Black hole pair creation and the entropy factor

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It is shown that in the instanton approximation the rate of creation of black holes is always enhanced by a factor of the exponential of the black hole entropy relative to the rate of creation of compact matter distributions (stars). This result holds for any generally covariant theory of gravitational and matter fields that can be expressed in Hamiltonian form. It generalizes the result obtained previously for the pair creation of magnetically charged black holes by a magnetic field in Einstein-Maxwell theory. The particular example of pair creation of electrically charged black holes by an electric field in Einstein-Maxwell theory is discussed in detail.

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I. INTRODUCTION

In a recent analysis of the pair creation of magnetically charged black holes by a magnetic field in Einstein-Maxwell theory, it was shown that the creation rate is enhanced by a factor of $\exp(\mathcal{S}_{\rm BH})$, where $\mathcal{S}_{\rm BH}$ is the black hole entropy, relative to the pair creation rate for grand unified theory (GUT) monopoles [1]. This result is important because it provides a clue to the problem of the origin of black hole entropy. In particular, it is consistent with the view that black holes have $\exp(\mathcal{S}_{\rm BH})$ internal or horizon quantum states.

In this paper the pair creation of nonextreme black holes (with horizons identified) is considered in a general setting. The result is always the same—the pair creation rate is enhanced by a factor of $\exp(\mathcal{S}_{\rm BH})$ relative to the creation rate for a pair of compact matter configurations (stars). This result holds for *any* generally covariant theory of gravitational and matter fields that can be expressed in Hamiltonian form. The enhancement in the black hole creation rate is derived solely from the formal mathematical framework in which the pair creation rate and the density of quantum states are expressed as path integrals.

The enhancement in the black hole creation rate applies, in particular, to the creation of electrically charged black holes by an electric field in Einstein-Maxwell theory. This result has been anticipated [2,3]: Since the creation rate for magnetically charged black holes is enhanced by the factor $\exp(\mathcal{S}_{\rm BH})$, by duality of the electromagnetic field the creation rate for electrically charged black holes should be enhanced by the same factor. Although this argument is correct physically, the details of the calculation for the electric case are not entirely obvious. The apparent difficulty stems from the use of instanton methods in which the leading-order approximation to the creation rate is related to the action of a classical solution, the instanton. For the case of magnetically charged black holes and magnetic fields [1–8], the instanton is obtained by the familiar substitution of $-it$ for t in the magnetic Ernst solution. The resulting instanton consists of a real Euclidean metric and a real electromagnetic vector potential. On the other hand, for the case of electrically charged black holes and electric fields, substitution of $-it$ for t in the electric Ernst solution yields an instanton that consists of a real Euclidean metric and an *imaginary* electromagnetic scalar potential. As shown here, this result is correct and leads to the expected pair creation rate for electrically charged black holes.

The appearance of an imaginary scalar potential is familiar from the path integral construction of the partition function for an electrically charged black hole [9,10]. If the black hole is rotating, the shift vector for the instanton is imaginary as well [11–13]. In general, instantons are stationary solutions with the following properties: All fields that appear in the Hamiltonian as Lagrange multipliers are imaginary, and the canonical variables are real. These are the essential properties that allow the instanton solution to match the corresponding Lorentzian solution along a stationary surface.

The pair creation of electrically charged black holes in Einstein-Maxwell theory is treated as a concrete example in this paper. Thus, I begin in Sec. II with a discussion of the electric Ernst solution and its relationship, through electromagnetic duality, to the magnetic Ernst solution. Section III contains a discussion of the connection between a general stationary Lorentzian solution of the Einstein-Maxwell equations of motion and its associated instanton. The instanton for the electric Ernst solution is displayed explicitly. In Sec. IV, the pair creation rate for electrically charged black holes in Einstein-Maxwell theory is computed relative to the pair creation rate for electrically charged stars. The analysis is generalized in Sec. V to apply to any generally covariant theory of gravitational and matter fields. The generalization requires a careful comparison of the formal path integral derivations of black hole pair creation and black hole entropy.

It should be emphasized that the results of this paper rely on the instanton approximation. Thus, the existence is assumed of instanton solutions that describe the cre-

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ation of black holes and compact matter distributions, in the appropriate physical contexts. On the other hand, it is not necessary that these classical solutions be known. Typically they are not. (An exception is the Ernst solution in Einstein-Maxwell theory.) The central result, the enhancement of black hole pair creation by the factor $\exp(\mathcal{S}_{\rm BH})$, does not depend on the details of the theory or the details of the instanton solutions.

II. DUALITY AND THE ELECTRIC ERNST SOLUTION

Let $\epsilon_{\mu\nu\rho\sigma}$ denote the totally antisymmetric tensor (volume element) with $\epsilon_{0123} = \sqrt{-g}$. The dual of the electromagnetic field $F_{\mu\nu}$ is defined by

$$ {}^*F^{\mu\nu} = (1/2)\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \Longleftrightarrow F^{\mu\nu} = -(1/2)\epsilon^{\mu\nu\rho\sigma}\,{}^*F_{\rho\sigma} \ . \tag{1} $$

The electric and magnetic fields are

$$ E^\mu = F^{\mu\nu}U_\nu \ , \qquad B^\mu = -{}^*F^{\mu\nu}U_\nu \ , \tag{2} $$

respectively, where U^μ is the unit normal vector field of a family of spacelike hypersurfaces. The electromagnetic stress tensor can be written either in terms of $F_{\mu\nu}$ or ${}^*F_{\mu\nu}$:

$$ 4\pi T_{\mu\nu} = F_{\mu\alpha}F_\nu{}^\alpha - (1/4)g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \tag{3a} $$

$$ = {}^*F_{\mu\alpha}{}^*F_\nu{}^\alpha - (1/4)g_{\mu\nu}{}^*F_{\alpha\beta}{}^*F^{\alpha\beta} \ . \tag{3b} $$

The classical equations of motion are the Einstein equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$ (with Newton's constant equal to 1) and the Maxwell equations $d\boldsymbol{F} = 0$ and $d{}^*\boldsymbol{F} = 0$. The electromagnetic field $F_{\mu\nu}$ and its dual ${}^*F_{\mu\nu}$ play a symmetric role. If $\{g_{\mu\nu}, F_{\mu\nu}\} = \{\tilde{g}_{\mu\nu}, \tilde{F}_{\mu\nu}\}$ is a solution of the Einstein-Maxwell equations, then $\{g_{\mu\nu}, F_{\mu\nu}\} = \{\tilde{g}_{\mu\nu}, -{}^*\tilde{F}_{\mu\nu}\}$ is also a solution. Here, the symbols $\tilde{g}_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ refer to specific tensors. Thus, $\{g_{\mu\nu}, F_{\mu\nu}\} = \{\tilde{g}_{\mu\nu}, \tilde{F}_{\mu\nu}\}$ implies that the electromagnetic field $F_{\mu\nu}$ is given by the tensor $\tilde{F}_{\mu\nu}$, whereas $\{g_{\mu\nu}, F_{\mu\nu}\} = \{\tilde{g}_{\mu\nu}, -{}^*\tilde{F}_{\mu\nu}\}$ implies that the electromagnetic field $F_{\mu\nu}$ is given by the tensor $-{}^*\tilde{F}_{\mu\nu}$ (which is minus the dual of $\tilde{F}_{\mu\nu}$). According to the definitions (2), the electric and magnetic fields for the solution $\{\tilde{g}_{\mu\nu}, -{}^*\tilde{F}_{\mu\nu}\}$ are $\tilde{B}^\mu$ and $-\tilde{E}^\mu$, respectively, where $\tilde{E}^\mu$ and $\tilde{B}^\mu$ are the electric and magnetic fields for the solution $\{\tilde{g}_{\mu\nu}, \tilde{F}_{\mu\nu}\}$.

The Ernst solution [14] describes a pair of oppositely charged black holes accelerating apart in an electric or magnetic field. The electric and magnetic cases are related by duality as described above. In both cases the metric for the spacetime region containing one black hole is

$$ \tilde{g}_{\mu\nu}dx^\mu dx^\nu = \frac{\Lambda^2}{A^2(x-y)^2}\Big[G(y)dt^2 - G^{-1}(y)dy^2 + G^{-1}(x)dx^2\Big] + \frac{G(x)}{A^2(x-y)^2\Lambda^2}d\varphi^2 \ , \tag{4} $$

where

$$ G(\xi) = (1 + r_- A\xi)(1 - \xi^2 - r_+ A\xi^3) \ , \tag{5a} $$

$$ \Lambda(x,y) = (1 + Bqx/2)^2 + \frac{B^2 G(x)}{4A^2(x-y)^2} \ , \tag{5b} $$

and $q^2 = r_+ r_-$. For the magnetic case, the electromagnetic field is $\tilde{F}_{\mu\nu} = \partial_\mu {}^M\!\tilde{A}_\nu - \partial_\nu {}^M\!\tilde{A}_\mu$ where

$$ {}^M\!\tilde{A}_\varphi = -\frac{2}{B\Lambda}(1 + Bqx/2) + k \ . \tag{6} $$

For the electric case, the electromagnetic field is $-{}^*\tilde{F}_{\mu\nu} = \partial_\mu {}^E\!\tilde{A}_\nu - \partial_\nu {}^E\!\tilde{A}_\mu$ where

$$ {}^E\!\tilde{A}_t = -\frac{B}{2A^2}\left[\frac{G(y)}{(x-y)^2}(1 + Bqx - Bqy/2) + (1 + r_- Ay)(1 + r_+ Ay)(1 - Bqy/2)\right] + qy + k \ . \tag{7} $$

The magnetic Ernst solution is $\{\tilde{g}_{\mu\nu}, {}^M\!\tilde{A}_\mu\}$ and the electric Ernst solution is $\{\tilde{g}_{\mu\nu}, {}^E\!\tilde{A}_\mu\}$.

For both the electric and magnetic Ernst solutions, certain restrictions must be imposed on the parameters r_-, r_+, A, and B [1–8]. In particular, assume $r_+ A < 2/(3\sqrt{3})$ so that the three roots of the cubic factor in $G(\xi)$ are real. For nonextreme black holes, the smallest of these roots, ξ_2, obeys $\xi_2 > -1/(r_- A)$. The angular coordinate x is restricted to $\xi_3 \le x \le \xi_4$, where ξ_3 and ξ_4 are the two larger roots of the cubic factor in $G(\xi)$. The poles $x = \xi_3$ and $x = \xi_4$ are free from conical singularities if $G'(\xi_3)\Lambda(\xi_3)^2 = -G'(\xi_4)\Lambda(\xi_4)^2$ and the period in φ is $4\pi\Lambda(\xi_3)^2/G'(\xi_3)$. [Note, $\Lambda(\xi_3) = \Lambda(\xi_3, y)$ is independent of y, and similarly for $\Lambda(\xi_4)$.] The black hole event horizon is the null surface $y = \xi_2$, and the acceleration horizon is the null surface $y = \xi_3$.

For the electric Ernst solution the magnitude of the electric field on the axis $x = \xi_3$ at spatial infinity ($y \to \xi_3$) is $BG'(\xi_3)/2\Lambda(\xi_3)^{3/2}$. The magnitude of the electric charge of the black hole is

$$ \frac{1}{4\pi}\oint {}^*d{}^E\!\tilde{\boldsymbol{A}} = \frac{q(\xi_4 - \xi_3)\Lambda(\xi_3)^{3/2}}{G'(\xi_3)(1 + Bq\xi_4/2)} \ . \tag{8} $$

The electric Ernst solution coincides with the electric Melvin solution [15] at spatial infinity, and also in the limit of vanishing r_- and r_+. The metric for the electric Melvin solution is the same as that for the magnetic Melvin solution, while the electromagnetic field is determined by the scalar potential ${}^E\!\tilde{A}_t = Bz$. (The notation is that of Ref. [1], and so here B is the value of the electric field on the z axis).

III. INSTANTON SOLUTIONS

The instanton that enters the calculation of the creation rate for electrically charged black holes is obtained by the substitution $t \to -it$ in the electric Ernst solution $\{\tilde{g}_{\mu\nu}, {}^E\!\tilde{A}_\mu\}$. It is useful to adopt a general notation and to consider this step from a Hamiltonian point of view. First, recall that the metric tensor and electromagnetic potential can be split in space and time according to

$$ds^2 = -(N\,dt)^2 + h_{ij}(dx^i + V^i dt)(dx^j + V^j dt)\ , \tag{9a}$$
$$\boldsymbol{A} = -\Phi dt + A_i(dx^i + V^i dt)\ . \tag{9b}$$

Here, N is the lapse function, V^i is the shift vector, h_{ij} is the spatial metric, $\Phi = -A_t + V^i A_i$ is the scalar potential, and A_i is the vector potential. The canonical coordinates are h_{ij} and A_i, and the canonically conjugate momenta are

$$P^{ij} = -\frac{\sqrt{h}}{32\pi N}(h^{ij}h^{k\ell} - h^{ik}h^{j\ell})(\dot{h}_{k\ell} - 2D_{(k}V_{\ell)})\ , \tag{10a}$$

$$\mathcal{E}^i = \frac{\sqrt{h}}{4\pi N}h^{ij}\left[\dot{A}_j + \partial_j(\Phi - V^k A_k) + 2V^k\partial_{[j}A_{k]}\right]\ . \tag{10b}$$

In Eq. (10a), D_k denotes the covariant derivative in space. The lapse N, shift V^i, and scalar potential Φ appear in the Hamiltonian formalism as Lagrange multipliers for the Hamiltonian, momentum, and Gauss's law constraints, respectively.

Let $\{\tilde{g}_{\mu\nu},\ \tilde{A}_\mu\}$ denote any stationary real Lorentzian solution of the Einstein-Maxwell equations, written in stationary coordinates. In terms of the space-time split (9), this solution is $\{N,\ V,\ h,\ \Phi,\ A\} = \{\tilde{N},\ \tilde{V},\ \tilde{h},\ \tilde{\Phi},\ \tilde{A}\}$, where the fields $\tilde{N}$, $\tilde{V}^i$, $\tilde{h}_{ij}$, $\tilde{\Phi}$, and $\tilde{A}_i$ are independent of t and are real. From this Lorentzian solution [i.e., Eq. (9) with tildes placed over the fields], the substitution $t \to -it$ generates another field configuration, namely, $\{N,\ V,\ h,\ \Phi,\ A\} = \{\bar{N},\ \bar{V},\ \bar{h},\ \bar{\Phi},\ \bar{A}\}$, where

$$\bar{N} = -i\tilde{N}\ , \qquad \bar{V}^i = -i\tilde{V}^i\ , \qquad \bar{\Phi} = -i\tilde{\Phi}\ , \tag{11a}$$
$$\bar{h}_{ij} = \tilde{h}_{ij}\ , \qquad \bar{A}_i = \tilde{A}_i\ . \tag{11b}$$

This is the instanton. Note that the lapse $\bar{N}$, shift $\bar{V}^i$, and scalar potential $\bar{\Phi}$ are imaginary. If $\bar{V}^i = 0$, the metric for the instanton is real Euclidean; otherwise, the metric is complex.

The instanton is a solution of the Einstein-Maxwell equations.[1] In the Hamiltonian setting this follows from a few simple observations. First, according to Eq. (11b), the canonical coordinates for the Lorentzian solution and the instanton coincide. Also, definition (10) shows that the canonical momenta for the Lorentzian solution are equal to the canonical momenta for the instanton solution, $\tilde{P}^{ij} = \bar{P}^{ij}$ and $\tilde{\mathcal{E}}^i = \bar{\mathcal{E}}^i$. Thus, under the substitution $t \to -it$, the canonical variables are unchanged and the Lagrange multipliers are multiplied by the factor $-i$. Now, the Einstein-Maxwell equations include the Hamiltonian, momentum, and Gauss's law constraints. The constraints are constructed entirely from the canonical variables—since they are satisfied for the Lorentzian solution, they are also satisfied for the instanton. The remaining equations of motion are the evolution equations $\dot{f} = \{f, H\}$. Here, the brackets are Poisson brackets, f denotes any function of the canonical variables, and the Hamiltonian H is a linear combination of constraints with Lagrange multipliers as coefficients (plus suitable boundary terms). For both the Lorentzian solution and the instanton, the left-hand side $\dot{f}$ vanishes by stationarity. Then for the Lorentzian solution the right-hand side $\{f, H\}$ vanishes. The right-hand side $\{f, H\}$ must vanish for the instanton case as well, since it just differs from the right-hand side in the Lorentzian case by an overall factor of $-i$. This shows that the equations of motion $\dot{f} = \{f, H\}$ are satisfied for the instanton configuration.

The stationary Lorentzian solution $\{N,\ V,\ h,\ \Phi,\ A\} = \{\tilde{N},\ \tilde{V},\ \tilde{h},\ \tilde{\Phi},\ \tilde{A}\}$ and its associated instanton $\{N,\ V,\ h,\ \Phi,\ A\} = \{\bar{N},\ \bar{V},\ \bar{h},\ \bar{\Phi},\ \bar{A}\}$ are characterized by the same canonical data, including the electric field $E^i = -4\pi\mathcal{E}^i/\sqrt{h}$. This is an essential feature of the instanton analysis. It ensures that the Lorentzian and instanton solutions match along a stationary surface. Also note that the value of the proper electrostatic potential as determined by an (Eulerian) observer who is at rest in the $t =$ const hypersurfaces, $-A_\mu U^\mu = \Phi/N$, is the same for the Lorentzian and instanton solutions. Likewise, the proper velocity of the spatial coordinate system, V^i/N, is the same for the Lorentzian and instanton solutions. In certain contexts this quantity has a physical meaning. For example, for the thermodynamical description of a rotating black hole [11,12] in corotating coordinates, V^ϕ/N is the angular velocity of the black hole with respect to the Eulerian observers.

The electric Ernst instanton solution is

$$\bar{g}_{\mu\nu}dx^\mu dx^\nu = \frac{-\Lambda^2}{A^2(x-y)^2}\left[G(y)dt^2 + G^{-1}(y)dy^2 - G^{-1}(x)dx^2\right] + \frac{G(x)}{A^2(x-y)^2\Lambda^2}d\varphi^2\ , \tag{12a}$$

$${}^E\!\bar{A}_t = \frac{iB}{2A^2}\left[\frac{G(y)}{(x-y)^2}(1 + Bqx - Bqy/2) + (1 + r_-Ay)(1 + r_+Ay)(1 - Bqy/2)\right] - iqy - ik\ . \tag{12b}$$

The metric (12a) is real Euclidean since the shift vector for the Ernst solution vanishes. The metric is regular for $\xi_2 \le y \le \xi_3$ if $G'(\xi_2) = -G'(\xi_3)$ and if the time coordinate is periodic with period $4\pi/G'(\xi_3)$ [1–8]. The vector field ${}^E\!\bar{A}_\mu$ is regular if ${}^E\!\bar{A}_t$ vanishes at both $y = \xi_2$ and $y = \xi_3$. These conditions are satisfied if ${}^E\!\bar{A}_\mu$ is defined separately in open neighborhoods of $y = \xi_2$ and $y = \xi_3$, and in each of these neighborhoods the constant k of Eq. (12b) is chosen appropriately.

Topologically, the Ernst instanton can be viewed as $\boldsymbol{R}^4$ with the interior of a "tube" $S^1 \times S^2$ removed, and points along the S^1 direction identified. This two-dimensional surface is $y = \xi_2$, and is referred to below as the Euclidean wormhole. The acceleration horizon of the Lorentzian

[1] It serves no purpose to introduce the terminology "Lorentzian equations of motion" and "Euclidean equations of motion," since a stationary Lorentzian solution and its associated instanton satisfy the *same* equations of motion.

Ernst solution corresponds to the two-dimensional surface $y = \xi_3$ of the instanton solution. For the instanton, the wormhole $y = \xi_2$ surrounds the surface $y = \xi_3$.

IV. BLACK HOLE PAIR CREATION IN EINSTEIN-MAXWELL THEORY

In the path integral for Einstein-Maxwell theory, each history $\{g_{\mu\nu}, A_\mu\}$ enters with a weight $\exp(\mathcal{S})$ where

$$\mathcal{S}[g_{\mu\nu}, A_\mu] = \frac{i}{16\pi}\int_{\mathcal{M}} d^4x\sqrt{-g}(R - F_{\mu\nu}F^{\mu\nu}) + (\text{boundary terms}) \ . \tag{13}$$

I will refer to $\mathcal{S}$ as the action. The path integral is ultimately defined as a sum over either Lorentzian metrics, or Euclidean metrics, or some other class of metrics. For the purpose of computing the leading-order (exponential) contribution to the path integral this issue is not important. In particular, the instanton can be viewed as a stationary point in a sum over real N, V, h, Φ, and A that lies off the axis of integration. Alternatively, one can rotate the integration contours for N, V, and Φ in the complex plane so that the instanton lies on the axis of integration.

The boundary terms in $\mathcal{S}$ depend on the boundary conditions that are appropriate for the problem at hand. Here, the pair creation rate for electrically charged black holes is computed relative to the pair creation rate for electrically charged stars. In the instanton approximation this is given by the exponential of $\mathcal{S}$ for the electric Ernst instanton (EEI) divided by the exponential of $\mathcal{S}$ for the charged star instanton (CSI). Thus, all that is required is the difference $\mathcal{S}[\text{EEI}] - \mathcal{S}[\text{CSI}]$. I will assume that the stars are compact, and that the matter, charge, and stress inside the stars are distributed in such a way that the gravitational and electromagnetic fields outside the stars coincide with the fields of the electric Ernst solution exterior to a pair of closed surfaces that surround the black holes. Then the instanton for the charged star is topologically $\mathbb{R}^4$ and coincides with the electric Ernst instanton everywhere except in the interior of a "tube" $S^1 \times S^2$ that encompasses the matter (for the charged star instanton) or wormhole (for the Ernst instanton). In this case the boundary terms that appear in Eq. (13) cancel in the calculation of $\mathcal{S}[\text{EEI}] - \mathcal{S}[\text{CSI}]$.

The calculation $\mathcal{S}[\text{EEI}] - \mathcal{S}[\text{CSI}]$ is easily carried out using the Hamiltonian approach with the electric Ernst and charged star instantons foliated along the surfaces of constant stationary time t. The Hamiltonian form of the action for Einstein-Maxwell theory is

$$\mathcal{S} = i\int dt\, d^3x\Big(P^{ij}\dot{h}_{ij} + \mathcal{E}^i\dot{A}_i - N\mathcal{H} - V^i\mathcal{H}_i + \Phi\mathcal{G}\Big) + (\text{boundary terms}) \ , \tag{14}$$

where $\mathcal{H}$, $\mathcal{H}_i$, and $\mathcal{G}$ are the Hamiltonian, momentum, and Gauss's law constraints. For the theory that describes the charged star solution, the matter fields contribute extra $p\dot{q}$ terms to $\mathcal{S}$ and also contribute terms to the constraints. (The matter fields might also contribute boundary terms at infinity. These will vanish for the charged star instanton since the matter distribution is compact.) The boundary terms in Eq. (14) include the boundary terms at infinity from Eq. (13) and also boundary terms that arise from total derivatives and integrations by parts in the space-time decomposition.[2] The surfaces $t = \text{const}$ extend from infinity to the "acceleration horizon" two-surface $y = \xi_3$, which serves as a boundary for the three-dimensional hypersurfaces. The boundary terms in Eq. (14) include boundary terms at $y = \xi_3$. These terms, like the boundary terms at infinity, cancel in the calculation of the difference $\mathcal{S}[\text{EEI}] - \mathcal{S}[\text{CSI}]$.

For the electric Ernst instanton, but not for the charged star instanton, the hypersurfaces $t = \text{const}$ intersect at the Euclidean wormhole. This two-dimensional surface constitutes an inner boundary b of topology S^2 for the hypersurfaces. In passing from the Lagrangian form (13) to the Hamiltonian form (14) of $\mathcal{S}$, the various total derivatives and integrations by parts introduce boundary terms at b. These boundary terms can be derived by cutting out a small region surrounding the wormhole and then taking the limit as the excised region vanishes. In this way an inner boundary B of topology $S^2 \times S^1$ is introduced into the Ernst instanton manifold. Under the simplifying assumption that the (outward pointing) unit normal vector field n^μ of B lies in the $t = \text{const}$ hypersurfaces, the boundary terms are [11–13]

$$\mathcal{S}\big|_B = -i\int dt\int_b d^2x\sqrt{\sigma}\Big[(n^i\partial_i N)/(8\pi) + V^i n^j(2P_{ij} + A_i\mathcal{E}_j)/\sqrt{h} - \Phi n_i\mathcal{E}^i/\sqrt{h}\Big] . \tag{15}$$

Here, σ denotes the determinant of the metric on b.

With $\mathcal{S}$ written in Hamiltonian form (14), all boundary terms except those displayed in Eq. (15) cancel in the calculation $\mathcal{S}[\text{EEI}] - \mathcal{S}[\text{CSI}]$. Furthermore, for both the electric Ernst instanton and the charged star instanton, the (four-dimensional) volume integral terms in $\mathcal{S}$ vanish—the $p\dot{q}$ terms vanish by stationarity and the remaining terms vanish because the constraints hold. Therefore $\mathcal{S}[\text{EEI}] - \mathcal{S}[\text{CSI}]$ is equal to the boundary term (15) evaluated at the electric Ernst instanton. Recall that for the Ernst instanton the shift vector $V^i = \bar{V}^i$ is zero and by regularity the scalar potential $\Phi = -{}^E\!\bar{A}_t$ vanishes at the wormhole b. Thus only the first term in Eq. (15) is nonzero. Setting $N = \bar{N} = -i\tilde{N}$, we have

$$\mathcal{S}[\text{EEI}] - \mathcal{S}[\text{CSI}] = -\int dt\int_b d^2x\sqrt{\sigma}(n^i\partial_i\tilde{N})/(8\pi) \ . \tag{16}$$

At each point of b, the quantity $-\int dt(n^i\partial_i\tilde{N})$ is the rate of change of proper circumference with respect to

[2] For Einstein gravity, a systematic derivation of the Hamiltonian action (14) from the covariant action (13), including boundary terms, was given in Ref. [12]. Electromagnetic, Yang-Mills, and other matter fields can be incorporated into that derivation in a straightforward manner.

proper radius for the circular trajectories of $\partial/\partial t$ in the neighborhood of B. (The minus sign appears because the normal n^μ points in the direction of decreasing radius.) By regularity of the metric this equals 2π. Therefore Eq. (16) becomes

$$\mathcal{S}[\mathrm{EEI}] - \mathcal{S}[\mathrm{CSI}] = A_{\mathrm{BH}}/4 , \tag{17}$$

where A_{BH} is the area $\int_b d^2x\sqrt{\sigma}$ of the wormhole. In turn, A_{BH} equals the horizon area of each black hole in the Lorentzian Ernst solution.

Equation (17) shows that, in the instanton approximation, the pair creation rate for electrically charged black holes is enhanced by a factor of $\exp(A_{\mathrm{BH}}/4)$ relative to the pair creation rate for electrically charged stars. Note that, with the calculation organized as above, the detailed forms of the electric Ernst solution and the charged star solution are not needed. Thus, the result (17) shows that the pair creation rate for black holes in Einstein-Maxwell theory is always enhanced by the factor $\exp(A_{\mathrm{BH}}/4)$ relative to the pair creation rate for matter distributions.

V. BLACK HOLE PAIR CREATION IN GENERAL

The inner boundary terms (15) that yield the black hole entropy factor for pair creation in Einstein-Maxwell theory are precisely the same terms that yield the black hole entropy in the path integral analysis of the partition functions [11–13]. The partition functions are obtained from the density of states ν by Laplace transforms—for example, the grand canonical partition function is obtained from ν by a Laplace transform that replaces energy with inverse temperature as the independent thermodynamical variable. For the purpose of deriving the entropy it is most convenient to work directly with the density of states ν. The entropy is given by the logarithm of ν.

The density of states is a function of the thermodynamical extensive variables such as energy, angular momentum, electric charge, etc. Expressed as a path integral, ν is a sum over all fields that fit inside an outer boundary (the periodically identified history of a "box") of topology $S^2 \times S^1$. The extensive variables are fixed as boundary conditions on this outer boundary [12]. Consider the action for such a path integral, for any generally covariant theory of gravitational and matter fields. For the moment, let the manifold $\mathcal{M}$ have topology $\mathcal{M} = \Sigma \times S^1$, where space Σ has boundary $\partial\Sigma = S^2$ and $\mathcal{M}$ has boundary $\partial\mathcal{M} = S^2 \times S^1$. I will assume that the action can be written in Hamiltonian form

$$\mathcal{S}[\lambda, q, p] = i \int dt\, d^3x \Big(p_a \dot{q}^a - \lambda^A \mathcal{C}_A \Big) + (\text{boundary terms}). \tag{18}$$

Notice that the volume integral contribution to the Hamiltonian is written as a linear combination of constraints $\mathcal{C}_A(q,p)$ with Lagrange multipliers λ^A. This form for the Hamiltonian follows from general covariance and the property that under reparametrizations in t the canonical variables p_a and q^a transform as scalars and the Lagrange multipliers transform as scalar densities [16]. I will assume that these conditions hold. It also follows that the boundary terms in (18) cannot depend solely on the canonical variables—each term, expressed as an integral over $\partial\mathcal{M}$, must include a Lagrange multiplier as a factor in its integrand in order to transform properly under reparametrizations in t.

The boundary terms in the action (18) must be correlated with the boundary conditions on $\partial\mathcal{M}$ in such a way that the boundary terms in the variation $\delta\mathcal{S}$ vanish. There are two types of boundary terms in $\delta\mathcal{S}$: namely, those that arise from variation of the boundary terms in $\mathcal{S}$, and those that arise from integration by parts. Integration by parts occurs when the constraints $\mathcal{C}_A$ contain spatial derivatives of the canonical variables. Thus, the boundary terms in $\delta\mathcal{S}$ that arise through integration by parts necessarily involve variations of the canonical variables. On the other hand, the explicit boundary terms in $\mathcal{S}$, upon variation, generate boundary terms in $\delta\mathcal{S}$ that involve variations of the Lagrange multipliers. These boundary terms can never be canceled by the boundary terms that come from integration by parts. Consequently, if the action $\mathcal{S}$ includes any explicit boundary terms, then $\delta\mathcal{S}$ will include boundary terms that involve variations of quantities that depend on the Lagrange multipliers. Because the boundary terms in $\delta\mathcal{S}$ must vanish by virtue of the boundary conditions, we see that the boundary data for this action will include fixation of quantities that depend on the Lagrange multipliers.

Armed with these observations it follows that the action appropriate for the path integral representation of the density of states ν (the "microcanonical action" [12]) is given by Eq. (18) with *no boundary terms*. Here is the reason: The density of states is a function of the thermodynamical extensive variables which, by definition, are properties of the states of the system. These variables appear at the classical level as functions of the canonical variables p_a and q^a. Thus, the path integral for ν must come from an action in which the fixed boundary data are functions only of the canonical variables—by the arguments above, such an action has no explicit boundary terms.

Now consider a stationary Lorentzian black hole solution $\{\tilde{\lambda}^A, \tilde{q}^a, \tilde{p}_a\}$ of the classical equations of motion. The black hole's entropy is found by evaluating the path integral for the density of states ν, with the boundary data fixed to those values that characterize the black hole. The path integral is given approximately by its integrand $\exp(\mathcal{S}[\lambda, q, p])$ evaluated at a complex black hole extremum $\{\bar{\lambda}^A, \bar{q}^a, \bar{p}_a\}$. The complex black hole is obtained from the Lorentzian black hole by the relations

$$\bar{\lambda}^A = -i\tilde{\lambda}^A , \qquad \bar{q}^a = \tilde{q}^a , \qquad \bar{p}_a = \tilde{p}_a . \tag{19}$$

The complex black hole satisfies the boundary conditions, since the boundary conditions involve only the canonical variables. The fact that the complex black hole is a solution of the classical equations of motion follows from an obvious generalization of the arguments given in Sec. III. Alternatively, observe that $\{\bar{\lambda}^A, \bar{q}^a, \bar{p}_a\}$ is obtained from $\{\tilde{\lambda}^A, \tilde{q}^a, \tilde{p}_a\}$ by a reparametrization $t \to -it$, where p_a and q^a transform as scalars and λ^A transforms as a scalar density.

The path integral constructed from the action (18) with no boundary terms yields the contribution to ν from the topological sector $\Sigma \times S^1$. However, the complex black hole is an extremum of the action on a manifold with topology $\mathcal{M} = S^2 \times \mathbb{R}^2$ and boundary $\partial\mathcal{M} = S^2 \times S^1$. Thus, the action (18) with no boundary terms is not correct as it stands for use in the approximation $\exp(\mathcal{S}[\bar{\lambda}, \bar{q}, \bar{p}])$ to the density of states. It is, however, correct with regard to the lack of boundary terms at $\partial\mathcal{M}$. The action to be used in the approximate evaluation of ν can be found by starting from (18), with no boundary terms, and writing this action in manifestly covariant (Lagrangian) form. The manifold can then be chosen to have topology $S^2 \times \mathbb{R}^2$. The resulting Lagrangian action can be written in Hamiltonian form if a small region surrounding the center of the disk $\mathbb{R}^2$, where the hypersurfaces intersect, is removed. This introduces an inner boundary $B = S^2 \times S^1$ in $\mathcal{M}$. In passing to the Hamiltonian form of the action, total derivatives and integrations by parts will generate boundary terms at the inner boundary B. There will be no boundary terms at the outer boundary, however, since none were present in the original action.

From the discussions above it follows that the black hole entropy is approximated by $\mathcal{S}_{\rm BH} \approx \mathcal{S}[\bar{\lambda}, \bar{q}, \bar{p}]$, and the action $\mathcal{S}[\lambda, q, p]$ has the form (18) where *only inner boundary terms are present*. In evaluating this action at the complex solution $\{\bar{\lambda}^A, \bar{q}^a, \bar{p}_a\}$, the limit is taken in which the excised region vanishes. Since $\partial\bar{q}/\partial t = 0$ (by stationarity) and $\mathcal{C}_A(\bar{q}, \bar{p}) = 0$, the only nonzero contribution to the entropy $\mathcal{S}_{\rm BH} \approx \mathcal{S}[\bar{\lambda}, \bar{q}, \bar{p}]$ comes from the inner boundary terms.[3] Note that the resulting entropy is real, since each inner boundary term must contain a Lagrange multiplier factor (in order to transform properly under reparametrizations) and the Lagrange multipliers $\bar{\lambda}^A$ are imaginary.

The inner boundary terms that yield the black hole entropy coincide with the inner boundary terms that yield the relative enhancement factor for black hole pair creation. This is not difficult to see. Recall from the example of black hole pair creation in Einstein-Maxwell theory that the action for the instanton that describes black hole pair creation, when written in Hamiltonian form, includes boundary terms at infinity as well as boundary terms at the acceleration and black hole horizons. The action for the instanton that describes pair creation of matter distributions (stars) contains the same boundary terms at infinity and at the acceleration horizon, but of course no horizon boundary terms. The volume integral contributions to the Hamiltonian actions for both the black hole instanton and the star instanton vanish because the instantons are stationary and satisfy the constraints. Thus, in taking the difference between the actions for the black hole instanton and the star instanton, only the inner boundary terms from the black hole horizon survive. Those inner boundary terms are derived by the same analysis as the inner boundary terms for black hole entropy. Namely, they arise when the Lagrangian action is expressed in Hamiltonian form in the presence of the boundary B of a small excised region around the black hole event horizon where the hypersurfaces intersect.

The enhancement factor for black hole pair creation is obtained by evaluating the inner boundary terms at the black hole instanton solution, while the entropy of a black hole is obtained by evaluating the *same* inner boundary terms at the complex black hole solution. But the black hole instanton is related to a real Lorentzian solution, which represents a physical black hole pair, by the substitution $t \to -it$. This relationship agrees precisely with the relationship between either of the two physical black holes and the complex solution that yields its entropy. We are therefore led to the main conclusion that the enhancement in the pair creation rate for black holes is given by the factor $\exp(\mathcal{S}_{\rm BH})$ for any generally covariant theory of gravitational and matter fields.

[3]It should be emphasized that the method discussed here, which generalizes the analysis of Refs. [11–13], can be used to derive an explicit expression for black hole entropy for any generally covariant theory of gravitational and matter fields that can be placed in Hamiltonian form. This method shows that the entropy $\mathcal{S}_{\rm BH} \approx \mathcal{S}[\bar{\lambda}, \bar{q}, \bar{p}]$ is a "geometrical" quantity (constructed from the gravitational and matter fields) defined locally at the black hole horizon. The local, geometrical character of black hole entropy has been examined in detail in Ref. [17] using Noether charge techniques.

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