defined as the quotient* $P(L_{\mathfrak{g}})/(P(L_{\mathfrak{g}})(K-k))$, *or equivalently:*

$$A^k_{\mathfrak{g}} = S(H\mathfrak{g}), \tag{4.3.19}$$

together with the pseudobracket given by (4.3.17) on generators with $K = k$. *As an* H*-differential algebra, it is isomorphic to the* H*-differential algebra of polynomials in* N *generators*

$$A^k_{\mathfrak{g}} \cong \mathbb{F}[\,\partial_I u_i \mid i = 1,\ldots,N], \tag{4.3.20}$$

where $\{u_1,\ldots,u_N\}$ *is a basis for* $\mathfrak{g}$, $\{\partial_1,\ldots,\partial_M\}$ *is a basis for* $\mathfrak{d}$, *and* ∂_I *is any ordered product of elements of* $\{\partial_1,\ldots,\partial_M\}$ *in* H.

Example 4.3.5. (W_M **PPA**) *Let* $\mathfrak{d}$ *be an* M*-dimensional Lie algebra with a basis* $\{\partial_1,\ldots,\partial_M\}$. *The analogue of the Virasoro Lie conformal algebra in higher dimensions is the Lie pseudoalgebra* $W(\mathfrak{d})$ *reviewed in Example 2.3.3 (cf.* [1]*). We will also call this Lie pseudoalgebra* W_M *in the following discussion. Note by* H*-bilinearity, the pseudobracket given by (2.3.9) is* *equivalent to the one below on the basis elements of* $\mathfrak{d}$*: for any* $1 \le i, j \le M$,

$$[(1\otimes\partial_i)*(1\otimes\partial_j)] = (1\otimes 1)\otimes_H(1\otimes[\partial_i,\partial_j]) - (1\otimes\partial_i)\otimes_H(1\otimes\partial_j) + (\partial_j\otimes 1)\otimes_H(1\otimes\partial_i). \quad (4.3.21)$$

In particular, for $i = j$ *and* $\mathfrak{d}$ *is abelian, we have:*

$$\begin{aligned}[(1\otimes\partial_i)*(1\otimes\partial_i)] &= (\partial_i\otimes 1 - 1\otimes\partial_i)\otimes_H(1\otimes\partial_i) \\ &= -(\Delta(\partial_i) - 2(\partial_i\otimes 1))\otimes_H(1\otimes\partial_i). \end{aligned} \quad (4.3.22)$$

If we identify $-\partial_i\otimes 1$ *with* λ_i *and the action* ∂_i *on* $1\otimes\partial_i$ *with* ∂*, (4.3.22) becomes the* λ_i*-bracket in the Virasoro conformal algebra (cf.(2.3.7)). This shows that each element of the basis of* $\mathfrak{d}$*, or rather any arbitrary element of* $\mathfrak{d}$ *gives a Virasoro algebra in* $W(\mathfrak{d})$ *after the identification above. The* W_M *PPA is defined to be the universal PPA over* $W(\mathfrak{d})$*:*

$$P(W_M) = S(W(\mathfrak{d})) = S(U(\mathfrak{d})\otimes\mathfrak{d}), \quad (4.3.23)$$

with the pseudobracket defined on the generators by (4.3.21) and extended uniquely to $P(W_M)$ *by the left and right Leibniz rules and* H*-bilinearity. One can see that* $P(W_M)$ *is a purely even* H*-differential algebra of polynomials in* M *generators:*

$$P(W_M) \cong \mathbb{F}[\,\partial_I u_i \mid i = 1,\dots,M], \quad (4.3.24)$$

where $u_i = 1\otimes\partial_i$ *and* ∂_I *is any ordered product of elements of* $\{\partial_1,\dots,\partial_M\}$ *in* H.

CHAPTER

5

SPECTRAL SEQUENCES

In this chapter, we review the definition and some properties of spectral sequences, which will be the main tool in proving the main theorem on the energy operators. The first two subsections in this chapter are adapted from [24] to fit into our discussion here. Other references are [27] or [29]. The last subsection explores the properties of the spectral sequence arising from an exact sequence of complexes.

Spectral sequences are usually constructed from exact couples or from filtered complexes. In fact, these two methods are equivalent [24]. Spectral sequences arising from double complexes serve as examples of the latter. After we review the construction of a spectral sequence from a filtered complex, we will focus on the spectral sequence associated to a double complex.

5.1 Definition of a spectral sequence

To define a spectral sequence, one of the key ingredients is H-differential bigraded modules. To define such modules, we start by defining a decreasing filtration on a left H-module.

Definition 5.1.1. *A decreasing filtration on a left H-module M is a family of submodules: $\{F^pM | p \in \mathbb{Z}\}$ such that:*

$$\cdots \subset F^{p+1}M \subset F^pM \subset F^{p-1}M \subset \cdots \subset M. \tag{5.1.1}$$

One can similarly define an increasing filtration of a left H-module M by reversing the direction of (5.1.1), that is replacing it with

$$M \supset \cdots \supset F^{p+1}M \supset F^pM \supset F^{p-1}M \supset \cdots \tag{5.1.2}$$

Here and further, we will focus on decreasing filtrations. Another notion related to the filtration above is the grading that is defined as follows.

Definition 5.1.2. *A left H-module M is called $\mathbb{Z}$-graded if $M = \bigoplus_{p\in\mathbb{Z}} M^p$, where the submodule M^p is called the degree p component of M.*

The notions of a filtration and a grading (both under $\mathbb{Z}$) are related. On one hand, given a filtration on M, one obtains the associated graded module $\mathrm{gr}(M)$ by setting $\mathrm{gr}^p(M) = F^pM/F^{p+1}M$ (or $\mathrm{gr}^p(M) = F^pM/F^{p-1}M$ for an increasing filtration). On the other hand, if a left H-module M is graded, then one naturally gets the corresponding decreasing (respectively increasing) filtration by setting $F^pM = \bigoplus_{k\geq p} M^k$ (respectively $F^pM = \bigoplus_{k\leq p} M^k$). However, it may be difficult to recover the filtration of M if only knowing the associated graded module $\mathrm{gr}(M)$. Similarly to the $\mathbb{Z}$-grading, we have a notion of bigrading on a left H-module:

Definition 5.1.3. *A left H-module M is called bigraded if $M = \bigoplus_{p,q\in\mathbb{Z}} M^{p,q}$, where the submodule $M^{p,q}$ is called the bidegree (p,q) component of M.*

Definition 5.1.4. *A differential bigraded left H-module is a pair (E,d), where E is a bigraded left H-module: $E = \bigoplus_{p,q\in\mathbb{Z}} E^{p,q}$, and $d : E \to E$ is an H-module homomorphism of bidegree $(s,-s+1)$ for some $s \in \mathbb{Z}$ such that $d^2 = 0$.*

Definition 5.1.5. *A first quadrant cohomological spectral sequence is a collection of differential bigraded left H-modules $\{(E_r,d_r)\}_{r\in\mathbb{Z}_+}$ such that the bidegree of d_r is $(r,-r+1)$ and $E_{r+1}^{p,q} \cong H^{p,q}(E_r,d_r)$ for all $p,q,r \in \mathbb{Z}_+$.*

For a first quadrant homological spectral sequence, one only needs to impose the bidegree of d_r to be $(-r, r-1)$ in the definition above. Here and further, unless otherwise specified, we will call a first quadrant cohomological spectral sequence a spectral sequence for short. In practice, one can visualize the spectral sequence as a notebook with pages indexed by $r = 0, 1, \ldots$ On the r-th page, one has $E_r^{p,q}$ on the lattice point (p, q) and the differential $d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$ can be denoted as an arrow from $E_r^{p,q}$ to $E_r^{p+r,q-r+1}$. For $r = 0$, the spectral sequence looks like:

$$
\begin{array}{c|ccccc}
3 & E_0^{0,3} & E_0^{1,3} & E_0^{2,3} & E_0^{3,3} & E_0^{4,3} \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
2 & E_0^{0,2} & E_0^{1,2} & E_0^{2,2} & E_0^{3,2} & E_0^{4,2} \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
1 & E_0^{0,1} & E_0^{1,1} & E_0^{2,1} & E_0^{3,1} & E_0^{4,1} \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
0 & E_0^{0,0} & E_0^{1,0} & E_0^{2,0} & E_0^{3,0} & E_0^{4,0} \\
\hline
 & 0 & 1 & 2 & 3 & 4
\end{array}
$$

Figure 5.1: E_0-page of a first quadrant cohomological spectral sequence

The arrows pointing upwards in the figure denote the differential d_0. One sees immediately that $E_1^{p,q}$ is just the usual cohomology $H^{p,q}(E_0, d_0)$. For $r = 1$, the E_1-page looks like:

$$
\begin{array}{c|ccccccccc}
3 & E_1^{0,3} & \longrightarrow & E_1^{1,3} & \longrightarrow & E_1^{2,3} & \longrightarrow & E_1^{3,3} & \longrightarrow & E_1^{4,3} \\
2 & E_1^{0,2} & \longrightarrow & E_1^{1,2} & \longrightarrow & E_1^{2,2} & \longrightarrow & E_1^{3,2} & \longrightarrow & E_1^{4,2} \\
1 & E_1^{0,1} & \longrightarrow & E_1^{1,1} & \longrightarrow & E_1^{2,1} & \longrightarrow & E_1^{3,1} & \longrightarrow & E_1^{4,1} \\
0 & E_1^{0,0} & \longrightarrow & E_1^{1,0} & \longrightarrow & E_1^{2,0} & \longrightarrow & E_1^{3,0} & \longrightarrow & E_1^{4,0} \\
\hline
 & 0 & & 1 & & 2 & & 3 & & 4
\end{array}
$$

Figure 5.2: E_1-page of a first quadrant cohomological spectral sequence

Now the differential $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ is pointing to the right. We also list the E_2 and E_3 pages below:

$$
\begin{tikzcd}
3 & E_2^{0,3} \arrow[drr] & E_2^{1,3} \arrow[drr] & E_2^{2,3} \arrow[drr] & E_2^{3,3} & E_2^{4,3} \\
2 & E_2^{0,2} \arrow[drr] & E_2^{1,2} \arrow[drr] & E_2^{2,2} \arrow[drr] & E_2^{3,2} & E_2^{4,2} \\
1 & E_2^{0,1} \arrow[drr] & E_2^{1,1} \arrow[drr] & E_2^{2,1} \arrow[drr] & E_2^{3,1} & E_2^{4,1} \\
0 & E_2^{0,0} & E_2^{1,0} & E_2^{2,0} & E_2^{3,0} & E_2^{4,0} \\
 & 0 & 1 & 2 & 3 & 4
\end{tikzcd}
$$

Figure 5.3: E_2-page of a first quadrant cohomological spectral sequence

$$
\begin{tikzcd}
3 & E_3^{0,3} \arrow[ddrrr] & E_3^{1,3} \arrow[ddrrr] & E_3^{2,3} & E_3^{3,3} & E_3^{4,3} \\
2 & E_3^{0,2} \arrow[ddrrr] & E_3^{1,2} \arrow[ddrrr] & E_3^{2,2} & E_3^{3,2} & E_3^{4,2} \\
1 & E_3^{0,1} & E_3^{1,1} & E_3^{2,1} & E_3^{3,1} & E_3^{4,1} \\
0 & E_3^{0,0} & E_3^{1,0} & E_3^{2,0} & E_3^{3,0} & E_3^{4,0} \\
 & 0 & 1 & 2 & 3 & 4
\end{tikzcd}
$$

Figure 5.4: E_3-page of a first quadrant cohomological spectral sequence

Spectral sequences are quite powerful computational tools and usually the spectral sequence will "stabilize" after the first few pages displayed above.

In the remaining of this subsection, we review the construction of a spectral sequence associated to a filtered complex. Consider a cochain complex $(C = \bigoplus_{n\geq 0} C^n, d)$, with a decreasing filtration $\{F^p C\}_{p\in\mathbb{Z}}$ of C such that the following three properties are satisfied:

1. Each subspace C^n gets a decreasing filtration:
$$\cdots \subset F^{p+1}C^n \subset F^p C^n \subset F^{p-1}C^n \subset \cdots \subset C^n, \quad n \geq 0, \tag{5.1.3}$$
where $F^p C^n = F^p C \cap C^n$.

2. The differential d is compatible with the filtration:
$$d F^p C^n \subset F^p C^{n+1}, \quad n \geq 0, \quad p \in \mathbb{Z}. \tag{5.1.4}$$

3. The filtration is separated:
$$\bigcap_{p\in\mathbb{Z}} F^p C^n = 0, \quad n \geq 0. \tag{5.1.5}$$

The spectral sequence $\{(E_r, d_r)\}_{r\geq 0}$ associated to the filtered complex (C, d) is constructed as follows. For $p, q \in \mathbb{Z}$ and $r \geq -1$, let
$$\begin{aligned} Z_r^{p,q} &= \{\alpha \in \mathrm{F}^p\, C^{p+q} \mid d\alpha \in \mathrm{F}^{p+r}\, C^{p+q+1}\} \\ &= \mathrm{F}^p\, C^{p+q} \cap d^{-1}(\mathrm{F}^{p+r}\, C^{p+q+1}), \end{aligned} \tag{5.1.6}$$

and
$$\begin{aligned} B_r^{p,q} &= \{\alpha \in \mathrm{F}^p\, C^{p+q} \mid \alpha = d\beta \ \text{ for some } \ \beta \in \mathrm{F}^{p-r}\, C^{p+q-1}\} \\ &= \mathrm{F}^p\, C^{p+q} \cap d(\mathrm{F}^{p-r}\, C^{p+q-1}) \\ &= d Z_r^{p-r,q+r-1}. \end{aligned} \tag{5.1.7}$$

By definition, one sees that $dB_r^{p,q} = 0$ and
$$B_r^{p,q} \subset B_s^{p,q} \subset Z_s^{p,q} \subset Z_r^{p,q}, \qquad -1 \leq r \leq s.$$

Since by definition $Z_{r-1}^{p+1,q-1} = Z_r^{p,q} \cap \mathrm{F}^{p+1}\, C^{p+q}$, one defines
$$E_r^{p,q} := Z_r^{p,q}/(Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}), \qquad p, q \in \mathbb{Z}, \quad r \in \mathbb{Z}_+. \tag{5.1.8}$$

Since
$$dZ_r^{p,q} = B_r^{p+r,q-r+1} \subset Z_r^{p+r,q-r+1}$$
and
$$d(Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}) = B_{r-1}^{p+r+1,q-r} \subset Z_{r-1}^{p+r+1,q-r} + B_{r-1}^{p+r,q-r+1},$$
letting $\eta_r^{p,q} : Z_r^{p,q} \to E_r^{p,q}$ be the quotient map, the differential $d : Z_r^{p,q} \to Z_r^{p+r,q-r+1}$ induces linear maps

$$d_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1}, \quad d_r[\alpha] = [d\alpha], \tag{5.1.9}$$

where $[\alpha] := \eta_r^{p,q}(\alpha) \in E_r^{p,q}$ denotes the equivalence class of $\alpha \in Z_r^{p,q}$. It follows immediately that $d_r^2 = 0$ since $d^2 = 0$. One checks that the cohomology of (E_r, d_r) is isomorphic to E_{r+1} (see e.g. [24]):
$$(E_r^{p,q} \cap \ker d_r)/d_r E_r^{p-r,q+r-1} \simeq E_{r+1}^{p,q}, \tag{5.1.10}$$

so indeed we have a spectral sequence associated to the filtered complex.

5.2 The spectral sequence associated to a double complex

In this section, as an example of a spectral sequence associated to a filtered complex, we review the method to construct a spectral sequence associated to a double complex. First recall the definition of a double complex [24].

Definition 5.2.1. *A double complex is a triple (M, d', d''), where M is a bigraded left H-module: $M = \bigoplus_{p,q\in\mathbb{Z}} M^{p,q}$, $d' : M \to M$ is the "horizontal" differential of bidegree $(1,0)$ satisfying $(d')^2 = 0$, $d'' : M \to M$ is the "vertical" differential of bidegree $(0,1)$ satisfying $(d'')^2 = 0$, and d', d'' anticommute, that is, for any $p, q \in \mathbb{Z}$,*

$$d''_{p+1,q} d'_{p,q} + d'_{p,q+1} d''_{p,q} = 0. \tag{5.2.1}$$

The graph below shows how d', d'' look like, where the horizontal arrows denote d' and the vertical ones denote d''.

For a double complex, one can define the associated total complex, denoted as $(T(M), D)$, to be the graded left H-module $T(M) = \bigoplus_{n\in\mathbb{Z}} T(M)^n$, where $T(M)^n = \bigoplus_{p+q=n} M^{p,q}$, and

$$
\begin{tikzcd}
 & {} & {} & \\
{} \arrow[r] & M^{p,q+1} \arrow[r, "d'"] \arrow[u] & M^{p+1,q+1} \arrow[r] \arrow[u] & {} \\
{} \arrow[r] & M^{p,q} \arrow[r, "d'"] \arrow[u, "d''"] & M^{p+1,q} \arrow[r] \arrow[u, "d''"] & {} \\
 & {} \arrow[u] & {} \arrow[u] &
\end{tikzcd}
$$

Figure 5.5: Part of a double complex

the total differential:

$$D_n = \sum_{p+q=n} d'_{p,q} + d''_{p,q}. \tag{5.2.2}$$

Note that $(d')^2 = (d'')^2 = 0$ and the anticommutativity of d', d'' imply that $D^2 = 0$. The graph below shows the concept of an associated total complex. The degree n component $T(M)^n$ can be visualized as the direct sum of all $M^{p,q}$'s on the line $p+q=n$.

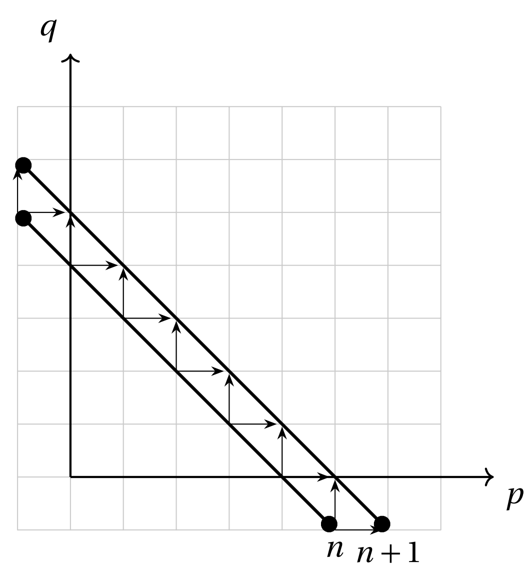


Figure 5.6: Total complex

Remark 5.2.1. *If we have a bigraded left H-module M but the differentials d', d'' commute instead of anticommute, one can check readily that by replacing $d''_{p,q}$ with $D''_{p,q} = (-1)^p d''_{p,q}$ (this is just alternating the signs by columns), (M, d', D'') is a double complex. This is useful when we apply the spectral sequence to an exact sequence in next section.*

In the following discussion, we focus on first quadrant double complexes, that is $M^{p,q} = 0$ whenever $p < 0$ or $q < 0$. Meanwhile, the cohomology $H(T(M), d)$ of the associated total complex of a double complex will be called the cohomology of the double complex. Before introducing the main theorem of the spectral sequence associated to a double complex, we review the construction of a filtration of $T(M)$ and the concept of convergence of a spectral sequence.

Let $F^p T(M)^n = \bigoplus_{i \geq p} M^{i,n-i}$. One checks that $\{F^p T(M) = \bigoplus_n F^p T(M)^n\}_{p \in \mathbb{Z}^+}$ is a decreasing filtration of $T(M)$, which satisfies conditions (5.1.4), (5.1.5), and $F^0 T(M) = T(M)$. In particular, for each n, $\{F^p T(M)^n\}_{p \in \mathbb{Z}^+}$ is a decreasing filtration of $T(M)^n$. In this filtration, one sees that $F^p T(M)^{p+q} / F^{p+1} T(M)^{p+q} = M^{p,q}$. Furthermore, if $(T(M), D)$ is a cochain complex, the 0-th page of the spectral sequence associated to the filtered complex $(T(M), D)$ is just the double complex. The other pages also follow from the construction of the spectral sequence associated to $(T(M), D)$. Similarly, one can define another decreasing filtration of $T(M)$ in the vertical direction by setting $F^q T(M)^n = \bigoplus_{j \geq q} M^{n-j,j}$.

Definition 5.2.2. *A spectral sequence is said to collapse at the N-th term if $d_r = 0$ for all $r \geq N$ but there exists a nontrivial d_{N-1}.*

Note that $d_N = 0$ implies that $E_{N+1}^{p,q} \cong H(E_N^{p,q}, d_N) = E_N^{p,q}$. Thus if a spectral sequence collapses at the N-th term, then $E_N^{p,q} \cong E_{N+1}^{p,q} \cong \cdots$ for all $p, q \geq 0$, and this common vector space is denoted by $E_\infty^{p,q}$.

Definition 5.2.3. *A spectral sequence is said to converge to a graded left H-module M if there exists a (decreasing) filtration F on M such that:*

$$E_\infty^{p,q} \cong F^p M^{p+q} / F^{p+1} M^{p+q}. \tag{5.2.3}$$

The main theorem on the cohomology of a double complex is the following:

Theorem 5.2.1 ([24]). *For any first quadrant double complex (M, d', d''), there exist two spectral sequences (E_r, d_r) and $(\widetilde{E}_r, \tilde{d}_r)$ whose E_2-pages are as follows: for any $p, q \in \mathbb{Z}_+$,*

$$E_2^{p,q} = H(H(M^{p,q}, d''), \overline{d'}) \quad \textit{and} \quad \widetilde{E}_2^{p,q} = H(H(M^{p,q}, d'), \overline{d''}) \quad \textit{respectively.} \tag{5.2.4}$$

Moreover, both spectral sequences converge to $H(T(M), D)$, the cohomology of the associated total complex of the double complex.

Remark 5.2.2. *By the anticommutativity of d' and d'' and the fact that $(\overline{d'})^2 = (\overline{d''})^2 = 0$ since they are induced from d' and d'' respectively, one can verify that $(H(M^{p,q}, d'), \overline{d''})$ and $(H(M^{p,q}, d''), \overline{d'})$ are differential bigraded modules. Theorem 5.2.1 provides two ways to compute the spectral sequences, which give us information of the associated total complex.*

5.3 Application of spectral sequence to an exact sequence

We apply the spectral sequence to an exact sequence in this section. As the first step, we naturally obtain a double complex associated to a cochain complex. To be specific, consider the following cochain complex of left H-modules:

$$0 \longrightarrow C_0 \xrightarrow{f_0} C_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{k-2}} C_{k-1} \xrightarrow{f_{k-1}} C_k \longrightarrow 0, \tag{5.3.1}$$

where for each $i \in \{0, \dots, k\}$, C_i itself is a cochain complex of left H-modules:

$$0 \longrightarrow C_i^0 \xrightarrow{d_{i,0}} C_i^1 \xrightarrow{d_{i,1}} C_i^2 \longrightarrow \cdots . \tag{5.3.2}$$

Since $f^2 = 0$, $d_i^2 = 0$ and f_j are chain maps, by doing a sign change given in Remark 5.2.1, we have the double complex associated to this cochain complex (see Figure 5.7 below).

From Section 5.2, we obtain a spectral sequence associated to this double complex, which will be called the spectral sequence associated to the cochain complex. The 0-th and 1-st page of this (cohomological) spectral sequence are shown below (see Figures 5.8 and 5.9 below). Note that for odd columns a minus sign is added; however, this does not affect the kernel and the image of the original differential, hence the cohomology stays the same as we see on the 1-st page. An important observation is the following lemma:

$$
\begin{tikzcd}
 & {} & {} & \\
{} \arrow[r] & C_p^{q+1} \arrow[u] \arrow[r, "f_p"] & C_{p+1}^{q+1} \arrow[u] \arrow[r] & {} \\
{} \arrow[r] & C_p^{q} \arrow[u, "(-1)^p d_{p,q}"] \arrow[r, "f_p"'] & C_{p+1}^{q} \arrow[u, "(-1)^{(p+1)} d_{p+1,q}"'] \arrow[r] & {} \\
 & {} \arrow[u] & {} \arrow[u] &
\end{tikzcd}
$$

Figure 5.7: Part of the double complex associated to a cochain complex

Lemma 5.3.1. *For the cochain complex given by* (5.3.1)*, the cohomological spectral sequence associated to it collapses at the* $(k+1)$*-th page.*

Proof. This is obvious since for the k-th page, considering the bidegree of the differential, the only nontrivial differentials are: $d_k : C_0^m \to C_k^{m-k+1}$, $m \geq k-1$, while all differentials on the $(k+1)$-th page are trivial. $\square$

$$
\begin{array}{c|ccccc}
3 & C_0^3 & C_1^3 & C_2^3 & C_3^3 & C_4^3 \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
2 & C_0^2 & C_1^2 & C_2^2 & C_3^2 & C_4^2 \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
1 & C_0^1 & C_1^1 & C_2^1 & C_3^1 & C_4^1 \\
 & \uparrow d_{0,0} & \uparrow -d_{1,0} & \uparrow d_{2,0} & \uparrow -d_{3,0} & \uparrow \\
0 & C_0^0 & C_1^0 & C_2^0 & C_3^0 & C_4^0 \\
\hline
 & 0 & 1 & 2 & 3 & 4
\end{array}
$$

Figure 5.8: 0-th page of a spectral sequence associated to a cochain complex

Next, we consider the case when the cochain complex (5.3.1) is exact. By exactness and Theorem 5.2.1, one immediately sees that the cohomology of the total complex is trivial since $H(C_p^q, f_p) = 0$. Moreover, by Lemma 5.3.1, the $(k+1)$-page is indeed a 0 page, that is all terms are 0, hence all the pages after are trivial. In the remainder of this section, we will explore the spectral sequence associated to an exact sequence given by (5.3.1) for $k = 2, 3$.

$$
\begin{array}{c|ccccccccc}
3 & H^{0,3} & \longrightarrow & H^{1,3} & \longrightarrow & H^{2,3} & \longrightarrow & H^{3,3} & \longrightarrow & H^{4,3} \\
2 & H^{0,2} & \longrightarrow & H^{1,2} & \longrightarrow & H^{2,2} & \longrightarrow & H^{3,2} & \longrightarrow & H^{4,2} \\
1 & H^{0,1} & \longrightarrow & H^{1,1} & \longrightarrow & H^{2,1} & \longrightarrow & H^{3,1} & \longrightarrow & H^{4,1} \\
0 & H^{0,0} & \xrightarrow{\bar{f}_0} & H^{1,0} & \xrightarrow{\bar{f}_1} & H^{2,0} & \xrightarrow{\bar{f}_2} & H^{3,0} & \xrightarrow{\bar{f}_3} & H^{4,0} \\
\hline
 & 0 & & 1 & & 2 & & 3 & & 4
\end{array}
$$

Figure 5.9: 1-st page of a spectral sequence associated to a cochain complex

For $k = 2$, (5.3.1) is just a short exact sequence:

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0, \tag{5.3.3}$$

where we use A, B, C in place of C_0, C_1, C_2, and i is injective and p is subjective. The following theorem is well known in homological algebra.

Theorem 5.3.1. *For a short exact sequence of cochain complexes (of left H-modules) given by (5.3.3), there exists a long exact sequence as follows:*

$$\cdots \to H^{n-1}(C) \xrightarrow{\partial_{n-1}} H^n(A) \xrightarrow{i_*} H^n(B) \xrightarrow{p_*} H^n(C) \xrightarrow{\partial_n} H^{n+1}(A) \to \cdots \tag{5.3.4}$$

where i_*, p_* are induced by i, p respectively, and $\partial_n : H^n(C) \to H^{n+1}(A)$ is the connecting homomorphism.

This theorem can be proved directly by verifying the exactness at each position using the definition of the maps. However, we give another proof here using the spectral sequence associated to the short exact sequence (5.3.3).

Proof. Consider the spectral sequence associated to (5.3.3). By Lemma 5.3.1, the spectral sequence collapses at the 3-rd page which is also a trivial page. Note that the 1-st page of the spectral sequence is the following:

$$
\begin{tikzcd}
0 \arrow[r] & H^{n+1}(A) \arrow[r, "i_*"] & H^{n+1}(B) \arrow[r, "p_*"] & H^{n+1}(C) \arrow[r] & 0 \\
0 \arrow[r] & H^{n}(A) \arrow[r, "i_*"] & H^{n}(B) \arrow[r, "p_*"] & H^{n}(C) \arrow[r] \arrow[ull, dashed, "\partial_n"'] & 0 \\
0 \arrow[r] & H^{n-1}(A) \arrow[r, "i_*"] & H^{n-1}(B) \arrow[r, "p_*"] & H^{n-1}(C) \arrow[r] \arrow[ull, dashed, "\partial_{n-1}"'] & 0
\end{tikzcd}
$$

Figure 5.10: 1-st page of the spectral sequence associated to a short exact sequence

where the connecting homomorphisms are added by dashed arrows. The theorem claims that this is indeed a long exact sequence. We first verify the exactness at $H^n(B)$. Note that on the 2-nd page of the spectral sequence, the differentials adjacent to $H(H^n(B))$ are both trivial. Hence $H(H^n(B)) = 0$ since the 3-rd page is a 0 page. This show that the long exact sequence is exact at $H^n(B)$.

$$
\begin{tikzcd}
0 \arrow[drr] & & & & & & \\
& 0 \arrow[drr] & H(H^{n+1}(A)) \arrow[drr, "d_2"] & H(H^{n+1}(B)) & H(H^{n+1}(C)) & & \\
& & H(H^{n}(A)) & H(H^{n}(B)) \arrow[drr] & H(H^{n}(C)) \arrow[drr] & & \\
& & H(H^{n-1}(A)) & H(H^{n-1}(B)) & H(H^{n-1}(C)) & 0 & 0
\end{tikzcd}
$$

Figure 5.11: 2-nd page of the spectral sequence associated to a short exact sequence

Another piece of information from the 2-nd page is that $d_2 : H(H^{n+1}(A)) \to H(H^n(C))$ is an isomorphism because the 3-rd page is a 0 page. Note that, in fact, $H(H^{n+1}(A)) = \ker(i_*)$

and $H(H^n(C)) = \mathrm{coker}(p_*)$. Consider the following sequence:

$$\cdots \xrightarrow{p_*} H^n(C) \xrightarrow{\rho_n} \mathrm{coker}(p_*) \overset{(d_2)^{-1}}{\cong} \ker(i_*) \xrightarrow{j_n} H^{n+1}(A) \xrightarrow{i_*} \cdots \tag{5.3.5}$$

where ρ_n is the projection and j_n is the injection. The connecting homorphism is defined as the composition: $\partial_n = j_n \circ (d_2)^{-1} \circ \rho_n : H^n(C) \to H^{n+1}(A)$. It is left to check the exactness at $H^n(C)$ and $H^{n+1}(A)$, which can be readily done:

$$\ker(\partial_n) = \ker(\rho_n) = \mathrm{im}(p_*) \quad \text{and} \quad \mathrm{im}(\partial_n) = \mathrm{im}(j_n) = \ker(i_*), \tag{5.3.6}$$

since ρ_n is subjective and j_n is injective. □

One sees that Theorem 5.3.1 is indeed a special application of the spectral sequence associated to the exact sequence for $k = 2$. Next, we prove a generalized theorem for $k = 3$ using the spectral sequence. This result will be necessary for our second main theorem on the energy operators. For $k = 3$, the exact sequence given by (5.3.1) is:

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \to 0, \tag{5.3.7}$$

where we replace C_i's with $A, \ldots, D$ and use f, g, h to denote the differentials. We will call this exact sequence a medium exact sequence.

Theorem 5.3.2. *For a medium exact sequence given by (5.3.7), there exists a medium exact sequence on the* 2*-nd page of the spectral sequence associated to (5.3.7) for any* $n \geq 0$*:*

$$0 \to H(H^{n+1}(B)) \xrightarrow{d_2^{1,n+1}} H(H^n(D)) \xrightarrow{\partial_n} H(H^{n+2}(A)) \xrightarrow{d_2^{0,n+2}} H(H^{n+1}(C)) \to 0, \tag{5.3.8}$$

where d_2 *is the differential on the* 2*-nd page of the spectral sequence (the double superscript denotes the starting position) and* ∂_n *is a connecting homomorphism.*

Compared to Theorem 5.3.1, the difference here is that we get a medium exact sequence on the 2-page now. It is tempting to construct a long exact sequence similar to Theorem 5.3.1 on the 1-st page; however, such a sequence is not exact anymore.

Proof. By Lemma 5.3.1, the 4-th page of the spectral sequence is a 0 page. The 3-rd page actually gives an isomorphism $d_3 : H(H(H^{n+2}(A))) \to H(H(H^n(D)))$ as shown in the Figure below.

$$
\begin{tikzcd}
H(H(H^{n+2}(A))) \arrow[ddrrr, "d_3"] & 0 & 0 & H(H(H^{n+2}(D))) \\
H(H(H^{n+1}(A))) & 0 & 0 & H(H(H^{n+1}(D))) \\
H(H(H^{n}(A))) & 0 & 0 & H(H(H^{n}(D)))
\end{tikzcd}
$$

Figure 5.12: 3-rd page of the spectral sequence associated to a medium exact sequence

The two columns in the middle are 0 columns since the differentials d_3 into and out of them are trivial and the 4-th page is a 0 page. This fact can be translated to the 2-nd page as: the differential map $d_2^{1,n+1}: H(H^{n+1}(B)) \to H(H^n(D))$ is injective and $d_2^{0,n+2}: H(H^{n+2}(A)) \to H(H^{n+1}(C))$ is surjective. Hence, we have shown the exactness of (5.3.8) at $H(H^{n+1}(B))$ and $H(H^{n+1}(C))$. Next, to define the connecting homomorphism ∂_n, note that $H(H(H^n(D))) =$

$$
\begin{tikzcd}
0 \arrow[drr] & H(H^{n+2}(A)) \arrow[drr, "d_2^{0,n+2}"] & H(H^{n+2}(B)) & H(H^{n+2}(C)) & H(H^{n+2}(D)) & \\
& H(H^{n+1}(A)) & H(H^{n+1}(B)) \arrow[drr, "d_2^{1,n+1}"] & H(H^{n+1}(C)) \arrow[drr] & H(H^{n+1}(D)) & \\
& H(H^{n}(A)) & H(H^{n}(B)) & H(H^{n}(C)) & H(H^{n}(D)) \arrow[uulll, dashed, "\partial_n"] & 0
\end{tikzcd}
$$

Figure 5.13: 2-nd page of the spectral sequence associated to a medium exact sequence

$\operatorname{coker}(d_2^{1,n+1})$ and $H(H(H^{n+2}(A))) = \ker(d_2^{0,n+2})$. Consider the following sequence:

$$
\cdots \xrightarrow{d_2^{1,n+1}} H(H^n(D)) \xrightarrow{\rho_n} \operatorname{coker}(d_2^{1,n+1}) \overset{(d_3)^{-1}}{\cong} \ker(d_2^{0,n+2}) \xrightarrow{j_n} H(H^{n+2}(A)) \xrightarrow{d_2^{0,n+2}} \cdots \tag{5.3.9}
$$

where ρ_n is the projection and j_n is the injection. One can define the connecting homomorphism $\partial_n := j_n \circ (d_3)^{-1} \circ \rho_n : H(H^n(D)) \to H(H^{n+2}(A))$. The exactness at $H(H^n(D))$ and $H(H^{n+2}(A))$ now follow from:

$$\ker(\partial_n) = \ker(\rho_n) = \operatorname{im}(d_2^{1,n+1}) \quad \text{and} \quad \operatorname{im}(\partial_n) = \operatorname{im}(j_n) = \ker(d_2^{0,n+2}). \tag{5.3.10}$$

□

Theorem 5.3.2 can be generalized to a general n based on Lemma 5.3.1 and the following observations:

1. On the k-th page, the spectral sequence gives an isomorphism d_k which allows one to define the connecting homomorphism ∂ (cf. Theorems 5.3.1 and 5.3.2);
2. One has the medium exact sequence involving the first and last two columns of the $(k-1)$-th page of the spectral sequence. This page is similar to Figure 5.13, but with $(k-3)$ zero columns in the middle.

We have indeed proved the following theorem:

Theorem 5.3.3. *Given a cochain complex (5.3.1) that is exact ($k \geq 3$), one has the following medium exact sequence on the $(k-1)$-th page of the spectral sequence associated to (5.3.1) for each $n \geq 0$,*

$$0 \to E_{k-1}^{n+k-2}(C_1) \xrightarrow{d_{k-1}^{1,n+k-2}} E_{k-1}^{n}(C_k) \xrightarrow{\partial_n} E_{k-1}^{n+k-1}(C_0) \xrightarrow{d_{k-1}^{0,n+k-1}} E_{k-1}^{n+1}(C_{k-1}) \to 0, \tag{5.3.11}$$

where $E_{k-1}^{m}(C_i)$ denotes the (i,m) term on the $(k-1)$-th page, d_{k-1} is the differential on the $(k-1)$-th page of the spectral sequence (the double superscript denotes the starting position), and ∂_n is the connecting homomorphism defined as $\partial_n := j_n \circ (d_k)^{-1} \circ \rho_n$, where $\rho_n : E_{k-1}^{n}(C_k) \to E_k^n(C_k)$ is the projection map and $j_n : E_k^{n+k-1}(C_0) \to E_{k-1}^{n+k-1}(C_0)$ is the injection map.

CHAPTER

6

COHOMOLOGY OF SOME POISSON PSEUDOALGEBRAS

We discuss the cohomology theory of PPAs in this chapter. After developing some general theory on the variational PPA cohomology, we describe the energy operators on the reduced and basic complexes and their cohomology assuming that H is commutative. Furthermore, using a spectral sequence, we show that, under certain assumptions on the PPA P and the P-module M, the energy operators N_1 and N_2 have only eigenvalues 0 and 1 on the reduced cohomology $H_{PP}(P, M)$ when $H = \mathbb{F}[\partial_1, \partial_2]$, which generalizes the energy operator on the PVA cohomology (cf. [5]). As an application of this result, we compute the cohomology of the free superfermion PPA $F_{\mathfrak{h}}$ for $H = \mathbb{F}[\partial_1, \partial_2]$.

6.1 General theory of cohomology of Poisson superpseudoalgebras

As in any cohomology theory, we start by defining the complexes in the sequel: both the basic and the reduced ones.

Definition 6.1.1 (**PPA modules**). *Given a cocommutative Hopf algebra H that is purely even, suppose P is a PPA with parity p, a module M over P is a vector superspace endowed with two module structures over P that are compatible: a module structure over P as an H-differential superalgebra, and as an LPA. Explicitly, let am denote the action of $a \in P$ on $m \in M$ in the differential superalgebra module and $a * m$ denote the action in the LPA module, then the following identities hold: suppose $a, b \in P$, $m \in M$ and $\sigma : H \otimes H \to H \otimes H$, $\sigma(h_1 \otimes h_2) = h_2 \otimes h_1$, then*

$$a(bm) = (ab)m, \tag{6.1.1}$$

$$a * (b * m) - (-1)^{p(a)p(b)}((\sigma \otimes 1) \otimes_H 1)(b * (a * m)) = [a * b]m, \tag{6.1.2}$$

and the compatibility conditions:

$$a * (bm) = [a * b]m + (-1)^{p(a)p(b)} b(a * m), \tag{6.1.3}$$

$$(ab) * m = a(b * m) + (-1)^{p(b)p(m)}(a * m)b, \tag{6.1.4}$$

where $[a * b]m$ is given by (4.2.2), $b(a * m)$ is given by (4.2.4) and $(a * m)b$ is defined as:

$$(a * m)b = (-1)^{p(a)p(b)+p(a)p(m)} \sum_i (g_{i(1)} \otimes f_i) \otimes_H (g_{i(-2)} b) v_i(a \otimes m), \tag{6.1.5}$$

if $a * m = \sum_i (f_i \otimes g_i) \otimes_H v_i(a \otimes m)$, and $v_i : P \otimes M \to M$ is a linear map for any i.

For example, the base field $\mathbb{F}$ is P-module with trivial actions: $am = a * m = 0$ for any $a \in P, m \in \mathbb{F}$. Another example is P is a P-module where am is just the usual product in P and $a * m = [a * m]$. In this case P is called the adjoint module.

6.1.1 Cohomology of Lie superpseudoalgebras

We review the cohomology of a LPA in this subsection. This material is adapted from [1]. In the following discussion, assume the LPA P has a parity p and let $\bar{p} = 1 - p$ denote the reverse parity. For a P-module M, the reduced cochain complex $C_{LP}(P, M)$ is defined as:

$$C_{LP}(P, M) := \bigoplus_{n \geq 0} C^n_{LP}(P, M), \tag{6.1.6}$$

where $C^n_{LP}(P, M)$ consists of all n-cochains $\gamma \in \mathrm{Hom}_{H^{\otimes n}}(P^{\otimes n}, H^{\otimes n} \otimes_H M)$, with the parity $\bar{p}$ induced from $\bar{p}$ of P and M, that satisfy some symmetry conditions. Explicitly, an n-cochain

$\gamma : P^{\otimes n} \to H^{\otimes n} \otimes_H M$ satisfies the H-multilinearity:

$$\gamma(h_1 a_1 \otimes \cdots \otimes h_n a_n) = ((h_1 \otimes \cdots \otimes h_n) \otimes_H 1)\, \gamma(a_1 \otimes \cdots \otimes a_n), \tag{6.1.7}$$

and the symmetry conditions:

$$\gamma(a_1 \otimes \cdots \otimes a_{i+1} \otimes a_i \otimes \cdots \otimes a_n) = (-1)^{\bar{p}(a_i)\bar{p}(a_{i+1})}(\sigma_{i,i+1} \otimes_H 1)\, \gamma(a_1 \otimes \cdots \otimes a_n), \tag{6.1.8}$$

for all $1 \le i \le n-1$, where $\sigma_{i,i+1} : H^{\otimes n} \to H^{\otimes n}$ is the transposition of the i-th and $(i+1)$-th factors in the tensor product.

By definition of $C^n_{LP}(P,M)$, we have $C^0_{LP}(P,M) = \mathbb{F} \otimes_H M \cong M/H^+M$ where $H^+ = \{h \in H \mid \epsilon(h) = 0\}$ is called the augmentation ideal. For $n = 1$, $C^1_{LP}(P,M) = \mathrm{Hom}_H(P,M)$ is the set of all H-module homomorphisms from P to M. For $n = 2$, $C^2_{LP}(P,M)$ consists of all $\gamma : P \otimes P \to (H \otimes H) \otimes_H M$ that satisfy the H-bilinearity and the symmetry condition $\gamma(a_1 \otimes a_2) = (-1)^{\bar{p}(a_1)\bar{p}(a_2)}(\sigma_{1,2} \otimes_H 1)\, \gamma(a_2 \otimes a_1)$.

The differential $d : C^n_{LP}(P,M) \to C^{n+1}_{LP}(P,M)$ for $n \ge 1$ is given by:

$$\begin{aligned}(d\gamma)(a_1 \otimes \cdots \otimes a_{n+1}) = &\sum_{1 \le i \le n+1} (-1)^{q_i}(\sigma_{1\to i} \otimes_H \mathrm{id}) a_i * \gamma(a_1 \otimes \cdots \otimes \hat{a}_i \otimes \cdots \otimes a_{n+1}) \\ &+ \sum_{1 \le i < j \le n+1} (-1)^{q_{ij}}(\sigma_{1\to i, 2\to j} \otimes_H \mathrm{id})\, \gamma([a_i * a_j] \otimes a_1 \otimes \cdots \otimes \hat{a}_i \otimes \cdots \otimes \hat{a}_j \otimes \cdots \otimes a_{n+1}),\end{aligned} \tag{6.1.9}$$

where $\hat{a}_i$ means a_i is omitted in the tensor product, $\sigma_{1\to i} : H^{\otimes(n+1)} \to H^{\otimes(n+1)}$ is the permutation

$$\sigma_{1\to i}(h_i \otimes h_1 \otimes \cdots \otimes h_{i-1} \otimes h_{i+1} \otimes \cdots \otimes h_{n+1}) = h_1 \otimes \cdots \otimes h_{n+1}, \tag{6.1.10}$$

$\sigma_{1\to i, 2\to j} : H^{\otimes(n+1)} \to H^{\otimes(n+1)}$ is the permutation

$$\sigma_{1\to i,2\to j}(h_i \otimes h_j \otimes h_1 \otimes \cdots \otimes h_{i-1} \otimes h_{i+1} \otimes \cdots \otimes h_{j-1} \otimes h_{j+1} \otimes \cdots \otimes h_{n+1}) = h_1 \otimes \cdots \otimes h_{n+1}, \tag{6.1.11}$$

and $a * \gamma(a_1 \otimes \cdots \otimes \hat{a}_i \otimes \cdots \otimes a_{n+1})$ is defined in the following way. It suffices to define $a * (f \otimes b)$

where $f \in H^{\otimes n}$. Suppose that

$$a * b = \sum_i f_i \otimes_H v_i \in H^{\otimes 2} \otimes_H M, \tag{6.1.12}$$

then we set

$$a * (f \otimes b) = \sum_i (1 \otimes f)(\mathrm{id} \otimes \Delta^{(n-1)})(f_i) \otimes_H v_i \in H^{\otimes(n+1)} \otimes_H M, \tag{6.1.13}$$

where $\Delta^{(n-1)}$ is the iterated coproduct. For $\gamma([a_i * a_j] \otimes a_1 \otimes \cdots \otimes \hat{a}_i \otimes \cdots \otimes \hat{a}_j \otimes \cdots \otimes a_{n+1})$, if

$$\gamma(a_1 \otimes \cdots \otimes a_n) = \sum_i g_i \otimes_H v_i \in H^{\otimes n} \otimes_H M, \tag{6.1.14}$$

and $g \in H^{\otimes 2}$, then we define

$$\gamma((g \otimes a_1) \otimes a_2 \otimes \cdots \otimes a_n) = \sum_i (g \otimes 1^{\otimes(n-1)})(\Delta \otimes \mathrm{id}^{\otimes(n-1)})(g_i) \otimes_H v_i \in H^{\otimes(n+1)} \otimes_H M. \tag{6.1.15}$$

The signs q_i and q_{ij} in (6.1.9) are given by:

$$q_i = \bar{p}(a_i)\big(\bar{p}(\gamma) + \bar{p}(a_1) + \cdots + \bar{p}(a_{i-1}) + 1\big) + 1, \tag{6.1.16}$$

$$\begin{aligned} q_{ij} = {} & \bar{p}(\gamma) + \bar{p}(a_i)\big(\bar{p}(a_1) + \cdots + \bar{p}(a_{i-1}) + 1\big) \\ & + \bar{p}(a_j)\big(\bar{p}(a_1) + \cdots + \bar{p}(\hat{a}_i) + \cdots + \bar{p}(a_{j-1})\big), \end{aligned} \tag{6.1.17}$$

where $\bar{p}(\hat{a}_i)$ means $\bar{p}(a_i)$ is omitted.

For $n = 0$, the differential $d : C^0_{LP}(P, M) \to C^1_{LP}(P, M)$ is defined as follows. Suppose that $a * m = \sum_i h_i \otimes_H v_i \in H^{\otimes 2} \otimes_H M$; then

$$(d(1 \otimes_H m))(a) = -(-1)^{\bar{p}(a)(\bar{p}(m)+1)} \sum_i (\mathrm{id} \otimes \epsilon)(h_i) v_i \in M. \tag{6.1.18}$$

By convention, for $n < 0$ we assume that $C^n_{LP}(P, M) = 0$, thus $d = 0$ when $n < 0$. One can verify that $d^2 = 0$, hence $C_{LP}(P, M)$ is a cochain complex. The cohomology of this complex is denoted as $H_{LP}(P, M) = \bigoplus_{n \geq 0} H^n_{LP}(P, M)$, where

$$H^n_{LP}(P, M) = \frac{\ker d_n}{\mathrm{im}\, d_{n-1}}, \qquad n \geq 0, \tag{6.1.19}$$

and is called the reduced cohomology of P (as an LPA) with coefficients in M. Similarly, one can define the basic cohomology of P (as an LPA) with coefficients in M by replacing $\otimes_H$ with $\otimes$ everywhere in the reduced cohomology above. For notation, we will use $\widetilde{C}_{LP}(P,M)=\bigoplus_{n\geq 0}\widetilde{C}^n_{LP}(P,M)$ to denote the basic complex and $\widetilde{H}_{LP}(P,M)=\bigoplus_{n\geq 0}\widetilde{H}^n_{LP}(P,M)$ to denote the basic cohomology.

For low degree cohomology of an LPA P, the following results are proved in [1]. Let M,N be two left H-modules, and $\mathrm{Chom}(M,N)$ denote the set of all $\mathbb{F}$-linear maps $\phi : M \to (H\otimes H)\otimes_H N$ such that

$$\phi(hm)=((1\otimes h)\otimes_H 1)\phi(m), \quad \text{for any} \quad h\in H,\ m\in M. \tag{6.1.20}$$

When M and N are P-modules, then $\mathrm{Chom}(M,N)$ has a natural structure of a P-module (see [1]).

Theorem 6.1.1 ([1]). *1. There exists a one-to-one correspondence between the elements of $H^1_{LP}(P, Chom(N,M))$ and the isomorphism classes of H-split extensions*

$$0\to M\to E\to N\to 0 \tag{6.1.21}$$

of finite modules over an LPA P.

2. *For a P-module C regarded as an LPA with trivial pseudobracket, there exists a bijective correspondence between the elements of $H^2(P,C)$ and the equivalence classes of H-split abelian extensions of P:*

$$0\to C\to \hat{P}\to P\to 0. \tag{6.1.22}$$

3. *There exists a bijective correspondence between the elements of $H^2(P,P)$ and the equivalence classes of the first-order deformations of P.*

In the remaining of this section, we will assume $H=U(\mathfrak{d})$ to be the universal enveloping algebra of a finite dimensional Lie algebra $\mathfrak{d}$ and fix a basis $\{\partial_1,\dots,\partial_N\}$ for $\mathfrak{d}$. The augmentation ideal H^+ consists of all sums of products of ∂_i's. Furthermore, we define an action of ∂_i

on $\widetilde{C}^n_{LP}(P,M)$ for each $i \in \{1,\dots,N\}$ as follows: for $\widetilde{\gamma} \in \widetilde{C}^n_{LP}(P,M)$,

$$(\partial_i \widetilde{\gamma})(a_1 \otimes \cdots \otimes a_n) = (1^{\otimes n} \otimes \partial_i - \partial_{i(1)} \otimes \cdots \otimes \partial_{i(n)} \otimes 1)\, \widetilde{\gamma}(a_1 \otimes \cdots \otimes a_n). \tag{6.1.23}$$

The right hand side is again an n-cochain if H is commutative, thus in this setting indeed $H = \mathbb{F}[\partial_1,\dots,\partial_N]$. Formula (6.1.23) gives an action of H on $\widetilde{C}_{LP}(P,M)$ and each ∂_i is an even endomorphism of $\widetilde{C}^n_{LP}(P,M)$ for all $n \geq 0$. Based on the discussion above, we will assume $H = \mathbb{F}[\partial_1,\dots,\partial_N]$ for the rest of this subsection.

Remark 6.1.1. *Since H is assumed to be commutative, one can rewrite the action (6.1.23) using the $\vec{\lambda}$ notation (cf. Example 4.3.1). Consider a cochain $\gamma : P^{\otimes n} \to H^{\otimes n} \otimes_H M$. For $i = 1,\dots,n$, let $\vec{\lambda}_i = (\lambda_{i1},\dots,\lambda_{iN})$ and $\vec{\partial} = (\partial_1,\dots,\partial_N)$. If we identify λ_{ij} with $-1 \otimes \cdots \otimes 1 \otimes \partial_j \otimes 1 \otimes \cdots \otimes 1$, where ∂_j is at the i-th position in the tensor product, then γ can be thought of as a map*

$$\gamma : P^{\otimes n} \to M[\vec{\lambda}_1,\dots,\vec{\lambda}_n]/\langle \partial_j + \lambda_{1j} + \cdots + \lambda_{nj}\rangle_{1\leq j\leq N} = M[\vec{\lambda}_1,\dots,\vec{\lambda}_n]/\langle \vec{\partial} + \vec{\lambda}_1 + \cdots + \vec{\lambda}_n\rangle.$$

This is similar to an LCA cochain other than the $\vec{\lambda}_i$'s are now vectors. Using the $\vec{\lambda}$-notation, the sesquilinearity conditions and symmetry conditions that γ should satisfy can be written as follows.
Sesquilinearity conditions: for all $1 \leq i \leq n$, $1 \leq j \leq N$,

$$\gamma_{\vec{\lambda}_1,\dots,\vec{\lambda}_n}(a_1 \otimes \cdots \otimes (\partial_j a_i) \otimes \cdots \otimes a_n) = -\lambda_{ij}\gamma_{\vec{\lambda}_1,\dots,\vec{\lambda}_n}(a_1 \otimes \cdots \otimes a_n), \tag{6.1.24}$$

and symmetry conditions:

$$\gamma_{\vec{\lambda}_1,\dots,\vec{\lambda}_{i+1},\vec{\lambda}_i,\dots,\vec{\lambda}_n}(a_1 \otimes \cdots \otimes a_{i+1} \otimes a_i \cdots \otimes a_n) = (-1)^{\bar{p}(a_i)\bar{p}(a_{i+1})}\gamma_{\vec{\lambda}_1,\dots,\vec{\lambda}_n}(a_1 \otimes \cdots \otimes a_n), \tag{6.1.25}$$

where we use the notation $\gamma_{\vec{\lambda}_1,\dots,\vec{\lambda}_N}(a_1 \otimes \cdots \otimes a_n)$ to denote the image of $a_1 \otimes \cdots \otimes a_n \in P^{\otimes n}$ under γ. Similarly to the reduced case, a cochain $\widetilde{\gamma} \in \widetilde{C}^n_{LP}(P,M)$ can be thought of as a map $\widetilde{\gamma} : P^{\otimes n} \to M[\vec{\lambda}_1,\dots,\vec{\lambda}_n]$. The sesquilinearity conditions and symmetry conditions are again given by (6.1.24) and (6.1.25) with γ's replaced by $\widetilde{\gamma}$'s. The actions of ∂_i's given by (6.1.23) on the basic complex $\widetilde{C}^n_{LP}(P,M)$ can be written in $\vec{\lambda}$-notation as: for all $i = 1,\dots,N$,

$$(\partial_i \widetilde{\gamma})_{\vec{\lambda}_1,\dots,\vec{\lambda}_n}(a_1 \otimes \cdots \otimes a_n) = (\partial_i + \lambda_{1i} + \cdots + \lambda_{ni})\, \widetilde{\gamma}_{\vec{\lambda}_1,\dots,\vec{\lambda}_n}(a_1 \otimes \cdots \otimes a_n), \tag{6.1.26}$$

or more concisely using the vector notation:

$$(\vec{\partial}\widetilde{\gamma})_{\vec{\lambda}_1,\dots,\vec{\lambda}_n}(a_1\otimes\cdots\otimes a_n)=(\vec{\partial}+\vec{\lambda}_1+\cdots+\vec{\lambda}_n)\,\widetilde{\gamma}_{\vec{\lambda}_1,\dots,\vec{\lambda}_n}(a_1\otimes\cdots\otimes a_n). \tag{6.1.27}$$

This observation allows one to translate many results of LCA over to the LPA case when H is commutative as some important ones given below.

Lemma 6.1.1. *For any $i\in\{1,\dots,N\}$, the action of ∂_i commutes with the differential d, thus $\partial_i\widetilde{C}^n_{LP}(P,M)$ is a subcomplex of $\widetilde{C}^n_{LP}(P,M)$.*

The proof of this lemma is a straightforward generalization of the proof of Lemma 2.2 in [6] if we regard the LPA as an LCA in higher dimensions and use the $\vec{\lambda}$-notation (cf. Remark 6.1.1 above).

Lemma 6.1.2. *For any $i\in\{1,\dots,N\}$, ∂_i is an injective endomorphism of $\widetilde{C}^n_{LP}(P,M)$ for all $n\geq 1$. In particular, it is an isomorphism between $\widetilde{C}^n_{LP}(P,M)$ and $\partial_i\widetilde{C}^n_{LP}(P,M)$ for all $n\geq 1$. Moreover, if M is a free left H-module, then $\widetilde{C}^n_{LP}(P,M)\cong\partial_i\widetilde{C}^n_{LP}(P,M)$ for all $n\geq 0$.*

As with the previous lemma, if we regard the LPA as an LCA in higher dimensions and use the $\vec{\lambda}$-notation, this is essentially proved in Proposition 2.1 in [6]. Note that if M is a free H-module, when $n=0$, then $\partial_i: M\to M$ is also injective.

Consider the following quotient map:

$$\pi: M[\vec{\lambda}_1,\dots,\vec{\lambda}_n]\to M[\vec{\lambda}_1,\dots,\vec{\lambda}_n]/\langle\vec{\partial}+\vec{\lambda}_1+\cdots+\vec{\lambda}_n\rangle, \tag{6.1.28}$$

with which one can define a linear map:

$$\epsilon:\widetilde{C}^n_{LP}(P,M)\to C^n_{LP}(P,M),\quad \epsilon(\widetilde{\gamma})=\pi\circ\widetilde{\gamma}. \tag{6.1.29}$$

Note that $\pi\circ(\partial_i\widetilde{\gamma})=0$ for all $i=1,\dots,N$ and $\widetilde{\gamma}\in\widetilde{C}^n_{LP}(P,M)$. Hence ϵ induces a linear map:

$$\widetilde{\epsilon}:\widetilde{C}^n_{LP}(P,M)/(\partial_1\widetilde{C}^n_{LP}(P,M)+\cdots+\partial_N\widetilde{C}^n_{LP}(P,M))\to C^n_{LP}(P,M),\quad \widetilde{\epsilon}([\widetilde{\gamma}])=[\pi\circ\widetilde{\gamma}]. \tag{6.1.30}$$

Proposition 6.1.1. *Suppose that $H = \mathbb{F}[\partial_1, \ldots, \partial_N]$. Then $\widetilde{\epsilon}$, defined by* (6.1.30)*, is injective for all $n \geq 0$. Moreover, if P is an LPA that is free as a left H-module, then $\widetilde{\epsilon}$ is an isomorphism for all $n \geq 0$.*

Proof. Note that $\epsilon(\widetilde{\gamma}) = \pi \circ \widetilde{\gamma}$ satisfies the sesquilinearity conditions (6.1.24) and the symmetry conditions (6.1.25), since $\widetilde{\gamma}$ satisfies these conditions. By definition, $\widetilde{\epsilon}$ is linear, so it suffices to prove that it is injective and surjective.

To see that $\widetilde{\epsilon}$ is injective, we prove that $\ker(\epsilon) = \partial_1 \widetilde{C}^n_{LP}(P, M) + \cdots + \partial_N \widetilde{C}^n_{LP}(P, M)$. The inclusion $\partial_1 \widetilde{C}^n_{LP}(P, M) + \cdots + \partial_N \widetilde{C}^n_{LP}(P, M) \subset \ker(\epsilon)$ is obvious since $\epsilon(\widetilde{\partial_i \gamma}) = \pi \circ (\partial_i \widetilde{\gamma}_i) = 0$. To prove the other direction, let $\widetilde{\gamma} \in \ker(\epsilon)$, that is $(\epsilon(\widetilde{\gamma})_{\vec{\lambda}_1, \ldots, \vec{\lambda}_n})(a_1 \otimes \cdots \otimes a_n) = \pi(\widetilde{\gamma}_{\vec{\lambda}_1, \ldots, \vec{\lambda}_n}(a_1 \otimes \cdots \otimes a_n)) = 0$ for any $a_1 \otimes \cdots \otimes a_n \in P^{\otimes n}$. This implies $\widetilde{\gamma}_{\vec{\lambda}_1, \ldots, \vec{\lambda}_n}(a_1 \otimes \cdots \otimes a_n) \in \langle \vec{\partial} + \vec{\lambda}_1 + \cdots + \vec{\lambda}_n \rangle \subset M[\vec{\lambda}_1, \ldots, \vec{\lambda}_n]$. Recall that $\langle \vec{\partial} + \vec{\lambda}_1 + \cdots + \vec{\lambda}_n \rangle = \langle \partial_j + \lambda_{1j} + \cdots + \lambda_{nj} | 1 \leq j \leq N \rangle$, hence one can find $(\widetilde{\gamma}_j)_{\vec{\lambda}_1, \ldots, \vec{\lambda}_n}(a_1 \otimes \cdots \otimes a_n)$, $j = 1, \ldots, N$, such that:

$$\widetilde{\gamma}_{\vec{\lambda}_1, \ldots, \vec{\lambda}_n}(a_1 \otimes \cdots \otimes a_n) = \sum_{j=1}^{N} (\partial_j + \lambda_{1j} + \cdots + \lambda_{nj})(\widetilde{\gamma}_j)_{\vec{\lambda}_1, \ldots, \vec{\lambda}_n}(a_1 \otimes \cdots \otimes a_n). \tag{6.1.31}$$

Note that in general $\{(\widetilde{\gamma}_j)_{\vec{\lambda}_1, \ldots, \vec{\lambda}_n}(a_1 \otimes \cdots \otimes a_n)\}_{1 \leq j \leq N}$ may not be unique. One can obtain a family of $\{(\widetilde{\gamma}_j)_{\vec{\lambda}_1, \ldots, \vec{\lambda}_n}(a_1 \otimes \cdots \otimes a_n)\}_{1 \leq j \leq N}$ with the following way of "division" algorithm. First write

$$\widetilde{\gamma}_{\vec{\lambda}_1, \ldots, \vec{\lambda}_n}(a_1 \otimes \cdots \otimes a_n) = (\partial_1 + \lambda_{11} + \cdots + \lambda_{n1})(\widetilde{\gamma}_1)_{\vec{\lambda}_1, \ldots, \vec{\lambda}_n}(a_1 \otimes \cdots \otimes a_n) + \widetilde{r}_1(a_1 \otimes \cdots \otimes a_n), \tag{6.1.32}$$

where $(\widetilde{\gamma}_1)_{\vec{\lambda}_1, \ldots, \vec{\lambda}_n}(a_1 \otimes \cdots \otimes a_n), (\widetilde{r}_1)_{\vec{\lambda}_1, \ldots, \vec{\lambda}_n}(a_1 \otimes \cdots \otimes a_n) \in M[\vec{\lambda}_1, \ldots, \vec{\lambda}_n]$ and $(\widetilde{r}_1)_{\vec{\lambda}_1, \ldots, \vec{\lambda}_n}(a_1 \otimes \cdots \otimes a_n) \notin \langle \partial_1 + \lambda_{11} + \cdots + \lambda_{n1} \rangle$. One can compute (6.1.32) for all elements in $P^{\otimes n}$, thus we have $\widetilde{\gamma}_1$ and $\widetilde{r}_1$ are linear maps from $P^{\otimes n}$ to $M[\vec{\lambda}_1, \ldots, \vec{\lambda}_n]$. To see that these are indeed cochains in $\widetilde{C}^n_{LP}(P, M)$, we check the sesquilinearity and symmetry conditions of $\widetilde{\gamma}_1$ and $\widetilde{r}_1$. For the sesquilinearity, by the sesquilinearity of $\widetilde{\gamma}$, one has for each $j = 1, \ldots, N$,

$$\begin{aligned}
&\widetilde{\gamma}_{\vec{\lambda}_1, \ldots, \vec{\lambda}_n}(a_1 \otimes \cdots \otimes \partial_j a_i \otimes \cdots \otimes a_n) \\
=&-\lambda_{ij} \widetilde{\gamma}_{\vec{\lambda}_1, \ldots, \vec{\lambda}_n}(a_1 \otimes \cdots \otimes a_n) \\
=&-\lambda_{ij}(\partial_1 + \lambda_{11} + \cdots + \lambda_{n1})\widetilde{\gamma}_1(a_1 \otimes \cdots \otimes a_n) - \lambda_{ij} \widetilde{r}_1(a_1 \otimes \cdots \otimes a_n) \\
=&(\partial_1 + \lambda_{11} + \cdots + \lambda_{n1})\widetilde{\gamma}_1(a_1 \otimes \cdots \otimes \partial_j a_i \otimes \cdots \otimes a_n) + \widetilde{r}_1(a_1 \otimes \cdots \otimes \partial_j a_i \otimes \cdots \otimes a_n),
\end{aligned} \tag{6.1.33}$$

which implies that $\widetilde{\gamma}_1$ and $\widetilde{r}_1$ satisfies the sesquilinearity conditions. By the symmetry conditions of $\widetilde{\gamma}$,

$$
\begin{aligned}
&\widetilde{\gamma}_{\vec{\lambda}_1,\dots,\vec{\lambda}_n}(a_1\otimes\cdots\otimes a_{i+1}\otimes a_i\otimes\cdots\otimes a_n)\\
=&(-1)^{\bar{p}(a_i)\bar{p}(a_{i+1})}\widetilde{\gamma}_{\vec{\lambda}_1,\dots,\vec{\lambda}_n}(a_1\otimes\cdots\otimes a_n)\\
=&(-1)^{\bar{p}(a_i)\bar{p}(a_{i+1})}(\partial_1+\lambda_{11}+\cdots+\lambda_{n1})(\widetilde{\gamma}_1)_{\vec{\lambda}_1,\dots,\vec{\lambda}_n}(a_1\otimes\cdots\otimes a_n)+(-1)^{\bar{p}(a_i)\bar{p}(a_{i+1})}\widetilde{r}_1(a_1\otimes\cdots\otimes a_n)\\
=&(\partial_1+\lambda_{11}+\cdots+\lambda_{n1})(\widetilde{\gamma}_1)_{\vec{\lambda}_1,\dots,\vec{\lambda}_n}(a_1\otimes\cdots\otimes a_{i+1}\otimes a_i\otimes\cdots\otimes a_n)+\widetilde{r}_1(a_1\otimes\cdots\otimes a_{i+1}\otimes a_i\otimes\cdots\otimes a_n),
\end{aligned}
\tag{6.1.34}
$$

which implies the symmetry conditions of $\widetilde{\gamma}_1$ and $\widetilde{r}_1$. Hence $\widetilde{\gamma}_1$ and $\widetilde{r}_1$ are cochains in $\widetilde{C}^n_{LP}(P,M)$. One can repeat the steps above on $\widetilde{r}_1$ (by factoring out $(\partial_2+\lambda_{12}+\cdots+\lambda_{n2})$ this time) and eventually writing $\widetilde{\gamma}_{\vec{\lambda}_1,\dots,\vec{\lambda}_n}(a_1\otimes\cdots\otimes a_n)$ in (6.1.31) with all $\widetilde{\gamma}_j\in\widetilde{C}^n_{LP}(P,M)$. This proves that $\widetilde{\gamma}\in\partial_1\widetilde{C}^n_{LP}(P,M)+\cdots+\partial_N\widetilde{C}^n_{LP}(P,M)$, so $\ker(\epsilon)\subset\partial_1\widetilde{C}^n_{LP}(P,M)+\cdots+\partial_N\widetilde{C}^n_{LP}(P,M)$. Together with the other direction proved earlier, we have $\ker(\epsilon)=\partial_1\widetilde{C}^n_{LP}(P,M)+\cdots+\partial_N\widetilde{C}^n_{LP}(P,M)$, thus $\ker(\widetilde{\epsilon})=0$ and $\widetilde{\epsilon}$ is injective.

It is left to show that $\widetilde{\epsilon}$ is surjective. By (6.1.30), it suffices to prove that ϵ is surjective. For $n=0$, $\epsilon: M\to M/(\partial_1M+\cdots+\partial_NM)$ is just the quotient map, hence surjective. For $n\geq 1$, suppose that $\gamma\in C^n_{LP}(P,M)$. Since P is a free left H-module, we can write $P=HU$ where U is a generating set of P as a left H-module. Note that γ is uniquely determined by its values on $U^{\otimes n}$ by sesquilinearity and satisfies the symmetry conditions given by (6.1.25). Consider the identification maps ($k=1,\dots,n$):

$$
\begin{aligned}
&\iota_k: M[\vec{\lambda}_1,\dots,\vec{\lambda}_n]/\langle\vec{\partial}+\vec{\lambda}_1+\cdots+\vec{\lambda}_n\rangle\to M[\vec{\lambda}_1,\dots,\vec{\lambda}_n],\\
&\iota_k\big([P(\vec{\lambda}_1,\dots,\vec{\lambda}_k,\dots,\vec{\lambda}_n)]\big)=\bar{P}(\vec{\lambda}_1,\dots,\vec{\lambda}_{k-1},\vec{\lambda}_{k+1},\dots,\vec{\lambda}_n),
\end{aligned}
\tag{6.1.35}
$$

where $\bar{P}(\vec{\lambda}_1,\dots,\vec{\lambda}_{k-1},\vec{\lambda}_{k+1},\dots,\vec{\lambda}_n)$ is obtained by replacing $\vec{\lambda}_k$ in $P(\vec{\lambda}_1,\dots,\vec{\lambda}_k,\dots,\vec{\lambda}_n)$ with $-\vec{\partial}-\sum_{i=1,i\neq k}^n\vec{\lambda}_i$, that is replacing λ_{kj} in $P(\vec{\lambda}_1,\dots,\vec{\lambda}_k,\dots,\vec{\lambda}_n)$ with $-\partial_j-\sum_{i=1,i\neq k}^n\lambda_{ij}$ for $j=1,\dots N$. One sees immediately that ι_k is well defined and

$$
\pi\circ\iota_k=\mathrm{id}\quad\text{for each}\quad k=1,\dots,n. \tag{6.1.36}
$$

Since any $\widetilde{\gamma}\in\widetilde{C}^n_{LP}(P,M)$ is determined by its value on $U^{\otimes n}$, for any $u_1\otimes\cdots\otimes u_n\in U^{\otimes n}$, set

$$
\widetilde{\gamma}_{\vec{\lambda}_1,\dots,\vec{\lambda}_n}(u_1\otimes\cdots\otimes u_n)=\frac{1}{n}\sum_{k=1}^n\iota_k(\gamma_{\vec{\lambda}_1,\dots,\vec{\lambda}_n}(u_1\otimes\cdots\otimes u_n)). \tag{6.1.37}
$$

The map $\widetilde{\gamma}$ defined this way satisfies the symmetry conditions since γ does. It can be extended to any value on $P^{\otimes n}$ by the sesquilinearity conditions, thus $\widetilde{\gamma} \in \widetilde{C}^n_{LP}(P, M)$. The identity $\epsilon(\widetilde{\gamma}) = \pi \circ \widetilde{\gamma} = \gamma$ on $U^{\otimes n}$ is readily checked by (6.1.36). Hence $\epsilon(\widetilde{\gamma}) = \gamma$, and ϵ is surjective, so is $\widetilde{\epsilon}$, completing the proof. $\square$

6.1.2 Variational Poisson superpseudoalgebra cohomology

In this subsection, let P be a PPA with parity p (in particular it is an LPA), and M be a P-module (PPA module). Consider the subspace $C^n_{PP}(P, M)$ of $C^n_{LP}(P, M)$ such that any $\gamma \in C^n_{PP}(P, M)$ satisfies the Leibniz rules:

$$\begin{aligned}\gamma(a_1 \otimes \cdots \otimes b_i a_i \otimes \cdots \otimes a_n) =& (-1)^{p(b_i)(\bar{p}(\gamma)+\bar{p}(a_1)+\cdots+\bar{p}(a_{i-1}))} b_i \gamma(a_1 \otimes \cdots \otimes a_i \otimes \cdots \otimes a_n) + \\ & (-1)^{p(a_i)(p(b_i)+\bar{p}(\gamma)+\bar{p}(a_1)+\cdots+\bar{p}(a_{i-1}))} a_i \gamma(a_1 \otimes \cdots \otimes b_i \otimes \cdots \otimes a_n) \end{aligned} \tag{6.1.38}$$

for all $i = 1, \ldots, n$ and $a_j, b_i \in P$. Here if $\gamma(a_1 \otimes \cdots \otimes a_i \otimes \cdots \otimes a_n)$ is given by

$$\gamma(a_1 \otimes \cdots \otimes a_i \otimes \cdots \otimes a_n) = \sum_j (g_{j1} \otimes \cdots \otimes g_{jn}) \otimes_H v_j(a_1 \otimes \cdots \otimes a_n), \tag{6.1.39}$$

then

$$b_i \gamma(a_1 \otimes \cdots \otimes a_i \otimes \cdots \otimes a_n) = \sum_j (g_{j1} \otimes \cdots \otimes g_{ji(1)} \otimes \cdots \otimes g_{jn}) \otimes_H (g_{ji(-2)} b_i) v_j(a_1 \otimes \cdots \otimes a_n). \tag{6.1.40}$$

The differential d is given by restricting (6.1.9) and (6.1.18) on $C_{PP}(P, M)$. For simplicity, we still denote it as d when its meaning is clear from the context. In the following discussion till the end of this subsection, we assume $H = \mathbb{F}[\partial_1, \ldots, \partial_N]$.

Lemma 6.1.3. *When $H = \mathbb{F}[\partial_1, \ldots, \partial_N]$, the differential map d satisfies: $dC^n_{PP}(P, M) \subset C^{n+1}_{PP}(P, M)$, hence $(C_{PP}(P, M), d)$ is a subcomplex of $(C_{LP}(P, M), d)$.*

This lemma is proved in [14] for the PVA case. If we regard our LPA as a higher-dimensional LCA, the proof also works here.

We call the cohomology $H_{PP}(P, M)$ of $(C_{PP}(P, M), d)$ the reduced variational PPA cohomology of P with coefficients in M. We also abbreviate it as the reduced cohomology when there is no ambiguity. As in the LPA case, we define the basic cohomology of P with coefficients in M, which is denoted as $\widetilde{H}_{PP}(P,M)$, to be the cohomology of $(\widetilde{C}_{PP}(P,M),d)$ obtained by replacing all $\otimes_H$ with $\otimes$ in $(C_{PP}(P,M),d)$.

By definition, one sees that $C^0_{PP}(P,M) = M/(\partial_1 M+\cdots+\partial_N M)$ and $C^1_{PP}(P,M) = \mathrm{Der}_H(P,M)$, where $\mathrm{Der}_H(P,M) \subset \mathrm{Hom}_H(P,M)$ consists of H-module homomorphisms from P to M that satisfy the Leibniz rule:

$$\gamma(ab) = (-1)^{p(a)\bar{p}(\gamma)} a\gamma(b) + (-1)^{p(b)(p(a)+\bar{p}(\gamma))} b\gamma(a). \tag{6.1.41}$$

For $n=2$, an element $\gamma \in C^2_{PP}(P,M)$ can be identified with a pseudobracket $[\cdot * \cdot] : P \otimes P \to H^{\otimes 2} \otimes_H M$ that satisfies the sesquilinearity (H-bilinearity), the symmetry condition with respect to $\bar{p}$ and the Leibniz rules given by (6.1.38) for $n=2$.

We state some results on low-degree cohomology $H^n_{PP}(P,M)$, $0 \le n \le 2$, when $H = \mathbb{F}[\partial_1,\dots,\partial_N]$. The pseudobracket will be identified with the $\vec{\lambda}$-bracket here (cf. Example 4.3.1 and Remark 6.1.1). Let $\int : M \to M/(\partial_1 M + \cdots + \partial_N M)$ be the quotient map. A Casimir element $\int m \in M/(\partial_1 M + \cdots + \partial_N M)$ is an element such that $a_{-\vec{\partial}}\, m = 0$ for all $a \in P$. We denote the set of all Casimir elements above as $\mathrm{Cas}(P,M)$. In the adjoint module, by the sesquilinearity of the $\vec{\lambda}$-bracket, the condition of being a Casimir element above can also be written as $[m_{\vec{\lambda}}\, a]\big|_{\vec{\lambda}=0} = 0$. A derivation $D : P \to M$ is an H-module homomorphism satisfying the Leibniz rule (6.1.41) and

$$D[a_{\vec{\lambda}}\, b] = (-1)^{p(D)p(a)} a_{\vec{\lambda}} D(b) + (-1)^{1+p(b)(p(D)+p(a))} b_{-\vec{\lambda}-\vec{\partial}} D(a), \tag{6.1.42}$$

for any $a, b \in P$ (here $\bar{p}(D) = p(D)$, see Remark 6.3.2). Moreover, the derivation D is called inner if there exists an element $\int m \in M/(\partial_1 M + \cdots + \partial_N M)$ such that for all $a \in P$,

$$D(a) = (-1)^{1+p(a)p(m)} a_{-\vec{\partial}}\, m. \tag{6.1.43}$$

For the adjoint module, (6.1.42) coincides with the usual definition of a derivation as being an H-module endomorphism of P such that:

$$D[a_{\vec{\lambda}}\, b] = [(D(a))_{\vec{\lambda}}\, b] + (-1)^{p(D)p(a)} [a_{\vec{\lambda}} (D(b))]. \tag{6.1.44}$$

Similarly, (6.1.43) in the definition of a derivation D being inner becomes: for any $b \in P$,

$$D(b) = [a_{\vec{\lambda}} b]\Big|_{\vec{\lambda}=0} = a_{(0)} b, \quad \text{for some} \quad a \in P. \tag{6.1.45}$$

We will denote the space of all derivations $D : P \to M$ by $\mathrm{Der}(P, M)$, while the inner ones by $\mathrm{Inder}(P, M)$. With the notations introduced above, one can prove the following theorem on low degree cohomology $H_{PP}(P, M)$, which is analogous to the PVA case [5; 14] and straightforward.

Theorem 6.1.2. *Let $H = \mathbb{F}[\partial_1, \ldots, \partial_N]$ and suppose P is a PPA and M is a P-module. Then the following are true:*

1. $H^0_{PP}(P, M) = \mathrm{Cas}(P, M)$;
2. $H^1_{PP}(P, M) = \mathrm{Der}(P, M)/\mathrm{Inder}(P, M)$;
3. $H^2_{PP}(P, M)$ *is the space of isomorphism classes of H-split extensions of the PPA P by the P-module M, where M is considered as a non-unital PPA with trivial products. In particular, $H^2_{PP}(P, P)$ parameterizes the equivalence classes of first-order deformations of P that preserve the product and the H-module structure (cf. [25]).*

The following two lemmas will be useful when we discuss a specific cochain.

Lemma 6.1.4. *Let P be a PPA and M a P-module, suppose $G \subset P$ generates P as an H-differential superalgebra, then any n-cochain $\gamma \in C^n_{PP}(P, M)$ is determined by its value on $G^{\otimes n}$.*

Proof. This claim is true because of the H-multilinearity (6.1.7) and the Leibniz rules (6.1.38). □

Lemma 6.1.5. *Under the same assumptions of the previous lemma and $H = \mathbb{F}[\partial_1, \ldots, \partial_N]$, an n-cochain $(n \geq 1)$ γ vanishes if any one of its n arguments is the unit $1 \in P$.*

Proof. By the symmetry conditions, assume 1 is the first argument, then for any $h \in H^+$, $0 = \gamma(h \cdot 1 \otimes a_2 \otimes \cdots \otimes a_n) = ((h \otimes 1 \otimes \cdots \otimes 1) \otimes_H 1)\gamma(1 \otimes a_2 \otimes \cdots \otimes a_n)$. Since for any $h \in H^+$ and $g \in H$, $hg = 0$ indicates $g = 0$, thus $\gamma(1 \otimes a_2 \otimes \cdots \otimes a_n) = 0$. □

As in the LPA case, after assuming $H=\mathbb{F}[\partial_1,\dots,\partial_N]$, one can define $\partial_k\widetilde{\gamma}$ by (6.1.23) for any $k\in\{1,\dots,N\}$ and $\widetilde{\gamma}\in\widetilde{C}^n_{PP}(P,M)$. We will also use the $\vec{\lambda}$-notation from Remark 6.1.1 if it is easier for the discussion, under which $\partial_k\widetilde{\gamma}$ is given by (6.1.26). It is straightforward to show that $\partial_k\widetilde{\gamma}$ satisfies the Leibniz rules (6.1.38) too; hence it is an element of $\widetilde{C}^n_{PP}(P,M)$. For any $k\in\{1,\dots,N\}$,∂_k is an injective endomorphism of $\widetilde{C}^n_{PP}(P,M)$ for all $n\geq 1$. It is an isomorphism between $\widetilde{C}^n_{PP}(P,M)$ and $\partial_k\widetilde{C}^n_{PP}(P,M)$ for all $n\geq 1$, which can be extended to $n=0$ if M is a free left H-module (cf. Lemma 6.1.2). We also have the following proposition as an analogue to Proposition 6.1.1.

Proposition 6.1.2. *If $H=\mathbb{F}[\partial_1,\dots,\partial_N]$, then $\widetilde{\epsilon}$ defined by* (6.1.30) *is injective when restricted to $\widetilde{C}^n_{PP}(P,M)$. Moreover, if P is a PPA such that as an H-differential superalgebra it is a superalgebra of differential polynomials in even or odd variables, then $\widetilde{\epsilon}$ is an isomorphism for all $n\geq 0$ when restricted to $\widetilde{C}^n_{PP}(P,M)$.*

Proof. Here we still use ϵ and $\widetilde{\epsilon}$ to denote their restrictions to $C^n_{PP}(P,M)$ and $\widetilde{C}^n_{PP}(P,M)$, respectively. The injectivity of $\widetilde{\epsilon}$ on $\widetilde{C}^n_{PP}(P,M)$ follows from the injectivity of $\widetilde{\epsilon}$ on $C^n_{LP}(P,M)$ proved in Proposition 6.1.1. The surjectivity here is proved in the same way as in Proposition 6.1.1, only noting that if P is a superalgebra of differential polynomials in M even or odd variables, then it can be written as $P=\mathbb{F}[\partial_j^k u_i | 1\leq j\leq N,\ k\geq 0,\ 1\leq i\leq M]$. One can pick the generating set $G=\{u_1,\dots,u_M\}$ that plays the role of U in the proof of Proposition 6.1.1. Then the preimage $\widetilde{\gamma}$ can be defined by (6.1.37) on the tensor products of generators and extended by sesquilinearity, symmetry conditions and Leibniz rules. □

For the PVA case, if P as a differential superalgebra is a superalgebra of differential polynomials in even or odd variables, then the isomorphism $\widetilde{\epsilon}:\widetilde{C}^n_{PP}(P,M)/\partial\,\widetilde{C}^n_{PP}(P,M)\to C^n_{PP}(P,M)$ gives a short exact sequence

$$0\to\widetilde{C}^n_{PP}(P,M)\xrightarrow{\partial}\widetilde{C}^n_{PP}(P,M)\to C^n_{PP}(P,M)\to 0,\tag{6.1.46}$$

which induces a long exact sequence that allows one to compute the reduced cohomology $H^n_{PV}(P,M)$. We would like to generalize (6.1.46) to higher dimensions, that is $H=\mathbb{F}[\partial_1,\dots,\partial_N]$ for $N>1$. In order to achieve this, as a first step, we discuss the Koszul complex in the following subsection.

6.1.3 The Koszul complex

The goal of this subsection is to review some facts about the Koszul complex (cf. [23]) that will be needed in the calculation later. The Koszul complex was first introduced by Jean-Louis Koszul to define a Lie algebra cohomology theory, and it turned out to be a useful tool in homological algebra later. The concept of a Koszul complex is related to that of regular sequences given below.

Definition 6.1.2 ([23])**.** *For a ring R and an R-module M, an element $x \in R$ is called M-regular (or just regular) if $xm = 0$ implies $m = 0$.*

Definition 6.1.3 ([23])**.** *Let R be a ring and M be an R-module. A sequence $x_1, \dots, x_n \in R$ is called an M-regular sequence if the following two conditions are satisfied:*

1. *x_i is $(M/\sum_{j=1}^{i-1} x_j M)$-regular for $i = 1, \dots, n$;*
2. *$M/\sum_{j=1}^{n} x_j M \neq 0$.*

Below is the definition of the Koszul complex:

Definition 6.1.4 ([23])**.** *Suppose that R is a ring, M is an R-module, and $x_1, \dots, x_n \in R$. The Koszul complex, denoted as $K(\underline{x}, M)$, is the complex below:*

$$0 \to K_n \xrightarrow{\phi_n} K_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_2} K_1 \xrightarrow{\phi_1} K_0 \to 0, \tag{6.1.47}$$

where $R^{\binom{n}{p}} = \bigoplus R e_{i_1 \dots i_p}$ is the free R-module of rank $\binom{n}{p}$ $(p = 1, \dots, n)$ with a basis $\{e_{i_1 \dots i_p} | 1 \leq i_1 < \cdots < i_p \leq n\}$, and $K_p = M^{\binom{n}{p}} = R^{\binom{n}{p}} \otimes_R M = \bigoplus e_{i_1 \dots i_p} M$. In addition, let $K_0 = M$ and $K_p = 0$ for $p < 0$ or $p > n$. The differential $\phi_p : K_p \to K_{p-1}$ is defined on the basis (and extended R-linearly to the whole space) as follows:

$$\phi_p(e_{i_1 \dots i_p}) = \sum_{j=1}^{p} (-1)^{j-1} e_{i_1 \dots \hat{i}_j \dots i_p} x_{i_j}, \tag{6.1.48}$$

for $1 < p \leq n$, where $\hat{i}_j$ means i_j is omitted, and $\phi_1(e_i) = x_i$.

By the definition of the differential (6.1.48), one can verify that $\phi_{p-1} \circ \phi_p = 0$ ($\phi_p = 0$ if $p > n$ or $p \leq 0$); hence (6.1.47) is indeed a complex.

Example 6.1.1. *With the notations in the definition of a Koszul complex above, for $n = 1$, the Koszul complex is just:*

$$0 \to M \xrightarrow{x} M \to 0, \tag{6.1.49}$$

where $M \xrightarrow{x} M$ denotes the left action by $x = x_1 \in R$. For $n = 2$, the Koszul complex is:

$$0 \to M \xrightarrow{\phi_2} M \oplus M \xrightarrow{\phi_1} M \to 0, \tag{6.1.50}$$

where $\phi_2 : M \to M \oplus M$, $\phi_2(m) = (-x_2 m, x_1 m)$ and $\phi_1 : M \oplus M \to M$, $\phi_1(m_1, m_2) = x_1 m_1 + x_2 m_2$.

The homology of the Koszul complex is denoted as $H(\underline{x}, M)$. By definition, one sees immediately that

$$H_0(\underline{x}, M) \cong M / \sum_{j=1}^{n} x_j M \quad \text{and} \quad H_n(\underline{x}, M) \cong \{m \in M | x_i m = 0, i = 1, \dots, n\}. \tag{6.1.51}$$

The following theorem is important for the cohomology of the Koszul complex.

Theorem 6.1.3 (cf. [23])**.** *Let R be a ring, M be an R-module, and $x_1, \dots, x_n \in R$ be an M-regular sequence. Then $H_p(\underline{x}, M) = 0$ if $p > 0$ and $H_0(\underline{x}, M) = M / \sum_{j=1}^{n} x_j M$.*

A direct consequence of this theorem is a free resolution of the quotient module $M / \sum_{j=1}^{n} x_j M$.

Corollary 6.1.1. *Under the assumptions of Theorem 6.1.3, we have a free resolution of $M / \sum_{j=1}^{n} x_j M$ as follows:*

$$0 \to M \xrightarrow{\phi_n} M^{\binom{n}{n-1}} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_2} M^{\binom{n}{1}} \xrightarrow{\phi_1} M \xrightarrow{\pi} M / \sum_{j=1}^{n} x_j M \to 0, \tag{6.1.52}$$

where $M^q = M \oplus \cdots \oplus M$ (q summands) and $\pi : M \to M / \sum_{j=1}^{n} x_j M$ is the quotient map.

Consider the Hopf algebra $H = \mathbb{F}[\partial_1, \dots \partial_N]$, which is also a commutative ring. The H-module structure on $\widetilde{C}^n_{PP}(P, M)$ is given by (6.1.23), or in $\vec{\lambda}$-notation by (6.1.26). Note that for each $n \geq 1$, $\partial_1, \dots, \partial_N$ is a $\widetilde{C}^n_{PP}(P, M)$-regular sequence (cf. Lemma 6.1.2). By Corollary 6.1.1 and Proposition 6.1.2, we have the following exact sequence:

$$0 \to \widetilde{C}^n_{PP}(P, M) \xrightarrow{\phi_N} \widetilde{C}^n_{PP}(P, M)^{\binom{N}{N-1}} \xrightarrow{\phi_{N-1}} \cdots \xrightarrow{\phi_2} \widetilde{C}^n_{PP}(P, M)^{\binom{N}{1}} \xrightarrow{\phi_1} \widetilde{C}^n_{PP}(P, M) \xrightarrow{\epsilon} C^n_{PP}(P, M) \to 0, \tag{6.1.53}$$

where ϵ is given by (6.1.29) restricted to $\widetilde{C}^n_{PP}(P,M)$. In particular, for $H=\mathbb{F}[\partial_1,\partial_2]$, the exact sequence is: for each $n\geq 1$,

$$0\to \widetilde{C}^n_{PP}(P,M)\xrightarrow{\phi_2}\widetilde{C}^n_{PP}(P,M)\oplus\widetilde{C}^n_{PP}(P,M)\xrightarrow{\phi_1}\widetilde{C}^n_{PP}(P,M)\xrightarrow{\epsilon}C^n_{PP}(P,M)\to 0, \tag{6.1.54}$$

where ϕ_1 and ϕ_2 are given in Example 6.1.1 with $x_1=\partial_1$ and $x_2=\partial_2$. This free resolution of $C^n_{PP}(P,M)$ is a direct generalization of the short exact sequence (6.1.46) for the PVA case in [5].

6.2 Energy operators

The energy operator approach is a useful tool in computing the variational cohomology of a PVA [5]. In this section, we generalize the energy operator to a family of energy operators on the variational PPA cohomology. In particular, we prove the second main theorem in this thesis: under certain assumptions on the PPA P and the P-module M, the energy operators N_1 and N_2 have only eigenvalues 0 and 1 on the reduced cohomology $H_{PP}(P,M)$ when $H=\mathbb{F}[\partial_1,\partial_2]$. Since H is commutative in this case, we will use the $\vec{\lambda}$ notation as discussed in Remark 6.1.1.

6.2.1 Definitions and basic properties of energy operators

Suppose $H=\mathbb{F}[\partial_1,\dots,\partial_N]$ and P is a PPA in the following discussion. Moreover, let $\{t_1,\dots,t_N\}$ denote the dual basis with respect to $\{\partial_1,\dots,\partial_N\}$.

Definition 6.2.1. *Fix $i\in\{1,\dots,N\}$. A Virasoro element L_i with respect to ∂_i is an even element of P such that if one identifies the pseudobracket of P with the $\vec{\lambda}$-bracket (cf. (4.3.1)), the following conditions are satisfied:*

$$[L_i * L_i]=(1\otimes\partial_i-\partial_i\otimes 1)\otimes_H L_i \iff [L_i{}_{\vec{\lambda}}L_i]=(\partial_i+2\lambda_i)L_i, \tag{6.2.1}$$

$$L_{i(0)}:=[L_i{}_{\vec{\lambda}}\cdot]\big|_{\vec{\lambda}=0}=\partial_i, \tag{6.2.2}$$

$$L_{i(t_i)}:=\frac{\partial}{\partial\lambda_i}[L_i{}_{\vec{\lambda}}\cdot]\big|_{\vec{\lambda}=0}\in\mathrm{End}(P)\quad \textit{is diagonalizable.} \tag{6.2.3}$$

An element $a\in P$ is said to have a conformal weight $\Delta_i(a)\in\mathbb{F}$ with respect to L_i if it is an eigenvector of $L_{i(t_i)}$ of eigenvalue $\Delta_i(a)$, that is $L_{i(t_i)}a=\Delta_i(a)a$. Note that (6.2.2) and

(6.2.3) may also work for a P-module M after changing the $\vec{\lambda}$-bracket to the $\vec{\lambda}$-action:

$$L^M_{i\ (0)} := (L_{i\,\vec{\lambda}}\,\cdot)\big|_{\vec{\lambda}=0} = \partial_i, \tag{6.2.4}$$

$$L^M_{i\ (t_i)} := \frac{\partial}{\partial\lambda_i}(L_{i\,\vec{\lambda}}\,\cdot)\big|_{\vec{\lambda}=0} \in \mathrm{End}(M) \quad \text{is diagonalizable.} \tag{6.2.5}$$

If (6.2.4) and (6.2.5) hold for a P-module M, we call M conformal with respect to L_i. An element $m \in M$ is said to have a conformal weight $\Delta_i(m) \in \mathbb{F}$ with respect to L_i if it is an eigenvector of $L^M_{i\ (t_i)}$ of eigenvalue $\Delta_i(m)$, that is $L^M_{i\ (t_i)} m = \Delta_i(m)m$.

Lemma 6.2.1. *For any $a \in P$, if $[L_{i\,\vec{\lambda}}L_j] = \partial_i L_j + \lambda_i L_j + \lambda_j L_i$ for $i, j \in \{1, \dots, N\}$, then we have $L_{i(t_i)}(L_{j(t_j)}a) = L_{j(t_j)}(L_{i(t_i)}a)$.*

Proof. Since the $\vec{\lambda}$-bracket is parity preserving, $L_{i(t_i)} \in \mathrm{End}(P)$ is even. By [6], P is a Lie conformal algebra in higher dimensions whose annihilation Lie algebra is W_N. In $W_N = \mathrm{Der}\,\mathbb{F}[t_1, \dots, t_N]$, $L_{i(t_i)}$ is identified with $-t_i\frac{\partial}{\partial t_i}$. Since $-t_i\frac{\partial}{\partial t_i}$ commutes with $-t_j\frac{\partial}{\partial t_j}$, it follows that $L_{i(t_i)}$ commutes with $L_{j(t_j)}$. □

For a monomial $a\lambda_1^{j_1}\cdots\lambda_N^{j_N}$ in $P[\vec{\lambda}]$ or $M[\vec{\lambda}]$, the conformal weight with respect to L_i is defined as:

$$\Delta_i(a\lambda_1^{j_1}\cdots\lambda_N^{j_N}) = \Delta_i(a) + j_i. \tag{6.2.6}$$

This is equivalent to say that the conformal weights with respect to L_i in $M[\vec{\lambda}]$ are the eigenvalues of the following operator:

$$N_i := L^M_{i\ (t_i)} + \lambda_i\frac{\partial}{\partial\lambda_i}. \tag{6.2.7}$$

The following properties of conformal weights are generalized from the PVA case and the proofs are similar too (cf. [5]). They are useful in the calculation.

Lemma 6.2.2. *Fix $i \in \{1, \dots, N\}$, and let $L_i \in P$ be a Virasoro element with respect to ∂_i. Suppose that $a \in P$ has a conformal weight $\Delta_i(a)$ and $m \in M$ has a conformal weight $\Delta_i(m)$. Then:*

1. *$\Delta_i(1) = 0$ where $1 \in P$ is the unit element;*

2. $\Delta_i(\partial_j a) = \Delta_i(a) + \delta_{ij}$ *and* $\Delta_i(\partial_j m) = \Delta_i(m) + \delta_{ij}$*, where* δ_{ij} *is the Kronecker delta:*

$$\delta_{ij} = \begin{cases} 1, & \text{if} \quad i = j, \\ 0, & \text{if} \quad i \neq j; \end{cases}$$

3. $\Delta_i(a_{\vec{\lambda}} m) = \Delta_i(a) + \Delta_i(m) - 1$;

4. $\Delta_i(am) = \Delta_i(a) + \Delta_i(m)$.

Proof. Part 1 follows from the definition of Δ_i and the fact that $[L_{i\vec{\lambda}} 1] = 0$.
Part 2: by the sesquilinearity (4.3.1) of the $\vec{\lambda}$-bracket,

$$\begin{aligned} L_{i(t_i)}(\partial_j a) &= \frac{\partial}{\partial \lambda_i}[L_{i\vec{\lambda}}(\partial_j a)]\Big|_{\vec{\lambda}=0} = \frac{\partial}{\partial \lambda_i}\big((\partial_j + \lambda_j)[L_{i\vec{\lambda}}\, a]\big)\Big|_{\vec{\lambda}=0} \\ &= \Delta_i(a)(\partial_j a) + \frac{\partial \lambda_j}{\partial \lambda_i}\big([L_{i\vec{\lambda}}\, a])\big|_{\vec{\lambda}=0}\big) \\ &= (\Delta_i(a) + \delta_{ij})(\partial_i a). \end{aligned}$$

One can prove $\Delta_i(\partial_j m) = \Delta_i(m) + \delta_{ij}$ in a similar way.
Prat 3: by the Jacobi identity (cf.(4.3.1)) and noting that $a_{\vec{\mu}} m \in M[\vec{\mu}]$, we get

$$\begin{aligned} L^M_{i\ (t_i)}(a_{\vec{\mu}} m) &= \frac{\partial}{\partial \lambda_i}[L_{i\vec{\lambda}}(a_{\vec{\mu}} m)]\Big|_{\vec{\lambda}=0} \\ &= \frac{\partial}{\partial \lambda_i}\big([L_{i\vec{\lambda}}\, a]_{\vec{\lambda}+\vec{\mu}} m\big)\Big|_{\vec{\lambda}=0} + \frac{\partial}{\partial \lambda_i}\big(a_{\vec{\mu}}(L_{i\vec{\lambda}} m)\big)\Big|_{\vec{\lambda}=0} \\ &= (L_{i(t_i)} a)_{\vec{\mu}} m + \frac{\partial}{\partial \mu_i}(L_{i(0)} a)_{\vec{\mu}} m + a_{\vec{\mu}}(L^M_{i\ (t_i)} m) \\ &= \Delta_i(a)(a_{\vec{\mu}} m) - \frac{\partial}{\partial \mu_i}(\mu_i(a_{\vec{\mu}} m)) + \Delta_i(m)(a_{\vec{\mu}} m) \\ &= (\Delta_i(a) + \Delta_i(m) - 1)(a_{\vec{\mu}} m) - \mu_i \frac{\partial}{\partial \mu_i}(a_{\vec{\mu}} m), \end{aligned}$$

which indicates

$$\big((L^M_i)_{(t_i)} + \mu_i \frac{\partial}{\partial \mu_i}\big)(a_{\vec{\mu}} m) = (\Delta_i(a) + \Delta_i(m) - 1)(a_{\vec{\mu}} m). \tag{6.2.8}$$

By (6.2.7), we have $\Delta_i(a_{\vec{\lambda}} m) = \Delta_i(a) + \Delta_i(m) - 1$.

Part 4: Using the $\vec{\lambda}$-notation, the Leibniz rule (cf.(4.2.1)) in the module setting is:

$$a_{\vec{\lambda}}(bm) = [a_{\vec{\lambda}}b]m + (-1)^{p(a)p(b)}b(a_{\vec{\lambda}}m). \tag{6.2.9}$$

With (6.2.9), one has:

$$\begin{aligned} L^M_{i\ (t_i)}(am) &= \frac{\partial}{\partial\lambda_i}(L_{i\vec{\lambda}}(am))\Big|_{\vec{\lambda}=0} \\ &= \frac{\partial}{\partial\lambda_i}[L_{i\vec{\lambda}}a]m\Big|_{\vec{\lambda}=0} + \frac{\partial}{\partial\lambda_i}\big(a(L_{i\vec{\lambda}}m)\big)\Big|_{\vec{\lambda}=0} \\ &= \Delta_i(a)am + \Delta_i(m)am \\ &= (\Delta_i(a)+\Delta_i(m))(am), \end{aligned} \tag{6.2.10}$$

hence $\Delta_i(am) = \Delta_i(a)+\Delta_i(m)$. □

Next we consider the energy operators N_i $(i = 1,\dots,N)$ on $M[\vec{\lambda}_1,\dots,\vec{\lambda}_n]$ defined as follows:

$$N_i := L^M_{i\ (t_i)} + \sum_{k=1}^{n}\lambda_{ki}\frac{\partial}{\partial\lambda_{ki}}. \tag{6.2.11}$$

By definition of N_i, one has the following lemma as a corollary of Lemma 6.2.1:

Lemma 6.2.3. *If* $[L_{i\vec{\lambda}}L_j] = \partial_i L_j + \lambda_i L_j + \lambda_j L_i$ *for any* $i,j \in \{1,\dots,N\}$, *then the energy operators* N_i $(i = 1,\dots,N)$ *on* $M[\vec{\lambda}_1,\dots,\vec{\lambda}_n]$ *commute with each other, that is* $N_iN_j = N_jN_i$ *for* $i,j \in \{1,\dots,N\}$.

Moreover, each N_i is an even endomorphism of $M[\vec{\lambda}_1,\dots,\vec{\lambda}_n]$ and is diagonalizable since both summands are. The following lemma allows one to pass the energy operators to the quotient space $M[\vec{\lambda}_1,\dots,\vec{\lambda}_n]/\langle\vec{\partial}+\vec{\lambda}_1+\cdots+\vec{\lambda}_n\rangle$.

Proposition 6.2.1. *The family of endomorphism* $\{N_i\}_{1\le i\le N}$ *of* $M[\vec{\lambda}_1,\dots,\vec{\lambda}_n]$ *satisfies the following identities:*

$$N_i\circ(\partial_j+\lambda_{1j}+\cdots+\lambda_{nj}) = (\partial_j+\lambda_{1j}+\cdots+\lambda_{nj})\circ(N_i+\delta_{ij}), \tag{6.2.12}$$

for any $i,j \in \{1,\dots,N\}$.

Proof. It suffices to show that (6.2.12) holds for $m\prod_{k,l=1}^{k=n,l=N}\lambda_{kl}^{t_{kl}}\in M[\vec{\lambda}_1,\dots,\vec{\lambda}_n]$, where m has conformal weight $\Delta_i(m)$. For simplicity, let $q=\prod_{k,l=1}^{k=n,l=N}\lambda_{kl}^{t_{kl}}$ in the calculation below. The left hand side is:

$$
\begin{aligned}
&\big(N_i\circ(\partial_j+\lambda_{1j}+\cdots+\lambda_{nj})\big)(mq)\\
=&(L_i^M{}_{(t_i)}(\partial_j m))q+\sum_{s=1}^{n}(\partial_j m)\lambda_{si}\frac{\partial q}{\partial\lambda_{si}}+\sum_{r=1}^{n}(L_i^M{}_{(t_i)}m)q\lambda_{rj}+\sum_{r=1}^{n}\sum_{s=1}^{n}m\lambda_{si}\frac{\partial(q\lambda_{rj})}{\partial\lambda_{si}}\\
=&(\Delta_i(m)+\delta_{ij})\partial_j mq+\partial_j mq\sum_{s=1}^{n}t_{si}+\Delta_i(m)mq\sum_{r=1}^{n}\lambda_{rj}+m\sum_{r=1}^{n}\sum_{s=1}^{n}(t_{si}q\lambda_{rj}+\delta_{rs}\delta_{ij}q\lambda_{si}),
\end{aligned}
\tag{6.2.13}
$$

while the right hand side is:

$$
\begin{aligned}
&\big((\partial_j+\lambda_{1j}+\cdots+\lambda_{nj})\circ(N_i+\delta_{ij})\big)(mq)\\
=&(\partial_j+\sum_{r=1}^{n}\lambda_{rj})\big((L_i^M{}_{(t_i)}m)q+m\sum_{s=1}^{n}\lambda_{si}\frac{\partial q}{\partial\lambda_{si}}\big)+\delta_{ij}(\partial_j+\sum_{r=1}^{n}\lambda_{rj})(mq)\\
=&\Delta_i(m)\partial_j mq+\Delta_i(m)mq\sum_{r=1}^{n}\lambda_{rj}+\partial_j mq\sum_{s=1}^{n}t_{si}+m\sum_{r=1}^{n}\sum_{s=1}^{n}t_{si}q\lambda_{rj}+\delta_{ij}\partial_j mq+\delta_{ij}mq\sum_{r=1}^{n}\lambda_{rj}
\end{aligned}
\tag{6.2.14}
$$

Note that (6.2.13) = (6.2.14), since $m\sum_{r=1}^{n}\sum_{s=1}^{n}\delta_{rs}\delta_{ij}q\lambda_{si}=\delta_{ij}mq\sum_{r=1}^{n}\lambda_{rj}$. □

Corollary 6.2.1. *The operators N_i $(1\le i\le N)$ given by (6.2.11) induce a family of even diagonalizable endomorphisms, still denoted by N_i, on $M[\vec{\lambda}_1,\dots,\vec{\lambda}_n]/\langle\vec{\partial}+\vec{\lambda}_1+\cdots+\vec{\lambda}_n\rangle$.*

The eigenvalues of the energy operator N_i will be called the conformal weight with respect to L_i and denoted as $\Delta_i(P(\vec{\lambda}_1,\dots,\vec{\lambda}_n))$ for $P(\vec{\lambda}_1,\dots,\vec{\lambda}_n)\in M[\vec{\lambda}_1,\dots,\vec{\lambda}_n]/\langle\vec{\partial}+\vec{\lambda}_1+\cdots+\vec{\lambda}_n\rangle$. With these notations, one can define a family of energy operators $\{N_i\}_{1\le i\le N}$ on the cochain complex $C_{PP}(P,M)$ by setting

$$(N_i\gamma)_{\vec{\lambda}_1,\dots,\vec{\lambda}_n}(a_1\otimes\cdots\otimes a_n)=(N_i+n)(\gamma_{\vec{\lambda}_1,\dots,\vec{\lambda}_n}(a_1\otimes\cdots\otimes a_n)) \tag{6.2.15}$$

$$-\sum_{k=1}^{n}\gamma_{\vec{\lambda}_1,\dots,\vec{\lambda}_n}(a_1\otimes\cdots\otimes L_{i(t_i)}a_k\otimes\cdots\otimes a_n), \tag{6.2.16}$$

for $\gamma \in C^n_{PP}(P,M)$. Using this definition, one can prove the following proposition by a straightforward calculation.

Proposition 6.2.2. *If* $[L_i{}_{\vec{\lambda}}L_j] = \partial_i L_j + \lambda_i L_j + \lambda_j L_i$ *for any* $i, j \in \{1,\dots,N\}$*, then on* $C^n_{PP}(P,M)$*,* $N_i N_j = N_j N_i$ *for all* $i, j \in \{1,\dots,N\}$*.*

Proof. By definition, one computes:

$$
\begin{aligned}
&(N_j(N_i\gamma))_{\vec{\lambda}_1,\dots,\vec{\lambda}_n}(a_1\otimes\cdots\otimes a_n)\\
=&(N_j+n)\big((N_i\gamma)_{\vec{\lambda}_1,\dots,\vec{\lambda}_n}(a_1\otimes\cdots\otimes a_n)\big)-\sum_{k=1}^{n}(N_i\gamma)_{\vec{\lambda}_1,\dots,\vec{\lambda}_n}(a_1\otimes\cdots\otimes L_{j(t_j)}a_k\otimes\cdots\otimes a_n)\\
=&(N_j+n)(N_i+n)\gamma_{\vec{\lambda}_1,\dots,\vec{\lambda}_n}(a_1\otimes\cdots\otimes a_n)-(N_j+n)\sum_{l=1}^{n}\gamma_{\vec{\lambda}_1,\dots,\vec{\lambda}_n}(a_1\otimes\cdots\otimes L_{i(t_i)}a_l\otimes\cdots\otimes a_n)\\
&-(N_i+n)\sum_{k=1}^{n}\gamma_{\vec{\lambda}_1,\dots,\vec{\lambda}_n}(a_1\otimes\cdots\otimes L_{j(t_j)}a_k\otimes\cdots\otimes a_n)\\
&+\sum_{k,l=1}^{n}\gamma_{\vec{\lambda}_1,\dots,\vec{\lambda}_n}(a_1\otimes\cdots\otimes L_{i(t_i)}a_k\otimes\cdots\otimes L_{j(t_j)}a_l\otimes\cdots\otimes a_n).
\end{aligned}
\tag{6.2.17}
$$

One gets $N_i(N_j\gamma)$ by switching i and j in (6.2.17). The first term in the last equality of (6.2.17) stays the same by Lemma 6.2.3, the two terms in the middle swap and stay the same as a whole, while the last term also stays the same by Lemma 6.2.1 (for $k=l$ case). □

Since N_i is diagonalizable on the right hand side of (6.2.15), it is diagonalizable on $C_{PP}(P,M)$. Furthermore, (6.2.15) implies the following identity on the conformal weights:

$$
\Delta_i(\gamma) = \Delta_i(\gamma_{\vec{\lambda}_1,\dots,\vec{\lambda}_n}(a_1\otimes\cdots\otimes a_n)) + n - \sum_{j=1}^{n}\Delta_i(a_j),
\tag{6.2.18}
$$

where $\Delta_i(\gamma)$ denote the conformal weight of $\gamma \in C^n_{PP}(P,M)$ with respect to L_i.

We next consider the interaction of the energy operators with the differential $d: C^n_{PP}(P,M) \to C^{n+1}_{PP}(P,M)$. Under the $\vec{\lambda}$-notation, the differential can be written as:

$$
\begin{aligned}
(d\gamma)_{\vec{\lambda}_1,\dots,\vec{\lambda}_{n+1}}(a_1\otimes\cdots\otimes a_{n+1}) =& \sum_{1\le i\le n+1}(-1)^{q_i}(a_i)_{\vec{\lambda}_i}\gamma_{\vec{\lambda}_1,\dots,\hat{\vec{\lambda}}_i,\dots,\vec{\lambda}_{n+1}}(a_1\otimes\cdots\otimes\hat{a}_i\otimes\cdots\otimes a_{n+1})\\
+\sum_{1\le i<j\le n+1}&(-1)^{q_{ij}}\gamma_{\vec{\lambda}_i+\vec{\lambda}_j,\vec{\lambda}_1,\dots,\hat{\vec{\lambda}}_i,\dots,\hat{\vec{\lambda}}_j,\dots,\vec{\lambda}_{n+1}}\big([a_i{}_{\vec{\lambda}_i}a_j]\otimes a_1\otimes\cdots\otimes\hat{a}_i\otimes\cdots\otimes\hat{a}_j\otimes\cdots\otimes a_{n+1}\big),
\end{aligned}
\tag{6.2.19}
$$

where q_i and q_{ij} are given by (6.1.16) and (6.1.17) respectively. The differential $\widetilde{d} : \widetilde{C}^n_{PP}(P, M) \to \widetilde{C}^{n+1}_{PP}(P, M)$ in the basic complex can be defined using the same formula above with d replaced by $\widetilde{d}$. One can prove the following proposition with (6.2.19), the proof is essentially the same as for the PVA case given in [5] (Lemma 3.25.).

Proposition 6.2.3. *The operators N_i given by (6.2.15) on $C_{PP}(P, M)$ commute with the differential d given by (6.2.19), that is:*

$$N_i \circ d = d \circ N_i, \quad \textit{for any} \quad i = 1, \ldots, N. \tag{6.2.20}$$

Corollary 6.2.2. *$\{N_i\}_{1\le i\le N}$ on $C^n_{PP}(P, M)$ induce a family of even diagonalizable endomorphisms, still denoted by $\{N_i\}_{1\le i\le N}$, on the cohomology $H^n_{PP}(P, M)$.*

On the other hand, one can define the family of energy operators on the basic complex $\widetilde{C}_{PP}(P, M)$ by:

$$\widetilde{N}_i = \frac{\partial}{\partial \lambda_i}\big(L_{i\vec{\lambda}}\,\cdot\big)\big|_{\vec{\lambda}=0}, \tag{6.2.21}$$

where L_i acts on the $\widetilde{C}_{PP}(P, M)$ by Lie derivative below: for $\widetilde{\gamma} \in \widetilde{C}^n_{PP}(P, M)$,

$$\begin{aligned}(L_{i\vec{\lambda}}\widetilde{\gamma})_{\vec{\lambda}_1,\ldots,\vec{\lambda}_n}(a_1\otimes\cdots\otimes a_n) &= L_{i\vec{\lambda}}\widetilde{\gamma}_{\vec{\lambda}_1,\ldots,\vec{\lambda}_n}(a_1\otimes\cdots\otimes a_n)\\ &\quad -\sum_{k=1}^{n}\widetilde{\gamma}_{\vec{\lambda}_1,\ldots,\vec{\lambda}+\vec{\lambda}_k,\ldots,\vec{\lambda}_n}(a_1\otimes\cdots\otimes[L_{i\vec{\lambda}}a_k]\otimes\cdots\otimes a_n).\end{aligned} \tag{6.2.22}$$

Lemma 6.2.4 (cf. [5])**.** *$\widetilde{N}_i$ defined by (6.2.21) satisfies the same formula as N_i given by (6.2.15), if we regard both sides of (6.2.15) as elements in $M[\vec{\lambda}_1, \ldots, \vec{\lambda}_n]$.*

The proof for the LPA case here is essentially the same as for the LCA case (for details see [5], Lemma 3.27.).

Lemma 6.2.5. *The following properties hold for the family of energy operators $\{\widetilde{N}_i\}_{1\le i\le N}$ on $\widetilde{C}_{PP}(P, M)$:*

1. *$\widetilde{N}_i$'s are diagonalizable on $\widetilde{C}^n_{PP}(P, M)$.*
2. *For a cochain $\widetilde{\gamma} \in \widetilde{C}^n_{PP}(P, M)$, $\Delta_i(\partial_j\widetilde{\gamma}) = \Delta_i(\widetilde{\gamma}) + \delta_{ij}$ if $\widetilde{\gamma}$ has a conformal weight $\Delta_i(\widetilde{\gamma})$.*

3. *For a cochain* $\widetilde{\gamma} \in \widetilde{C}^n_{PP}(P, M)$,

$$\pi \circ (\widetilde{N}_i \widetilde{\gamma}) = N_i(\pi \circ \widetilde{\gamma}), \tag{6.2.23}$$

where $\pi : M[\vec{\lambda}_1, \ldots, \vec{\lambda}_n] \to M[\vec{\lambda}_1, \ldots, \vec{\lambda}_n]/\langle \vec{\partial} + \vec{\lambda}_1 + \cdots + \vec{\lambda}_n \rangle$ *is the quotient map.*

4. *For a cochain* $\widetilde{\gamma} \in \widetilde{C}^n_{PP}(P, M)$, *if* $\widetilde{N}_i \widetilde{\gamma} = \Delta_i(\widetilde{\gamma})\widetilde{\gamma}$, *then*

$$N_i(\pi \circ \widetilde{\gamma}) = \Delta_i(\widetilde{\gamma})(\pi \circ \widetilde{\gamma}). \tag{6.2.24}$$

Proof. Part 1 follows from Lemma 6.2.4 and the fact that N_i's are diagonalizable. Parts 2 and Part 3 follow from Proposition 6.2.1, Lemma 6.2.4 and (6.2.15). Part 4 follows immediately from Part 3. □

The energy operators N_i above in fact act trivially on the basic cohomology $\widetilde{H}^n_{LP}(P, M)$. This follows from Cartan's formula below.

Proposition 6.2.4 (cf. [5], [6])**.** *The following Cartan's formula holds for any* $a \in P$ *(*P *as an LPA):*

$$a_{\vec{\lambda}} = [\iota_{\vec{\lambda}}(a), \widetilde{d}] =: \iota_{\vec{\lambda}}(a)\widetilde{d} - (-1)^{\bar{p}(a)} \widetilde{d} \iota_{\vec{\lambda}}(a), \tag{6.2.25}$$

where $\widetilde{d}$ *is the differential map given by (6.2.19) (considered as in the basic complex) and* $\iota_{\vec{\lambda}}(a)\widetilde{\gamma} : P^{\otimes(n-1)} \to M[\vec{\lambda}_1, \ldots, \vec{\lambda}_{n-1}, \vec{\lambda}]$ *for any cochain* $\widetilde{\gamma} \in \widetilde{C}^n_{LP}(P, M)$ *is called the contraction of* $\widetilde{\gamma}$ *by* a, *which is defined as:*

$$(\iota_{\vec{\lambda}}(a)\widetilde{\gamma})_{\vec{\lambda}_1, \ldots, \vec{\lambda}_{n-1}}(a_1 \otimes \cdots \otimes a_{n-1}) = (-1)^{\bar{p}(a)\bar{p}(\widetilde{\gamma})} \widetilde{\gamma}_{\vec{\lambda}, \vec{\lambda}_1, \ldots, \vec{\lambda}_{n-1}}(a \otimes a_1 \otimes \cdots \otimes a_{n-1}). \tag{6.2.26}$$

Corollary 6.2.3. *For any* $a \in P$, $a_{\vec{\lambda}} \circ \widetilde{d} = -(-1)^{\bar{p}(a)} \widetilde{d} \circ a_{\vec{\lambda}}$, *and it induces a trivial action on* $\widetilde{H}_{LP}(P, M)$. *In particular, it acts trivially on the basic cohomology* $\widetilde{H}_{PP}(P, M)$.

Proof. The equality follows immediately from Cartan's formula (6.2.25). To see the action of $a_{\vec{\lambda}}$ is trivial on $\widetilde{H}_{LP}(P, M)$, let $\widetilde{\gamma} \in \widetilde{C}^n_{LP}(P, M)$ be a cocycle, that is $\widetilde{d}\widetilde{\gamma} = 0$. One computes:

$$a_{\vec{\lambda}}[\widetilde{\gamma}] = [a_{\vec{\lambda}}\widetilde{\gamma}] = [\iota_{\vec{\lambda}}(a)\widetilde{d}\widetilde{\gamma} - (-1)^{\bar{p}(a)} \widetilde{d}\iota_{\vec{\lambda}}(a)\widetilde{\gamma}] = [-(-1)^{\bar{p}(a)} \widetilde{d}\iota_{\vec{\lambda}}(a)\widetilde{\gamma}] = 0, \tag{6.2.27}$$

thus $a_{\vec{\lambda}}$ acts trivially on $\widetilde{H}_{LP}(P, M)$. Moreover, note that in (6.2.27) we have $a_{\vec{\lambda}}\widetilde{\gamma} = \widetilde{d}(-(-1)^{\bar{p}(a)}\iota_{\vec{\lambda}}(a)\widetilde{\gamma})$. If $\widetilde{\gamma}$ satisfies the Leibniz rules, so does $-(-1)^{\bar{p}(a)}\iota_{\vec{\lambda}}(a)\widetilde{\gamma}$ (by (6.2.26)), hence $a_{\vec{\lambda}}$ acts trivially on $\widetilde{H}_{PP}(P, M)$. □

6.2.2 The main theorem on energy operators when $H = \mathbb{F}[\partial_1, \partial_2]$

The goal of this subsection is to prove our second main theorem of this thesis as follows:

Theorem 6.2.1. *Let $H = \mathbb{F}[\partial_1, \partial_2]$, let P be a PPA endowed with two Virasoro elements L_1, L_2 with respect to ∂_1, ∂_2 respectively, and M be a P-module which is conformal with respect to both L_1 and L_2. If as a differential superalgebra P is a superalgebra of differential polynomials in even or odd variables, then the energy operators $N_1, N_2 \in \operatorname{End} H_{PP}(P, M)$ are diagonalizable and both have only eigenvalues 0 and 1, that is $N_1^2 = N_1$, $N_2^2 = N_2$.*

In order to prove the theorem, we construct the spectral sequence associated to the Koszul complexes given at the end of Subsection 6.1.3: for each $n \geq 1$,

$$0 \to \widetilde{C}^n_{PP}(P, M) \xrightarrow{\phi_2} \widetilde{C}^n_{PP}(P, M) \oplus \widetilde{C}^n_{PP}(P, M) \xrightarrow{\phi_1} \widetilde{C}^n_{PP}(P, M) \xrightarrow{\epsilon} C^n_{PP}(P, M) \to 0, \tag{6.2.28}$$

where ϕ_1, ϕ_2 are given by:

$$\phi_1(\tilde{\gamma}_1, \tilde{\gamma}_2) = \partial_1 \tilde{\gamma}_1 + \partial_2 \tilde{\gamma}_2, \qquad \phi_2(\tilde{\gamma}) = (-\partial_2 \tilde{\gamma}, \partial_1 \tilde{\gamma}),$$

and ϵ is the quotient map. We study the spectral sequence associated to (6.2.28) and prove a few lemmas needed before proving the theorem itself.

Lemma 6.2.6. *For $H = \mathbb{F}[\partial_1, \dots, \partial_N]$, the differential $\tilde{d}$ commutes with the action of ∂_i for any $i = 1, \dots, N$ on $\widetilde{C}_{PP}(P, M)$, that is $\tilde{d}(\partial_i \tilde{\gamma}) = \partial_i(\tilde{d} \tilde{\gamma})$ for any $\tilde{\gamma} \in \widetilde{C}^n_{PP}(P, M)$.*

This is proved in [6] Lemma 2.2 for the LCA in the even case. The proof is essentially the same for individual i in the super case, if we regard P as a higher-dimensional LPA. As a direct consequence, one sees that the 0-th page of the spectral sequence associated to (6.2.28) is indeed a double complex (see the graph below, note we alternate the signs by columns).

In the figure, $\widetilde{C}^n$ is short for $\widetilde{C}^n_{PP}(P, M)$ and C^n is short for $C^n_{PP}(P, M)$. The anticommutativity in the definition of a double complex holds because of Lemma 6.2.6 and $d \circ \epsilon = \epsilon \circ \tilde{d}$. We label these 4 nontrivial columns as the 0-th to the 3-rd column from left to right. The 1-st page of the spectral sequence looks like Figure 6.2 below.

$$
\begin{array}{cccccc}
0 & \widetilde{C}^{n+2} & \widetilde{C}^{n+2}\oplus\widetilde{C}^{n+2} & \widetilde{C}^{n+2} & C^{n+1} & 0 \\
 & \Big\uparrow{\scriptstyle \widetilde{d}} & \Big\uparrow{\scriptstyle -\widetilde{d}\oplus\widetilde{d}} & \Big\uparrow{\scriptstyle \widetilde{d}} & \Big\uparrow{\scriptstyle -d} & \\
0 & \widetilde{C}^{n}+1 & \widetilde{C}^{n+1}\oplus\widetilde{C}^{n+1} & \widetilde{C}^{n+1} & C^{n} & 0 \\
 & \Big\uparrow{\scriptstyle \widetilde{d}} & \Big\uparrow{\scriptstyle -\widetilde{d}\oplus\widetilde{d}} & \Big\uparrow{\scriptstyle \widetilde{d}} & \Big\uparrow{\scriptstyle -d} & \\
0 & \widetilde{C}^{n} & \widetilde{C}^{n}\oplus\widetilde{C}^{n} & \widetilde{C}^{n} & C^{n-1} & 0
\end{array}
$$

Figure 6.1: 0-th page of a spectral sequence associated to the Koszul complexes

$$0 \longrightarrow H^{n+2}(\widetilde{C}) \xrightarrow{(\phi_2)_*} H^{n+2}(\widetilde{C})\oplus H^{n+2}(\widetilde{C}) \xrightarrow{(\phi_1)_*} H^{n+2}(\widetilde{C}) \xrightarrow{\epsilon_*} H^{n+2}(C) \longrightarrow 0$$

$$0 \longrightarrow H^{n+1}(\widetilde{C}) \xrightarrow{(\phi_2)_*} H^{n+1}(\widetilde{C})\oplus H^{n+1}(\widetilde{C}) \xrightarrow{(\phi_1)_*} H^{n+1}(\widetilde{C}) \xrightarrow{\epsilon_*} H^{n+1}(C) \longrightarrow 0$$

$$0 \longrightarrow H^{n}(\widetilde{C}) \xrightarrow{(\phi_2)_*} H^{n}(\widetilde{C})\oplus H^{n}(\widetilde{C}) \xrightarrow{(\phi_1)_*} H^{n}(\widetilde{C}) \xrightarrow{\epsilon_*} H^{n}(C) \longrightarrow 0$$

Figure 6.2: 1-st page of a spectral sequence associated to the Koszul complexes

In the figure, $(\phi_2)_*$, $(\phi_1)_*$ and ϵ_* are induced by ϕ_2, ϕ_1 and ϵ respectively. The notation $H^n(\widetilde{C})$ is short for $H^n(\widetilde{C}_{PP}(P,M),\widetilde{d})$ and $H^n(C)$ for $H^n(C_{PP}(P,M),d)$. We also identify the space $H^n(\widetilde{C}\oplus\widetilde{C},-\widetilde{d}\oplus\widetilde{d})$ with $H^n(\widetilde{C})\oplus H^n(\widetilde{C})$. One sees that the energy operator $\widetilde{N}_i$ is trivial on $H^n(\widetilde{C})$ in the 0-th column (cf. Lemma 6.2.3). Note that the action of energy operators are not compatible with $(\phi_2)_*$ and $(\phi_1)_*$ in this case. Next we examine the 2-nd page of the spectral sequence below.

Here $\ker^n(\phi_2)_*$ is the kernel of $(\phi_2)_*$ and the superscript n only means in Row n and similarly $\mathrm{im}^n\epsilon_*$ denotes the image of ϵ_* in Row n. Furthermore, by definition, $H(H^n(\widetilde{C})) = \ker^n\epsilon_*/\mathrm{im}^n(\phi_1)_*$ and $H(H^n(\widetilde{C})\oplus H^n(\widetilde{C})) = \ker^n(\phi_1)_*/\mathrm{im}^n(\phi_2)_*$. By Theorem 5.3.2, one has an exact sequence on the 2-nd page of the spectral sequence as follows:

$$0\to H(H^{n+1}(\widetilde{C})\oplus H^{n+1}(\widetilde{C})) \xrightarrow{d_2^{1,n+1}} H^n(C)/\mathrm{im}^n\epsilon_* \xrightarrow{\theta_n} \ker^{n+2}(\phi_2)_* \xrightarrow{d_2^{0,n+2}} H(H^{n+1}(\widetilde{C}))\to 0, \tag{6.2.29}$$

$$\begin{tikzcd}
0 & \ker^{n+2}(\phi_2)_* \arrow[drr, "d_2^{0,n+2}"] & H(H^{n+2}(\widetilde{C}) \oplus H^{n+2}(\widetilde{C})) & H(H^{n+2}(\widetilde{C})) & H^{n+2}(C)/\mathrm{im}^{n+2}\epsilon_* & 0 \\
0 & \ker^{n+1}(\phi_2)_* & H(H^{n+1}(\widetilde{C}) \oplus H^{n+1}(\widetilde{C})) \arrow[drr, "d_2^{1,n+1}"] & H(H^{n+1}(\widetilde{C})) & H^{n+1}(C)/\mathrm{im}^{n+1}\epsilon_* & 0 \\
0 & \ker^{n}(\phi_2)_* & H(H^{n}(\widetilde{C}) \oplus H^{n}(\widetilde{C})) & H(H^{n}(\widetilde{C})) & H^{n}(C)/\mathrm{im}^{n}\epsilon_* & 0
\end{tikzcd}$$

Figure 6.3: 2-nd page of a spectral sequence associated to the Koszul complexes

with the connecting homomorphism $\theta_n : H^n(C)/\mathrm{im}^n \epsilon_* \to \ker^{n+2}(\phi_2)_*$ given by $\theta_n = j_n \circ (d_3)^{-1} \circ \rho_n$, where $j_n : \ker(d_2^{0,n+2}) \to \ker^{n+2}(\phi_2)_*$ is the injection map, $\rho_n : H^n(C)/\mathrm{im}^n \epsilon_* \to \mathrm{coker}(d_2^{1,n+1})$ is the projection (quotient) map and d_3 is the differential on the 3-rd page of the spectral sequence, which is an isomorphism. (cf. (5.3.9)).

By Proposition 6.2.1, the energy operators are not compatible with ϕ_2 and ϕ_1. In order to fix this, for $i = 1, 2$, we consider the following modified operators, which are related to energy operators, on the Koszul complex as shown below:

$$0 \to \underset{\widetilde{N}_i+1}{\widetilde{C}^n_{PP}(P,M)} \xrightarrow{\phi_2} \underset{(\widetilde{N}_i+\delta_{1i} \quad , \quad \widetilde{N}_i+\delta_{2i})}{\widetilde{C}^n_{PP}(P,M) \oplus \widetilde{C}^n_{PP}(P,M)} \xrightarrow{\phi_1} \underset{\widetilde{N}_i}{\widetilde{C}^n_{PP}(P,M)} \xrightarrow{\epsilon} \underset{N_i}{C^n_{PP}(P,M)} \to 0. \qquad (6.2.30)$$

The modified operator on each complex is labeled below it. For example, for $i = 1$, consider N_1 on $C^n_{PP}(P,M)$, which is compatible with $\widetilde{N_1}$ on $\widetilde{C}^n_{PP}(P,M)$, that is $N_1\epsilon(\widetilde{\gamma}) = \epsilon(\widetilde{N}_1\widetilde{\gamma})$. To keep the compatibility, we shall have $(\widetilde{N}_1 + 1, \widetilde{N}_1)$ on $\widetilde{C}^n_{PP}(P,M) \oplus \widetilde{C}^n_{PP}(P,M)$, which is again compatible with $\widetilde{N}_1 + 1$ on $\widetilde{C}^n_{PP}(P,M)$. One can write out the modified operators explicitly for $i = 2$ in a similar way. In fact, if we view these modified operators on each term of the double complex as an operator on the double complex, this operator is also compatible with the differential map d_r for any $r \geq 0$, which is formulated in a more general setting in the following proposition.

Proposition 6.2.5. *Consider the spectral sequence associated to a double complex (M, d', d''). Suppose that for each p and q, we have an endomorphism $A^{p,q}$ on the (p,q)-term $E_0^{p,q}$ of the 0-th page of the spectral sequence, such that they commute with the differentials d' and d''.*

If we view them as an endomorphism $A=\bigoplus_{p,q} A^{p,q}$ *on the* 0*-th page* $E_0=\bigoplus_{p,q} E_0^{p,q}$*, then* A *induces an endomorphism* $\bar{A}_r$ *on the* r*-th page of the spectral sequence for all* $r\geq 0$*, and* $\bar{A}_r\circ d_r = d_r\circ \bar{A}_r$*.*

Proof. We use the notation $(E_r^{p,q}, d_r)$ to denote the r-th page of the spectral sequence and $F^pT(M)^{p+q}$ to denote the decreasing filtration of the associated total complex $T(M)$. For $r\geq 1$, recall that $d_r: E_r^{p,q}\to E_r^{p+r,q-r+1}$ is induced by the total differential $D=d'+d''$:

$$d_r[\alpha]=[D\alpha], \qquad \text{for all} \quad [\alpha]\in E_r^{p,q}. \tag{6.2.31}$$

By definition, $E_r^{p,q}=Z_r^{p,q}/(Z_{r-1}^{p+1,q-1}+B_{r-1}^{p,q})$, $Z_r^{p,q}=F^pT(M)^{p+q}\cap D^{-1}(F^{p+r}T(M)^{p+q+1})$, and $B_r^{p,q}=F^pT(M)^{p+q}\cap D(F^{p-r}T(M)^{p+q-1})$. Note that $Z_r^{p,q}$ and $B_r^{p,q}$ are invariant under A, since A commutes with d' and d'' and hence with D. So A induces an endomorphism $\bar{A}_r$ on the r-th page E_r. Then $\bar{A}_r$ commutes with d_r from the calculation:

$$\bar{A}_r(d_r[\alpha])=\bar{A}_r[D\alpha]=[AD\alpha]=[D(A\alpha)]=d_r[A\alpha]=d_r(\bar{A}_r[\alpha]). \tag{6.2.32}$$

It is worth mentioning that by construction $E_r^{p,q}\cong H^{p,q}(E_{r-1},d_{r-1})$, and the isomorphism is induced by the map $\eta: Z_r^{p,q}\to H^{p,q}(E_{r-1},d_{r-1})$, whose kernel is exactly $Z_{r-1}^{p+1,q-1}+B_{r-1}^{p,q}$. This implies that the isomorphism $\eta_*[\alpha]=[\alpha]$ is just the identity map. So A induces an endomorphism on $H^{p,q}(E_{r-1},d_{r-1})$ for each $r\geq 1$. □

By Proposition 6.2.5 and the modified operators given in (6.2.30), the induced endomorphism on the 1-st page of the spectral sequence is indeed given by the same formula as in (6.2.30):

$$0\to \underset{\widetilde{N}_i+1}{H^n(\widetilde{C})}\xrightarrow{(\phi_2)_*} \underset{(\widetilde{N}_i+\delta_{1i}\quad,\quad}{H^n(\widetilde{C})}\oplus \underset{\widetilde{N}_i+\delta_{2i})}{H^n(\widetilde{C})}\xrightarrow{(\phi_1)_*} \underset{\widetilde{N}_i}{H^n(\widetilde{C})}\xrightarrow{\epsilon_*} \underset{N_i}{H^n(C)}\to 0. \tag{6.2.33}$$

And on the exact sequence (6.2.29), which is on the 2-nd page of the spectral sequence, the induced action is given by:

$$0\to H(\underset{(\widetilde{N}_i+\delta_{1i}\quad,\quad}{H^{n+1}(\widetilde{C})}\oplus \underset{\widetilde{N}_i+\delta_{2i})}{H^{n+1}(\widetilde{C})})\xrightarrow{d_2^{1,n+1}} \underset{N_i}{H^n(C)}/\operatorname{im}^n\epsilon_* \xrightarrow{\theta_n} \underset{\widetilde{N}_i+1}{\ker^{n+2}(\phi_2)_*}\xrightarrow{d_2^{0,n+2}} H(\underset{\widetilde{N}_i}{H^{n+1}(\widetilde{C})})\to 0. \tag{6.2.34}$$

Lemma 6.2.7. *In (6.2.34), we have* $(\widetilde{N}_i+1)\circ\theta_n=\theta_n\circ N_i$ *for* $i=1,2$*.*

Proof. By construction, $\theta_n = j_n \circ (d_3)^{-1} \circ \rho_n$ which is the composition:

$$\underset{N_i}{H^n(C)/\operatorname{im}^n \epsilon_*} \xrightarrow{\rho_n} \underset{N_i}{\operatorname{coker}(d_2^{1,n+1})} \xrightarrow{(d_3)^{-1}} \underset{\widetilde{N}_i+1}{\ker(d_2^{0,n+2})} \xrightarrow{j_n} \underset{\widetilde{N}_i+1}{\ker^n(\phi_2)_*}. \tag{6.2.35}$$

Since $\ker(\rho_n) = \operatorname{im}(d_2^{1,n+1})$, we have $\rho_n(N_i\gamma_1) = 0$ for any $\gamma_1 \in \operatorname{im}(d_2^{1,n+1}) \subset H^n(C)/\operatorname{im}^n \epsilon_*$, because $N_i\gamma_1 \in \operatorname{im}(d_2^{1,n+1})$. Thus N_i induces an endomorphism, which is still denoted as N_i, on $\operatorname{coker}(d_2^{1,n+1})$. For $\gamma \in H^n(C)/\operatorname{im}^n \epsilon_*$, one can check that:

$$N_i(\rho_n(\gamma)) = N_i[\gamma] = [N_i\gamma] = \rho_n(N_i\gamma). \tag{6.2.36}$$

For the next step, by Proposition 6.2.5, we have:

$$N_i \circ d_3 = d_3 \circ (\widetilde{N}_i + 1), \tag{6.2.37}$$

which implies:

$$(d_3)^{-1} \circ N_i = (\widetilde{N}_i + 1) \circ (d_3)^{-1}. \tag{6.2.38}$$

For the last step, since j_n is the injection map, it is restricted to an identity map from $\ker(d_2^{0,n+2})$ to $\operatorname{im} j_n \subset \ker(\phi_2)_*$. So we have

$$(\widetilde{N}_i + 1) \circ j_n = j_n \circ (\widetilde{N}_i + 1), \tag{6.2.39}$$

which completes the proof of the lemma. □

Proof of Theorem 6.2.1. We consider the two cases $n \geq 1$ and $n = 0$ separately. For $n \geq 1$, to see the eigenvalues of N_i on $H^n(C)$, note that we have a short exact sequence:

$$0 \to \underset{N_i}{\operatorname{im}^n \epsilon_*} \xrightarrow{i} \underset{N_i}{H^n(C)} \xrightarrow{q} \underset{N_i}{H^n(C)/\operatorname{im}^n \epsilon_*} \to 0, \tag{6.2.40}$$

where i is the inclusion map, q is the quotient map and N_i commutes with both i and q. One also checks that N_i is trivial on $\operatorname{im}^n \epsilon_*$ as follows: for any $[\widetilde{\gamma}] \in H^n(\widetilde{C})$, by (6.2.33), we have

$$N_i(\epsilon_*[\widetilde{\gamma}]) = [N_i(\epsilon(\widetilde{\gamma}))] = [N_i(\pi \circ \widetilde{\gamma})] = [\pi \circ (\widetilde{N}_i(\widetilde{\gamma}))] = \epsilon_*(\widetilde{N}_i[\widetilde{\gamma}]) = 0. \tag{6.2.41}$$

In particular, this also implies that $N_i(N_i - 1)$ acts trivially on $\operatorname{im}\epsilon_*$. Next, we examine the eigenvalues of N_1 and N_2 on $H^n(C)/\mathrm{im}^n \epsilon_*$. Take N_1 as an example.

Since $\widetilde{N}_1$ is trivial on the basic cohomology $H^n(\widetilde{C})$, the compatible operators in (6.2.33) become:

$$0 \to \underset{1}{H^n(\widetilde{C})} \xrightarrow{(\phi_2)_*} \underset{(1\quad,\quad 0)}{H^n(\widetilde{C}) \oplus H^n(\widetilde{C})} \xrightarrow{(\phi_1)_*} \underset{0}{H^n(\widetilde{C})} \xrightarrow{\epsilon_*} \underset{N_1}{H^n(C)} \to 0. \tag{6.2.42}$$

Moreover, (6.2.34) becomes:

$$0 \to \underset{(1\quad,\quad 0)}{H(H^{n+1}(\widetilde{C}) \oplus H^{n+1}(\widetilde{C}))} \xrightarrow{d_2^{1,n+1}} \underset{N_1}{H^n(C)\big/\mathrm{im}^n \epsilon_*} \xrightarrow{\theta_n} \underset{1}{\ker^{n+2}(\phi_2)_*} \xrightarrow{d_2^{0,n+2}} \underset{0}{H(H^{n+1}(\widetilde{C}))} \to 0. \tag{6.2.43}$$

On the other hand, if we replace N_1 with $N_1 - 1$ and $\widetilde{N}_1$ with $\widetilde{N}_1 - 1$ in (6.2.30), we have another set of operators that are compatible with ϕ_2, ϕ_1 and ϵ:

$$0 \to \underset{\widetilde{N}_1}{\widetilde{C}^n_{PP}(P,M)} \xrightarrow{\phi_2} \underset{(\widetilde{N}_1\quad,\quad \widetilde{N}_1-1)}{\widetilde{C}^n_{PP}(P,M) \oplus \widetilde{C}^n_{PP}(P,M)} \xrightarrow{\phi_1} \underset{\widetilde{N}_1-1}{\widetilde{C}^n_{PP}(P,M)} \xrightarrow{\epsilon} \underset{N_1-1}{C^n_{PP}(P,M)} \to 0. \tag{6.2.44}$$

Again since $\widetilde{N}_1$ is trivial on the basic cohomology $H^n(\widetilde{C})$, we have:

$$0 \to \underset{0}{H^n(\widetilde{C})} \xrightarrow{(\phi_2)_*} \underset{(0\quad,\quad -1)}{H^n(\widetilde{C}) \oplus H^n(\widetilde{C})} \xrightarrow{(\phi_1)_*} \underset{-1}{H^n(\widetilde{C})} \xrightarrow{\epsilon_*} \underset{N_1-1}{H^n(C)} \to 0. \tag{6.2.45}$$

and

$$0 \to \underset{(0\quad,\quad -1)}{H(H^{n+1}(\widetilde{C}) \oplus H^{n+1}(\widetilde{C}))} \xrightarrow{d_2^{1,n+1}} \underset{N_1-1}{H^n(C)\big/\mathrm{im}^n \epsilon_*} \xrightarrow{\theta_n} \underset{0}{\ker^{n+2}(\phi_2)_*} \xrightarrow{d_2^{0,n+2}} \underset{-1}{H(H^{n+1}(\widetilde{C}))} \to 0. \tag{6.2.46}$$

From (6.2.43) and (6.2.46), we see that the endomorphism $N_1(N_1 - 1)$ on $H^n(C)/\mathrm{im}^n \epsilon_*$ is trivial, since it is compatible with trivial endomorphisms on both $H(H^{n+1}(\widetilde{C}) \oplus H^{n+1}(\widetilde{C}))$ and $\ker^{n+2}(\phi_2)_*$. Putting this information back to the short exact sequence:

$$0 \to \underset{0}{\mathrm{im}^n \epsilon_*} \xrightarrow{i} \underset{N_1(N_1-1)}{H^n(C)} \xrightarrow{q} \underset{0}{H^n(C)/\mathrm{im}^n \epsilon_*} \to 0. \tag{6.2.47}$$

one sees that $N_1(N_1 - 1)$ is the trivial endomorphism on $H^n(C)$ because it is compatible with the trivial endomorphisms on both $\mathrm{im}^n \epsilon_*$ and $H^n(C)/\mathrm{im}^n \epsilon_*$. Furthermore, since N_1 is diagonalizable on $H^n(C)$, the eigenvalues of N_1 on $H^n(C)$ is either 0 or 1. The same reasoning also works for N_2 with a slight modification and we see that N_2 also only has eigenvalues 0 and 1.

Now we are left with the case $n=0$. Note that $\widetilde{C}^0_{PP}(P,M)=M$; however, ∂_1,∂_2 may fail to be an M-regular sequence in this case, so we give a direct computation for the eigenvalues of N_1 and N_2 on $H^0_{PP}(P,M)$. Let $\alpha\in C^0_{PP}(P,M)=M/(\partial_1 M+\partial_2 M)$ be a cocycle. Since $\epsilon: M\longrightarrow M/(\partial_1 M+\partial_2 M)$ is surjective, there exists $\widetilde{\beta}\in\widetilde{C}^0_{PP}(P,M)$ such that $\epsilon(\widetilde{\beta})=\alpha$. Then $d\alpha=0$ implies that $\epsilon(\widetilde{d}\widetilde{\beta})=d(\epsilon(\widetilde{\beta}))=d\alpha=0$. By the exactness of the Koszul complex for $n=1$, there exist $(\widetilde{\gamma}_1,\widetilde{\gamma}_2)\in\widetilde{C}^1_{PP}(P,M)\oplus\widetilde{C}^1_{PP}(P,M)$ such that $\partial_1\widetilde{\gamma}_1+\partial_2\widetilde{\gamma}_2=\widetilde{d}\widetilde{\beta}$. Note that $d(\partial_1\widetilde{\gamma}_1+\partial_2\widetilde{\gamma}_2)=\partial_1(d\widetilde{\gamma}_1)+\partial_2(d\widetilde{\gamma}_2)=0$; hence $(\widetilde{d}\widetilde{\gamma}_1,\widetilde{d}\widetilde{\gamma}_2)\in\ker(\phi_1)$. By the exactness of the Koszul complex for $n=2$, there exists $\widetilde{\gamma}\in\widetilde{C}^2_{PP}(P,M)$ such that $(-\partial_2\widetilde{\gamma},\partial_1\widetilde{\gamma})=(\widetilde{d}\widetilde{\gamma}_1,\widetilde{d}\widetilde{\gamma}_2)$. Obviously $\widetilde{d}\widetilde{\gamma}=0$; hence $\widetilde{\gamma}$ is a cocycle in $\widetilde{C}^2_{PP}(P,M)$. Since N_i is trivial on the basic cohomology $H_{PP}(P,M)$, which means that the image under N_i of a cocycle is a coboundary, we have

$$N_1\widetilde{\gamma}=\widetilde{d}\widetilde{\sigma}_0,\quad\text{for some}\quad\widetilde{\sigma}_0\in\widetilde{C}^1_{PP}(P,M).\tag{6.2.48}$$

Thus

$$\widetilde{d}(\widetilde{N}_1\widetilde{\gamma}_1)=\widetilde{N_1}(\widetilde{d}\widetilde{\gamma}_1)=\widetilde{N}_1(-\partial_2\widetilde{\gamma})=-\partial_2(\widetilde{N}_1\widetilde{\gamma})=-\partial_2(\widetilde{d}\widetilde{\sigma}_0)=\widetilde{d}(-\partial_2\widetilde{\sigma}_0),\tag{6.2.49}$$

and $\widetilde{N}_1\widetilde{\gamma}_1+\partial_2\widetilde{\sigma}_0$ is a cocycle. Again its image under $\widetilde{N}_1$ is a coboundary:

$$\widetilde{N}_1(\widetilde{N}_1\widetilde{\gamma}_1+\partial_2\widetilde{\sigma}_0)=\widetilde{d}\widetilde{\sigma}_1\quad\text{for some}\quad\widetilde{\sigma}_1\in\widetilde{C}^0_{PP}(P,M).\tag{6.2.50}$$

Equivalently,

$$\widetilde{N}_1^2\widetilde{\gamma}_1=-\widetilde{N}_1\partial_2\widetilde{\sigma}_0+\widetilde{d}\widetilde{\sigma}_1.\tag{6.2.51}$$

We replace $\widetilde{N}_1(-\partial_2\widetilde{\gamma})$ with $\widetilde{N}_1(\partial_1\widetilde{\gamma})$ in (6.2.49) and compute again:

$$\widetilde{d}(\widetilde{N}_1\widetilde{\gamma}_2)=\widetilde{N_1}(\widetilde{d}\widetilde{\gamma}_2)=\widetilde{N}_1(\partial_1\widetilde{\gamma})=\partial_1(\widetilde{N}_1+1)\widetilde{\gamma}=\partial_1(\widetilde{d}\widetilde{\sigma}_0)+\widetilde{d}\widetilde{\gamma}_2=\widetilde{d}(\partial_1\widetilde{\sigma}_0+\widetilde{\gamma}_2),\tag{6.2.52}$$

which implies that $(\widetilde{N}_1-1)\widetilde{\gamma}_2-\partial_1\widetilde{\sigma}_0$ is a cocycle. Its image under $\widetilde{N}_1$ is a coboundary:

$$\widetilde{N}_1((\widetilde{N}_1-1)\widetilde{\gamma}_2-\partial_1\widetilde{\sigma}_0)=\widetilde{d}\widetilde{\sigma}_2\quad\text{for some}\quad\widetilde{\sigma}_2\in\widetilde{C}^0_{PP}(P,M),\tag{6.2.53}$$

or equivalently

$$\widetilde{N}_1(\widetilde{N}_1-1)\widetilde{\gamma}_2=\widetilde{N}_1\partial_1\widetilde{\sigma}_0+\widetilde{d}\widetilde{\sigma}_2. \tag{6.2.54}$$

Now apply the operator $\widetilde{N}_1(\widetilde{N}_1-1)^2$ to $\partial_1\widetilde{\gamma}_1+\partial_2\widetilde{\gamma}_2$. By (6.2.51) and (6.2.54), we get:

$$\begin{aligned}\widetilde{N}_1(\widetilde{N}_1-1)^2(\partial_1\widetilde{\gamma}_1+\partial_2\widetilde{\gamma}_2)=&\partial_1(\widetilde{N}_1+1)\widetilde{N}_1^2\widetilde{\gamma}_1+\partial_2\widetilde{N}_1(\widetilde{N}_1-1)^2\widetilde{\gamma}_2\\ =&\partial_1(\widetilde{N}_1+1)(-\widetilde{N}_1\partial_2\widetilde{\sigma}_0+\widetilde{d}\widetilde{\sigma}_1)+\partial_2(\widetilde{N}_1-1)(\widetilde{N}_1\partial_1\widetilde{\sigma}_0+\widetilde{d}\widetilde{\sigma}_2)\\ =&-\widetilde{N}_1(\widetilde{N}_1-1)\partial_1\partial_2\widetilde{\sigma}_0+\widetilde{d}\widetilde{N}_1\partial_1\widetilde{\sigma}_1+(\widetilde{N}_1-1)\widetilde{N}_1\partial_2\partial_1\widetilde{\sigma}_0+\widetilde{d}(\widetilde{N}_1-1)\partial_2\widetilde{\sigma}_2\\ =&\widetilde{d}(\widetilde{N}_1\partial_1\widetilde{\sigma}_1+(\widetilde{N}_1-1)\partial_2\widetilde{\sigma}_2).\end{aligned} \tag{6.2.55}$$

This is equivalent to:

$$\widetilde{N}_1(\widetilde{N}_1-1)^2(\widetilde{d}\widetilde{\beta})=\widetilde{d}(\widetilde{N}_1(\widetilde{N}_1-1)^2\widetilde{\beta})=\widetilde{d}(\widetilde{N}_1\partial_1\widetilde{\sigma}_1+(\widetilde{N}_1-1)\partial_2\widetilde{\sigma}_2). \tag{6.2.56}$$

So $\widetilde{N}_1(\widetilde{N}_1-1)^2\widetilde{\beta}-\widetilde{N}_1\partial_1\widetilde{\sigma}_1-(\widetilde{N}_1-1)\partial_2\widetilde{\sigma}_2$ is a cocycle in $\widetilde{C}^0_{PP}(P,M)$. Its image under $\widetilde{N}_1$ is 0:

$$\widetilde{N}_1^2(\widetilde{N}_1-1)^2\widetilde{\beta}-\widetilde{N}_1^2\partial_1\widetilde{\sigma}_1-\widetilde{N}_1(\widetilde{N}_1-1)\partial_2\widetilde{\sigma}_2=0. \tag{6.2.57}$$

Now if we apply ϵ to both sides of (6.2.57), we obtain

$$N_1^2(N_1-1)^2\alpha=0, \tag{6.2.58}$$

since $\epsilon(\partial_i\widetilde{\gamma})=0$ for any $\widetilde{\gamma}\in\widetilde{C}_{PP}(P,M)$. Recall that by assumption M is conformal with respect to L_1, which implies that N_1 is diagonalizable on $M/(\partial_1 M+\partial_2 M)$. Hence N_1 has only eigenvalues 0 and 1. One can prove the same conclusion for N_2 similarly. □

6.3 Cohomology of the free superfermion Poisson superpseudoalgebra when $H=\mathbb{F}[\partial_1,\partial_2]$

In this section we compute the cohomology of free superfermion PPA for $H=\mathbb{F}[\partial_1,\partial_2]$ as an application of Theorem 6.2.1. The proofs here are similar to the PVA case in [5]. Let $H=\mathbb{F}[\partial_1,\partial_2]$ through the rest of this section. Recall that the free superfermion PPA defined in Example 4.3.3 is:

$$F_{\mathfrak{h}} = S(H\mathfrak{h}), \tag{6.3.1}$$

with the pseudobracket on the generators:

$$[a * b] = (\gamma(a,b) \otimes 1) \otimes_H 1, \quad \text{for any} \quad a, b \in \mathfrak{h}, \tag{6.3.2}$$

which is extended to $F_{\mathfrak{h}}$ by H-bilinearity and Leibniz rules. Here $\mathfrak{h}$ is a dimension N vector superspace with parity p and equipped with a super-skewsymmetric nondegenerate bilinear form $\gamma : \mathfrak{h} \times \mathfrak{h} \to \mathbb{F}$. Moreover, we also impose the condition that $\gamma(a,b) = 0$ if $p(a) \neq p(b)$. Note that the super-skewsymmetry of γ means:

$$\gamma(a,b) = -(-1)^{p(a)p(b)}\gamma(b,a), \tag{6.3.3}$$

for any homogeneous $a, b \in \mathfrak{h}$. Pick a homogeneous basis $\{u_1, \dots, u_N\}$ of $\mathfrak{h}$ and its dual basis $\{u^1, \dots, u^N\}$ of $\mathfrak{h}$ with respect to γ, that is:

$$\gamma(u_i, u^j) = \delta_{ij}. \tag{6.3.4}$$

Then for any $a \in \mathfrak{h}$, one can write a as a linear combination of $\{u^1, \dots, u^N\}$ uniquely: $a = c_1 u_1 + \cdots + c_N u_N$ for some $c_1, \dots, c_N \in \mathbb{F}$, which implies:

$$\gamma(a, u^j) = \gamma(\sum_{i=1}^N c_i u_i, u^j) = \sum_{i=1}^N c_i \gamma(u_i, u^j) = c_j. \tag{6.3.5}$$

Similarly, if $a = b_1 u^1 + \dots b_N u^N$, the same reasoning works for $\gamma(u_i, a) = b_i$. So we have:

$$a = \sum_{i=1}^N \gamma(a, u^i) u_i = \sum_{i=1}^N \gamma(u_i, a) u^i. \tag{6.3.6}$$

Another equation that is useful is $\sum_{j=1}^N u^j u_j = 0$. The left hand side of the equation is independent of the choice of the basis. By (6.3.3), $\gamma(u^j, (-1)^{p(u^j)p(u_i)+1} u_i) = \delta_{ij}$ so the dual basis of $\{u^1, \dots, u^N\}$ is $\{(-1)^{p(u^j)p(u_i)+1} u_i\}_{1 \le i \le N}$. Thus

$$\sum_{j=1}^N u^j u_j = \sum_{j=1}^N (-1)^{p(u^j)p(u_i)+1} u_j u^j = -\sum_{j=1}^N u^j u_j, \tag{6.3.7}$$

which concludes:

$$\sum_{j=1}^{N} u^j u_j = 0. \tag{6.3.8}$$

The pseudobracket (6.3.2) can be written as a $\vec{\lambda}$-bracket as follows:

$$[a_{\vec{\lambda}} b] = \gamma(a, b) \quad \text{for any} \quad a, b \in \mathfrak{h}. \tag{6.3.9}$$

This $\vec{\lambda}$-bracket satisfies all the axioms listed in Example 4.3.1. For the rest of this section, let $F_{\mathfrak{h}}$ denote the free superfermion PPA for $H = \mathbb{F}[\partial_1, \partial_2]$, with the $\vec{\lambda}$-bracket in (6.3.9).

Proposition 6.3.1. *For the free superfermion PPA $F_{\mathfrak{h}}$, we have that*

$$L_i = \frac{1}{2}\sum_{j=1}^{N} (\partial_i u^j) u_j, \quad i = 1, 2, \tag{6.3.10}$$

is a Virasoro element with respect to ∂_i. Furthermore, any $a \in \mathfrak{h}$ has a conformal weight $\Delta_i(a) = 1/2$ with respect to L_i.

Proof. To see that L_i is a Virasoro element with respect to ∂_i, we first compute $[a_{\vec{\lambda}} L_i]$ for any $a \in \mathfrak{h}$:

$$\begin{aligned}
[a_{\vec{\lambda}} L_i] = \frac{1}{2}\sum_{j=1}^{N} [a_{\vec{\lambda}} (\partial_i u^j) u_j] &= \frac{1}{2}\sum_{j=1}^{N} \left(((\partial_i + \lambda_i)[a_{\vec{\lambda}} u^j]) u_j + (-1)^{p(u^j)p(u_j)} [a_{\vec{\lambda}} u_j] \partial_i u^j \right) \\
&= \frac{1}{2}\sum_{j=1}^{N} \left(\lambda_i \gamma(a, u^j) u_j + (-1)^{p(u_j)p(u^j)} \gamma(a, u_j) \partial_i u^j \right)
\end{aligned} \tag{6.3.11}$$

where we use the Leibnize rule (4.3.5) and sesquilinearity in the second equality. By some simple linear algebra calculation $\sum_{j=1}^{N} (-1)^{p(u_j)p(u^j)} \gamma(a, u_j) u^j = -a$, thus

$$[a_{\vec{\lambda}} L_i] = \frac{1}{2}(\lambda_i - \partial_i) a. \tag{6.3.12}$$

By super-skewsymmetry, since L_i is even by definition, we have:

$$[L_i\, {}_{\vec{\lambda}} a] = -[a_{-\vec{\lambda}-\vec{\partial}} L_i] = \left(\partial_i + \frac{1}{2}\lambda_i\right) a, \tag{6.3.13}$$

from which one sees that on $\mathfrak{h}$, $L_{i(0)} = \partial_i$ and $L_{i(t_i)}$ is diagonalizable. In particular $\Delta_i(a) = \dfrac{1}{2}$

for any $a \in \mathfrak{h}$. Furthermore for any $a, b \in \mathfrak{h}$:

$$\begin{aligned}
[L_i\,{}_{\vec{\lambda}}(\partial_j a)] =& (\partial_j + \lambda_j)\big(\partial_i + \frac{1}{2}\lambda_i\big)a \\
=& \partial_i(\partial_j a) + \frac{1}{2}\lambda_i \partial_j a + \lambda_j \partial_i a + \frac{1}{2}\lambda_i \lambda_j a; \qquad (6.3.14) \\
[L_i\,{}_{\vec{\lambda}}\, a b] =& [L_i\,{}_{\vec{\lambda}}\, a]b + (-1)^{p(a)p(b)}[L_i\,{}_{\vec{\lambda}}\, b]a \\
=& \big(\big(\partial_i + \frac{1}{2}\lambda_i\big)a\big)b + (-1)^{p(a)p(b)}\big(\big(\partial_i + \frac{1}{2}\lambda_i\big)b\big)a \\
=& \partial_i(ab) + \lambda_i ab, \qquad (6.3.15)
\end{aligned}$$

which implies that $L_{i(0)} = \partial_i$ and $L_{i(t_i)}$ is diagonalizable on $\partial_j \mathfrak{h}$ and any product of elements in $\mathfrak{h}$. Thus $L_{i(0)} = \partial_i$ and $L_{i(t_i)}$ is diagonalizable on $F_{\mathfrak{h}}$. Next we compute $[L_i\,{}_{\vec{\lambda}}\, L_i]$:

$$\begin{aligned}
[L_i\,{}_{\vec{\lambda}}\, L_i] =& \frac{1}{2}\sum_{j=1}^{N}[L_i\,{}_{\vec{\lambda}}((\partial_i u^j)u_j)] \\
=& \frac{1}{2}\sum_{j=1}^{N}\big([L_i\,{}_{\vec{\lambda}}(\partial_i u^j)]u_j + (\partial_i u^j)[L_i\,{}_{\vec{\lambda}}\, u_j]\big) \\
=& \frac{1}{2}\sum_{j=1}^{N}\big((\partial_i + \lambda_i)\big(\partial_i + \frac{1}{2}\lambda_i\big)u^j\big)u_j + \frac{1}{2}\sum_{j=1}^{N}(\partial_i u^j)\big(\big(\partial_i + \frac{1}{2}\lambda_i\big)u_j\big) \\
=& \frac{1}{2}\sum_{j=1}^{N}(\partial_i^2 u^j)u_j + \frac{1}{2}\lambda_i L_i + \lambda_i L_i + \frac{1}{4}\lambda_i^2 \sum_{j=1}^{N} u^j u_j + \frac{1}{2}\sum_{j=1}^{N}(\partial_i u^j)(\partial_i u_j) + \frac{1}{2}\lambda_i L_i \\
=& (\partial_i + 2\lambda_i)L_i, \qquad (6.3.16)
\end{aligned}$$

since $\sum_{j=1}^{N} u^j u_j = 0$ by (6.3.8). We have verified that for $i = 1, 2$, L_i is is a Virasoro element with respect to ∂_i. $\square$

Remark 6.3.1. *Similarly to (6.3.16), we check that* $[L_i\,{}_{\vec{\lambda}}\, L_j] = \partial_i L_j + \lambda_i L_j + \lambda_j L_i$ *below:*

$$\begin{aligned}
[L_i\,{}_{\vec{\lambda}}\, L_j] =& \frac{1}{2}\sum_{k=1}^{N}\big([L_i\,{}_{\vec{\lambda}}(\partial_j u^k)]u_k + (\partial_j u^k)[L_i\,{}_{\vec{\lambda}}\, u_k]\big) \\
=& \frac{1}{2}\sum_{k=1}^{N}\big((\partial_j + \lambda_j)\big(\partial_i + \frac{1}{2}\lambda_i\big)u^k\big)u_k + \frac{1}{2}\sum_{k=1}^{N}(\partial_j u^k)\big(\big(\partial_i + \frac{1}{2}\lambda_i\big)u_k\big) \\
=& \frac{1}{2}\sum_{k=1}^{N}(\partial_i\partial_j u^k)u_k + \frac{1}{2}\lambda_i L_j + \lambda_j L_i + \frac{1}{4}\lambda_j\lambda_i \sum_{k=1}^{N} u^k u_k + \frac{1}{2}\sum_{k=1}^{N}(\partial_j u^k)(\partial_i u_k) + \frac{1}{2}\lambda_i L_j \\
=& \partial_i L_j + \lambda_i L_j + \lambda_j L_i. \qquad (6.3.17)
\end{aligned}$$

By Proposition 6.2.17, N_1 and N_2 commute on $C^n_{PP}(F_\mathfrak{h}, F_\mathfrak{h})$. Since they are also diagonalizable on $C^n_{PP}(F_\mathfrak{h}, F_\mathfrak{h})$, they can be diagonalized simultaneously on $C^n_{PP}(F_\mathfrak{h}, F_\mathfrak{h})$, hence also on $H^n_{PP}(F_\mathfrak{h}, F_\mathfrak{h})$. Pick a basis B_n of $C^n_{PP}(F_\mathfrak{h}, F_\mathfrak{h})$, under which both N_1 and N_2 are diagonal. There exists a subset $\bar{B}_n \subset B_n$ such that it is a basis of $\ker(d_n)$. Then $[\bar{B}_n]$ contains a basis H_n for $H^n_{PP}(F_\mathfrak{h}, F_\mathfrak{h})$ such that N_1 and N_2 are again both diagonal on H_n.

Lemma 6.3.1. *For an equivalence class $[Y] \in H^n_{PP}(P, M)$, if $N_i[Y] = 0$, we can pick a representative X such that $N_i X = 0$. If $N_i[Y] = [Y]$, we can pick a representative Z such that $N_i Z = Z$. In other words, one can assume without loss of generality that $N_i Y = 0$ if $N_i[Y] = 0$ and $N_i Y = Y$ if $N_i[Y] = [Y]$.*

Proof. Take $N_i[Y] = 0$ as an example. Since N_i is diagonalizable on $C^n_{PP}(P, M)$, Y can be written as a sum of linearly independent eigenvectors of N_i:

$$Y = Y_0 + Y_1 + \cdots + Y_M, \tag{6.3.18}$$

where we assume that Y_0 is an eigenvector of eigenvalue 0 and Y_j are eigenvectors of different nonzero eigenvalues for all $j \geq 1$. Then

$$dY = dY_0 + dY_1 + \cdots + dY_M = 0, \tag{6.3.19}$$

since Y is a cocycle. Moreover if $dY_j \neq 0$, since N_i commutes with d, it is an eigenvector of N_i with the same eigenvalue as Y_j, thus their sum must not be 0. Hence $dY = 0$ implies $dY_j = 0$ for all j. Now we replace Y with Y_0 and show that $[Y] = [Y_0]$. By construction $[N_i(Y - Y_0)] = 0$, so there exist $Z \in C^{n-1}_{PP}(P, M)$ such that $N_i(Y - Y_0) = N_i(Y_1 + \cdots + Y_M) = dZ$. Since $N_i Y_j$'s are linearly independent and N_i commutes with d, it is true that Z can be written as a sum:

$$Z = Z_1 + \cdots + Z_M, \quad Z_j \in C^{n-1}_{PP}(P, M), \tag{6.3.20}$$

such that $N_i Y_j = dZ_j$, which implies Y_j is in fact a coboundary for $j \geq 1$. This shows that $[Y] = [Y_0]$ so one can assume $N_i Y = 0$ if $N_i[Y] = 0$. A similar proof works for $[N_i Y] = [Y]$. The trick is to consider $N_i - 1$ instead of N_i. So we can also assume that $N_i Y = Y$ if $[N_i Y] = [Y]$. □

Now we compute the cohomology of the free superfermion PPA $F_\mathfrak{h}$ with coefficients in the adjoint module.

Theorem 6.3.1. *For the free superfermion PPA $F_{\mathfrak{h}}$ (for $H = \mathbb{F}[\partial_1, \partial_2]$), the only nontrivial cohomology group with coefficients in the adjoint module is $H^0_{PP}(F_{\mathfrak{h}}, F_{\mathfrak{h}}) \cong \mathbb{F}\int 1$, where $\int$ is the quotient map: $\int : F_{\mathfrak{h}} \to F_{\mathfrak{h}}/(\partial_1 F_{\mathfrak{h}} + \partial_2 F_{\mathfrak{h}})$.*

The proof is adapted from the proof for the PVA case given in [5]. We divide the proof into a few lemmas below.

Lemma 6.3.2. $H^n_{PP}(F_{\mathfrak{h}}, F_{\mathfrak{h}}) = 0$ *when* $n \geq 3$.

Proof. For $[Y] \in H^n_{PP}(F_{\mathfrak{h}}, F_{\mathfrak{h}})$, by Theorem 6.2.1 and Lemma 6.3.1, pick a representative Y such that $\Delta_1(Y) = 0$ or 1. Recall that Y is uniquely determined by its value on $\mathfrak{h}^{\otimes n}$ (cf. Lemma 6.1.4). On the other hand, by (6.2.18) and Proposition 6.3.1, we have:

$$\Delta_i(Y_{\vec{\lambda}_1,\dots,\vec{\lambda}_n}(a_1 \otimes \cdots \otimes a_n)) + \frac{n}{2} = \Delta_i(Y) = 0 \text{ or } 1, \quad a_j \in \mathfrak{h}. \tag{6.3.21}$$

Hence if $n \geq 3$, the cocycle Y is trivial since $\Delta_i(Y_{\vec{\lambda}_1,\dots,\vec{\lambda}_n}(a_1 \otimes \cdots \otimes a_n)) \geq 0$ by (6.2.11). □

Next we consider the cases $n = 0, 1, 2$ separately. To compute explicitly, we review the following material from [5]. Since $\mathfrak{h}$ is finite dimensional, the bilinear form γ on $\mathfrak{h}$ defined earlier gives an isomorphism $\Phi : \mathfrak{h} \to \mathfrak{h}^*$ by $\Phi(v)(\cdot) = \gamma(v, \cdot)$. On the other hand, $\mathfrak{gl}(\mathfrak{h}) \cong \mathfrak{h} \otimes \mathfrak{h}^*$ and $\mathfrak{gl}(\mathfrak{h})$ acts on $\mathfrak{h}$ by $(u \otimes \phi) \cdot a = \phi(a)u$. Under the identification of Φ, we have $\mathfrak{gl}(\mathfrak{h}) \cong \mathfrak{h} \otimes \mathfrak{h}$ and the action above becomes $(u \otimes v) \cdot a = \gamma(v, a)u$. Let $\mathfrak{spo}(\mathfrak{h})$ denote the subalgebra of $\mathfrak{gl}(\mathfrak{h})$ defined in [16]:

$$\mathfrak{spo}(\mathfrak{h}) = \{x \in \mathfrak{gl}(\mathfrak{h}) \mid \gamma(x \cdot a, b) + (-1)^{p(x)p(a)} \gamma(a, x \cdot b) = 0 \text{ for any } a, b \in \mathfrak{h}\}. \tag{6.3.22}$$

Then $S^2\mathfrak{h} \cong \mathfrak{spo}(\mathfrak{h})$ after we embed $S^2\mathfrak{h}$ into $\mathfrak{h} \otimes \mathfrak{h}$ via the map: $uv \to u \otimes v + (-1)^{p(u)p(v)} v \otimes u$. The following lemma is proved for the PVA case in [5]; it is also true for the PPA $F_{\mathfrak{h}}$ here with the same proof.

Lemma 6.3.3. *By identifying $S^2\mathfrak{h}$ with $\mathfrak{spo}(\mathfrak{h})$, the action of $S^2\mathfrak{h}$ on $F_{\mathfrak{h}}$ by $\eta : S^2\mathfrak{h} \to \operatorname{End} F_{\mathfrak{h}}$, $\eta(x) = x_{(0)} := [x_{\vec{\lambda}} \cdot]|_{\vec{\lambda}=0}$ corresponds to an action of $\mathfrak{spo}(\mathfrak{h})$ on $F_{\mathfrak{h}}$, which coincides with the standard action when restricted to $\mathfrak{h}$.*

We will use the fact that N_1 and N_2 can be diagonalized simultaneously on $C^n_{PP}(F_{\mathfrak{h}}, F_{\mathfrak{h}})$ under certain basis B_n (cf. Remark 6.3.1) to simplify the proofs of the following lemmas. Recall in Remark 6.3.1 we also have H_n as a basis for $H^n_{PP}(F_{\mathfrak{h}}, F_{\mathfrak{h}})$.

Lemma 6.3.4. $H^0_{PP}(F_{\mathfrak{h}}, F_{\mathfrak{h}}) \cong \mathbb{F}\int 1$.

Proof. For an equivalence class $[Y]\in H_0\subset H^0_{PP}(F_{\mathfrak{h}},F_{\mathfrak{h}})$, by construction, $N_1[Y]=[N_1Y]=[c_1Y]=c_1[Y]=0$ or $[Y]$ for some $c_1\in\mathbb{F}$, thus $\Delta_1(Y)=0$ or 1 respectively. If $\Delta_1(Y)=0$, $Y\in\mathbb{F}\int 1$ since $\Delta_1(u)=\dfrac{1}{2}$ and $\Delta_1(\partial_i u)=\dfrac{1}{2}+\delta_{ij}\geq\dfrac{1}{2}$ for any $u\in\mathfrak{h}$. If $\Delta_1(Y)=1$, since N_1 and N_2 are diagonalized simultaneously, one has $\Delta_2(Y)=0$ or 1. If $\Delta_2(Y)=0$, then $Y\in\mathbb{F}\int 1$, while $\Delta_2(Y)=1$ forces $Y=\int x$ for some $x\in S^2\mathfrak{h}$. Recall that, by definition, $dY=0$ implies $[a_{\vec{\lambda}}Y]=0$ for all $a\in F_{\mathfrak{h}}$, thus $x_{(0)}=0$. However, Lemma 6.3.3 shows that $x_{(0)}\neq 0$ for any nonzero $x\in S^2\mathfrak{h}$. Hence $H^0(F_{\mathfrak{h}},F_{\mathfrak{h}})\cong\mathbb{F}\int 1$. □

Lemma 6.3.5. $H^1_{PP}(F_{\mathfrak{h}},F_{\mathfrak{h}})=0$.

Proof. For an equivalence class $[Y]\in H_1\subset H^1_{PP}(F_{\mathfrak{h}},F_{\mathfrak{h}})$, if $N_1[Y]=0$, as in the previous lemma, then $\Delta_1(Y)=0$. If Y is nontrivial, then

$$\Delta_1(Y_{\vec{\lambda}}(u))+\frac{1}{2}=\Delta_1(Y)=0,\quad a\in\mathfrak{h},\tag{6.3.23}$$

which contradicts $\Delta_1(Y_{\vec{\lambda}}(u))\geq 0$. Hence to find a nontrivial Y, one must have $N_1[Y]=[Y]$ and $\Delta_1(Y_{\vec{\lambda}}(a))=\dfrac{1}{2}$. By symmetry, we see this is also true for N_2, and thus $N_2[Y]=[Y]$. Because N_2 and N_1 are diagonalized simultaneously, $N_2Y=Y$, which implies $\Delta_2(Y)=1$ and $\Delta_2(Y_{\vec{\lambda}}(a))=\dfrac{1}{2}$. Since both $\Delta_1(Y_{\vec{\lambda}}(a))$ and $\Delta_2(Y_{\vec{\lambda}}(u))$ are $\dfrac{1}{2}$, we have $Y_{\vec{\lambda}}(u)\in\mathfrak{h}$. Let $Y_{\vec{\lambda}}(u)=\beta(u)$ for some linear map $\beta:\mathfrak{h}\to\mathfrak{h}$. For $a_1,a_2\in\mathfrak{h}$, by (6.2.19) and Lemma 6.1.5, one computes:

$$\begin{aligned}(dY)_{\vec{\lambda}_1,\vec{\lambda}_2}(a_1\otimes a_2)&=(-1)^{q_1}\gamma(a_1,\beta(a_2))+(-1)^{q_2}\gamma(a_2,\beta(a_1))+\langle\vec{\lambda}_1+\vec{\lambda}_2\rangle\\&=(-1)^{q_1}\gamma(a_1,\beta(a_2))+(-1)^{q_2+1+p(a_2)}\gamma(\beta(a_1),a_2)+\langle\vec{\lambda}_1+\vec{\lambda}_2\rangle\\&=(-1)^{p(a_2)}\big((\gamma(\beta(a_1),a_2)+(-1)^{p(\beta)p(a_1)}\gamma(a_1,\beta(a_2))\big)+\langle\vec{\lambda}_1+\vec{\lambda}_2\rangle,\end{aligned}\tag{6.3.24}$$

where q_1 and q_2 are given by (6.1.16) and we use the fact that $p(\beta)=\bar{p}(\beta)$ and $p(a_1)=p(\beta)+p(a_2)$ (otherwise one get 0 for $\gamma(a_1,\beta(a_2))$ and $\gamma(a_2,\beta(a_1))$ by the definition of γ). From the calculation, $dY=0$ is equivalent to $\beta\in\mathfrak{spo}(\mathfrak{h})$. On the other hand, any element of $\mathfrak{spo}(\mathfrak{h})$ corresponds to an inner derivation under the identification in Lemma 6.3.3, thus Y is a coboundary. This shows that $H^1_{PP}(F_{\mathfrak{h}},F_{\mathfrak{h}})=0$. □

Remark 6.3.2. *In the proof above, $\bar{p}(Y)=\bar{p}(\beta)$ is induced by $\bar{p}$ of $F_{\mathfrak{h}}$, and one has $\bar{p}(Y)=\bar{p}(Y_{\vec{\lambda}}(u))-\bar{p}(u)=p(Y_{\vec{\lambda}}(u))-p(u)=p(Y)$, where $p(Y)$ is the parity of Y induced by p of $F_{\mathfrak{h}}$. This is why we claim that $\bar{p}(\beta)=p(\beta)$ in the proof. In general, using the same argument, for an n-cochain Y, one has: $\bar{p}(Y)=p(Y)$ if n is odd ($n=1$ in the case above) and $\bar{p}(Y)=1-p(Y)$*

if n is even, where p, $\bar{p}$ on the cochains are induced by p, $\bar{p}$ of the PPA P respectively.

Lemma 6.3.6. $H^2_{PP}(F_\mathfrak{h}, F_\mathfrak{h}) = 0$.

Proof. For an equivalence class $[Y] \in H_2 \subset H^2_{PP}(F_\mathfrak{h}, F_\mathfrak{h})$, as in the case $n = 1$, one must have $N_1[Y] = [Y]$, so $N_1 Y = Y$. By

$$\Delta_1(Y_{\vec{\lambda}_1,\vec{\lambda}_2}(a_1 \otimes a_2)) + 1 = \Delta_1(Y) = 1, \quad a_1, a_2 \in \mathfrak{h}, \tag{6.3.25}$$

we have $\Delta_1(Y_{\vec{\lambda}_1,\vec{\lambda}_2}(a_1 \otimes a_2)) = 0$. Since N_1 and N_2 are diagonalized simultaneously, one has $N_2 Y = Y$, hence $\Delta_2(Y_{\vec{\lambda}_1,\vec{\lambda}_2}(a_1 \otimes a_2)) = 0$. Because both weights are 0, we have $Y_{\vec{\lambda}_1,\vec{\lambda}_2}(a_1 \otimes a_2) \in \mathbb{F}$ and Y can be written as:

$$Y_{\vec{\lambda}_1,\vec{\lambda}_2}(a_1 \otimes a_2) = \rho(a_1 \otimes a_2) + \langle \vec{\lambda}_1 + \vec{\lambda}_2 \rangle, \tag{6.3.26}$$

for some linear map $\rho : S^2(\Pi\mathfrak{h}) \to \mathbb{F}$. One can easily check that $dY = 0$ in this case. On the other hand, (6.3.24) implies that the vector superspace of coboundaries is isomorphic to $\mathfrak{gl}(\mathfrak{h})/\mathfrak{spo}(\mathfrak{h})$. Note the dimension of the vector superspace of cocycles is $\dim((S^2(\Pi\mathfrak{h}))^*) = \dim(S^2(\Pi\mathfrak{h}))$, while the dimension of the vector superspace of coboundaries is $\dim(\mathfrak{gl}(\mathfrak{h})) - \dim(\mathfrak{spo}(\mathfrak{h}))$. Since it is true in general that

$$\dim(S^2(\Pi\mathfrak{h})) = \dim(\mathfrak{gl}(\mathfrak{h})) - \dim(\mathfrak{spo}(\mathfrak{h})), \tag{6.3.27}$$

all cocycles here must be coboundaries, proving that $H^2_{PP}(F_\mathfrak{h}, F_\mathfrak{h}) = 0$. □

This completes the proof of Theorem 6.3.1.

CHAPTER

7

OPEN PROBLEMS

At the end of this thesis, we list some interesting open problems for future reference:

1. Generalize Theorem 6.2.1 to $H = \mathbb{F}[\partial_1, \ldots, \partial_N]$ for $N > 2$. One may want to exploit the tool of spectral sequences to study the eigenvalues of the energy operators in dimensions bigger than 2. As the dimension grows, to prove a similar result, one may need to go through more pages of the spectral sequence.

2. Study the cohomology of other examples in Section 4.3: the cohomology of the free superboson PPA, affine PPA and W_2 PPA for $H = \mathbb{F}[\partial_1, \partial_2]$. One may want to try the method of energy operators to compute them.

3. The cohomology of PVAs can be used to study the integrability of the Hamiltonian systems of evolution equations [14]. Such cohomological approaches to integrability of classical Hamiltonian PDEs were initiated in [18; 26]. One may want to generalize this approach to the cohomology of PPAs.

4. Study the cohomology of some Lie pseudoalgebras in [1]. In this thesis, we use the information on the basic cohomology and the relation of basic and reduced cohomology of a PPA to study the reduced cohomology of a PPA. This technique

may also be applied to the LPA case. It is proved in [1] (see Proposition 15.2 and Corollary 15.3) that the basic cohomology of a Lie pseudoalgebra is isomorphic to the Gelfand–Fuchs cohomology of its annihilation Lie algebra. This may allow one to compute the reduced cohomology of a Lie pseudoalgebra if the cohomology of its annihilation Lie algebra is already known.

5. Generalize the definition of energy operators to the case when $H = U(\mathfrak{d})$ for a finite-dimensional Lie algebra $\mathfrak{d}$, where $\mathfrak{d}$ is not abelian anymore. Use these operators to study the cohomology of PPAs like the free superfermions $P = S(H\mathfrak{h})$.

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