The Coupled Dynamical Problem of Thermoelasticity in Case of Large Temperature Differences

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ABSTRACT

In the tasks of thermoelasticity in general, also in dynamical problems it is common to suppose small temperature differences. The equations used in scientific literature refer to these. It arises the thought what is the influence on the dynamical problems of taking account the large temperature changes.

To investigate this first we present the general equation of heat conduction in case of small temperature differences according to Nowacki and Biot. On this basis we introduce the general equation of heat conduction with large temperature changes. Some remarks show the connection between the two cases.

Using the latter in the equations of thermoelasticity we write down the expressions of the problem for the thermal shock of a long bar.

Finally we show the results of the numerical example and the experimental opportunity to measure some of the constants.

1. INTRODUCTION

In thermoelastic problems in general, and thus also in dynamic problems, it is customarary to assume small changes in temperature as compared to normal conditions. All the relationships used in the literature relate to such assumption. The idea of investigating what will be the difference if considerable changes in temperature are assumed in dynamic problems has come up.

In doing so, let us first see the general heat conduction equation for small and then for considerable changes in temperature, then use the latter in writing the fundamental thermoelastic equations and, finally, with the problem of thermal shock of a long rod solved, draw the conclusion from the numerical results and shov how can the constants be determined experimentally.

The thermoelastic equation system consists of the following equations: geometric equation, equation of motion, Constitutive equation, lst and 2nd Law of Thermodynamics, and heat conduction equation.
There is no known solution of a closed shape, applicable to the general case, to the partial differential equation system set up of the above six equations. For some special problems, the equations become more simple and treatable from a mathematical point of view while for quite simple problems, even solutions of closed shape can be obtained.

For the sake of convenience, the lst Law of Thermodynamics can be written /with the intrinsic heat source neglected/ as

$$s = c \ln \frac{T_1}{T_0} - \frac{1}{\varrho} \beta^{k\ell} \varepsilon_{k\ell} \qquad /2/$$

instead of the well-known formula

$$\varrho \dot{u} = t^{k\ell} v_{k\ell} - h^{k}{}_{;k} . \qquad /1/$$

With this, the general equation of heat conduction for coupled problem and for small changes in temperature will be

$$c \varrho \dot{T} - T_0 \beta^{k\ell} \dot{\varepsilon}_{k\ell} = (\lambda^{k\ell} T_{,k})_{;\ell} . \qquad /3/$$

2. GENERAL EQUATION OF HEAT CONDUCTION FOR SMALL CHANGES IN TEMPERATURE

Equation /3/ can be set up e.g. on the basis of [1] and [2] accoding to NOWACKI and BIOT. Here only two details of the deduction are quoted.

In case of small changes in temperature when $T/T_0 \ll 1$ i.e.

$$\frac{T_1}{T_0} = \frac{T_0 + T}{T_0} \cong 1 ,$$

it is a good approach to assume that

$$s = c \frac{T}{T_0} - \frac{1}{\varrho} \beta^{k\ell} \varepsilon_{k\ell} . \qquad /2'/$$

The next step in the deduction is rather important in respect of what comes later. Both authors say that, in case of small changes in volume and temperature,

$$q \cong T_0 \varrho s \qquad /4/$$

or, making use of /2'/,

$$q = c\varrho T - T_o \beta^{k\ell} \varepsilon_{k\ell} . \qquad /5/$$

However, according to the Fourier theory of heat conduction as well as to the definition of amount of heat,

$$\dot{q} = h^k{}_{;k} = (\lambda^{k\ell} T_{,k})_{;\ell} , \qquad /6/$$

thus we obtain equation /3/ on the basis of /5/ and /6/.

3. GENERAL EQUATION OF HEAT CONDUCTION FOR CONSIDERABLE CHANGES IN TEMPERATURE

According to the authors referred to, $T_o \varrho s$ in equation /4/ is the amount of heat accumulated locally in the continuum, which, due to the small change in temperature i.e. $T_o \cong T_1$, can also be written as

$$q \cong T_1 \varrho s . \qquad /4'/$$

This may also mean that, in accordance with the 2nd Law,

$$dq = \varrho T_1 (s, \dots) ds \qquad /7/$$

and, in integrating, relationship $dT/ds \cong 0$ is taken into consideration, on the basis of which we obtain /4/.

For considerable changes in temperature, let us take relationship

$$d(T_1 s) = d T_1 s + T_1 ds \qquad /8/$$

as a basis for writing the integrated formula for the 2nd Law. Assuming that T_1 is a function of entropy and other variables i.e.

$$T_1 = T_1 (s, \dots) ,$$

then, on the basis of /7/, one might write:

$$\frac{1}{\varrho} dq = d(T_1 s) - d T_1 s = d(T_1 s) - \frac{\partial T_1}{\partial s} s ds . \qquad /9/$$

With this integrated, and assuming that $q_o = 0$ and $s_o = 0$ we obtain

$$\frac{1}{\varrho} q = T_1 s - \int_0^s \frac{\partial T_1}{\partial s} s ds . \qquad /10/$$

Note that, on the basis of $dT_1/ds \cong 0$ which corresponds to small changes in temperature, /4'/ follows from this directly. To calculate the integral, let us now write temperature as a functions of entropy as follows:

$$T_1 = \sum_{i=0}^{\infty} \alpha_i s^i . \qquad /11/$$

With this instead of /10/ we obtain

$$\frac{1}{\varrho} q = T_1 s - \sum_{i=0}^{\infty} \alpha_i \frac{i}{i+1} s^{i+1} . \qquad /12/$$

With the use of /12/, /2/ and /6/ we obtain after reduction

$$\dot{T} \left[\varrho c \left(\ln \frac{T_1}{T_o} + 1 \right) - \beta^{k\ell} \varepsilon_{k\ell} \right] - T_1 \beta^{k\ell} \dot{\varepsilon}_{k\ell} - \left(\varrho c \frac{\dot{T}}{T_1} - \right.$$

$$\left. - \beta^{k\ell} \dot{\varepsilon}_{k\ell} \right) \sum_{i=0}^{\infty} i \alpha_i \left(c \ln \frac{T_1}{T_o} - \frac{1}{\varrho} \beta^{k\ell} \varepsilon_{k\ell} \right)^i = (\lambda^{k\ell} T_{,k})_{;\ell} . \qquad /13/$$

which, for coupled problem and for considerable changes in temperature, gives the general equation of heat conduction.

4. SOME COMMENTS CONCERNING EQUATION /13/

Should only the term of zero order in equation /11/ be taken into consideration, then we will obtain /4'/ instead of /12/. As a matter of fact, $\alpha_0 = T_0$ in this case.

This is anyway the condition for small changes in temperature, with which we obtain from /13/

$$c\varrho\dot{T} - T_1\beta^{k\ell}\dot{\varepsilon}_{k\ell} - \beta^{k\ell}\varepsilon_{k\ell}\dot{T} = (\lambda^{k\ell}T_{,k})_{;\ell}. \qquad /3'/$$

A comparison of this relationship with /3/ shows that, in one term, the left sides of the two equations differ from each other.

Should also the first order term of the equation be taken into consideration, then

$$\frac{1}{\varrho}q = T_1 s - \frac{1}{2}\alpha_1 s^2 \qquad /12'/$$

will apply instead of /12/. Making use of the fact that the change in temperature lies within reasonable limits even for very quick coupled problems i.e. the assumption of average derivative

$$\alpha_1 \cong \frac{T_1 - T_0}{s} \qquad /14/$$

is a good approach, we may write:

$$\frac{1}{\varrho}q = (T_0 + \frac{T}{2})s. \qquad /15/$$

5. THERMOELASTIC EQUATIONS FOR CONSIDERABLE CHANGES IN TEMPERATURE

From among the thermoelastic equations mentioned introductorily, the 1st and 2nd Law of Thermodynamics and the equation of heat conduction were taken as a basis for obtaining relationship /13/, the general equation of heat conduction for considerable changes in temperature. From the other three relationships, we obtain for small deformation and for a homogeneous, isotropic continuum of a material characteristic independent of temperature, with the body force neglected and the equation reduced that

$$\mu u_{i;jj} + (\mu + \lambda)u_{k;ki} - \gamma T_{,i} = \varrho\ddot{u}_i, \qquad /16/$$

which is essentially the general equation of motion for a material meeting the above requirements, the general equation of heat conduction taking the following shape:

$$\dot{T}\left[\varrho c\left(\ln\frac{T_1}{T_0} + 1\right) + \gamma u_{k;k}\right] + T_1\gamma\dot{u}_{k;k} - \left(\varrho c\frac{\dot{T}}{T_1} + \right.$$
$$\left. + \gamma\dot{u}_{k;k}\right)\sum_{i=0}^{\infty} i\alpha_i\left(c\ln\frac{T_1}{T_0} + \frac{\gamma}{\varrho}u_{k;k}\right)^i = \varkappa T_{,kk}. \qquad /17/$$

By means of the equation system set up of /16/ and /17/ we may investigate the dynamic coupled problem of thermoelasticity for considerable changes in temperature. To describe the problem such a way has the advantage that the boundary and initial conditions refer to temperature and displacement

and to their derivatives thus on the basis of experiments can be taken into account.

6. THERMAL SHOCK OF LONG ROD

Let us apply these to the thermal shock of long rod that is fixed on one end, thermally insulated, homogeneouos, isotropic. If heat is added at the free end of the rod in such a way that the temperature at the end of the rod changes in accordance with a definite time function, then the equation system describing the conditions will take the following shape [3]:

$$\frac{E}{\varrho} u_{xx} - u_{tt} - \frac{E\alpha_t}{\varrho} T_x = 0, \qquad /18/$$

$$\dot{T}\left[\varrho c\left(\ln\frac{T_1}{T_0}+1\right)+\gamma u_x\right]+\gamma T_1 \dot{u}_x - \left(\varrho c\,\frac{\dot{T}}{T_1}+\gamma \dot{u}_x\right)\sum_{i=0}^{\infty} i\alpha_i\left(c\ln\frac{T_1}{T_0}+\right. \qquad /19/$$

$$\left.+\frac{\gamma}{\varrho}u_x\right)^i = \varkappa T_{xx},$$

$$T(0,t)=\nu t, \qquad u_x(0,t)=\alpha_t \nu t, \qquad /20/$$

$$T(l,t)=0, \qquad u(l,t)=0, \qquad /21/$$

$$T(x,0)=0, \qquad u(x,0)=0, \qquad \dot{u}(x,0)=0. \qquad /22/$$

After conversion into difference equations, equation system /18/ thru /22/ was solved by means of a digital computer [3].

Two ways of approach were used to obtain the numerical results. First only the terms in equation /19/ corresponding to $i = 0$ were taken into consideration, then those corresponding to $i = 0,1$. The calculations showed [4] that, for given starting parameters, displacement function $u\,/x,t/$ and temperature function $T\,/x,t/$ were only slightly affected by the number of terms in equation /19/.

7. POSSIBLE MEASUREMENT METHOD TO DETERMINE CONSTANTS α_i

To determine unanbiguously what an accuracy could be achieved by taking the relationships concerning considerable changes in temperature into consideration, it would be necessary to use the terms of higher order in equation /19/. This, however, would require the knowledge of coefficients α_i. A measurement method offers itself for the determination of these coefficients.

Let us take formula

$$\varrho \dot{s} \geq -\left(\frac{1}{T}h^k\right)_{,k} \qquad /23/$$

of the 2nd Law of Thermodynamics as a basis, where the intrinsic heat source was neglected. Assuming that the process is reversible, we obtain for a one-dimensional problem after reduction that

$$\varrho \dot{s} = \frac{1}{T^2} T_x h - \frac{1}{T} h_x. \qquad /24/$$

However, according to the Fourier's law of heat conduction /similarly on the basis of the above assumptions/,

$$h = -\alpha T_x \,. \qquad /25/$$

With h eliminated from the latter two equations, we obtain after reduction

$$\dot{s} = \frac{\alpha}{\rho}\left[-\left(\frac{T_x}{T}\right)^2 + \frac{T_{xx}}{T}\right] . \qquad /26/$$

With temperature difference function T /x,t/ measured /using e.g. the method described in [5] /, $\dot{s}$ /x,t/ can be calculated on the basis of /26/.

LITERATURE

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SYMBOLS

ρ = mass density

u = internal energy density

t^{kl} = Cauchy's stress tensor

v_{kl} = strain rate tensor

h^k = intensity of heat flux

s = entropy density

α_{ij} = matrix of linear expansion coefficients

T_1 = temperature

T_o = temperature in normal state

T = $T_1 - T_o$

β_{ij} = $-(3\lambda + 2\mu)\alpha_{ij}$

λ, μ = Lamé constants

α_t = coefficient of linear expansion

γ = $\alpha_t (3\lambda + 2\mu)$

ε_{kl} = strain tensor

q = heat

c = specific heat

u_k = displacement vector

λ^{kl} = matrix of heat conduction coefficients

λ = coefficient of heat conduction

α_i = constants of $T_1 = f/s/$

E = Young's modulus

ν = temperature change rate

u = displacement

x = locus

t = time

$( )_x = \frac{\partial ( )}{\partial x}$