A COMPREHENSIVE ANALYTICAL APPROACH TO THE FREE AND FORCED VIBRATION ANALYSIS OF LARGE STEAM GENERATOR U-TUBES

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SUMMARY

While the theory of vibration of straight tubes has been known for many years, it is only recently that thorough studies of the vibration of curved tubes have been completed. The object of the work described in this paper has been to combine the two theories and thereby obtain analytical solutions for the natural frequencies and modal shapes of inverted U-tubes utilized in large industrial steam generators. The analytical solutions for the straight and curved sections are coupled using established techniques and boundary conditions already discussed in the literature. Separate solutions are obtained for the in-plane and out-of-plane vibrations.

It will be appreciated that the analytical solutions offer vast advantages over those hitherto obtained by finite element methods. The solutions are not only much more tractable, but they present distinct advantages in the obtaining of integrals required for establishing fluid elastic instability velocities, random tube response to turbulence in the flow, and tube response due to vortex shedding. The strain-displacement relationships are, of course, obtained in analytical form thereby giving highly accurate relations between tube vibration amplitude and bending stresses.

In the second part of the paper, a step-by-step vibration design analysis of a large industrial steam generator is conducted utilizing contemporary design criteria. Generalized patterns for the matrix array resulting from satisfaction of the boundary conditions are introduced and discussed. The entire presentation is conducted in such a way as to permit the method to be immediately applied to steam generators of various geometries and boundary conditions, etc.

Utilizing existing tabulated eigenvalues for straight tubes and curved tubes, it is shown how the frequency regions in which actual U-tube eigenvalues must fall can be predicted in advance. It is believed that the method described will find wide application in the design of U-tube type steam generators and heat exchangers.

1. INTRODUCTION

The theory of free vibration of single span and multi-span plane curved tubes of constant raidus of curvature is very well developed by Lee in References |1| and |2|, respectively. The theory of multi-span straight tubes is discussed extensively by Gorman in Reference |3|. For this reason, the theory will not be discussed here, but only the formulations relating displacements, bending moments and shear forces etc., will be introduced for use in this paper. It has been shown by Lee |1| that there are two distinct mode families of free vibration which a curved tube can undergo. These are "in-plane" vibrations in which all tube displacements take place in the plane of the curve, and "perpendicular-to-plane" vibrations in which all displacements are perpendicular to the plane of the curve. It is highly advantageous to discuss these two mode families separately as each can occur independently of the other.

2. U-TUBE FREE VIBRATION ANALYSIS

2.1 In-Plane Free Vibrations.

Using the nomenclature shown in Figure 1, it is shown that only the displacements W and U can take on non-zero values during in-plane vibrations. All displacements are functions of the angular variable θ as shown in Figure 1. It is shown that

$$W(\theta) = \sum_{j=1}^{6} C_J e^{\lambda_J \theta} \tag{1}$$

where the C_J's are constants to be determined through the boundary conditions and the quantities λ_J are the six roots of the algebraic equation

$$\lambda^6 + 2\lambda^4 + (1 - p)\lambda^2 + p = 0 \tag{2}$$

where p, denoted herein as the eigenvalue, equals $mR^4 \omega^2/EI$.

Lee |1| has shown that from the point of view of mathematics, three possible sets of roots exist for equation (2) depending on the tube geometries, etc. Fortunately, it is found that for the U-tubes of current steam generators where the angles between the antivibration bars are not too large (See Figure 2) and where the ratio between the straight tube spans and the radius of curvature of the curved tubes is not too large, one encounters only the solution

$$W(\theta) = C_1 \sin(G_1\theta) + C_2 \cos(G_2\theta) + C_3 \sinh(G_3\theta)$$
$$+ C_4 \cosh(G_4\theta) + C_5 \sinh(G_5\theta) + C_6 \cosh(G_6\theta) \tag{3}$$

This solution correponds to the situation described by Lee |1| where $p > 17.637$.

It will be appreciated that equation (2) is a cubic equation involving the quantity λ^2. Lee has shown that λ^2 has one negative real root and two positive real roots for the solution under study. One may find the negative root by numerical means and denoting it as "a", it is shown that the two positive roots "b" and "c" are given by

$$b = (x_2 - x)/2 \; ; \quad \text{and} \quad c = -(x_2 + x)/2$$

where $\quad x = a + 2 \quad \text{and} \quad x_2 = \sqrt{x^2 + 4\,p/a}$

It then follows that

$$G_1 = G_2 = +\sqrt{|a|} \; ; \quad G_3 = G_4 = +\sqrt{b} \; ;$$
$$G_5 = G_6 = +\sqrt{c} \, .$$

Following the practice of Lee, we assume that extension of the tube along its axis is zero, i.e.,

$$U = \frac{\partial W}{\partial \theta} \tag{4}$$

we may therefore write,

$$U(\theta) = C_1\ G_1 \cos(G_1\theta) - C_2\ G_2 \sin(G_2\theta) + C_3\ G_3 \cosh(G_3\theta)$$
$$+ C_4\ G_4 \sinh(G_4\theta) + C_5\ G_5 \cosh(G_5\theta) + C_6\ G_6 \sinh(G_6\theta) \tag{5}$$

Finally, we introduce the expressions for the bending moments and shear forces for the in-plane motion as follows |1|,

$$M_y = \frac{EI}{R^2} \left\{ \frac{\partial^2 U(\theta)}{\partial \theta^2} + \frac{\partial W(\theta)}{\partial \theta} \right\}$$
$$Q_x = \frac{EI}{R^3} \left\{ \frac{\partial^3 U(\theta)}{\partial \theta^3} + \frac{\partial^2 W(\theta)}{\partial \theta^2} \right\} \tag{6}$$

2.2 Perpendicular-to-Plane Free Vibrations

For this mode family of vibrations only the displacements involving tube transverse displacement $V(\theta)$ and the tube rotation $\gamma(\theta)$ are non-zero. Introducing $B(\theta)$ (where $B(\theta) = R\,\gamma(\theta)$ we have

$$B(\theta) = \sum_{j=1}^{6} C_J\, e^{\lambda_J \theta} \tag{7}$$

where

$$\lambda^6 + 2\lambda^4 + (1 - pK)\lambda^2 + p = 0 \tag{8}$$

For this family of modes we define the eigenvalue p as follows:

$$p = \frac{m\, R^4\, \omega^2}{K\, E\, I}, \text{ where } K = \frac{G\, I_o}{E\, I}.$$

We observe that equation (8) differs from equation (2) through the introduction of the factor K, only.

It is shown that the cubic equation in λ^2, equation 8, will again have three real roots, one being negative, so long as $p > p_2$ where

$$p_2 = \frac{(2\,K^2 + 9K + 27/4) + \sqrt{(2\,K^2 + 9K + 27/4)^2 - 4(K^4 + K^3)}}{2K^3}$$

It is found that this condition is generally satisfied for conventional steam generators as described earlier. The roots, a, b, and c are therefore obtained in a manner identical to that described for the in-plane vibrations and in fact, the expression for $B(\theta)$ is identical to that given for $W(\theta)$ in equation (3). It is shown by Lee |1| that $V(\theta)$ and $B(\theta)$ are related as follows:

$$V(\theta) = \frac{1}{1+K} \left\{ \int \left(\int B(\theta)\, d\theta \right) d\theta - K\, B(\theta) \right\} \tag{9}$$

Integrating, it can be shown that the constants of integration are zero and we obtain

$$V(\theta) = - C_1\, G_{1_K} \sin(G_1\theta) - C_2\, G_{2_K} \cos(G_2\theta)$$
$$+ C_3\, G_{3_K} \sinh(G_3\theta) + C_4\, G_{4_K} \cosh(G_4\theta) + C_5\, G_{5_K} \sinh(G_5\theta)$$
$$+ C_6\, G_{6_K} \cosh(G_6\theta) \tag{10}$$

where $G_{1_K} = \{K + \frac{1}{G_1^2}\}$; $G_{2_K} = \{K + \frac{1}{G_2^2}\}$; $G_{3_K} = \{\frac{1}{G_3^2} - K\}$

$$G_{4K} = \{ \frac{1}{G_4^2} - K\} \; ; \; G_{5K} = \{ \frac{1}{G_5^2} - K\} \; ; \; G_{6K} = \{ \frac{1}{G_6^2} - K\}.$$

For this family of modes, the expressions for bending moments and shear forces become (1)

$$M_x = \frac{EI}{R^2} \{B(\theta) - \frac{\partial^2 V}{\partial \theta^2}\}$$

$$Q_y = \frac{EI}{R^3} \{(1+K) \frac{\partial B}{\partial \theta} + K \frac{\partial V}{\partial \theta} - \frac{\partial^3 V}{\partial \theta^3}\}$$

$$M_z = K \frac{EI}{R^2} \{ \frac{\partial B}{\partial \theta} + \frac{\partial V}{\partial \theta} \} \tag{11}$$

3. THE STEAM GENERATOR U-TUBE PROBLEM

3.1 U-Tube Geometry

In Figure 2, we are given the geometry and support spacing of a typical industrial steam generator tube. We will use this tube to illustrate the free vibration and flow induced vibration analysis. A further simplification will be made here in view of the complete geometric similarity of the tubes with respect to the center plane of the steam generator. It will be appreciated that all vibratory modes, in-plane or perpendicular-to-plane, will be symmetric or anti-symmetric with respect to the center plane of the steam generator.

3.2 In-Plane Tube Vibrations

The requirement of the analyst here is to take the solutions of known form for each span of the U-tube, curved or straight; enforce the appropriate boundary conditions, and look for those values of the eigenvalues which permit non trivial solutions for the displacement. We begin by considering the first span of the straight tube and working our way around to the center plane of the steam generator.

Here it is convenient to non-dimensionalize the straight span lengths through division by the radius of curvature R of the curved tube. Following the procedure discussed by Gorman |3|, one may write for the first span

$$r_1(\xi) = A_1 \{\sin(\beta\xi) - \sinh(\beta\xi) + \phi_1(\cos(\beta\xi) - \cosh(\beta\xi))\} \tag{12}$$

where $\phi_1 = \dfrac{\sin(\beta\mu_1) - \sinh(\beta\mu_1)}{\cosh(\beta\mu_1) - \cos(\beta\mu_1)}$

$$\mu_1 = \ell_1/R \;, \quad \xi = x/R \;, \quad \beta = (p)^{\frac{1}{4}}$$

Similarly for the nth straight span $(1 < n < 10)$ we may write

$$r_n(\xi) = A_n\{\sin(\beta\xi) - \theta_n \sinh(\beta\xi)\} + B_n \{\cos(\beta\xi) - \cosh(\beta\xi) + \phi_n \sinh(\beta\xi)\} \tag{13}$$

where $\mu_n = \ell_n/R \; ; \; \xi = x/\ell_n$

$$\theta_n = \frac{\sin(\beta\mu_n)}{\sinh(\beta\mu_n)} \; ; \quad \phi_n = \frac{\cosh(\beta\mu_n) - \cos(\beta\mu_n)}{\sinh(\beta\mu_n)}$$

Finally, for span $N (N = 10)$, which is the last straight span, we may write

$$r_N(\xi) = A_N \sin(\beta\xi) + B_N(\cos(\beta\xi) - \cosh(\beta\xi)) + C_N \sinh(\beta\xi) \tag{14}$$

where $\xi = x/\ell_N$. It is seen that there are 9 interior straight span tube junctions which provide 18 equations (continuity of slope and bending moment) for relating the constants A_1, A_2, B_2 ... etc.

For the curved spans, the expressions for $W(\theta)$ and $U(\theta)$ are already known. We must, however, enforce the boundary conditions at the extremities of each span. We now look at these conditions, beginning with the junction between the final straight span and the first curved span. It is evident that we must satisfy the conditions of continuity of displacement, slope, bending moment and shear force. We also impose the condition of zero axial displacement at the junction. The equations resulting from fulfillment of these conditions are best understood by studying Figure 3. The junction in question lies between spans μ_N and θ_1. Factors common to both sides of the equations are deleted. Prime symbols indicate differentiation with respect to θ.

At the curved span junctions we satisfy the conditions of zero displacement, axial and radial; and continuity of slope and bending moment. Finally, we come to the conditions to be satisfied at the end of the 4th curved span, i.e., at the central plane of the steam generator. Two sets of conditions are possible depending on whether we are studying symmetric or antisymmetric modes of vibration. They can be expressed as follows,

(i) Antisymmetric case: $U(\theta) = \frac{\partial^2 U(\theta)}{\partial \theta^2} = \frac{\partial^4 U(\theta)}{\partial \theta^4} = 0$

(ii) Symmetric case: $W(\theta) = \frac{\partial U(\theta)}{\partial \theta} = \frac{\partial^3 U(\theta)}{\partial \theta^3} = 0$

We find that for the tube in question, there are a total of 44 unknown constants and 44 homogeneous algebraic equations relating these constants. Eigenvalues are obtained following standard procedures. One seeks those values of the eigenvalues which permit the determinant of the associated matrix to vanish. By setting one of the constants equal to unity, say A_1, it is then possible to solve for the other constants and hence obtain the analytic expressions for the mode shapes.

3.3 Perpendicular-to-Plane Tube Vibrations

Establishment of the matrix for this family of modes is achieved in a manner identical to that described for the in-plane family. The boundary conditions to be satisfied are depicted with the aid of Figure 4. There are a few minor differences related to this second family and they will be described in detail.

Firstly, for the straight spans, the parameter β must be formulated as follows:

$$\beta = (Kp)^{\frac{1}{4}} \tag{15}$$

Furthermore, in view of the sign conventions which have been chosen in the literature, it is necessary to take the bending moment and shear force for the straight tube at the "straight tube-curved tube" junction as,

$$M_x = \frac{-EI}{R^2} \frac{d^2\, r(\xi)}{d\, \xi^2} \; ; \text{ and } \; Q_x = \frac{-EI}{R^2} \frac{d^3\, r(\xi)}{d\, \xi^3} \tag{16}$$

respectively.

As indicated in Figure 4, we must impose continuity of transverse displacement, slope, bending moment and shear at the above junction. A further requirement is that the torsion moment be continuous at the junction. Lee |2| has discussed this condition and proposes that the inertia effects of the straight tube in torsion may be neglected. It can then be shown that the last boundary condition formulated for this junction in Figure (4) is applicable.

Boundary conditions to be imposed at interior curved tube junctions are also depicted in Figure (4). These involve conditions of zero lateral displacement as well as continuity of slope and bending moment related to transverse displacement. Continuity of rotation and torsion moment is also enforced. Finally, the boundary conditions to be imposed for symmetric or antisymmetric modes are indicated. Again, we find that 44 homogeneous algebraic equations are available to relate the 44 unknown constants.

4. RESULTS OF THE FREE VIBRATION ANALYSIS

Eigenvalues computed for in-plane and perpendicular-to-plane vibration modes are presented in Tables I and II, respectively. Eigenvalues inferred from a previous study which had been conducted by means of a finite element approach are also tabulated. We begin by considering the in-plane results.

It is noted that there is no difference in computed symmetric and antisymmetric mode eigenvalues for the first fifteen modes and that there is good agreement with the results of the finite element study where the two families of frequencies were also found to be equal. An examination of Figure 5 reveals the reason why the two mode families have the same frequencies. In Figure 5, the three spans on the right hand side are the curved spans, leading up to the steam generator central plane. It is noted that up to the fifth mode they do not participate in the displacement. In fact, their participation is found to be minimal up to the 19th mode. Since stretching of the tubes is forbidden, it will be appreciated that there must be at least one nodal point along each of the curved spans during in-plane vibration. The same is not true for vibration perpendicular to the plane. For in-plane vibration, the curved tube section is not capable of responding at the relatively low frequencies characteristic of the straight section. In fact, if one were to neglect the curved section and consider the straight tube to be clamped at each end, one would not introduce much error in the frequencies for the in-plane mode family.

We next consider the family of modes perpendicular to the plane of the U-tubes. The first five antisymmetric mode shapes are presented in Figure 6. One notes that there are no nodal points along the curved spans and, furthermore, these curved spans now participate significantly in the tube displacement. This participation is even greater at the higher modes and, in fact, is found to dominate for the eleventh symmetric and antisymmetric modes. One must be particularly concerned about these latter modes during studies of cross-flow induced vibration in the U-bend region.

It will be noted that agreement between the frequency results for the analytical study and the finite element study in Table II is not quite as good as for the results of Table I. In fact, we do not find the fifth mode frequencies obtained in the finite element study. Apart from this discrepancy, the agreement would be fairly good. One can, in fact, demonstrate from analytical arguments that the fifth mode frequencies obtained in the finite element study do not exist. Utilizing References |1| and |3| one can easily find the fundamental frequencies of the curved and straight portions of the tubes if they were considered to be isolated. The curved tube section has a much higher fundamental frequency. It can be shown that frequencies of the real tube must lie between the clamped-simply supported, and clamped-clamped frequencies associated with the straight tube section, until a higher mode frequency is reached which exceeds the fundamental frequency of the isolated curved tube. In fact, this is true of all frequencies computed by the analytic means, but is not true of the fifth mode frequency computed by the finite element method. This illustrates how tabulated analytical results can be used to verify ranges within which frequencies must lie.

5. SOME RESULTS OF THE FLOW INDUCED VIBRATION STUDY

5.1 Fluid-elastic Instability.

The problem of obtaining critical velocities at which fluid-elastic instability of a tube bundle can be expected is well discussed in the literature. It is shown in Reference |4|, for example, that based on current theories regarding this phenomenon, we may write

$$V_g = k\, f \sqrt{\frac{M\,\delta}{\rho}}\ \sqrt{I_1/I_2} \tag{17}$$

where I_1 = Integral along the tube of the displacement squared.

I_2 = Integral along the excited portion of the tube of the displacement squared.

and k is a factor determined experimentally.

Equation (17) is valid for uniform homogeneous cross-flow. It will be appreciated now, that the integrals I_1 and I_2 are easily obtained from the analytical expressions which give mode shapes for the spans. We can, of course, examine numerous modes to find the lowest critical gap velocity.

Two studies have been performed here; one relating to instabilities associated with liquid cross-flow at the lower inlet to the steam generator, and one relating to two-phase cross-flow at the U-bend section using uniform densities and velocities based on the homogeneous model. One could consider these two flow conditions simultaneously, but the results would not be significantly different since strong excitation in the two regions will occur for different modes.

Results of the study are presented in Tables III and IV. A value of 3.3 has been used for the constant "k" and a value of 0.250 for the damping ratio. This relatively high value for the damping ratio has been used because of the fact that almost all of the U-tube is submerged in two-phase flow. It should be appreciated that use of the above value for the quantity "k" incorporates a safety factor (See Reference |5|, for example), nevertheless, a ratio η introduced in Tables III and IV which gives the ratio of the calculated acceptable velocity limits to the actual velocities.

It is found that the most serious excitation due to two-phase flow is associated with 11-th mode symmetric vibration perpendicular to the plane of the U. This is not surprising as we have already noted that for this mode, almost all of the displacement occurs in the curved region of the tube. The fact that there is almost no displacement in the straight section means that almost no energy is dissipated in this non-excited region.

One finds, as well, that the most serious excitation, due to liquid cross-flow, occurs with the ninth antisymmetric mode of vibration perpendicular to the plane of the U. There is considerable displacement of the curved tube portion for this mode, but because inlet flow occurs over only a small portion of the inlet span, the ninth mode is important because it permits relatively large displacement over this short excited region near the tube sheet. One could anticipate different results if the inlet flow was distributed over the entire inlet span.

5.2 Vortex Shedding Studies

It is generally agreed in the literature that vortex shedding does not occur in two-phase cross-flow. We focus our attention, therefore, on possible vortex excitation as a result of liquid cross-flow at the steam generator inlet. It can be shown, assuming conservatively, that the vortex driving forces are completely correlated along the excited area of the tube that for resonance in mode "i" the displacement of the tube is given by

$$y(\xi) = \frac{F_o\, \phi_i(\xi) \int_o^{Ex} |\phi_i(\xi)|\, d\xi}{2\rho_f \omega_i^2\, m \int_o^{L} \phi_i(\xi)^2\, d\xi} \tag{17}$$

when $F_o = \frac{C_F \rho_f V_g^2 d}{2}$

and the upper and lower integrals of Eq. (17) are taken along the excited length of the tube and the entire length of the half tube, respectively. Values for the lift coefficient C_F are discussed in Reference |5| and taken here as 0.1. It will be appreciated that both of the above integrals are easily obtained with the analytical solution available.

In the study reported here, we have considered the straight tubes to be uncoupled from the curved tubes at their respective junctions. This is a conservative assumption, but simplifies the analysis since energy dissipated in the curved region is neglected. The largest amplitude of displacement was found to be .099 mm occurring in the ninth mode.

5.3 Vibration Resulting from Turbulence in the Flow

Response of tubes to excitation forces related to fluid turbulence is discussed in the literature (see for example Reference |5|). It can be shown that the time mean square of response contributed through J'th mode participation is given by

$$\frac{\phi_J^2 \, f_J \, \pi \, I_{1_J}^2 < f(t)^2 >'}{\omega_J^2 \, 4 \, \zeta_J \, m^2 \, I_{2_J}^2} \qquad (18)$$

where $< f(t)^2 >'$ is the time average of the exciting force per unit length of tube per frequency band width centered at frequency f_J, and

$$I_{1_J} = \int_0^{\ell Ex} \phi_J(\xi)\, d\xi \; ; \quad I_{2_J} = \int_0^{\xi} \phi_J^2(\xi)\, d\xi$$

the first interval being taken over the excitation distance, only, the second along the entire half tube. Values for the excitation term, $< f(t)^2 >'$, have been discussed in Reference |5|. Using values extracted from this report, it has been possible to obtain modal displacement contributions from Eq. (18). It will be appreciated that all of the integrals are available. In an earlier study, using Eq. (18), and considering the straight portion of the tube separately, it was found that amplitudes of vibration induced by liquid inlet cross-flow had a maximum value of 0.07 mm (R.M.S.). It will be evident that with the analytical solutions available for the U-tube displacement, it is now possible to apply Eq. (18) to the entire tube. While information regarding the driving forces in the two-phase flow is currently very limited, some measurement experience has been discussed in Reference |5|.

6. SUMMARY

Analytical solutions for U-tube free vibration are readily available using previously developed theory. They allow analysts to obtain a good insight into the interaction between the curved and straight tube sections and to verify that no frequencies have been overlooked. Integrals required for establishing the response of tubes are readily obtained. Bending stresses are accurately established using the classical displacement-bending moment relationships. The analysis conducted here pertained to a particular steam generator with geometric and flow symmetry about its central plane. It will be appreciated that systems lacking this symmetry can also be analyzed and different boundary conditions can be enforced.

TABLE I. SQUARE ROOTS OF EIGENVALUES FOR IN-PLANE VIBRATIONS.

Mode No.	Symmetric	Antisymmetric	Finite Element Results
1	24.37	24.37	24.37
2	26.57	26.57	26.44
3	29.87	29.87	29.63
4	33.93	33.93	33.64
5	38.44	38.44	38.12
6	43.09	43.09	42.67
7	47.50	47.50	46.86
8	51.11	51.11	50.33
9	53.21	53.21	52.73
10	95.91	95.91	95.93
11	100.08	100.08	99.66
12	106.10	106.10	105.44
13	113.23	113.24	112.63
14	121.02	121.02	120.34
15	128.98	128.98	128.11

TABLE II. SQUARE ROOTS OF EIGENVALUES FOR PERPENDICULAR-TO-PLANE STUDIES.

	Results of Analytical Study		Finite Element Results	
Mode No.	Symmetric	Anti Symmetric	Symmetric	Anti Symmetric
1	28.03	28.03	27.92	27.92
2	30.28	30.29	29.98	29.99
3	33.72	33.73	33.41	33.44
4	38.04	38.05	37.60	37.69
5	42.92	42.96	40.56	41.03
6	48.08	48.15	42.91	42.99
7	53.07	53.30	47.80	47.80
8	56.63	57.87	52.40	52.88
9	58.88	61.07	54.38	56.88
10	61.36	62.31	57.28	60.02
11	68.29	78.91	60.38	61.97
12	92.52	107.00	70.54	80.97
13	110.85	111.04	92.17	103.59
14	115.74	116.15	110.19	110.19
15	120.54	123.19	113.13	114.97

TABLE III. SAFETY FACTORS RELATED TO FLUID-ELASTIC INSTABILITY.

	In-Plane Vibrations				Vibrations Perpendicular-to-Plane			
	Anti-Symmetric		Symmetric		Anti-Symmetric		Symmetric	
Mode #	$\eta_{(Liq)}$	$\eta_{(2\phi)}$	$\eta_{(Liq)}$	$\eta_{(2\phi)}$	$\eta_{(Liq)}$	$\eta_{(2\phi)}$	$\eta_{(Liq)}$	$\eta_{(2\phi)}$
1	.042	.009	.041	.008	.038	.044	.038	.045
2	.011	.008	.077	.008	.081	.098	.071	.083
3	.103	.032	.103	.021	.096	.103	.096	.106
4	.122	.017	.122	.016	.115	.115	.115	.117
5	.136	.013	.136	.013	.130	.120	.129	.127
6	.147	.018	.147	.017	.142	.124	.141	.137
7	.157	.014	.157		.153	.127	.147	.172
8	.164	.016	.164	.015	.164	.135	.116	.284
9	.168	.003	.169	.003	.183	.134	.147	.171
10	.032	.006	.032	.006	.052	.241	.115	.044
11	.058	.013	.058	.013	.0001	.267	.001	.289
12	.076	.015	.076	.014	.002	.191	.001	.230
13	.089	.016	.089	.016	.035	.043	.034	.019
14	.098	.017	.098	.017	.060	.039	.057	.043
15	.106	.016	.016	.016	.077	.033	.038	.155
16	.114	.015	.114	.015	.089	.028	.071	.082
17	.121	.011	.121	.011	.098	.025	.088	.035
18	.125	.005	.125	.005	.106	.022	.098	.027
19	.0001	.095	.0001	.094	.113	.018	.105	.022
20	.023	.010	.022	.015	.120	.013	.113	.018

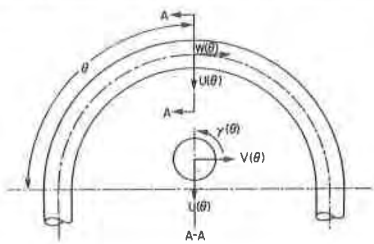

FIG. I COORDINATES & DISPLACEMENT FOR THE CURVED TUBE

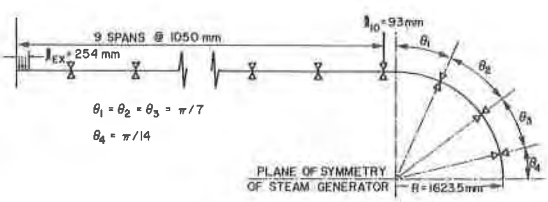

FIG. 2 GEOMETRY OF OUTER U-TUBE

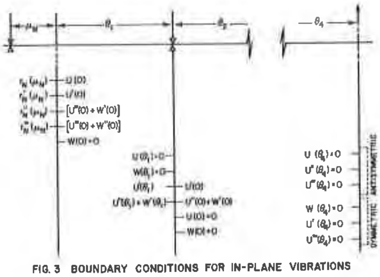

FIG. 3 BOUNDARY CONDITIONS FOR IN-PLANE VIBRATIONS

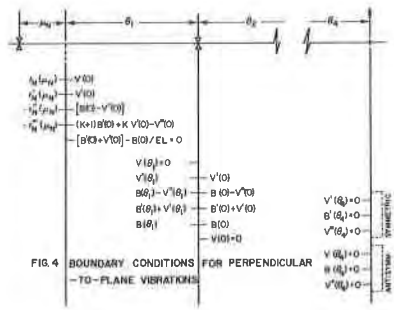

FIG. 4 BOUNDARY CONDITIONS FOR PERPENDICULAR -TO-PLANE VIBRATIONS

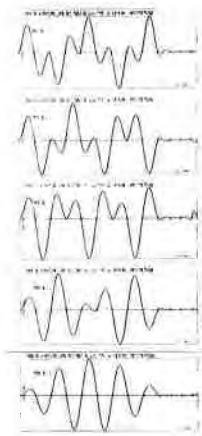

Fig. 5. First Five Mode Shapes for In-Plane Vibration.

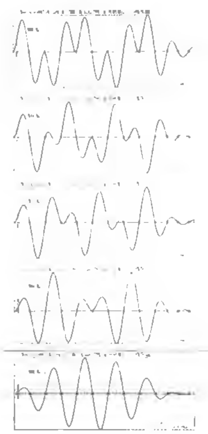

Fig. 6. First Five Mode Shapes for Free Vibration Perpendicular-to-Plane.

Nomenclature

d	Tube outer diameter
E	Young's modulus
f	Vibration frequency
I	Moment of inertia (bending)
I_o	Moment of inertia (torsion)
L	Span length
m	Effective mass/unit length
R	Radius of curvature
V_g	Gap velocity
x	Distance along tube
ω	Circular frequency
ζ	Damping ratio
ρ_f	Fluid density
$\phi_{(x)}$	Mode shape
$\eta_{(Liq)}$	Ratio gap velocity to critical gap velocity (liquid)
$\eta_{(2\phi)}$	Ratio gap velocity to critical gap velocity (2 phase)

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