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SOME THEOREMS ON COVERINGS

by

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I. INTRODUCTION

Recently several papers have appeared discussing covering and related problems, [6, 7, 10, 11]. In this note, we treat these topics using graph-theoretical notions. Let $S_n = \{1, 2, \ldots, n\}$ and define for $1 \leq \ell \leq r$ the following two finite undirected graphs $L_{r,n}^{\ell}$, $T_{r,n}^{\ell}$: The vertex-set of $T_{r,n}^{\ell}(L_{r,n}^{\ell})$ consists of all ordered (unordered) r-tuples of elements of S_n with repetitions (without repetitions), and two vertices in $L_{r,n}^{\ell}(T_{r,n}^{\ell})$ are adjacent iff the corresponding r-tuples have at least ℓ coordinates (symbols) in common.

We quote three sets of problems concerning the graphs $L_{r,n}^{\ell}$ and $T_{r,n}^{\ell}$ that bear special significance for coverings and designs.

(a) Evaluation of the internal stability number α.

(b) Evaluation of the external stability number β.

(c) Characterization problems.

Of these, we propose to discuss (a) and (b) in this paper. As to (c), attention has been focused on related association schemes and uniqueness in terms of their parameters. For reference, see e.g. [1, 3, 5, 8, 12], in particular [2].

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II. THE INTERNAL STABILITY NUMBER α .

Let us first investigate $L_{r,n}^{\ell}$. An internally stable set M will consist of ordered r- tuples in which no ordered ℓ- tuple appears more than once. An orthogonal array $OA(r, \ell, \lambda, n)$ with r constraints, strength ℓ, index λ and n levels is an $r \times \lambda.n^{\ell}$ - matrix where each $\ell \times \lambda.n^{\ell}$ submatrix contains all possible $\ell \times 1$ column-vectors with the same frequency λ .

<u>Theorem 1</u>: $\alpha(L_{r,n}^{\ell}) \leqq n^{\ell}$ with equality iff there exists an orthogonal array $OA(r, \ell, 1, n)$.

<u>Proof</u>: Let M be an internally stable set of $L_{r,n}^{\ell}$ and let f denote the number of times an arbitrary ordered $(\ell-1)$ - tuple appears in M . Clearly $f \leqq n$. Considering all possible $(\ell-1)$ - tuples we obtain the inequality

$$\binom{r}{\ell-1} \cdot |M| \leqq \binom{r}{\ell-1} \cdot n^{\ell-1} \cdot n \ .$$

Equality in the above formula then means that every ordered ℓ-tuple appears exactly once, hence the theorem.

Examination of the frequency with which an element or coordinate can appear leads to the following two bounds for $\alpha(L_{r,n}^{\ell})$.

<u>Theorem 2</u>:
$$\alpha(L_{r,n}^{\ell}) \leqq n. \ \alpha(L_{r-1,n}^{\ell-1}),$$
$$\alpha(L_{r,n-1}^{\ell}) \geqq \alpha(L_{r,n}^{\ell}) - \sum_{i=1}^{\ell-1} \binom{r}{i}\alpha(L_{r-i,n}^{\ell-i}) - \binom{r}{\ell} \ .$$

<u>Proof</u>: Given a maximal internally stable set M for $L_{r,n}^{\ell}$, it is clear that an arbitrary element can appear as, say, first coordinate at most $\alpha(L_{r-1,n}^{\ell-1})$ times. As to the second inequality, the quantity on the right hand side which is being subtracted plainly constitutes an upper bound for the number of r-tuples which contain a single element at least once.

Let us now evaluate $\alpha(L^{\ell}_{r,n})$ for small values of ℓ.

$\ell = 1$. $\alpha(L^{1}_{r,n}) = n$, the set $(1, \ldots, 1), \ldots (n, \ldots, n)$ being a maximal internally stable set.

$\ell = 2$. It is well known that the existence of an OA(r, 2, 1, n) is equivalent to the existence of r-2 mutually orthogonal Latin squares of order n. Thus we have $\alpha(L^{2}_{r,n}) = n^2$ for $r \leqq \min(p^t) + 1$, where $n = \Pi p^t$, and $\alpha(L^{2}_{r,n}) < n^2$ for $r > n+1$. For $r = 4$ more information is available, namely, $\alpha(L^{2}_{r,n}) = n^2$ except for $n = 2, 6$.

From the theory of orthogonal arrays (see [4]), we know that $r \leqq n+\ell-1$, and in case $\ell \geqq 3$, n odd that $r \leqq n+\ell-2$. Furthermore, it is known that OA(n+1, 3, 1, n) exists for $n = p^t$ (p prime) and OA(n+2, 3, 1, n) for $n = 2^t$.

$\ell = 3$. $\alpha(L^{3}_{r,n}) < n^3$ for $r > n+2$ and $r > n+1$ when n is odd, $\alpha(L^{3}_{r,n}) = n^3$ for $r \leqq n+2$ and $n = 2^t$, $\alpha(L^{3}_{r,n}) = n^3$ for $r \leqq n+1$ and $n = p^t$.

The corresponding theorems for the graph $T^{\ell}_{r,n}$ read as follows, [9, 11].

<u>Theorem 1a</u>:

$$\alpha(T^{\ell}_{r,n}) \leqq \frac{\binom{n}{\ell}}{\binom{r}{\ell}},$$

where now $n \geqq r$, with equality iff there exists a tactical system TS(r, ℓ, n).

<u>Theorem 2a</u>:

$$\alpha(T^{\ell}_{r,n}) \leqq \left[\frac{n}{r} \cdot \alpha(T^{\ell-1}_{r-1,\, n-1})\right],$$

whence by induction

$$\alpha(T^{\ell}_{r,n}) \leqq \left[\frac{n}{r}\left[\frac{n-1}{r-1}\left[\ldots\left[\frac{n-\ell+1}{r-\ell+1}\right]\ldots\right]\right]\right],$$

$$\alpha(T^{\ell}_{r,n}) \geqq \alpha(T^{\ell}_{r,\, n+1}) - \alpha(T^{\ell-1}_{r-1,\, n}).$$

The known values of α for small ℓ are:

$\ell = 1$. $\alpha(T^1_{r,n}) = [\frac{n}{r}]$.

$\ell = 2$. $\alpha(T^2_{r,n}) = [\frac{n}{r}[\frac{n-1}{r-1}]]$ for $r = 2$, $r = 3$ and $n \not\equiv 5 \mod 6$, $r = 4$ and $n \equiv 0, 1, 3, 4 \mod 12$, $r = 5$ and $n \equiv 0, 1, 4, 5 \mod 20$, and

$$\alpha(T^2_{r,n}) = [\frac{n}{r}[\frac{n-1}{r-1}]] - 1$$

for $r = 3$ and $n \equiv 5 \mod 6$.

$\ell = 3$. $\alpha(T^3_{r,n}) = [\frac{n}{r}[\frac{n-1}{r-1}[\frac{n-2}{r-2}]]]$ for $r = 3$, $r = 4$ and $n \equiv 1, 2, 3, 4 \mod 12$.

III. THE NUMBER $\bar{\alpha}$.

Let us define the numbers $\bar{\alpha}(L^\ell_{r,n})$ and $\bar{\alpha}(T^\ell_{r,n})$ as the minimal cardinality of a set N of ordered (unordered) r-tuples of elements taken from S_n such that each ordered (unordered) ℓ-tuple appears at least once in N. This notion will prove, apart from its own interest, particularly important with regard to the evaluation of the external stability number β, since by definition of β, we will be concerned with minimal sets P of r-tuples such that an arbitrary r-tuple coincides in at least one ℓ-tuple with some member of P.

It is easy to derive the following theorems analogous to theorems 1 - 2a.

<u>Theorem 3</u>: $\bar{\alpha}(L^\ell_{r,n}) \geqq n^\ell$ with equality iff there exists an orthogonal array OA(r, ℓ, 1, n).

<u>Theorem 4</u>: $\bar{\alpha}(L^\ell_{r,n}) \geqq n \cdot \bar{\alpha}(L^{\ell-1}_{r-1,\, n})$,

$$\bar{\alpha}(L^\ell_{r,n}) \leqq \bar{\alpha}(L^\ell_{r,\, n-1}) + \sum_{i=1}^{\ell-1} \binom{r}{i} \bar{\alpha}(L^{\ell-i}_{r-i,\, n-1}) + 1 .$$

<u>Theorem 3a</u>: $\bar{\alpha}(T_{r,n}^{\ell}) \geq \frac{\binom{n}{\ell}}{\binom{r}{\ell}}$ with equality iff there exists a tactical system $TS(r, \ell, n)$.

<u>Theorem 4a</u>: $\bar{\alpha}(T_{r,n}^{\ell}) \geq \{ \frac{n}{r} \bar{\alpha}(T_{r-1,\ n-1}^{\ell-1}) \}^{(1*)}$, whence, by induction

$$\bar{\alpha}(T_{r,n}^{\ell}) \geq \{ \frac{n}{r} \{ \frac{n-1}{r-1} \{ \ldots \{ \frac{n-\ell+1}{r-\ell+1} \} \ldots \}\}\} ,$$

$$\bar{\alpha}(T_{r,n}^{\ell}) \leq \bar{\alpha}(T_{r,\ n-1}^{\ell}) + \bar{\alpha}(T_{r-1,\ n-1}^{\ell-1}) .$$

Some known values, [6, 10]:

$\ell = 1$. $\bar{\alpha}(L_{r,n}^{1}) = n$, $\bar{\alpha}(T_{r,n}^{1}) = \{ \frac{n}{r} \}$.

$\ell = 2$. $r = 3$: $\bar{\alpha}(L_{3,n}^{2}) = n^2$, $\bar{\alpha}(T_{3,n}^{2}) = \{ \frac{n}{3} \{ \frac{n-1}{2} \}\}$.

$r = 4$: $\bar{\alpha}(L_{4,n}^{2}) = n^2$ for $n \neq 2, 6$.

$\bar{\alpha}(T_{4,n}^{2}) = \{ \frac{n}{4} \{ \frac{n-1}{3} \}\}$ for $n = 1, 2, 4, 5 \bmod 12$.

$\ell = 3$. $r = 4$: $\bar{\alpha}(T_{4,n}^{3}) = \{ \frac{n}{4} \{ \frac{n-1}{3} \{ \frac{n-2}{2} \}\}\}$ $n \equiv 2, 3, 4, 5 \bmod 6$.

IV. THE EXTERNAL STABILITY NUMBER β .

An externally stable set in an undirected graph is a set of vertices with the property that any vertex of the graph either lies in the set or is adjacent to at least one member of the set. With regard to our special graphs $L_{r,n}^{\ell}$, $T_{r,n}^{\ell}$, we are then faced with the problem of constructing sets of r-tuples which cover an arbitrary r-tuple in at least an ℓ-tuple . Since the number of vertices adjacent to an arbitrary vertex of $L_{r,n}^{\ell}$ $(T_{r,n}^{\ell})$ clearly is

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(1*) $\{u\}$ stands for the smallest integer not smaller than u .

$\sum_{i=\ell}^{r-1} \binom{r}{i}(n-1)^{r-i}$ and $\sum_{i=\ell}^{r-1} \binom{r}{i}\binom{n-\ell}{r-\ell}$, respectively, we obtain

<u>Theorem 5</u>:

$$\frac{n^r}{\sum_{i=\ell}^{r} \binom{r}{i}(n-1)^{r-i}} \leqq \beta(L_{r,n}^{\ell}) \leqq n^{\ell} , \quad (2^*)$$

$$\frac{\binom{n}{r}}{\sum_{i=\ell}^{r} \binom{r}{i}\binom{n-\ell}{r-\ell}} \leqq \beta(T_{r,n}^{\ell}) \leqq \frac{\binom{n}{\ell}}{\binom{r}{\ell}} .$$

Two more inequalities analogous to Theorems 4 and 4a are spelled out in

<u>Theorem 6</u>:

$$\beta(L_{r,n}^{\ell}) \leqq n\beta(L_{r-1,\ n}^{\ell-1}) ,$$

$$\beta(T_{r,n}^{\ell}) \leqq \beta(T_{r,\ n-1}^{\ell}) + \beta(T_{r-1,\ n-1}^{\ell-1}) .$$

$\ell = 1$. $\beta(L_{r,n}^{1}) = n$, $\beta(T_{r,n}^{1}) = [\frac{n}{r}]$, since for smaller values of β we could cover at most n^r-1 and $\binom{n}{r}-1$ vertices, respectively.

$\ell = 2$.

<u>Theorem 7</u>: $\beta(L_{r,n}^{2}) \geqq \frac{n^2}{r-1}$.

<u>Proof</u>: We use induction on r . For $r = 2$, the theorem is trivially satisfied since no two vertices are adjacent. Let us take now a minimal externally stable set P in $L_{r,n}^{2}$ and let us assume $|P| \leqq \frac{n^2}{r-1}$. Let x be an element that appears with minimum frequency f as a coordinate, say the first coordinate.

(2^*) The inequality $\beta \leqq \alpha$ is true for all graphs.

We then have

$$f \leqq \frac{|P|}{n} \leqq \frac{n}{r-1} .$$

Denoting by $s_2, s_3, \ldots, s_r$ the numbers of distinct elements appearing in the second, third, ..., r-th position of the f r-tuples headed by x, we obtain $s_i \leqq f$ $(i = 2, \ldots, r)$. Suppose without loss of generality $s_2 \geqq s_j$ $(j \neq 2)$ and $s_2 = \frac{n}{r-1} - t$ $(t \geqq 0)$. Since f was the minimal frequency of any coordinate, we note that the s_2 distinct elements appear in a combined total of at least $f \cdot s_2 \geqq s_2^2$ r-tuples of P. Further, any ordered (r-1)-tuple with first coordinate distinct from the s_2 elements from above, with second coordinate distinct from the s_3 elements, etc., must coincide in at least one ordered pair with some (r-1)-tuple of 2nd, ..., r-th coordinates taken from the subset of P not containing x as first coordinate. Since we have at least $n-s_2$ i-th coordinates $(i \geqq 2)$ other than the s_i specified above, we obtain the inequality

$$|P| \geqq \left(\frac{n}{r-1} - t\right)^2 + \frac{\left(\frac{r-2}{r-1} n + t\right)^2}{r-2}$$

$$= \frac{n^2}{r-1} + \frac{r-1}{r-2} t^2 ,$$

thus proving the theorem.

<u>Corollary</u>: Let $n \equiv u \bmod r-1$, $0 \leqq u < r-1$ and suppose orthogonal arrays $OA(r, 2, 1, [\frac{n}{r-1}])$ and $OA(r, 2, 1, \{\frac{n}{r-1}\})$ exist, then

$$\beta = \left\{ \frac{n^2}{r-1} \right\} ,$$

provided that $u(r-1-u) \leqq v$, where $n^2 \equiv - v \bmod r-1$, $0 \leqq v < r-1$.

Proof: We split the set S_n into $r-1$ components, u of these containing $\frac{n-u+r-1}{r-1}$ elements, $r-1-u$ containing $\frac{n-u}{r-1}$ elements. For each one of these, we construct an orthogonal array, thus obtaining

$$u\left(\frac{n-u+r-1}{r-1}\right)^2 + (r-1-u)\left(\frac{n-u}{r-1}\right)^2 =$$

$$\frac{n^2}{r-1} + \frac{u(r-1-u)}{r-1} \leqq \frac{n^2+v}{r-1} = \left\{\frac{n^2}{r-1}\right\} \quad \text{r-tuples} .$$

Now, since an arbitrary r-tuple must have at least two coordinates in the same component, it follows that the hereby constructed set constitutes an externally stable set, thus establishing the corollary.

Remark: The problem of evaluating the external stability number was first formulated in graph theory-language by Taussky-Todd [13] for the case $L^{\ell}_{\ell+1,\, n}$, the general problem seems not to have been treated so far.

For $r \leqq p \leqq n$, let β_p be the minimal cardinality of a set P of unordered p-tuples taken from S_n, such that every r-tuple has at least a pair of elements in common with some member of P. Clearly we have $\beta_r = \beta(T^2_{r,n})$.

Theorem 8:

$$\beta_p \geqq \frac{n(n-r+1)}{p(p-1)(r-1)} .$$

Proof: For $r = 2$, the condition of the theorem states that every element must appear with every other element in some p-tuple of P, thus the minimal frequency of an arbitrary element is at least $\frac{n-1}{p-1}$. Since every p-tuple contributes p appearances to the count, we obtain the inequality

$$\beta_p \geqq \frac{n(n-1)}{p(p-1)} .$$

Suppose the assertion is correct for $r-1$, and let P be a set of minimal cardinality for r, $|P| \leqq \frac{n(n-r+1)}{p(p-1)(r-1)}$. Let x be an element of minimal frequency f, then

$$f \leqq \frac{n-r+1}{(p-1)(r-1)} .$$

If s denotes the number of distinct elements appearing together with x in some p-tuple , we have

$$s \leqq (p-1)\, f \leqq \frac{n-r+1}{r-1} , \quad \text{or}$$

$$s = \frac{n}{r-1} - 1 - t , \quad t \geqq 0 ,$$

$$f \geqq \frac{n-r+1-t(r-1)}{(r-1)(p-1)} .$$

Now clearly every (r-1)-tuple taken from the set of elements different from the s elements and x must coincide in at least one pair with some p-tuple in P , and further everyone of the s elements appears at least f times. Hence, by induction-hypothesis, we obtain

$$|P| \geqq \frac{1}{p} \cdot (s+1) \cdot f + \frac{(n-s-1)(n-s-r+1)}{p(p-1)(r-2)}$$

$$\geqq \frac{n(n-r+1)}{p(p-1)(r-1)} + \frac{(r-1)t^2}{p(p-1)(r-2)} . \qquad (1)$$

<u>Corollary</u>: $\beta(T^2_{r,n}) \geqq \frac{n(n-r+1)}{r(r-1)^2}$ with equality iff $\frac{n}{r-1}$ is an integer and a tactical system $TS(r, 2, \frac{n}{r-1})$ exists.

<u>Proof</u>: From the argument used in Theorem 8, it follows that equality in (1) implies $t = 0$, i.e., $s = \frac{n}{r-1} - 1 = (r-1) \cdot f$. Hence, $\frac{n}{r-1}$ must be an integer and by using induction on r again, the necessity-part is easily established. As to the sufficiency, we split S_n into $r-1$ components of $\frac{n}{r-1}$ elements each, and construct $r-1$ tactical systems $TS(r, 2, \frac{n}{r-1})$, thus obtaining

$$(r-1) \cdot \frac{\frac{n}{r-1}(\frac{n}{r-1} - 1)}{r(r-1)} = \frac{n(n-r+1)}{r(r-1)^2}$$

r-tuples altogether. Since every r-tuple must have at least two elements in the same component, the corollary follows.

V. $\beta(L_{3,n}^2)$ AND $\beta(T_{3,n}^2)$.

To give an application of Theorems 7 and 8, the external stability number of $L_{3,n}^2$ and $T_{3,n}^2$ shall be evaluated, and properties of the minimal externally stable sets discussed.

Theorem 9: $\beta(L_{3,n}^2) = \{\frac{n^2}{2}\}$ and P is a minimal externally stable set iff it can be split into two components of $[\frac{n}{2}]^2$ and $\{\frac{n}{2}\}^2$ elements, respectively, such that the two components have no coordinates in common and form orthogonal arrays OA(3, 2, 1, $[\frac{n}{2}]$), OA(3, 2, 1, $\{\frac{n}{2}\}$) upon suitable relabeling of the elements.

Proof: Since Latin squares exist for all n, the corollary to Theorem 7 immediately establishes β. If we are given two Latin squares of orders $[\frac{n}{2}]$, $\{\frac{n}{2}\}$, respectively, then by invoking the same corollary, we find P to be a minimal externally stable set. (The statement of the theorem simply means that, since repetitions are permitted, we may relabel the coordinates arbitrarily, making sure that the Latin squares remain coordinate-disjoint.) Let P now be an arbitrary minimal externally stable set and let x be an element that appears with minimum frequency f as a coordinate. Using the same notation as in Theorem 7, it follows that

$$|P| \geqq f \cdot s_2 + (n-s_2)(n-s_3) ,$$

hence

$$f = s_2 = s_3 = [\frac{n}{2}] .$$

Let us split P into two components, the first containing the s_2 and s_3 coordinates which appear together with x, the second comprising all remaining triples. The cardinalities of these two sets are then $[\frac{n}{2}]^2$, $\{\frac{n}{2}\}^2$, respectively, and the same minimal frequency argument applied to one of the s_2 second coordinates (appearing with frequency f) establishes the desired decomposition.

To compute $\beta(T^2_{3,n})$, it appears convenient to treat the cases n even and n odd separately.

Theorem 10a: Let n be even, then

$$\beta(T^2_{3,n}) = \{\frac{n(n-2)}{12}\} \qquad \text{for } n \equiv 2, 4, 6 \bmod 12$$

$$= \{\frac{n(n-2)}{12}\} + 1 \qquad n \equiv 0, 8, 10 \bmod 12 . \qquad (2)$$

Any minimal externally stable set P consists of two element-disjoint components A, B with A containing $\frac{n}{2}$, B containing $\frac{n}{2}$ elements for $n \equiv 2, 6 \bmod 12$; A containing $\frac{n}{2} - 1$, B containing $\frac{n}{2} + 1$ elements for $n \equiv 0, 4, 8 \bmod 12$; A containing $\frac{n}{2}$, B containing $\frac{n}{2}$ elements or A containing $\frac{n}{2} - 1$, B containing $\frac{n}{2} + 1$ elements for $n \equiv 10 \bmod 12$, such that A, B contain all possible pairs of their respective sets of elements at least once, and are minimal with respect to this property.

Proof: It is a simple computation to show that the cardinality of a set P as described in the theorem attains the bound (2) by quoting the result of Section III, [6]. By the argument employed in the Corollary to Theorem 8, these sets are readily seen to be externally stable.

To give one example, suppose $n = 12k + 10$. Let us split S_n into two sets containing $\frac{n}{2}$ elements each. It then follows that

$$|A| + |B| = 2 \cdot \{\frac{\frac{n}{2}}{3}\{\frac{\frac{n}{2}-1}{2}\}\} = 12k^2 + 18k + 8 = \{\frac{n(n-2)}{12}\} + 1 .$$

It remains to verify that β can not be smaller than (2), and that every minimal set P is of the specified form. For $n \equiv 2, 6 \bmod 12$, (2) is the exact bound of Theorem 8, and so the corollary applies. Hence we may assume $n \equiv 0, 4, 8$ or $10 \bmod 12$. Let P now be such a minimal set and f the minimal frequency. Since we already know an upper bound for β, we must have $f \leqq \frac{n-2}{4}$.

Case A. $n \equiv 0 \bmod 4$,

Here we have $f \leqq \frac{n}{4} - 1$, and thus $s \leqq 2f \leqq \frac{n}{2} - 2$ (using the same notation as in Theorem 8).

Let $s = \frac{n}{2} - 2 - t \quad (t \geqq 0)$, $v = n - s - 1$, then

$$|P| \geqq \frac{1}{3}\left((s+1)f + v \cdot \{\frac{v-1}{2}\} \right), \tag{3}$$

since the expression in parantheses gives a lower bound for the sum of the frequencies.

Let us denote the subset of S_n consisting of the s elements appearing with x plus the element x S_1, the complementary set S_2. A triple of P containing elements of both S_1 and S_2 shall be called a mixed triple. (3) now becomes

$$|P| \geqq \frac{1}{3}\left((\frac{n}{2} - 1 - t) \cdot (\frac{n}{4} - 1 - [\frac{t}{2}]) + (\frac{n}{2} + 1 + t)\cdot (\frac{n}{4} + \{\frac{t}{2}\})\right)$$

$$= \frac{n(n-2)}{12} + \frac{2n}{12}\left(\{\frac{t}{2}\} - [\frac{t}{2}]\right) + \frac{4(t+1)}{12}\left(\{\frac{t}{2}\} + [\frac{t}{2}] + 1\right) . \tag{4}$$

Since for $t \geqq 1$, (4) exceeds the known bound (2), we must have $t = 0$, i.e.,

$$|P| \geqq \frac{n(n-2)}{12} + \frac{4}{12} . \tag{5}$$

For $n \equiv 4 \bmod 12$, (5) is an integer, and since we clearly cannot have any mixed triples (we would add at least two appearances), the result follows.

For $n \equiv 0, 8 \bmod 12$, we have to add $\frac{2}{3}$, i.e., two appearances to (5), in order to obtain an integer. Suppose we had mixed triples, then they are of the form $(ab'b'')$ or $(a'a''b)$ $(a \in S_1, b \in S_2)$, but since both $|S_1|$ and $|S_2|$ are odd and $|S_1| \geqq 3$ $(n \geqq 8)$, we would obtain 3 additional appearances. (The additional summand $\frac{2}{3}$ to (5) is accounted for by the fact that one of the two sets S_1 or S_2 has cardinality $C \equiv 5 \bmod 6$, and that a covering of such sets (see [6]) contains exactly two elements appearing $\frac{C+1}{2}$ times, whereas all the others appear $\frac{C-1}{2}$ times.)

<u>Case B</u>. $n \equiv 10 \bmod 12$.

In this case, $f \leqq \frac{n-2}{4}$, $s \leqq 2f \leqq \frac{n-2}{2}$.

With $s = \frac{n}{2} - 1 - t$, $v = n - s - 1$, we obtain

$$|P| \geqq \frac{1}{3}\left[(\frac{n}{2} - t)(\frac{n-2}{4} - [\frac{t}{2}]) + (\frac{n}{2} + t) \cdot (\frac{n-2}{4} + \{\frac{t}{2}\})\right]$$

$$= \frac{n(n-2)}{12} + \frac{2n}{12}\left(\{\frac{t}{2}\} - [\frac{t}{2}]\right) + \frac{4t}{12}\left(\{\frac{t}{2}\} + [\frac{t}{2}]\right) . \qquad (6)$$

Comparing (6) with the previously established bound (2), we readily infer $t = 0$ or 2 . Suppose $t = 0$, then $|S_2| \equiv 5 \bmod 6$, and quoting [6] again, we see that in this case (2) cannot be improved, and by a similar argument as before no mixed triple can occur. For $t = 2$, the bound (2) is attained exactly, no additional appearances are possible, hence the second possibility cited in the theorem results.

Theorem 10b: For n odd

$$\beta(T_{3,n}^{2}) = \{\frac{n(n-1)}{12}\} \qquad \text{for } n \equiv 1, 3, 5, 7, 11 \bmod 12$$

$$= \{\frac{n(n-1)}{12}\} + 1 \qquad n \equiv 9 \qquad \bmod 12 . \qquad (7)$$

Any minimal externally stable set P consists of two element-disjoint components A, B with element sets S_1 and S_2, respectively, such that A and B contain all pairs taken from their element sets at least once and are minimal with respect to this property, where

$$|S_1| = \frac{n-1}{2}, \quad |S_2| = \frac{n+1}{2} \qquad \text{for } n \equiv 1, 5, 7 \bmod 12$$

$$\left.\begin{array}{l} |S_1| = \frac{n-1}{2}, \quad |S_2| = \frac{n+1}{2} \\ \text{or} \quad |S_1| = \frac{n-3}{2}, \quad |S_2| = \frac{n+3}{2} \end{array}\right\} \text{for } n \equiv 3, 9, 11, \bmod 12 . \qquad (8)$$

If one of the two sets S_1, S_2 in (8) has cardinality $C \equiv 2$ or $4 \bmod 6$, P may contain one mixed triple with two elements from the set of cardinality C, one from the other.

Proof: Sets P as specified in (8) have cardinality (7) and thus provide an upper bound for β . Using the notation of Theorem 10a, we then have

$$f \leqq \frac{n-1}{4} .$$

Case A. $n \equiv 1 \bmod 4$.

Writing $s = \frac{n-1}{2} - t$, $t \geqq 0$, $f \geqq \frac{n-1}{4} - [\frac{t}{2}]$, $v = \frac{n-1}{2} + t$, we obtain

$$|P| \geqq \frac{1}{3}\left((\frac{n+1}{2} - t)(\frac{n-1}{4} - [\frac{t}{2}]) + (\frac{n-1}{2} + t)(\frac{n-1}{4} + [\frac{t}{2}])\right)$$

$$= \frac{n(n-1)}{12} + \frac{8t-4}{12}[\frac{t}{2}] . \qquad (9)$$

In order not to exceed bound (7), t = 0, 1 or 2 . Since for t = 1, v equals the value of s+1 for t = 0 and vice versa, it suffices to consider one of the two posibilities, e.g., $t = 1$. For $n \equiv 1 \bmod 12$, (9) is an integer and, since we cannot have any mixed triples (i.e., no additional appearances), the decomposition (8) results. If $n = 12k + 5$, we have $|S_1| = 6k + 2$, $|S_2| = 6k + 3$, and since $\{\frac{n(n-1)}{12}\} = \frac{n(n-1)}{12} + \frac{4}{12}$, we have one additional appearance at our disposal. If there are no mixed triples, we are led to (8), if we had a triple $(ab'b'')$, $a \in S_1$, $b', b'' \in S_2$, three additional appearances would occur. Suppose now we have a triple $(a'a''b)$. Since b must occur with all other elements of S_2 , we have to construct a set of triples A' , which contains all possible pairs of elements, taken from S_1, except $a'a''$, such that $|A'| = \frac{(n-1)^2 - 16}{24}$. This is done as follows. $S_1'' = S_1 - \{a''\}$ contains $\frac{n-3}{2} = 6k + 1$ elements. We construct a Steiner triple system A'' on S_1'' and then define A' to be:

$$A' = A'' \cup (a_1, a_2, a'') \cup (a_3, a_4, a'') \cup \ldots \cup (a_{6k-1}, a_{6k}, a'') ,$$

where $S_1'' = \{a_1, a_2, \ldots, a_{6k}, a'\}$.

Now A' plainly covers all pairs distinct from $a'a''$, and $|A'| = \frac{(n-3)}{2} \cdot \frac{(n-5)}{12} + \frac{n-5}{4} = \frac{(n-1)^2 - 16}{24}$. Since we clearly cannot have more than one mixed triple, the desired result follows.

The case $n = 12k + 9$ can be dealt with similarly, noting that $|S_1| = 6k + 4$, $|S_2| = 6k + 5$ and that, since neither S_1 nor S_2 permits a Steiner triple system, the bound (7) cannot be improved.

For $t = 2$, (9) becomes $\frac{n(n-1)}{12} + 1 > \{\frac{n(n-1)}{12}\}$, hence we only have to consider the case $n = 12k + 9$. Here

$$|S_1| = 6k + 3 , \quad |S_2| = 6k + 6 .$$

No additional appearances are possible, i.e., no mixed triples can occur, and we arrive at (8).

Case B. $n \equiv 3 \ (4)$.

Here $f \leqq \frac{n-3}{4}$, $s = \frac{n-3}{2} - t$, $v = \frac{n+1}{2} + t$, and

$$|P| \geqq \frac{1}{3}\left((\frac{n-1}{2} - t)(\frac{n-3}{4} - [\frac{t}{2}]) + (\frac{n+1}{2} + t)(\frac{n+1}{4} + [\frac{t}{2}])\right)$$

$$= \frac{n(n-1)}{12} + \frac{(4t + 2)(2[\frac{t}{2}] + 1)}{12} . \qquad (10)$$

Since in this case $\frac{n(n-1)}{12}$ is not an integer, for (10) not to exceed the already established bound (7), $t = 0$ or 1 . We are then faced with three possibilities as to whether $n \equiv 3, 7, 11 \bmod 12$, and since the argument follows the pattern of Case A, the proof has been omitted.

Let us conclude with an example to illustrate (8), $n = 15$. $\beta(T^2_{3,15}) = 18$ and according to (8), there are essentially three different decompositions of a minimal externally stable set P .

(a) $|S_1| = 7$, $|S_2| = 8$, no mixed triples.

$A = \{(1, 2, 3), (1, 4, 5), (1, 6, 7), (2, 4, 6), (2, 5, 7), (3, 4, 7), (3, 5, 6)\}$,

$B = \{(8, 9, 10), (8, 11, 12), (8, 13, 14), (9, 11, 13), (9, 12, 14), (10, 11, 14), (10, 12, 13), (8, 9, 15), (10, 11, 15), (12, 13, 15), (13, 14, 15)\}$.

(b) $|S_1| = 7$, $|S_2| = 8$, one mixed triple.

A as in (a), B as in above minus (13, 14, 15), mixed triple (1, 14, 15).

(c) $|S_1| = 6$, $|S_2| = 9$, no mixed triple.

$A = \{(1, 2, 3), (1, 2, 4), (3, 4, 5), (3, 4, 6), (5, 6, 1), (5, 6, 2)\}$,

$B = \{(7, 8, 9), (7, 10, 11), (7, 12, 13), (7, 14, 15), (8, 10, 12), (8, 11, 14), (8, 13, 15), (9, 10, 15), (9, 11, 13), (9, 12, 14), (10, 13, 14), (11, 12, 15)\}$.

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