

AN OPTIMAL METHOD FOR COMPUTING EIGENVECTORS USING THE INVERSE POWER TECHNIQUE

R. ROSEN

*Computer Systems Department,
Mechanics Research Inc., Los Angeles, California 90045, U.S.A.*

SUMMARY

The inverse power method of eigenvalue/eigenvector extraction is ideally suited to large structural dynamic analysis computer programs. Many of the computer codes currently available to the engineering community are successfully employing this technique. The particular method described in this paper has been successfully employed in the MRI/STARDYNE Structural Analysis System. The basic method as well as the extraction of close roots has been presented by Rosen and Rubinstein ("Dynamic Analysis By Matrix Decomposition", ASCE, *Journal of Engineering Mechanics Division*, V. 94, NEM 2, pp. 385-395) in April 1968.

This present paper examines in detail the nature of the assumed eigenvector as it changes during the iteration process. Making use of the multiple eigenvector extraction method (as present in "Dynamic Analysis By Matrix Decomposition") and prescribing a convergence criterion of N significant digits in the "to be computed" eigenvector an equation is derived to compute the number of iterations required to reach convergence. At this point an algorithm is formulated to extract the eigenvalues/eigenvectors using a minimum amount of computer time. This algorithm uses the number of iterations required for convergence as well as the computer times associated with different phases of the inverse power method to internally decide when to cease iterating at the current shift point and move to the next shift point. Using the information determined during the current iteration the algorithm also decides on the location of the next shift point. By utilizing this technique the program user is no longer burdened with the "a priori" decisions of shift point selection and the number of iterations to be attempted.

INVERSE POWER METHOD

In order to develop the new concepts presented in this paper it is first necessary to explain the basic method. (For a complete derivation of the inverse power technique and the extraction of close eigenvalues the reader is referred to reference [1]).

Consider a structural system with displacement coordinates q_i ($i = 1, \dots, n$), a mass matrix $[m]$, and a stiffness matrix $[K]$. The equations of motion for free vibration of the system are

$$[K] \{q\} = \omega^2 [m] \{q\} \quad (1)$$

Any scalar ω^2 and corresponding vector $\{q\}$ which satisfy eq. (1) are termed an eigenvalue and eigenvector of the system. Physically ω_i is the i^{th} natural angular frequency of the structural system and $\{q^{(i)}\}$ is the associated displaced configuration of the system vibrating at ω_i .

The utility of the inverse power method is in being able to extract eigenvalues/eigenvectors closest to any prescribed frequency. Let λ_0 be the frequency in the neighborhood of which we wish to extract the eigenvalues/eigenvectors.

Write the relationship

$$\omega^2 = \lambda_0^2 + \lambda^2 \quad (2)$$

and substitute eq. (2) into eq. (1)

$$[K] \{q\} = (\lambda_0^2 + \lambda^2) [m] \{q\} \quad (3)$$

Equation (3) can be rewritten as

$$[K]^* \{q/\lambda^2\} = [m] \{q\} \quad (4)$$

where

$$[K]^* = [K] - \lambda_0^2 [m] \quad (5)$$

The iteration procedure begins with assuming an initial vector $\{v_1\}$ and solving for a new vector $\{v_2\}$. The new vector $\{v_2\}$ is used as the starting vector for the second iteration. The iteration equation can be written as

$$[K]^* \{v_{i+1}\} = [m] \{v_i\} \quad (6)$$

Comparing eq. (6) to eq. (4) it is seen that convergence is achieved when $\{v_i\}$ and $\{v_{i+1}\}$ differ by a scalar multiple, which in fact is $\frac{1}{\lambda^2}$.

SOLUTION CONVERGENCE

Let the vector $\{v\}$ be algebraically represented as a linear combination of the eigenvectors

$$\{v\} = \sum_{i=1,n} c_i \{q^{(i)}\} \tag{7}$$

(The eigenvectors form a basis in n space.)

Now consider the vector $\{v_1\}$ as the starting vector in the iteration process then after one iteration the new vector $\{v_2\}$ will be

$$\{v_2\} = \sum_{i=1,n} \frac{c_i}{\lambda_i^2} \{q^{(i)}\}$$

and likewise after n + 1 iterations

$$\{v_{n+1}\} = \sum_{i=1,n} \frac{c_i}{\lambda_i^{2n}} \{q^{(i)}\} \tag{8}$$

Assuming

$$\lambda_1 < \lambda_2 < \dots < \lambda_n$$

then

$$\lambda_1^{2n} \ll \lambda_1^{2n} \ll \dots \ll \lambda_n^{2n}$$

hence after a sufficient number of iterations

$$\{v_{n+1}\} \cong \frac{c_1}{\lambda_1^{2n}} \{q^{(1)}\} \tag{9}$$

From eq. (9) it is obvious that convergence will be achieved to that eigenvalue/eigenvector closest to the originally specified λ_0 .

PREDICTION OF CONVERGENCE

Let eq. (7) represent the current vector in the iteration process where λ_1 and λ_2 are also known. [The values of the two most dominant λ 's are easily obtained from both the double and triple root extraction procedures. (Reference [2]). Convergence to the eigenvalues occurs much more rapidly than for the eigenvectors]. Now the question is posed "How many iterations are required to reach convergence as defined by eq. (9)".

Stated mathematically it is required that the factor multiplying $\{q^{(1)}\}$, $\frac{c_1}{\lambda_1^{2n}}$, be N orders of magnitude greater than the factor multiplying $\{q^{(2)}\}$, $\frac{c_2}{\lambda_2^{2n}}$.

In equation form this relationship is

$$\frac{c_1}{\lambda_1^{2n}} - \frac{c_2}{\lambda_2^{2n}} > 10^N \tag{10}$$

Taking logarithms of both sides of eq. (10)

$$\log_{10} \left(\frac{c_1}{c_2} \frac{\lambda_2^{2n}}{\lambda_1^{2n}} \right) > N \tag{11}$$

or

$$\log_{10} \left(\frac{c_1}{c_2} \right) + n \log_{10} \left(\frac{\lambda_2^2}{\lambda_1^2} \right) > N \tag{12}$$

It can be reasonably assumed that if the double root extraction procedure yielded meaningful values of λ_1 and λ_2 then c_1/c_2 is reasonably close to unity.

Solving for the required number of iterations "n" from eq. (12) yields

$$n > \frac{N}{\log_{10} \left(\frac{\lambda_2^2}{\lambda_1^2} \right)} \tag{13}$$

Table I gives the number of iterations required for convergence based on the eigenvalue ratio $\frac{\lambda_2^2}{\lambda_1^2}$ and the convergence criteria N.

Table I Number of Iterations Required for Convergence

Eigenvalue Ratio $\frac{\lambda_2^2}{\lambda_1^2}$	N		
	5	6	7
1.1	119	143	170
1.3	44	53	62
1.5	29	35	40
2.0	17	20	24
4.0	9	10	12
10.0	5	6	7
100.0	3	3	4

OPTIMAL CONVERGENCE ALGORITHM

The iteration algorithm has been described by eq. (6). After at least three iterations have been performed the following information can be computed at each step.

- a) Current eigenvalue: This is obtained from the normalization factor for the computed vector.
- b) Two lowest eigenvalues; This is obtained from the quadratic convergence procedure. (Requires last two computed vectors.)
- c) Three lowest eigenvalues; This is obtained from the cubic convergence procedure. (Requires last three computed vectors.)

In addition the following times can be obtained from reading the internal clock.

- 1) Time to go to next shift point. (Choose a new λ_0^2). This is the time required to decompose (triangularize) the $[K]^*$ matrix.
- 2) Time required to perform one complete iteration.

When agreement has been achieved between the quadratic and cubic extraction procedures regarding the two lowest eigenvalues, eq. (15) may be used to predict the number of iterations required for convergence. If the time required for convergence is greater than the time required to go to a new shift point and perform three iterations then the algorithm should choose the current estimate of the lowest eigenvalue as the next shift point. Since convergence to the lowest eigenvalue is very rapid, this value is known fairly accurately and a shift to its immediate vicinity will result in convergence within three iterations.

CONCLUSIONS

A technique for predicting the number of iterations required for convergence in the inverse power method for extracting eigenvalues/eigenvectors has been presented. Using information available during the computations the most efficient strategy for iterating or shifting can be utilized. Since this information is not available to the engineer using the program, the computer code itself can best specify the shift point selection to extract all of the desired eigenvalues/eigenvectors in a given frequency range.

REFERENCES

- [1] Rosen, R., Ragle, R. E., Curtis, R. E., et al, S'TARDYNE User's Manual, Mechanics Research, Inc., Los Angeles, Calif., 1973.
- [2] Rosen, R. and Rubinstein, M. F., "Dynamic Analysis By Matrix Decomposition", V. 94, NEM 2, pp. 385-395, April 1968.

