REMARKS ON RENEWAL THEORY WHEN THE QUALITY OF RENEWALS VARIES

by

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1. Introduction.

In spite of its simple "raw ingredients," the theory of renewal processes has produced many very subtle mathematical investigations and supports a very elaborate structure of generalizations, extensions, and applications. The "raw ingredients" to which we refer consist of an infinite sequence of independent and identically distributed, non-negative random variables $\{X_n\}$; in the original renewal theory these random variables represent lifetimes of successive renewals of some industrial item subject to failure. An item is installed at the initial instant $(t = 0)$ and renewed at time $t = X_1$; this second item is renewed in turn at time $t = S_2 = X_1 + X_2$; and so on. Thus the n<u>th</u> <u>renewal</u> occurs at time $t = S_n = X_1 + \cdots + X_n$ and the number of renewals to occur by time t is the maximum integer k such that $S_k \le t$.

In actual applications the assumption that the $\{X_n\}$ are identically distributed may be unrealistic; there may be steady technical improvements in the quality of the item; there may be a steady diminution in those forces (friction, for instance) which bring about the failure of the item. It therefore seems desirable to explore the consequences of various assumptions that could be made to allow of progressive improvement (or degradation) in the quality of renewals.

A large number of different models are worthy of study in the present context, but in this paper we shall be primarily concerned with the model we shall call the model of <u>sequentially geometric improvement</u> (s.g.i.). Let $V(x)$ be a fixed non-degenerate distribution function (d.f.) of a non-negative random variable and let $\lambda > 1$ be some finite constant. The model s.g.i.

supposes that the $\{X_n\}$ are still independent but that $P\{X_n \leq x\} = V(x/\lambda^n)$. An equivalent, but perhaps more vivid, formulation is given in the following specification.

<u>Model A</u> <u>Let $\{Y_n\}$ be an infinite sequence of independent and identically distributed random variables with d.f. V(x), assumed non-degenerate, and suppose that $X_n = \lambda^n Y_n$, $n = 1,2,\ldots$, for some constant $\lambda > 1$.</u>

In studying this model a certain series plays an important role. This series is

$$Z = \frac{Y_1}{\lambda} + \frac{Y_2}{\lambda^2} + \frac{Y_3}{\lambda^3} + \cdots \tag{1.1}$$

It is important to know when this series converges almost surely; we shall find that a necessary and sufficient condition for this is that

$$\int_1^\infty \frac{1 - V(x)}{x}\, dx < \infty \tag{1.2}$$

Notice that (1.2) is equivalent to the condition $\mathcal{E} \log^+ Y_n < \infty$. When (1.2) is satisfied it follows that Z is a proper random variable with a non-defective d.f. K(x), say. In these circumstances it is not hard to see that a renewal process satisfying the requirements of Model A will necessarily satisfy the requirements of the more general model below:

<u>Model B</u> <u>Let there be a non-degenerate and non-defective d.f. K(x) and suppose that, as $n \to \infty$,</u>

$$P\left\{ \frac{X_1 + X_2 + \cdots + X_n}{\lambda^{n+1}} \leq x \right\} \to K(x) \tag{1.3}$$

<u>at all continuity points.</u>

We shall see that when Model B holds the d.f. K(x) must be the d.f. of a random variable Z given by a series such as (1.1), the random variables

$\{Y_n\}$ of this series must have a d.f. $V(x)$ satisfying (1.2); we shall also see that the d.f. $K(x)$ must be continuous (it may be singular of course).

In our work, and in stating our main result (Theorem 1) below, it is necessary to make use of the following notation. For all $t \geq 1$ we define $n(t)$ to be the least integer not less than $\log_\lambda t$. We then define $d(t) = n(t) - \log_\lambda t$. We shall prove:

Theorem 1 Let $N(t)$ be the number of renewals to occur by time t, including any that occur at time t. Suppose the renewal process satisfies Model B. Then (a) for any integer value of j (positive, zero, or negative),

$$(1.4) \qquad P\{N(t) - n(t) \geq j\} - K(\lambda^{-j-d(t)}) \to 0, \quad \text{as } t \to \infty.$$

(b) for any integer value of $\nu \geq 1$,

$$(1.5) \qquad \mathcal{E}\left[\frac{N(t)}{n(t)} - 1\right]^{\nu} \to 0, \quad \text{as } t \to \infty.$$

(c) if there is a d.f. $W(x) \leq P\{X_n \leq \lambda^n x\}$ for all n and all x, and if, for some integer $\nu \geq 1$,

$$(1.6) \qquad \int_1^\infty \frac{(\log x)^\nu}{x} [1 - W(x)]\, dx < \infty$$

it follows that

$$(1.7) \qquad \mathcal{E}[N(t) - n(t)]^\nu - R_\nu(d(t)) \to 0, \quad \text{as } t \to \infty,$$

where

$$(1.8) \quad R_\nu(d) = \sum_{j=1}^{\infty} [j^\nu - \overline{j-1}^{\,\nu}] K(\lambda^{-j-d}) - \sum_{j=1}^{\infty} [j^\nu - \overline{j-1}^{\,\nu}][1 - K(\lambda^{j-d})].$$

Concerning Model A we shall prove the following.

Theorem 2 If the renewal process satisfies the requirements of Model A and if (1.2) holds, then parts (a) and (b) of Theorem 1 again hold. If the d.f. $V(x)$ is such that for integer $\nu \geq 1$

$$\int_1^\infty \frac{(\log x)^\nu}{x} [1 - V(x)]dx < \infty \tag{1.9}$$

then part (c) of Theorem 1 also applies.

A feature of these results, at first surprising, is that, under suitable conditions, the distribution and moments of $N(t) - \log_\lambda t$ do not approach a strict limit as $t \to \infty$ but approach more and more closely a periodic function of $\log_\lambda t$. An important special case of our results is that, as $t \to \infty$,

$$\mathcal{E}N(t) = \log_\lambda t + R_1(d(t)) + d(t) + o(1) \tag{1.10}$$

where

$$R_1(d) = \sum_{j=1}^\infty K(\lambda^{-j-d}) - \sum_{j=1}^\infty [1 - K(\lambda^{j-d})]. \tag{1.11}$$

In more familiar problems in renewal theory such periodic behavior of the renewal function $\mathcal{E}N(t)$ is a consequence of a lattice-structure of the variables $\{X_n\}$. In the present problem no such lattice-structure need be involved.

Of course, before the present results can be applied, the computation of the remainder functions $R_\nu(d)$ is necessary. Whilst this does not seem to be a difficult numerical matter, a theoretical study of these functions is desirable; time and space compel us to postpone this study, however.

The methods of this paper will deal with more general set-ups. For instance one might imagine a more general version of Model A in which

$$X_n = \lambda^n n^\gamma Y_n, \quad n = 1,2,\cdots,$$

for some constant γ. Only an increase in algebraic complexity is involved; no changes in the essential argument are needed.

Some results along the present lines were presented by the author in an invited paper, presented at the meeting of the Society for Industrial and Applied Mathematics in November 1965 at Seattle, Washington, entitled "On renewal theory with improving or degrading quality of replacements." The results then available were much less complete than the ones now given.

2. Consequences of Model A.

It will be most convenient to prove Theorem 2 first, and then in the next section we shall discuss modifications of our argument which are necessary for the establishment of Theorem 1. In this section we shall therefore suppose Model A applies, and we begin by discussing the relationship between the distribution functions $K(x)$ and $V(x)$. As usual we denote the Laplace - Stieltjes transform of a d.f. $V(x)$ thus: $V^*(s)$.

Lemma 2.1 The d.f. $K(x) = P\{Z \leq x\}$, where Z is defined by (1.1), is non-defective if and only if (1.2) holds.

Proof By reference to the zero-one law, we see that convergence of (1.1) is almost sure if and only if the infinite product $\Pi_1^\infty V^*(\lambda^{-n})$ converges (i.e. does not diverge to zero). This occurs if and only if

$$\Sigma_1^\infty \{1 - V^*(\lambda^{-n})\} < \infty,$$

which is equivalent to

$$\int_0^\infty \psi_o(x)\{1 - V(x)\}dx < \infty$$

where

$$\psi_o(x) = \Sigma_1^\infty \lambda^{-n} e^{-x/\lambda^n}.$$

But it can be shown (by comparison with suitable integrals) that for some finite constants $A_2 > A_1 > 0$, and for all $x \geq 0$,

$$\frac{A_1}{1+x} \leq \psi_o(x) \leq \frac{A_2}{1+x} .$$

Thus the lemma is proved.

<u>Lemma 2.2</u> <u>If V(x) and K(x) are the distribution functions already defined then for $\nu = 0,1,2,\cdots$,</u>

$$\int_1^\infty \frac{(\log x)^\nu}{x} \{1 - K(x)\}dx < \infty$$

<u>if and only if</u>

$$\int_1^\infty \frac{(\log x)^{\nu+1}}{x} \{1 - V(x)\}dx < \infty$$

<u>Proof</u> Let $\{Z_n\}$ be an infinite sequence of independent and identically distributed variables with d.f. K(x). Then we see from Lemma 2.1 that $\mathcal{E} \log^+ Z_n < \infty$ if and only if $\Sigma Z_n \lambda^{-n}$ converges almost surely; this is equivalent to the almost sure convergence of

$$\frac{Y_1}{\lambda} + \frac{Y_2 + Y_3}{\lambda^2} + \frac{Y_4 + Y_5 + Y_6}{\lambda^3} + \cdots$$

Pursuing the same argument as in the proof of Lemma 2.1 leads us to the condition

$$\int_0^\infty \psi_1(x)\{1 - V(x)\}dx < \infty$$

where

$$\psi_1(x) = \Sigma_1^\infty n\lambda^{-n} e^{-x/\lambda^n}.$$

However, there will be constants $B_2 > B_1 > 0$ such that, for all $x \geq 0$,

$$\frac{B_1(1 + \log^+ x)}{(1 + x)} \leq \psi_1(x) \leq \frac{B_2(1 + \log^+ x)}{(1 + x)} .$$

Thus $\mathcal{E} \log^+ Z_n < \infty$ if and only if

$$\int_1^\infty \frac{(\log x)}{x}\{1 - V(x)\}dx < \infty ,$$

or equivalently, if and only if $\mathcal{E}(\log^+ Y_n)^2 < \infty$.

This argument can be continued sequentially to prove the lemma. For example, we have that $\mathcal{E}(\log^+ Z_n)^2 < \infty$ if and only if

$$\frac{Z_1}{\lambda} + \frac{Z_2 + Z_3}{\lambda^2} + \frac{Z_4 + Z_5 + Z_6}{\lambda^3} + \cdots$$

converges almost surely. This is equivalent to the almost sure convergence of

$$\frac{Y_1}{\lambda} + \frac{Y_2 + Y_3 + Y_4}{\lambda^2} + \frac{Y_5 + \cdots + Y_{10}}{\lambda^3} + \cdots$$

where there are n(n+1)/2 Y's in the numerator of the nth term. This leads to requiring

$$\int_0^\infty \psi_2(x)\{1 - V(x)\}dx < \infty$$

where

$$\psi_2(x) = \Sigma_1^\infty n^2 \lambda^{-n} e^{-x/\lambda^n} .$$

One can show that, for some constants $C_2 > C_1 > 0$, and for all $x \geq 0$,

$$\frac{C_1(1 + \log^+ x)^2}{(1 + x)} \leq \psi_2(x) \leq \frac{C_2(1 + \log^+ x)^2}{(1 + x)} .$$

Thus the lemma is proved.

Theorem 3 If V(x) is not degenerate then K(x) must be continuous.

Proof Let $\alpha_1, \alpha_2, \cdots$ be distinct points such that $P\{Y_n = \alpha_i\} > 0$ and let $\mathfrak{D}$ be the set of all these α-points. It is not hard to see that if $P\{Y_n \varepsilon \mathfrak{D}\} < 1$ then K(x) must be continuous. Thus we may suppose that $P\{Y_n \varepsilon \mathfrak{D}\} = 1$ and hence that $\mathfrak{D}$ has at least two members.

Suppose there is a β such that $P\{Z = \beta\} = b > 0$. Then

$$\sum_{i=1}^\infty P\{Z - \frac{Y_n}{\lambda^n} = \beta - \frac{\alpha_i}{\lambda^n}\} P\{Y_n = \alpha_i\} = b.$$

Thus we can find a finite $m \geq 2$ such that

$$\sum_{i=1}^m P\{Z - \frac{Y_n}{\lambda^n} = \beta - \frac{\alpha_i}{\lambda^n}\} P\{Y_n = \alpha_i\} = c > 0.$$

In what follows we shall let $\tilde{\alpha}_i = \alpha_{i+1}$, $1 \leq i < m$, and $\tilde{\alpha}_m = \alpha_1$. Let

$$\tilde{\omega} = \min_{1 \leq i \leq m} \frac{P\{Y_n = \tilde{\alpha}_i\}}{P\{Y_n = \alpha_i\}} ,$$

then $\tilde{\omega} > 0$ and

$$\sum_{i=1}^{m} P\left\{Z - \frac{Y_n}{\lambda^n} = \beta - \frac{\alpha_i}{\lambda^n}\right\} P\{Y_n = \tilde{\alpha}_i\} \geq \tilde{\omega}c.$$

Thus

$$\sum_{i=1}^{m} P\left\{Z = \beta + \frac{\tilde{\alpha}_i - \alpha_i}{\lambda^n}\right\} \geq \tilde{\omega}c. \tag{2.1}$$

Let there be k distinct values taken by the numbers $(\tilde{\alpha}_i - \alpha_i)$ for $i = 1,2, \cdots m$. Then $k \geq 2$, and we write $\gamma_1, \gamma_2, \cdots, \gamma_k$ for these values, none of which is zero. Then (2.1) implies (if we make due allowance for possibly repeated terms) that

$$\sum_{i=1}^{k}{}' P\left\{Z = \beta + \frac{\gamma_i}{\lambda^n}\right\} \geq \frac{\tilde{\omega}c}{m}. \tag{2.2}$$

Suppose we choose the integer ν so that λ^{ν} exceeds every number $|\gamma_i/\gamma_j|$ and also that $\lambda^{-\nu}$ is less than every such number. Then, if r and s are integers, the equation

$$\beta + \frac{\gamma_i}{\lambda^{r\nu}} = \beta + \frac{\gamma_j}{\lambda^{s\nu}}$$

implies that $|\gamma_i/\gamma_j| = \lambda^{(r-s)\nu}$, and this requires that $r = s$ and that $i = j$. Let $\mathcal{Y}_r$ be the set of points

$$\beta + \frac{\gamma_1}{\lambda^{r\nu}},\ \beta + \frac{\gamma_2}{\lambda^{r\nu}},\ \cdots,\ \beta + \frac{\gamma_k}{\lambda^{rk}}.$$

Then (2.2) shows

$$P\{Z \varepsilon \mathcal{Y}_r\} \geq \frac{\tilde{\omega}c}{m} > 0,$$

and we have that $\mathcal{Y}_1, \mathcal{Y}_2, \cdots$ are disjoint sets. This leads to the impossible conclusion $P\{Z \in \Sigma \mathcal{Y}_r\} = \infty$. Thus the theorem is proved.

We are now ready to prove Theorem 2. In the discussion that follows we shall, as already stated, write n(t) for the least integer not exceeding log t and define d(t) = n(t) - log t, so that $0 \le d(t) < 1$. We shall be first concerned with proving some lemmas about the following functions of n and d; we define

$$W_n(d) = \sum_{j=1}^{\infty} j^{(\nu-1)} P\{X_1 + X_2 + \cdots + X_{n+j} \le \lambda^{n-d}\}$$

$$T_n(d) = \sum_{j=1}^{(n-1)} j^{(\nu-1)} P\{X_1 + X_2 + \cdots + X_{n-j} \ge \lambda^{n-d}\}.$$

For large integer C > 0 we also need the following multilations of the above functions.

$$W_n^C(d) = \sum_{j=C}^{\infty} j^{(\nu-1)} P\{X_1 + X_2 + \cdots + X_{n+j} \le \lambda^{n-d}\}$$

$$T_n^C(d) = \sum_{j=C}^{(n-1)} j^{(\nu-1)} P\{X_1 + X_2 + \cdots + X_{n-j} \ge \lambda^{n-d}\}.$$

Lemma 2.3 On the sole assumption that V(0+) < 1 we can, for any $\varepsilon > 0$, choose large enough C so that

$$W_n^C(d) < \varepsilon$$

for all large n, $0 \le d \le 1$.

Proof If $\Sigma_1^{(n-j)} X_i \le \lambda^{n-d}$ then $X_i \le \lambda^{n-d}$ for every i = 1,2,···, n+j. Thus $Y_i \le \lambda^{(n-d-i)}$ for every i = 1,2,···, n+j. Therefore

$$P\{X_1 + X_2 + \cdots + X_{n+j} \le \lambda^{n-d}\} \le \prod_{i=1}^{(n+j)} V(\lambda^{n-d-i}). \tag{2.3}$$

Now there must be constants $\zeta > 0$, $\eta > 0$, such that $V(x) \leq \eta < 1$ if $x \leq \zeta$. Furthermore $\lambda^{(n-d-i)} \leq \zeta$ if and only if $i \geq n - d - \log_\lambda \zeta$. Thus, for large j,

$$P\{X_1 + X_2 + \cdots + X_{n+j} \leq \lambda^{n-d}\} \leq \eta^{j-d-1-\log_\lambda \zeta}$$

and so

$$W_n^C(d) \leq \sum_{j=C}^{\infty} j^{(\nu-1)} \eta^{j-d-1-\log_\lambda \zeta}$$

$$< \varepsilon \qquad , \text{ for large } C.$$

<u>Lemma 2.4</u> <u>Under the assumption that $V(0+) < 1$ and that (1.2) holds, as $n \rightarrow \infty$,</u>

$$W_n(d) \rightarrow W_\infty(d) \equiv \sum_{j=1}^{\infty} j^{(\nu-1)} K(\lambda^{-(j+d)})$$

<u>uniformly with respect to d, $0 \leq d \leq 1$, and the limit $W_\infty(d)$ is necessarily finite.</u>

<u>Proof</u> That $W_n(d) \rightarrow W_\infty(d)$ as $n \rightarrow \infty$, follows from the fact that

$$P\{X_1 + X_2 + \cdots + X_{n+j} \leq \lambda^{n-d}\} \rightarrow K(\lambda^{-(j+d)})$$

which is a consequence of Lemma 2.3 and the fact that Model A implies Model B when (1.2) holds. The functions $W_n(d)$ are non-increasing functions of $d \geq 0$. Hence the uniformity follows.

<u>Lemma 2.5</u> <u>If Model B holds (and, in particular, K(x) is non-defective), then for any $\varepsilon > 0$ we can choose C so large that</u>

$$\limsup_{n\to\infty} \frac{T_n^C(d)}{n^\nu} < \varepsilon \qquad (2.4)$$

Proof Evidently,

$$T_n^C(d) \le n^\nu P\{X_1 + \cdots + X_{n-C} \ge \lambda^{n-d}\}.$$

Thus

$$\limsup_{n\to\infty} \frac{T_n^C(d)}{n^\nu} \le 1 - K(\lambda^{C-d})$$

$$< \varepsilon$$

for all $0 \le d \le 1$, if C is large enough.

Lemma 2.6 If Model A holds, with $\mathcal{E}(\log^+ Y_n)^{\nu+1} < \infty$, then for any $\varepsilon > 0$ we can choose C so large that

$$T_n^C(d) < \varepsilon \tag{2.5}$$

for all $0 \le d \le 1$, and all large n.

Proof We have that, if $S_n = X_1 + \cdots + X_n$,

$$\begin{aligned} P\{S_{n-j} \ge \lambda^{n-d}\} &= P\{\lambda Y_1 + \cdots + \lambda^{(n-j)} Y_{n-j} \ge \lambda^{n-d}\} \\ &= P\{\frac{Y_1}{\lambda} + \frac{Y_2}{\lambda^2} + \cdots + \frac{Y_{(n-j)}}{\lambda^{(n-j)}} \ge \lambda^{j-1-d}\} \\ &\le 1 - K(\lambda^{(j-1-d)}). \end{aligned}$$

Thus

$$T_n^C(d) \le \sum_{j=C}^{\infty} j^{(\nu-1)}[1 - K(\lambda^{j-1})].$$

The fact that the right-hand member can be made small by choice of C follows from the convergence of

$$\int_1^\infty \frac{(\log x)^{(\nu-1)}}{x} [1 - K(x)]dx$$

which is assured by our assumptions and Lemma 2.2.

Lemma 2.7 (i) Under the assumption that Model B holds, as $n \to \infty$,

$$\frac{T_n(d)}{n^{\nu}} \to 0 .$$

(ii) If Model A holds with $\mathcal{E}(\log^+ Y_n)^{(\nu+1)} < \infty$, then as $n \to \infty$,

$$T_n(d) \to T_\infty(d) \equiv \sum_{j=1}^{\infty} j^{(\nu-1)} [1 - K(\lambda^{(j-d)})]$$

uniformly with respect to d, $0 \leq d \leq 1$, and the limit $T_\infty(d)$ is necessarily finite.

Proof Part (i) follows easily from Lemma 2.5. Part (ii) follows from Lemma 2.6 and the fact that Model A implies Model B in the same way as Lemma 2.4 was proved. Note that the functions $T_n(d)$ are non-decreasing functions of d. Let us now turn to the proof of Theorem 2. We have that

$$\mathcal{E}[N(t) - n(t)]^{\nu} = \sum_{j=1}^{\infty} [j^{\nu} - \overline{j-1}^{\nu}]P\{N(t) \geq j + n(t)\}$$

$$- \sum_{j=-1}^{-\infty} [|j|^{\nu} - |j-1|^{\nu}]P\{N(t) \leq j + n(t)\}$$

$$= \sum_{j=1}^{\infty} [j^{\nu} - \overline{j-1}^{\nu}]P\{S_{n(t)+j} \leq t\}$$

$$- \sum_{j=1}^{n(t)-1} [j^{\nu} - \overline{j-1}^{\nu}]P\{S_{n(t)-j} \geq t\}$$

$$= S_{1n}(d) - S_{2n}(d), \text{ say.}$$

Let us first suppose only that Model B holds. Then it follows easily from Lemma 2.4 and Lemma 2.7 (i) that

$$\frac{\mathcal{E}[N(t) - n(t)]^{\nu}}{[n(t)]^{\nu}} \to 0$$

or, in other words,

$$\mathcal{E}\left[\frac{N(t)}{n(t)} - 1\right]^{\nu} \to 0.$$

This is the first part of the theorem. Let us now suppose that $\mathcal{E}(\log^{+} Y_n)^{\nu+1} < \infty$ for some integer $\nu \geq 1$. Then it follows from Lemma 2.4 and Lemma 2.7(ii) that

$$\mathcal{E}[N(t) - n(t)]^{\nu} \to \sum_{j=1}^{\infty} [j^{\nu} - \overline{j-1}^{\nu}]K(\lambda^{-(j+d)}) - \sum_{j=1}^{\infty} [j^{\nu} - \overline{j-1}^{\nu}][1 - K(\lambda^{j-d})]$$

as $t \to \infty$ through a sequence which keeps $d(t) = d$ fixed. However, since the convergence is uniform with respect to d, we can infer that

$$\mathcal{E}[N(t) - n(t)]^{\nu} - R_{\nu}(d(t)) \to 0$$

as $t \to \infty$ continuously, where

$$R_{\nu}(d) = \sum_{j=1}^{\infty} [j^{\nu} - \overline{j-1}^{\nu}]K(\lambda^{-(j+d)}) - \sum_{j=1}^{\infty} [j^{\nu} - \overline{j-1}^{\nu}][1 - K(\lambda^{j-d})]$$

This completes the proof.

3. Consequences of Model B.

In this section we shall discuss the modifications necessary to be made in the arguments of the previous section in order to establish Theorem 1 under the hypothesis that Model B holds (with, be it noted, K(x) non-defective).

Let us write $G_n(x)$ for the d.f. of X_n. Then, from (1.3),

$$G_1^*\left(\frac{s}{\lambda^{n+1}}\right) G_2^*\left(\frac{s}{\lambda^{n+1}}\right) \cdots G_n^*\left(\frac{s}{\lambda^{n+1}}\right) \to K^*(s)$$

Thus

$$G_1^*\left(\frac{s}{\lambda^{n+1}}\right) \cdots G_{n-1}^*\left(\frac{s}{\lambda^{n+1}}\right) \longrightarrow K^*(\tfrac{s}{\lambda})$$

and so

$$G_n^*\left(\frac{s}{\lambda^{n+1}}\right) \longrightarrow \frac{K^*(s)}{K^*(\frac{s}{\lambda})} = V^*(\tfrac{s}{\lambda}) \quad \text{, say.}$$

Since $V^*(s) \longrightarrow 1$ as $s \downarrow 0+$, it follows that $V^*(s)$ is the Laplace-Stieltjes transform of a non-defective d.f. $V(x)$, and

$$G_n(\lambda^{n+1} x) \longrightarrow V(\lambda x) \quad , \quad \text{as } n \longrightarrow \infty, \tag{3.1}$$

at continuity points.

Further,

$$\prod_{j=1}^{n} V^*\left(\frac{s}{\lambda^j}\right) = \frac{K^*(s)}{K^*(s/\lambda^n)} \quad .$$

Therefore

$$\prod_{j=1}^{\infty} V^*\left(\frac{s}{\lambda^j}\right) = K^*(s),$$

since $K^*(s/\lambda^n) \longrightarrow 1$ as $n \longrightarrow \infty$. From the proof of Lemma 2.1 it then follows that $V(x)$ must satisfy (1.2) if $K(x)$ is to be non-defective.

Suppose we now introduce the "domination" assumption

$$W(x) \leq G_n(\lambda^n x) \tag{3.2}$$

where $W(x)$ is such that (1.6) holds for some integer $\nu \geq 1$. These assumptions imply that $V(x) \geq W(x)$ for all x and so that

$$\int_1^\infty \frac{(\log x)^\nu}{x} [1 - V(x)]dx < \infty. \tag{3.3}$$

From Lemma 2.2 we deduce that

$$\Pi_1^\infty V^*(s/\lambda^n)$$

converges, and from Theorem 3 that K(x) is a continuous function of x.

The proof of Theorem 1 now can be delineated. In order to save space we shall only list the changes necessary in the arguments of the last section.

In the proof of Lemma 2.3 we replace (2.3) by

$$P\{X_1 + \cdots + X_{n+j} \leq \lambda^{n-d}\} \leq \prod_{i=1}^{n+j} G_i(\lambda^{n-d}).$$

In view of (3.2) we have, for an arbitrary $\varepsilon > 0$, $G_n(\lambda^{n+1} \zeta) \leq (\eta + \varepsilon)$ for all sufficiently large n; we use ζ and η here as in the proof of Lemma 2.3. The rest of that proof goes through as before with η replaced by $(\eta + \varepsilon)$.

The proofs of Lemmas 2.4, 2.5, and 2.7 need no change, nor does the final argument at the close of Section 2 in which the theorem is established. All that remains is to comment upon Lemma 2.6.

Let $\{\tilde{Y}_n\}$ be an infinite sequence of independent and identically distributed random variables with distribution function W(x) and let $\tilde{K}(x)$ be the distribution function of

$$\sum_{n=1}^\infty \frac{\tilde{Y}_n}{\lambda^n}$$

Then Lemma 2.6 applies to a Model A based upon the $\{\tilde{Y}_n\}$. But the domination assumption (3.2) implies $G_n(x) \geq W(x/\lambda^n)$ for all x, and hence that

$$P\{X_1 + \cdots + X_{n-j} \geq \lambda^{n-d}\} \leq P\{\lambda\tilde{Y}_1 + \cdots + \lambda^{n-j} \tilde{Y}_{n-j} \geq \lambda^{n-d}\}$$

$$\leq 1 - \tilde{K}(\lambda^{j-1-d}).$$

This observation should be sufficient to show that

$$\sum_{j=C}^{(n-1)} P\{X_1 + \cdots + X_{n-j} \geq \lambda^{n-d}\}$$

can be made arbitrarily small for all large n by choice of C.