Constitutive Scheme of Anisotropic Elastic-Plastic Hysteresis

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ABSTRACT

The pure hysteresis model (SMIRT 4, L1/9; SMIRT 5, L1/6) is at the origin of a class of well founded thermomechanical schemes (of discrete memory form) which takes into account the essential features of the behaviour of metals, alloys and shape memory alloys (SMIRT 8, L1/8 and L5/7; SMIRT 9, p.163; SMIRT 10, p.7). Moreover, the model has been expanded to include the isotropic-deviatoric coupling effects: thus, it takes into account the pressure dependent behaviour of elastic-plastic continua such as granular media subjected to arbitrary cyclic loading. Nevertheless, at this stage of the analysis, the theory is still of isotropic type due to the fact that the problem of the physically relevant preferred reference frame has been left open. The aim of the paper is to suggest how to overcome this difficulty and to outline the associated anisotropic generalization of the elastic-plastic pure hysteresis model.

1 INTRODUCTION

Let us consider, in the cyclic case, the variations $\Delta_R^t\sigma$ and $\Delta_R^t\varepsilon$ of the stress and strain from the inversion point (time t_R) to the current state (time t). Previous studies (see references) are based on elastic-plastic scheme such as :

$$\Delta_R^t\sigma = \omega\ \mathrm{th}\left(\frac{\mu_0\ \Delta_R^t\varepsilon}{\omega\ S_0}\right) \Longleftrightarrow \frac{\partial}{\partial t}\Delta_R^t\sigma \equiv \dot{\sigma} = \mu_0\dot{\varepsilon} + \left(\frac{-\mu_0}{(\omega S_0)^2}\right)\Phi\,\Delta_R^t\sigma\ ;\ \omega = 1 \text{ or } 2$$

generalized to the three-dimensional case following a form :

$$\dot{\sigma}^i{}_j = \lambda_0 I_D \delta^i{}_j + 2\mu_0 D^i{}_j + \beta\,\bar{\Phi}\,\Delta_R^t\bar{\sigma}^i{}_j\ ;\ \Delta_R^t\bar{\sigma}^i{}_j = \Delta_R^t\sigma^i{}_j - \frac{1}{3}\Delta_R^t\sigma^k{}_k\delta^i{}_j \qquad (1.1)$$

which is linear with respect to the strain rate D and where β is not only defined like $-\mu_0/(\omega S_0)^2$ but is generalized by $\beta(S_0, \omega, \alpha_\Delta)$ with $\alpha_\Delta = \varphi_\Delta - \varphi_R$ where φ_Δ and φ_R are the phases in the stress space (notation θ_Δ and θ_R in the $\mathbb{R}2$ case of § 3). Moreover $\bar{\Phi}$ is not only taken as $\bar{M} = \Delta_R^t\bar{\sigma}^i{}_j\,\bar{D}^j{}_i$ but is also generalized (Pegon 1988) by :

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$$\bar{\Phi} = \bar{M} + \gamma\,\dot{\alpha}_{\Delta} = \bar{M} + \gamma\,\dot{\varphi}_{\Delta} \quad ; \quad \left(\dot{\varphi}_R \equiv 0\right) \tag{1.2}$$

The simplest plastic yield which may be taken into account with (1.1) through the definition of β is the Mises one :

$$\mathrm{trace}(\bar{\sigma})^2 = 2\ S_0^{\ 2} = Q_0^{\ 2}$$

2 GENERAL FEATURES OF THE ANISOTROPIC PURE HYSTERESIS PATTERN

The usual Euclidian space E_3 is referred to an orthonormal fixed frame $\left(O,\ \vec{e}_{s\alpha}\right)$ (α =1,2,3) "associated" with the loading process (see §.3 devoted to a E_2 example). One considers three fields of (objectively defined) frames which are initially (at time t = 0) identical with the orthonormal field $\left(M,\ \vec{e}_{\alpha}\right)$ "associated" with the directions of material anisotropy (cf §.3 for the E_2 example of homogeneous type). The first basic field of Reference Frames (RF) is the "usual" convected one $\left(M,\ \vec{a}_{\alpha}\right)$ associated with the dragged along material coordinates x^{α} (of current metric tensor $a_{\alpha b}$). The second basic field of RF $\left(M,\ \vec{b}_{\alpha}\right)$, such as :

$$\vec{b}_{\alpha} = (1 - \lambda_b)\ \vec{A}_{\alpha} + \lambda_b \vec{a}_{\alpha}\ ;\ 0 \leqslant \lambda_b \leqslant 1\ ;\quad \dot{\lambda}_b = 0 \quad \forall\ t > 0 \tag{2.1}$$

is semi-convected "between" $\left(M,\ \vec{a}_{\alpha}\right)$ and the third RF field $\left(M,\ \vec{A}_{\alpha}\right)$ which is the Preferred Reference Frame (PRF). This PRF is orthonormal and deduced from $\left(M,\ \vec{e}_{\alpha}\right)$ (defined at time 0) by the product α of two rotations α_d and α_{φ} defined below. Owing to the fact that (2.1) is a clue for the analysis , the scalar λ_b is taken constant for sake of simplicity. The rotation α_d is that of the principal axes ($\vec{u}$, $\vec{v}$, $\vec{w}$) of the strain rate tensor D :

$$D_{\alpha\beta} = \frac{1}{2}\frac{\partial a_{\alpha\beta}}{\partial t} = \frac{\partial \Delta_R^t \varepsilon_{\alpha\beta}}{\partial t} \quad \text{with } 2\Delta_R^t \varepsilon_{\alpha\beta} = a_{\alpha\beta} - {}_R^t a_{\alpha\beta} \quad \text{in } \left(M,\ \vec{a}_{\alpha}\right)$$

The rotation α_{φ} is infered from that of an orthonormal thermomechanical frame (OTF) (M, $\vec{\lambda}$, $\vec{\mu}$, $\vec{\nu}$). The inference is obtained making the α_{φ} intensity a small functional of the intensity of that of the OTF, the reduction factor being of the order of S_0/μ (Pegon 1988). The OTF unit vectors are the principal axes of a field of thermomechanical tensors : the simplest form of this field is $\sum_{\alpha=1}^{3} \vec{a}_{\alpha} \otimes \vec{a}_{\alpha}$ (see §.3 for the E2 example of homogeneous type). The quasi reversible and quasi perfectly plastic properties are then defined with the aid of two fourth order tensors A(t=0) and $\bar{P}$(t=0) dragged along in $\left(M,\ \vec{b}_{\alpha}\right)$ and $\left(M,\ \vec{A}_{\alpha}\right)$ respectively, what lead therefore to :

$$\left|\begin{array}{ll} A(t) = A(t{=}0)^{\alpha\beta\gamma\delta}\ \vec{b}_{\alpha}(t)\otimes\vec{b}_{\beta}(t)\otimes\vec{b}_{\gamma}(t)\otimes\vec{b}_{\delta}(t) & \text{in } \left(M,\ \vec{b}_{\alpha}\right) \quad \forall t > 0 \\ \\ \bar{P}(t) = \bar{P}(t{=}0)_{\alpha\beta\gamma\delta}\ \vec{A}_{\alpha}(t)\otimes\vec{A}_{\beta}(t)\otimes\vec{A}_{\gamma}(t)\otimes\vec{A}_{\delta}(t) & \text{in } \left(M,\ \vec{A}_{\alpha}\right) \quad \forall t > 0 \end{array}\right. \tag{2.2}$$

The plastic tensor defines the yield limit following :

$$\bar{\sigma}^{\alpha\beta}\ \bar{P}_{\alpha\beta\gamma\delta}\ \bar{\sigma}^{\gamma\delta} = Q_0^2 \quad \text{in } \left(M, \vec{A}_\alpha\right) \quad \forall t > 0 \tag{2.3}$$

Consequently, the basic operational hinge of the definition of the HARP pattern ("Hysteresis pure Anisotrope en Repères Privilégiés") is threefold (through the $\left(M, \vec{a}_\alpha\right)$, $\left(M, \vec{b}_\alpha\right)$, $\left(M, \vec{A}_\alpha\right)$ reference frames respectively). Moreover, the operational hinge is as follows : the HARP scheme is "almost of isotropic pure hysteresis" in the PRF where the basic tensor field are the stress rate $\dot{\sigma}^{\alpha\beta}$ and a strain rate $\dot{E}_{\alpha\beta}$ associated with $D_{\alpha\beta}$ with the aid of the fourth order tensor A (HARP is only "almost isotropic" because eq 2.3 - see §.3 for an example). The formalism of HARP is then three-fold through three sets of definition in $\left(M, \vec{A}_\alpha\right)$, $\left(M, \vec{b}_\alpha\right)$ and $\left(M, \vec{a}_i\right)$ respectively. This gives firstly :

$$\left.\begin{aligned} &{}^{t}_{R}\dot{\sigma} = 0 \ ; \ \dot{\sigma}^{\alpha\beta} = \dot{E}^{\alpha\beta} + \beta\Phi\, \Delta^t_R \sigma^{\alpha\beta} \quad \text{with} \quad \Phi = \Delta^t_R \sigma^{\alpha\beta}\ \dot{E}_{\alpha\beta} + \gamma\, \dot{\varphi}_\Delta + \ldots \\ &\bar{\sigma}^{\alpha\beta}\ \bar{P}_{\alpha\beta\gamma\delta}\ \bar{\sigma}^{\gamma\delta} = Q_0^2 \qquad \text{in} \quad \left(M, \vec{A}_\alpha\right) \end{aligned}\right\} \tag{2.4}$$

and secondly :

$$\dot{E}^{\alpha\beta} = A^{\alpha\beta\gamma\delta}\, D_{\gamma\delta} \quad \text{in} \quad \left(M, \vec{b}_\alpha\right) \tag{2.5}$$

Finally, the introduction of $\left(M, \vec{a}_\alpha\right)$ is necessary to define both $\vec{b}_\alpha$ (through (2.1)) and the rotation α_φ. In the case of homogeneous problems and if α_φ is neglected, it is clear that the role of the third RF field is confined to the derivation of the components $D_{\gamma\delta}$ in (2.5) from the expression $D_{s\alpha\beta}\ \vec{e}_{s\alpha} \otimes \vec{e}_{s\beta}$ of D in $\left(O, \vec{e}_s\right)$ (see §3 for an example).

3 THE EXAMPLE OF THE TWO-DIMENSIONAL HOMOGENEOUS CASE

In order to illustrate how the various steps of definition of the pattern are combined in order to obtain an operational rate form constitutive scheme, one considers the <u>homogeneous</u> kinematics of strain rate :

$$D = \sum_{\alpha=1}^{2} \frac{\dot{K}_\alpha}{J_\alpha}\ \vec{e}_{s\alpha} \otimes \vec{e}_{s\alpha} + \left(\vec{e}_{s1} \otimes \vec{e}_{s2} + \vec{e}_{s2} \otimes \vec{e}_{s1}\right) \left(\frac{\dot{\tau}}{J_1} - \frac{\tau\ \dot{K}_2}{J_1 J_2}\right) \quad \text{in} \quad \left(O, \vec{e}_s\right)$$

where $J_{\alpha=1,2}$ denote as previously the stretches $1 + K_\alpha$ and 2τ denotes the shear along $\vec{e}_{s2}$. The three components $D_S^{\alpha\beta}$ of D in $\left(O, \vec{e}_s\right)$ are then related with the rates $\dot{K}_1$, $\dot{K}_2$ and $\dot{K}_3$ ($\equiv \dot{\tau}$) by the matricial form :

$$|D_S| = \boxed{J}\, |\dot{K}| \ ; \ |D_S|^T = \left|D_S^{11},\ D_S^{22},\ D_S^{12}\right| ; \ |\dot{K}|^T = \left|\dot{K}_1, \dot{K}_2, \dot{K}_3\right| \ \text{in} \ \left(O, \vec{e}_s\right)$$

where $\boxed{J}$ is a 3x3 matrice of line by line listing : $1/J_1$, 0, 0; 0, $1/J_2$, 0; 0, $-\tau/J_1 J_2$, $1/J_1$. Let θ_0 and α denote the angles of $\left(O, \vec{e}_s\right)$ and of the PRF $(M, \vec{A})$ respectively with respect to the RF field $(M, \vec{e})$ associated with the direction of the material anisotropy of the <u>homogeneous</u> sheet of two-dimensional material. Making use of the matricial notation introduced above, one obtains (cf appendix) :

$$\left| D_{..\vec{b}} \right| = \boxed{E} \, |\dot{K}| \quad ; \quad \boxed{E} = \boxed{b2} \, \boxed{H2}^{-1} \, \boxed{\theta_0 2} \, \boxed{J} \tag{3.1}$$

In the $\mathbb{R}2$ case, four systems (instead of seven in the $\mathbb{R}3$ case) must be considered : monoclinic, rhombic, cubic, hexagonal-isotropic. The associated A tensor contain 6, 4, 3, and 2 physical parameters respectively. In the rhombic case (for example) the parameters are c_1, c_2, c and μ ; (2.2) gives :

$$A = \sum_{\alpha=1}^{2} c_\alpha \vec{b}_\alpha \otimes \vec{b}_\alpha \otimes \vec{b}_\alpha \otimes \vec{b}_\alpha + c\left(\vec{b}_1 \otimes \vec{b}_1 \otimes \vec{b}_2 \otimes \vec{b}_2 + \vec{b}_2 \otimes \vec{b}_2 \otimes \vec{b}_1 \otimes \vec{b}_1\right) + \mu\left(\vec{b}_1 \otimes \vec{b}_2 \otimes \vec{b}_1 \otimes \vec{b}_2 + .. + \vec{b}_2 \otimes \vec{b}_1 \otimes \vec{b}_2 \otimes \vec{b}_1\right)$$

The matricial form of (2.5) in (M, $\vec{b}$) is then : $\left| \dot{E}^{..}_{\vec{b}} \right| = \boxed{A} \, |D_{..}|$ where $\boxed{A}$ is of simple structure : c_1, c, 0; c, c_2, 0; 0, 0, 2μ. Then :

$$\left| \dot{E}_{\vec{A}} \right| = \boxed{E_d} \, |\dot{K}| \quad ; \quad \boxed{E_d} = \boxed{B2} \, \boxed{A} \, \boxed{E} \tag{3.2}$$

Consequently, one obtains in the PRF :

$$\left| \dot{\sigma}_{\vec{A}} \right| = \boxed{\beta S2} \left| \dot{E}_{\vec{A}} \right| + \gamma \beta \left| \Delta\bar{\sigma}_{\vec{A}} \right| \dot{\theta}_\Delta = \boxed{CS}^{-1} \boxed{\beta S2} \left| \dot{E}_{\vec{A}} \right| = \boxed{L} \left| \dot{E}_{\vec{A}} \right| \tag{3.3}$$

with :

$$\boxed{\beta S2} = \begin{pmatrix} 1+\beta\Delta\bar{\sigma}^{11}\Delta\bar{\sigma}^{11} & \beta\Delta\bar{\sigma}^{22}\Delta\bar{\sigma}^{11} & 2\beta\Delta\bar{\sigma}^{12}\Delta\bar{\sigma}^{11} \\ \beta\Delta\bar{\sigma}^{11}\Delta\bar{\sigma}^{22} & 1+\beta\Delta\bar{\sigma}^{22}\Delta\bar{\sigma}^{22} & 2\beta\Delta\bar{\sigma}^{12}\Delta\bar{\sigma}^{22} \\ \beta\Delta\bar{\sigma}^{11}\Delta\bar{\sigma}^{12} & \beta\Delta\bar{\sigma}^{22}\Delta\bar{\sigma}^{12} & 1+2\beta\Delta\bar{\sigma}^{12}\Delta\bar{\sigma}^{12} \end{pmatrix} ; \quad \boxed{CS} = \begin{pmatrix} \frac{2-c}{2} & \frac{c}{2} & c\tan\theta_\Delta \\ \frac{c}{2} & \frac{2-c}{2} & -c\tan\theta_\Delta \\ \frac{-c}{2\tan\theta_\Delta} & \frac{c}{2\tan\theta_\Delta} & 1+c \end{pmatrix}$$

and :

$$2c = \beta\gamma \sin 2\theta_\Delta ; \quad \beta = \frac{-1}{(\omega\, \Omega'(S_0, \alpha_\Delta))^2} ; \quad \alpha_\Delta = \theta_\Delta - \theta_R$$

As previously, the scalar functional γ is obtained accordingly to the definition of the plastic yield locus through the tensor $\bar{P}$. In $\mathbb{R}2$, this tensor is built introducing a limit shear stress and 3, 2, 1 or 0 shape parameters following the type of the anisotropic system. In the rhombic case, $\bar{P}$ leads to an elliptic yield locus :

$$\left(\frac{\bar{\sigma}^{12}}{a}\right)^2 + \left(\frac{\bar{\sigma}^{11}}{b}\right)^2 = S_0^2 \quad ; \quad b \geqslant 1 \geqslant a \; ; \; e = \frac{\sqrt{b^2 - a^2}}{b} \; ; \quad 0 \leqslant e \leqslant 1 \tag{3.4}$$

Therefore, γ is defined through the introduction of neutral paths which are ellipses of <u>constant excentricity e.</u> Then (3.3) leads to :

$$\boxed{(\theta_0 - \alpha)2} \left| \dot{\sigma}_S \right| = \boxed{L} \, \boxed{E_d} \, |\dot{K}| = \boxed{1} \, |\dot{K}| \text{ with } \boxed{1} = \boxed{CS}^{-1} \boxed{\beta S2} \, \boxed{B2} \, \boxed{A} \, \boxed{E} \tag{3.5}$$

Regarding α_φ, the use of the simplest form of thermomechanical tensor given above (§2) results in a tilting angle $\Omega^\varphi = \operatorname{atan}\left(-\tau + \sqrt{1+\tau^2}\right)$ and α_φ is a small functional of Ω_φ, previously defined (Pegon 1988).

4 SOME REMARKS

The PH pattern has been expanded through the α functional and the associated implicit definition of the stress in the PRF. As a consequence of the two dimensional analysis of §.3, the following points may be outlined; the simplest form of the pattern being obtained in the simple shear case, the cyclic off-axis shear tests are distinguished as specially interesting (parameter identification, validation and calibration); the proposed scheme implies that a gradual shear strain appears under hydrostatic loading: this effect is continuously exhibited from the initial quasi-reversible states to the quasi perfectly plastic states. To our knowledge, this gradual effect is not described by any other current theory; the proposed scheme presents an outstanding property regarding the push-pull test performed in "almost in axis" orientations. In these conditions, the sensitivity of the theory to the definition of the preferred reference frame may result in a strong orientation dependence of the usual 0.2pct yield stresses. This may be compared with actual experimental results regarding the 0.2 pct yield stress data obtained on single-crystal superalloys. To our knowledge, this effect is not foreseen by any of the current approaches; the two-dimensional case is of straightforward interest regarding the tests on fibre reinforced sheets (identification validation and calibration); there is no basic theoretical drawbacks, neither with respect to the thermomechanical foundations, nor regarding the micro-macro correlations associated with the pure hysteresis concept (Favier et al 1990). It is now possible to go deeply in the previous studies devoted to the questions of second order effects, hardenings, thermoplasticity and shape memory effects and granular media (Pegon et al 1990).

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APPENDIX

$$\vec{e}_s = \boxed{\theta_0}\ \vec{e} \qquad \text{with} \qquad \boxed{\theta_0} = \begin{pmatrix} \cos\theta_0 & \sin\theta_0 \\ -\sin\theta_0 & \cos\theta_0 \end{pmatrix}$$

$$\left|D_{\vec{e}}\right| = \boxed{\theta_0 2}\ \left|D_{\vec{e}_s}\right| \qquad \text{with} \qquad \boxed{\theta_0 2} = \begin{pmatrix} \cos^2\theta_0 & \sin^2\theta_0 & -\sin 2\theta_0 \\ \sin^2\theta_0 & \cos^2\theta_0 & +\sin 2\theta_0 \\ \frac{1}{2}\sin 2\theta_0 & \frac{-1}{2}\sin 2\theta_0 & \cos 2\theta_0 \end{pmatrix}$$

$$\vec{A} = \boxed{\alpha}\ \vec{e} \text{ with } \boxed{\alpha} = \begin{pmatrix} +\cos\alpha & +\sin\alpha \\ -\sin\alpha & +\cos\alpha \end{pmatrix} ;\ \vec{a}_{.} = \boxed{G}\ \vec{e} \text{ with } \boxed{G} = \begin{pmatrix} 1+\delta_1^1 & \delta_1^2 \\ \delta_2^1 & 1+\delta_2^2 \end{pmatrix}$$

$$\delta_1^1 = K_1\cos^2\theta_0 + K_2\sin^2\theta_0 - \tau\sin 2\theta_0 ;\ \delta_2^2 = K_1\cos^2\theta_0 + K_2\sin^2\theta_0 + \tau\sin 2\theta_0$$

$$\delta_1^2 = \frac{1}{2}(K_2-K_1)\sin 2\theta_0 + 2\tau\cos^2\theta_0 ;\ \delta_2^1 = \frac{1}{2}(K_2-K_1)\sin 2\theta_0 - 2\tau\cos^2\theta_0$$

$$\vec{b}_{.} = \boxed{H}\ \vec{e} \qquad \text{with} \qquad \boxed{H} = (1-\lambda)\ \boxed{\alpha} + \lambda\ \boxed{G}$$

$$\left|D_{\vec{e}}\right| = \boxed{H2}\ \left|D_{\vec{b}}^{\cdot\cdot}\right| \qquad \text{with} \qquad \boxed{H2} = \begin{pmatrix} H_1^1H_1^1 & H_2^1H_2^1 & 2H_1^1H_2^1 \\ H_1^2H_1^2 & H_2^2H_2^2 & 2H_1^2H_2^2 \\ H_1^1H_1^2 & H_2^1H_2^2 & H_1^1H_2^2 + H_1^2H_2^1 \end{pmatrix}$$

$$\vec{b}_{.} = \boxed{b}\ \vec{b}^{\cdot} \qquad \text{with} \qquad \boxed{b} = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix}$$

$$\left|D_{..\vec{b}^{\cdot}}\right| = \boxed{b2}\ \left|D_{\vec{b}_{.}}^{\cdot\cdot}\right| \qquad \text{with} \qquad \boxed{b2} = \begin{pmatrix} b_{11}b_{11} & b_{12}b_{12} & 2b_{11}b_{12} \\ b_{12}b_{12} & b_{22}b_{22} & 2b_{12}b_{22} \\ b_{11}b_{12} & b_{12}b_{22} & b_{11}b_{22} + b_{12}b_{12} \end{pmatrix}$$

$$\vec{b}_{.} = \boxed{B}\ \vec{A} \qquad \text{with} \qquad \boxed{B} = (1-\lambda)\ \boxed{1} + \lambda\ \boxed{G}\ \boxed{\alpha}^{-1}$$

$$\left|\dot{E}_{\vec{A}}\right| = \boxed{B2}\ \left|\dot{E}_{\vec{b}_{.}}^{\cdot\cdot}\right| \qquad \text{with} \qquad \boxed{B2} = \begin{pmatrix} B_1^1B_1^1 & B_2^1B_2^1 & 2B_1^1B_2^1 \\ B_1^2B_1^2 & B_2^2B_2^2 & 2B_1^2B_2^2 \\ B_1^1B_2^1 & B_1^2B_2^2 & B_1^1B_2^2 + B_1^2B_2^1 \end{pmatrix}$$