Finite Element Analysis of Temperature Field of Shells

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Abstract

A temperature finite element suitable for the analysis of temperature field in shells is presented. For this element, the variation in temperature in the direction of thickness is assumed as a quadratic function, and then the various boundary conditions in inner and outer surfaces can be exactly satisfied, whereas the variation in temperature in mid-surface is expressed by general interpolation function. By the use of this element, both data preparation and computer run time can be saved to a great extent, and the practical examples for axisymmetric shells show that the accuracy obtained is satisfactory.

1. Introduction

The analysis of temperature field of two- and three-dimensional and axisymmetric solid have been become a routine work due to the development of finite element method and its successful applications to field problems. Generally, the analysis program of temperature field is taken as a portion of structure analysis program, and is combined with the stress analysis program. For a particular problem, first the analysis of temperature can be conducted, then based on the obtained results, the thermal stress analysis can be performed with the same finite element mesh.

However, for the analysis of temperature and stress of shell structure, widely employed in nuclear reactors and petrochemical equipment and etc., some difficulties have been encountered with. Because up to now no temperature element suitable to shell structures has been reported in literatures, the combined analysis of temperature and stress cannot be accomplished by using one same mesh. In order to overcome the above difficulty, a method and formulation of the finite element analysis of shell temperature field, following our previous work[1], is developed in this paper.

In this paper, the variation in temperature across thickness of shell is assumed as quadric. All types of boundary conditions in inner and outer sides of shell can be exactly satisfied by this assumption. The order of error within shell is $(t/R)^3$, where t is the thickness of shell, and R the radius of curvature. The problems may be solved by finite element method.

The shell temperature element suggested in this paper can be conveniently combined with shell stress element. Data preparations and computer run time can be saved to a great extent, because the same mesh may be used to analyse both the temperature and the stress field of a particular problem.

2. Equation and boundary conditions of shell temperature field

(1) Three-dimensional problems of the steady-state temperature field without source.

The equation of the field can be written as

$$\frac{\partial}{\partial X_i}\left(K\frac{\partial T}{\partial X_i} \right) = 0 \qquad \text{in } \Omega \qquad (1)$$

and the boundary conditions can be expressed as

$$T = \bar{T} \qquad \text{on } S_1 \qquad (2\text{-}1)$$

$$K\frac{\partial T}{\partial n} = \bar{q} \qquad \text{on } S_2 \qquad (2\text{-}1)$$

$$K\frac{\partial T}{\partial n} = h\,(T_a - T) \qquad \text{on } S_3 \qquad (2\text{-}3)$$

where X_i (i=1,2,3) are Cartesian coordinates in three dimensions, the summation convention is used in Eq.(1), K is conductivity and h is the heat transfer coefficient, $\bar{T}$ and $\bar{q}$ are the prescribed values of temperature and heat flow on the boundaries, T_a is the environment temperature, S_1 , S_2 and S_3 are the boundary surfaces of the first, 2nd and 3rd boundary conditions respectively, and n is in the direction of the outward normal.

(2) Equations and boundary conditions of the temperature field in shells with invariable thickness.

For shells, a curved coordinate system of ξ_1, ξ_2 and ξ_3 may be established. The first two are on the midsurface, ξ_3 is normal to the midsurface, and set $\xi_3 = -\frac{1}{2}$ and $\xi_3 = \frac{1}{2}$ on the inner and outer surfaces of the shell respectively. The Cartesian coordinates of the point in the midsurface, X_{10}, X_{20} and X_{30}, may be expressed, for ξ_1 and ξ_2 , as

$$X_{i0} = X_{i0}(\xi_1, \xi_2) \qquad (i=1,2,3) \qquad (3)$$

So the Cartesian coordinates of a point in the shell, X_1 , X_2 and X_3 , may be expressed as

$$X_i = X_{i0}(\xi_1, \xi_2) + l_{3i}\, t\, \xi_3 \qquad (4)$$

where l_{3i} is the cosine of normal ξ_3, and can be written as

$$l_{3i} = \frac{1}{A_3}\, e_{ijk}\, \frac{\partial X_{j0}}{\partial \xi_1}\, \frac{\partial X_{k0}}{\partial \xi_2}$$

$$A_3 = \left(e_{ijk}\, \frac{\partial X_{j0}}{\partial \xi_1}\, \frac{\partial X_{k0}}{\partial \xi_2}\, e_{imn}\, \frac{\partial X_{m0}}{\partial \xi_1}\, \frac{\partial X_{n0}}{\partial \xi_2} \right)^{1/2}$$

e_{ijk} is the permutation symbol.

The differential relation between two coordinate systems can be written as

$$\frac{\partial}{\partial \xi_i} = \frac{\partial x_j}{\partial \xi_i}\, \frac{\partial}{\partial x_j} \qquad (5)$$

where

$$\frac{\partial X_i}{\partial \xi_i} = J_{ij} = \overset{0}{J}_{ij} + \overset{1}{J}_{ij}\xi_3$$

$$\overset{0}{J}_{ij} = \begin{cases} \dfrac{\partial X_{j0}}{\partial \xi_i} & (i=1,2) \\ l_{3j}\, t & (i=3) \end{cases}$$

$$\overset{1}{J}_{ij} = \frac{\partial l_{3j}}{\partial \xi_i}\, t$$

$$\frac{\partial}{\partial X_i} = J^{-1}_{ij}\, \frac{\partial}{\partial \xi_j} \qquad (6)$$

The volume element can be expressed, for ξ_1, ξ_2 and ξ_3, as

$$d\Omega = |J_{ij}|\, d\xi_1\, d\xi_2\, d\xi_3 \qquad (7)$$

where $|J_{ij}|$ is Jacobian, and can computed as follows

$$|J_{ij}| = D_0 + D_1\xi_3 + D_2\xi_3^2 \qquad (8)$$

$$D_0 = |\overset{0}{J}_{ij}| = e_{ijk}\overset{0}{J}_{i1}\overset{0}{J}_{j2}\overset{0}{J}_{k3}$$

$D_1 = e_{ijk}\overset{1}{J}_{i1}\overset{0}{J}_{j2}\overset{0}{J}_{k3} + e_{ijk}\overset{0}{J}_{i1}\overset{1}{J}_{j2}\overset{0}{J}_{k3}$

$D_2 = e_{ijk}\overset{1}{J}_{i1}\overset{1}{J}_{j2}\overset{0}{J}_{k3}$

The area element for ξ_1, ξ_2 and ξ_3, when ξ_3= const., can be computed as

$$dS = A_3 d\xi_1 d\xi_2 \qquad (9)$$

Because ξ_3 is in the normal direction of the midsurface, we have the properties, for $\overset{0}{J}_{ij}$, $\overset{1}{J}_{ij}$ J_{ij}, J^{-1}_{ij} and so forth, as follows :

$$\overset{0}{J}_{ji}\overset{0}{J}_{3i} = \delta_{3j}t^2 \quad , \quad \overset{1}{J}_{ji}\overset{0}{J}_{3i} = 0$$

$$J_{ji}J_{3i} = \delta_{3j}t^2 \quad , \quad J^{-1}_{ij}J^{-1}_{i3} = \delta_{3j}\cdot\frac{1}{t^2} \qquad (10)$$

By the use of Eq.(6), Eq.(1) can be rewritten as

$$J^{-1}_{ij}\frac{\partial}{\partial\xi_j}\left(J^{-1}_{ik}\frac{\partial T}{\partial\xi_k} \right) = 0 \qquad \text{in } \Omega \qquad (11)$$

For the boundary surfaces of the shell, i.e. $\xi_3 = \pm\frac{1}{2}$, the boundary conditions (2) can be written as

$$T = \bar{T} \qquad \text{on } S_1 \qquad (12\text{-}1)$$

$$\pm\frac{K}{t}\frac{\partial T}{\partial\xi_3} = \bar{q} \qquad \text{on } S_2 \qquad (12\text{-}2)$$

$$\pm\frac{K}{t}\frac{\partial T}{\partial\xi_3} = h\,(T_a - T) \qquad \text{on } S_3 \qquad (12\text{-}3)$$

In the last two equations, the sign "+" is used for the surface of $\xi_3 = \frac{1}{2}$, and the sign "-" for the surface of $\xi_3 = -\frac{1}{2}$.

3. Variational formulation of shell temperature field

The variational formulation equivalent to the field equation (11) with the boundary conditions (12), by reference to Eq.(10), is to find the stationary conditions of functional

$$\chi = \iiint_{-\frac{1}{2}}^{\frac{1}{2}} \frac{K}{2}\Big\{ J^{-1}_{i1}J^{-1}_{i1}\left(\frac{\partial T}{\partial\xi_1}\right)^2 + J^{-1}_{i2}J^{-1}_{i2}\left(\frac{\partial T}{\partial\xi_2}\right)^2 + 2J^{-1}_{i1}J^{-1}_{i2}\left(\frac{\partial T}{\partial\xi_1}\right)\left(\frac{\partial T}{\partial\xi_2}\right)$$
$$+ \frac{1}{t^2}\left(\frac{\partial T}{\partial\xi_3}\right)^2\Big\}|J_{ij}|\,d\xi_3 d\xi_1 d\xi_2 - \iint_{S_2}\bar{q}\,T A_3\,d\xi_1 d\xi_2 + \iint_{S_3}\frac{1}{2}h\,(T_a - T)^2 A_3\,d\xi_1 d\xi_2$$
$$- \iint_{S_2\,-\frac{1}{2}}^{\frac{1}{2}}\bar{q}\,T A_c\,d\xi_3 dC + \iint_{S_3\,-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{2}h\,(T_a - T)^2 A_c\,d\xi_3 dC \qquad (13)$$

where C is on shell contour made up of normal . When C is the boundary surface of ξ_1=const. for example, $A_c = A_1$ and $dC = d\xi_2$.

The variation of the last functional equals to zero. By considering the integration by parts of the last item of the volume integral and, in the meanwhile, recognizing that

(1) On the surfaces of $\xi_3 = \pm\frac{1}{2}$, $|J_{ij}| = tA_3$ (14)

(2) From Eq.(8), we have $\frac{\partial}{\partial\xi_3}|J_{ij}| = D_1 + 2D_2\xi_3$ (15)

(3) The chosen interpolation satisfies beforehand the boundary conditions on S_1, and on S_2 and S_3 of $\xi_3 = \pm\frac{1}{2}$,

we find

$$\delta\chi = \iiint_{-\frac{1}{2}}^{\frac{1}{2}}\Big\{\left(J^{-1}_{i1}J^{-1}_{i1}\frac{\partial T}{\partial\xi_1} + J^{-1}_{i1}J^{-1}_{i2}\frac{\partial T}{\partial\xi_2}\right)\frac{\partial\delta T}{\partial\xi_1} + \left(J^{-1}_{i2}J^{-1}_{i2}\frac{\partial T}{\partial\xi_2} + J^{-1}_{i1}J^{-1}_{i2}\frac{\partial T}{\partial\xi_1}\right)\frac{\partial\delta T}{\partial\xi_2} - \frac{1}{t^2}\frac{\partial^2 T}{\partial\xi_3}\delta T$$
$$- \frac{1}{t^2}\frac{\partial T}{\partial\xi_3}\delta T\,(D_1 + 2D_2\xi_3)\frac{1}{|J_{ij}|}\Big\}|J_{ij}|\,d\xi_3 d\xi_1 d\xi_2 - \iint_{S_2\,-\frac{1}{2}}^{\frac{1}{2}}\bar{q}\,\delta T A_c\,d\xi_3 dC$$
$$- \iint_{S_3\,-\frac{1}{2}}^{\frac{1}{2}} h\,(T_a - T)\,\delta T A_c\,d\xi_3 dC = 0 \qquad (16)$$

4. The FEM analysis of shell temperature field

Assume the variation in temperature in the direction of ξ_3 to be a quadric function,

$$T(\xi_1, \xi_2, \xi_3) = T_0(\xi_1, \xi_2) + T_1(\xi_1, \xi_2)\xi_3 + T_2(\xi_1, \xi_2)\xi_3^2 \qquad (17)$$

The order of the errors for this assumption is $O\,(t\,/\,R)^3$, where R is the radius of curvature of shells.

From the last equation, in terms of $\zeta_3 = 0$ on the midsurface we have

$$T_o(\zeta_1, \zeta_2) = T(\zeta_1, \zeta_2, 0) \tag{18}$$

where $T_o(\zeta_1, \zeta_2)$ is the temperature of the midsurface, $T_1(\zeta_1, \zeta_2)$ and $T_2(\zeta_1, \zeta_2)$ can be found by the boundary conditions on the surfaces of $\zeta_3 = \pm\frac{1}{2}$. For generality, all the boundary conditions on surfaces of $\zeta_3 = \pm\frac{1}{2}$ can be assumed as convection conditions, i.e.

$$\begin{aligned} \frac{K}{t}\frac{\partial T}{\partial \zeta_3} &= h_1 (T_{a1} - T) & \zeta_3 &= \tfrac{1}{2} \\ -\frac{K}{t}\frac{\partial T}{\partial \zeta_3} &= h_2 (T_{a2} - T) & \zeta_3 &= -\tfrac{1}{2} \end{aligned} \tag{19}$$

By introducing Eqs.(17) and (18) into Eq. (19), we obtain

$$\begin{aligned} T_1(\zeta_1, \zeta_2) &= C_1 T_o(\zeta_1, \zeta_2) + C_3(\zeta_1, \zeta_2) \\ T_2(\zeta_1, \zeta_2) &= C_2 T_o(\zeta_1, \zeta_2) + C_4(\zeta_1, \zeta_2) \end{aligned} \tag{20}$$

where

$$C_1 = (\bar{h}_2 - \bar{h}_1)/\Delta \quad , \quad C_2 = (\bar{h}_1 + \bar{h}_2 + \bar{h}_1\bar{h}_2)/\Delta$$

$$C_3(\zeta_1, \zeta_2) = (1 + \tfrac{1}{2}\bar{h}_2)\bar{h}_1 T_{a1}(\zeta_1, \zeta_2) - (1 + \tfrac{1}{2}\bar{h}_1)\bar{h}_2 T_{a2}(\zeta_1, \zeta_2)/\Delta$$

$$C_4(\zeta_1, \zeta_2) = (1 + \tfrac{1}{2}\bar{h}_2)\bar{h}_1 T_{a1}(\zeta_1, \zeta_2) + (1 + \tfrac{1}{2}\bar{h}_1)\bar{h}_2 T_{a2}(\zeta_1, \zeta_2)/\Delta$$

$$\Delta = 2 + \tfrac{3}{4}\bar{h}_1 + \tfrac{3}{4}\bar{h}_2 + \tfrac{1}{4}\bar{h}_1\bar{h}_2$$

$$\bar{h}_1 = th_1/K \qquad \bar{h}_2 = th_2/K$$

Then we substitute Eqs. (20) and (18) into Eq.(17), corresponding to the covection conditions, we find the temperature distribution in the element as

$$T(\zeta_1, \zeta_2, \zeta_3) = (1 + C_1\zeta_3 + C_2\zeta_3^2)T_o(\zeta_1, \zeta_2) + C_3(\zeta_1, \zeta_2)\zeta_3 + C_4(\zeta_1, \zeta_2)\zeta_3^2 \tag{21}$$

It should be pointed out that if the boundary condition on the surface of $\zeta_3 = \frac{1}{2}$ is the prescribed one of heat flow, there is a meed for setting $\bar{h}_1 = 0$ and $\bar{h}_1 T_{a1}(\zeta_1, \zeta_2) = \frac{t}{K}\bar{q}(\zeta_1, \zeta_2)$ in the equations above, and that if the condition on the surface of $\zeta_3 = \frac{1}{2}$ is the prescribed temperature, there is a meed for setting $\bar{h}_1 = H$ (H is arbitrary constant, just in need of $H \gg \bar{h}_2$ and $H \gg 1$) and $T_{a1}(\zeta_1, \zeta_2) = \bar{T}(\zeta_1, \zeta_2)$

$T_o(\zeta_1, \zeta_2)$ in the temperature expression (21) can be expressed as

$$T_o(\zeta_1, \zeta_2) = \sum_{i=1}^{n} N_i(\zeta_1, \zeta_2) T_i \tag{22}$$

where $N_i(\zeta_1, \zeta_2)$ is the general interpolation function, T_i is the temperature of nodes and n is the number of nodes in the element.

By substituting Eq.(22) into Eq. (21), from Eq.(16) we can find matrix equation

$$H\,T = Q \tag{23}$$

where H is heat conduction matrix, Q is the heat flow , and T the vector of nodal temperature.

From Eq.(23) we can obtain T_i. By substituting T_i into Eqs.(21) and (22) we can find $T(\zeta_1, \zeta_2, \zeta_3)$, the description of temperature distribution in shells.

5. The FEM formulation of axisymmetric shell temperature field

Now we can extend the method presented above to axisymmetric shells.

Consider the element shown in Fig.1. The coordinate of ζ is normal to midsurface of shell, and η along the meridian curve of midsurface of revolution shell. Set $-\frac{1}{2} \le \zeta \le \frac{1}{2}$ and $0 \le \eta \le 1$.

The relations between the system of coordinates r and z and that of ζ and η are

$$\begin{aligned} r(\zeta, \eta) &= r_o(\eta) + t\zeta \cos\theta \\ z(\zeta, \eta) &= z_o(\eta) + t\zeta \sin\theta \end{aligned} \tag{24}$$

where $r_0(\eta)$ and $z_0(\eta)$ are the coordinates of a point on mid-surface in the system of r and z, and

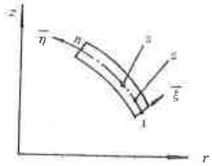

Fig. 1

$$\begin{aligned} \sin\theta &= -\frac{1}{A}\frac{dr_0}{d\eta} \\ \cos\theta &= \frac{1}{A}\frac{dz_0}{d\eta} \\ A &= \sqrt{(dr_0/d\eta)^2+(dz_0/d\eta)^2} \end{aligned} \tag{25}$$

Following the steps pressented in Sec.2 and 3, corresponding to Eq.(16), for the axisymmetric shell temperature field we can find

$$\delta\chi = \iint_{-\frac{1}{2}}^{\frac{1}{2}} K\Big\{\frac{tr}{A+tf\xi}\frac{\partial T}{\partial\eta}\frac{\partial\delta T}{\partial\eta} - \frac{r(A+tf\xi)}{t}\frac{\partial^2 T}{\partial\xi^2}\delta T - [\cos\theta(A+tf\xi)+rf]\frac{\partial T}{\partial\xi}\delta T\Big\}d\xi d\eta - \int_{S_2\,-\frac{1}{2}}^{\frac{1}{2}} r\bar{q}t\delta T d\xi - \int_{S_3\,-\frac{1}{2}}^{\frac{1}{2}} rth(T_a-T)\delta T d\xi = 0$$

where

$$f = \frac{d\theta}{d\eta} = \frac{1}{A^2}\left(\frac{d^2z_0}{d\eta^2}\frac{dr_0}{d\eta} - \frac{dz_0}{d\eta}\frac{d^2r_0}{d\eta^2}\right) \tag{26}$$

All the equations from (17) to (22) are applicable for the present case, if we substitute ξ for ξ_3, and η for ξ_1 and ξ_2. For example, Eqs.(21) and (22) can be rewritten as

$$T(\eta,\xi)=(1+C_1\xi+C_2\xi^2)T_0(\eta)+C_3(\eta)\xi+C_4(\eta)\xi^2 \tag{27}$$

$$T_0(\eta)=\sum_{i=1}^{n}N_i(\eta)T_i \tag{28}$$

Lagrange's interpolation formula can be used as $N_i(\eta)$ directly.

It should be pointed out that for axisymmetric shell, the integrals in heat conduction matrix and heat flow vector can be obtained by general analytic method in the direction of ξ, and if we do so, computer run time can be saved to a very great extent and the accuracy obtained will be much better than that by the numerical methods.

6. Examples

Example 1

Consider an axisymmetric shell (Fig.2) with meridian consisting of two straight segments and one circular are(this shell is one part of a practical experiment device). The prescribed boundary conditions are convection on outer surface, heat flow on inner surface and temperature on two top surfaces. The obtained average temperature on the sections and the difference beteeen inner and outer surfaces are shown in Fig.3 and Fig.4. For comparison.the the results obtained by the use of axisymmetric solid element (599 triangle element with 354 nodes) are presented in Fig.3 and Fig.4 as well. From the figures we can know that the results are consistent favourably with each other, the maximum error of average temperature on section is 1%,and the maximum error of trmperature difference between inner and outer surface 3%. This also shows that this element is efficient to thick curved axisymmetric shell (t/ R= 1/1.5), and the errors are much less than the estimated order of O(t/ R).

Example 2

Consider an axisymmetric composite structure containing solid parts (Fig.5). The connection between the shell elements and the solid elements is established by the equations of multi-points constraints ssuggested in reference[1].

The part of 1-34 in Fig.5 are divided into 12 temperature curved shell elements and the part of 79-149 into 35 elements. The part of 34-79 is handled as axisymmetric solid consisting of 360 axisymmetric solid triangle elements. The results given by the use of axisymmetric solid elements(1184 triangle elements for all the parts) show that the maximum difference of temperature between the method suggested in this paper and the existing method is 0.5°C, and relative error less than 0.4%. It has been shown here that the accuracy of the shell temperature element suggested in this paper is satisfactory.

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