ABSTRACT

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Tannaka's Theorem states that a linear algebraic group G is determined by the category of finite dimensional G-modules and the forgetful functor. We extend this result to linear differential algebraic groups by introducing a category corresponding to their representations and show how this category determines such a group.

We also provide conditions for a category with a fiber functor to be equivalent to the category of representations of a linear differential algebraic group. This generalizes the notion of a neutral Tannakian category used to characterize the category of representations of a linear algebraic group [9, 4].

Tannakian Categories and Linear Differential Algebraic Groups

by

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To my parents.

Biography

Alexey Ovchinnikov was born in the middle of 1982, in Moscow, Soviet Union. His parents are Igor Ovchinnikov, originally from Moscow, and Elena Novikova, from Moscow as well. Following graduation from his high school, Alexey entered Moscow State University. He graduated in 2004 with a specialist degree in mathematics and applied mathematics, which is equivalent to B.S. Alexey earned an M.S. in mathematics while at his Ph.D. program at North Carolina State University.

Alexey's interest in differential algebra was highly influenced by Eugeny Pankratiev, who is his advisor at Moscow State University and he started working in the field in 2001 and continued at NC State under supervision of Michael Singer who has extended the view of Alexey on the field very much.

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Contents

Chapter 1

Introduction

Given a linear algebraic group G, a rational representation (or finite dimensional G-module) is a finite dimensional vector space V together with a morphism $\rho_V : G \to \mathbf{GL}(V)$. The collection of such objects forms a rigid, abelian, tensor category and Tannaka's theorem ([9, Theorem 1],[4, Theorem 2.11],[10, Theorems 2.5.3 and 2.5.7]) states that one can recover the group G as an affine variety together with the morphisms corresponding to multiplication and inverse (or equivalently, its coordinate ring and its structure as a Hopf algebra) from the knowledge of this category $\mathbf{Rep}_G$ and the forgetful functor from $\mathbf{Rep}_G$ to finite dimensional vector spaces.

In this work, we consider linear differential algebraic groups. These are groups of invertible matrices with entries in a differential field $\mathbf{k_0}$ with derivation ∂ that are, in addition, differential varieties, that is, they are defined by the vanishing of differential polynomials. A representation of such a group is a finite dimensional vector space V over $\mathbf{k_0}$ together with a differential polynomial morphism from G to $\mathbf{GL}(V)$.

The study of these groups and their representations was initiated by Cassidy in [1, 2]. In this work, we introduce differentiation on vector spaces over differential fields such that representations of G correspond to this construction. We then show that for a linear differential group G, the category of objects corresponding to its representations completely determines G as a differential variety together with its

morphisms for multiplication and inverse, that is, we show that one can recover its *differential* coordinate ring together with its Hopf algebra and *differential* structure.

We note that the categorical approach to representations of linear algebraic groups leads to the theory of Tannakian categories. This theory has found many uses and, in particular, one can develop the Galois theory of linear differential equations using this categorical approach. Recently, a theory of parameterized linear differential equations has been developed by Cassidy and Singer [3] where the Galois groups are linear differential algebraic groups. The category $\mathcal{V}$ defined in this work was motivated by a desire to give a similar categorical development of the Galois theory of parameterized differential equations.

Recall that in the non-differential case Tannaka's Theorem states that a linear algebraic group is determined by its category of representations. The problem of recognizing when a category is the category of representations of a linear algebraic group (or more generally, an affine group scheme) is attacked via the theory of neutral Tannakian categories (see [9, 4]). This theory allows one to detect the underlying presence of a linear algebraic group in various settings. For example, the Galois theory of linear differential equations can be developed in this context (see [4, 11]).

Building on the work of Cassidy [1, 2], we proved an analogue of Tannaka's Theorem for linear differential algebraic groups. Here, we develop the notion of a neutral differential Tannakian category and show that this plays the same role for linear differential algebraic groups that the theory of neutral Tannakian categories plays for linear algebraic groups. As an application, we are able to give a categorical development of the theory of parameterized linear differential equations that was introduced in [3].

The rest of the paper is organized as follow. Chapter 2 gives formal definitions and properties of linear differential algebraic groups. In Chapter 3, we introduce the the category $\mathcal{V}$ and show how a representation of a linear differential algebraic group corresponds to an object of $\mathcal{V}$. In Chapter 4, we give various representation

theoretic properties of the objects of $\mathcal{V}$ as well as some consequences (e.g., any representation can be constructed from a faithful representation using the operations of linear algebra). In Chapter 5, we show how to recover the group from its associated category.

In Chapter 7 we define and develop basic properties of a category which is a differential analogue of a neutral Tannakian category. In Section 7.7 we prove one of the main result of the thesis: namely, there is a (pro-)linear differential algebraic group such that a given neutral differential tannakian category is the category of representations of the group. We give an application of this result to the theory of parametrized linear differential equations in Chapter 8. The techniques we use are built on the techniques introduced in [9, 4] but the presence of the differential structure introduces new subtleties and several new constructions.

Chapter 2

Basic definitions

2.1 Differential algebra

Let $\mathbf{k_0}$ be a differential ∂-field. This means that $\mathbf{k_0}$ is a field with the operator ∂ satisfying for all a, $b \in \mathbf{k_0}$ we have

$$\partial(a+b) = \partial(a) + \partial(b), \quad \partial(ab) = \partial(a)b + a\partial(b).$$

For example, $\mathbb{Q}$ is a differential ring with a unique possible differentiation (which is the zero one). The field $\mathbb{C}(t)$ is also a differential ring with $\partial_t(t) = f$, and this f can be any rational function in $\mathbb{C}(t)$.

In Chapter 8 we require that every consistent system of algebraic differential equations with coefficient in $\mathbf{k_0}$ has a solution in $\mathbf{k_0}$. Such a field is called differentially closed. In characteristic zero for a differential field one can construct is differential algebraic closure unique upto an isomorphism (see [3, Definition 3.2]).

Let $Y = \{y_1, \ldots, y_n\}$ be a set of variables. We differentiate them:

$$\Delta Y := \left\{ \partial^{i_1} y_1 \partial^{i_2} y_2 \cdot \ldots \cdot \partial^{i_n} y_n \;\middle|\; (i_1, \ldots, i_n) \in \mathbb{Z}^n_{\geqslant 0} \right\}.$$

The ring of differential polynomials $\mathbf{k_0}\{Y\}$ in differential indeterminates Y is the ring of commutative polynomials $\mathbf{k_0}[\Delta Y]$ in infinitely many algebraically independent variables ΔY with the differentiation ∂.

We also use the ring of differential operators $\mathbf{k_0}[\partial]$ which is the ring of non-commutative polynomials in the variable ∂ with the following commutation rule:

$$\partial \cdot a = a \cdot \partial + \partial(a).$$

We denote the set of operators from this ring of order less than or equal to p by $\mathbf{k_0}[\partial]_{\leqslant p}$. A homomorphism of differential objects, such as differential rings, modules, is a usual homomorphism which commutes with differentiations.

The category of ∂-differential algebras over a ∂_t-differential field $\mathbf{k_0}$ is denoted by $\mathrm{Alg_k}(\partial)$. The differential algebra homomorphisms are denoted by simply Hom or $\mathrm{Hom}_{\mathbf{k_0}[\partial]-\mathrm{alg}}$. If we mean $\mathbf{k_0}[\partial]$-module homomorphisms then we write $\mathrm{Hom}_{\mathbf{k_0}[\partial]}$ or $\mathrm{Hom}_{\mathbf{k_0}[\partial]-\mathrm{lin}}$.

2.2 Linear differential algebraic groups

We shall recall some definitions and results from differential algebra (see for more detailed information [1, 7]) leading up to the "classical definition" of a linear differential algebraic group and its representative functions. Later in the paper we will give a Hopf-theoretic treatment and provide an equivalent definition in terms of representable functors.

Let $\mathbf{k_0} \subset \mathcal{U}$ be a universal differential field over $\mathbf{k_0}$, that is, a differential field such that if K is a differential field extension of $\mathbf{k_0}$, finitely generated in the differential sense, then there exists a $\mathbf{k_0}$-isomorphism of K into $\mathcal{U}$. We will assume that all differential fields we consider are subfields of $\mathcal{U}$ (the Hopf-theoretic treatment will *not* use universal differential extensions).

Definition 2.1 *For a differential field extension $K \supset \mathbf{k_0}$ a Kolchin closed subset $W(K)$ of K^n over $\mathbf{k_0}$ is a set of common zeros of a system of differential algebraic equations with coefficients in $\mathbf{k_0}$, that is, for $f_1, \ldots, f_k \in \mathbf{k_0}\{Y\}$ we define*

$$W(K) = \{a \in K^n \mid f_1(a) = \ldots = f_k(a) = 0\}.$$

There is a bijective correspondence between Kolchin closed subsets W of $\mathcal{U}^n$ defined over $\mathbf{k_0}$ and radical differential ideals $\mathbf{I}(W) \subset \mathbf{k_0}\{y_1, \ldots, y_n\}$ (that is, radical ideals that are stable under the action of ∂) generated by the differential polynomials $f_1, \ldots, f_k$ that define W. Given a Kolchin closed subset W of $\mathcal{U}^n$ defined over $\mathbf{k_0}$ we let the *coordinate ring* $\mathbf{k_0}\{W\}$ be:

$$\mathbf{k_0}\{W\} = k\{y_1, \ldots, y_n\} \big/ \mathbf{I}(W).$$

A *differential polynomial* map $\varphi : W_1 \to W_2$ between Kolchin closed subsets of $\mathcal{U}^n$, defined over $\mathbf{k_0}$, is given in coordinates by differential polynomials from $\mathbf{k_0}\{y_1, \ldots, y_n\}$. To give $\varphi : W_1 \to W_2$ is equivalent to defining $\varphi^* : \mathbf{k_0}\{W_2\} \to \mathbf{k_0}\{W_1\}$.

Definition 2.2 *[1, Chapter II, Section 1, page 905] A linear differential algebraic group is a Kolchin closed subgroup G of $\mathbf{GL}_n(\mathcal{U})$, that is, an intersection of a Kolchin closed subset of $\mathcal{U}^{n^2}$ with $\mathbf{GL}_n(\mathcal{U})$ which is closed under the group operations.*

Note that we identify $\mathbf{GL}_n(\mathcal{U})$ with a Zarisky closed subset of $\mathcal{U}^{n^2+1}$ given by

$$\left\{(A, a) \;\middle|\; (\det(A)) \cdot a - 1 = 0\right\}.$$

Definition 2.3 *[2] A differential polynomial group homomorphism $\rho : G \to \mathbf{GL}(V)$ is called a differential representation of a linear differential algebraic group G, where V is a finite dimensional vector space over $\mathbf{k_0}$. Such space is called a differential G-module.*

A Hopf algebra A is a commutative associative algebra together with comultiplication $\Delta : A \to A \otimes A$, coinverse $S : A \to A$, and counit $\varepsilon : A \to \mathbf{k_0}$ satisfying certain axioms [10, 2.1.2].

In [12, page 23] representations of an algebraic group G are viewed as comodules over the Hopf algebra A of regular functions on G. In Section 3 we will develop a similar technique to look at differential representations as *differential comodules* (Definition 3.9) over *differential* Hopf algebras (Definition 3.5).

Following [12, page 5] one interprets algebraic groups as representable functors from the category of $\mathbf{k_0}$-algebras to groups, that is, there must be a $\mathbf{k_0}$-algebra A such that for any $\mathbf{k_0}$-algebra B the B-points of G, denoted by $G(B)$, are just $\mathrm{Hom}(A, B)$. Then [12, Theorem, page 6] says that natural maps from one group G to another one G', viewed as functors, correspond to the algebra homomorphisms $A' \to A$. We will develop this in differential setting in Section 3.2.

2.3 Representative functions

Let $W \subset \mathcal{U}^n$ be a Kolchin closed subset defined over $\mathbf{k_0}$. We define $\mathbf{k_0}\langle W\rangle$ to be the complete ring of quotients of $\mathbf{k_0}\{W\}$, that is $\mathbf{k_0}\{W\}$ is localized with respect to the set of non-zero divisors.

A linear differential algebraic group G acts on the rational functions $\mathbf{k_0}\langle G\rangle$ by right translations (see [1, page 901], [2, page 227]). According to [2, page 227] the functions with a finite dimensional orbit are called *representative functions* $R(G)$ and ([2, page 230, Theorem]) that the totality of these functions form the algebra of functions $\mathbf{k_0}\{G\}$.

We show how these functions (and, hence, the algebra) can be connected with finite dimensional differential representations of G.

Proposition 2.4 *Representative functions are the same as the coordinate functions of finite dimensional representations of G.*

Proof. Let $\rho : G \to \mathbf{GL}(V)$. Let also φ_{ij} be a coordinate function. We have

$$G \xrightarrow{\ \rho\ } \mathbf{GL}(V) \xrightarrow{\ \varphi_{ij}\ } \mathbf{k_0}$$

According to [2, Corollary 1, page 231] we have $\varphi_{ij} \circ \rho \in \mathbf{k_0}\{G\}$. By [2, Theorem, page 230] we have $R(G) = \mathbf{k_0}\{G\}$. We show that any $f \in R(G)$ is of the form $\varphi_{ij} \circ \rho_f$ for a finite dimensional representation ρ_f of G.

Take any $f \in R(G)$ and consider its G-orbit $Gf =: V$, which is finite dimensional, under the action $r : G \to \mathbf{GL}(\mathbf{k_0}\{G\})$ with

$$\rho(g)(f)(x) = f(xg).$$

Let V be spanned by $\{f = f_1, \ldots, f_n\}$ over $\mathbf{k_0}$. So,

$$\rho(g)(f_1) = \sum_{i=1}^{n} c_i(g) f_i.$$

Evaluating the last equality at the point $e \in G$ for all $g \in G$ we get

$$f(g) = (\rho(g)(f))(e) = \sum_{i=1}^{n} c_i(g) \cdot f_i(e).$$

It remains to change the basis which correspond to the conjugation of the representation matrices. Assume that $f_1(e) \neq 0$. A conjugation matrix is

$$C = \begin{pmatrix} f_1(e) & f_2(e) & f_3(e) & \ldots & f_n(e) \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

If $B(g)$ is a representation matrix then it goes to $CB(g)C^{-1}$ under such a change of coordinates. Thus, we have obtained the function $f(g)$ as a coordinate function of some finite dimensional differential representation of the linear differential algebraic group G. ■

Chapter 3

The category $\mathcal{V}$ and representations

We start with introducing a derivative of a finite dimensional vector space and investigate its essential properties.

3.1 Definition

Definition 3.1 *The category $\mathcal{V}$ over a differential field $\mathbf{k_0}$ is the category of finite dimensional vector spaces over $\mathbf{k_0}$ with operations $\otimes$, $\oplus$, $*$, and additional operations:*

$$\partial^p : V \mapsto V^{(p)} := \mathbf{k_0}[\partial]_{\leqslant p} \otimes V.$$

Remark 3.2 *Note that $\mathbf{k_0}[\partial]$ is a non-commutative $\mathbf{k_0}$-algebra, so in this tensor product we think of $\mathbf{k_0}[\partial]$ as a right $\mathbf{k_0}$-space and V as a left $\mathbf{k_0}$-space.*

Remark 3.3 *The category $\mathcal{V}$ is still the category of vector spaces. We just add a new operation ∂ to it. Note that ∂ requires the ground field $\mathbf{k_0}$ to be differential and is essential to recover the differential structure.*

Remark 3.4 *If $\{v_1, \dots, v_n\}$ is a basis of V then $\{v_1, \dots, v_n, \dots, \partial^p \otimes v_1, \dots, \partial^p \otimes v_n\}$ is a basis of $V^{(p)}$.*

We denote $\partial \otimes v$ simply by ∂v.

3.2 Linear differential algebraic groups

In the following Section 3.3 we will introduce another definition of a differential representation of a linear differential algebraic group (see Definition 3.9). For this purpose we will view linear differential algebraic groups as *representable functors* from the category of differential $\mathbf{k_0}$-algebras to groups (see (3.2)).

We take a Hopf-theoretic approach to the study of linear differential algebraic groups as in [2]. Let A be a (finitely generated) differential algebra over the field $\mathbf{k_0}$. Following [2, page 226] one defines the set

$$G(\mathcal{U}) = \mathrm{Hom}(A, \mathcal{U}),$$

where $\mathcal{U}$ is the universal differential field as before. Assume that A is supplied with the following operations:

- differential algebra homomorphism $m : A \otimes A \to A$ is the multiplication map on A,
- differential algebra homomorphism $\Delta : A \to A \otimes A$ which is a comultiplication corresponding to the group multiplication $G \times G \to G$,
- differential algebra homomorphism $\varepsilon : A \to k$ which is a counit corresponding to $e \in G$,
- differential algebra homomorphism $S : A \to A$ which is a coinverse corresponding to the group inverse $G \to G$.

We also assume that these maps satisfy commutative diagrams (see [2, page 225]):

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \mathrm{id}_A \otimes \Delta} \\ A \otimes A & \xrightarrow{\Delta \otimes \mathrm{id}_A} & A \otimes A \otimes A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \downarrow{\scriptstyle \mathrm{id}_A} & & \downarrow{\scriptstyle \mathrm{id}_A \otimes \varepsilon} \\ A & \xrightarrow{\sim} & A \otimes k \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle m \circ (S \otimes \mathrm{id}_A)} \\ k & \xrightarrow{\hookrightarrow} & A \end{array} \tag{3.1}$$

Definition 3.5 *Such a commutative associative differential* $\mathbf{k_0}$*-algebra* A *with the unity and operations* m, Δ, S, *and* ε *satisfying axioms* (3.1) *is called a differential Hopf algebra.*

Remark 3.6 *Introduced in this way* $G(\mathcal{U})$ *with operations corresponding to* Δ, S, *and* ε *is a Kolchin-closed subset of some* $\mathcal{U}^n$ *and a group at the same time such that all operations are continuous in the Kolchin topology.*

Recall that the group G as a differential variety is given by a system of differential algebraic equations $F = 0$ or by the radical differential ideal I of $\mathbf{k_0}\{y_1, \ldots, y_n\}$ generated by F (see [1, page 895]). If we represent $0 \to I \to \mathbf{k_0}\{y_1, \ldots, y_n\} \to A \to 0$ then we can consider G in a differential $\mathbf{k_0}$-algebra K, that is,

$$G(K) = \mathrm{Hom}_{\mathbf{k_0}[\partial]}(A, K) =: \mathrm{Hom}(A, K).$$

We again may identify elements of $G(K)$ with n-tuples of elements of K that annihilate I. For convenience we gather all such $G(K)$ and form the following *representable* functor

$$G : \{\text{differential } \mathbf{k_0}\text{-algebras}\} \to \{\text{Groups}\}, \quad B \mapsto \mathrm{Hom}_{\mathbf{k_0}[\partial]}(A, B), \tag{3.2}$$

which we call the *affine differential algebraic group* defined by A.

Among all differential algebraic groups we distinguish the differential general linear group. Consider a finite dimensional vector space V over $\mathbf{k_0}$ of dimension m. We define $\mathbf{GL}(V)$ by the functor

$$\mathbf{GL}(V) : B \mapsto \mathrm{Hom}_{\mathbf{k_0}[\partial]}(\mathbf{k_0}\{X_{11}, \ldots, X_{mm}, 1/\det\}, B).$$

One may consider $\mathbf{GL}(V)(B)$ as the set of $m \times m$ invertible matrices with coefficients in the differential $\mathbf{k_0}$-algebra B. The linear differential algebraic groups we defined earlier correspond to subgroups of $\mathbf{GL}(V)$. Unlike the situation for algebraic groups, there are affine differential algebraic groups that are not isomorphic to linear differential algebraic groups (see [1, page 911]).

Morphisms of differential algebraic groups then in our sense are *morphisms of their representable functors*. We need the following result (a corollary of Yoneda's Lemma) from the theory of categories to see that this corresponds to the morphisms of algebras defining the groups.

Lemma 3.7 *[8, Corollary 2, page 44],[6, 30.7, Corollary, page 224] Let $\mathcal{C}$ be a category such that* $\mathrm{Hom}(A, B)$ *is a set for all objects A, B of $\mathcal{C}$. Let E and F be functors from some category $\mathcal{C}$ to the category of sets represented by some objects A and B, that is, $E = \mathrm{Hom}\,(A, \_)$ and $F = \mathrm{Hom}\,(B, \_)$. Then morphisms of functors E and F correspond to homomorphisms of B and A.*

It remains to note that the category of differential $\mathbf{k_0}$-algebras satisfies the assumptions of Lemma 3.7.

3.3 Representations

We will take a careful look at differential representations of a linear differential algebraic group G. These are differential algebraic group homomorphisms

$$\Phi : G \to \mathbf{GL}\,(V) =: \mathrm{Aut}\,(V)$$

for some finite dimensional $\mathbf{k_0}$-vector space V. By Lemma 3.7 the morphism Φ corresponds to the homomorphism of the algebras. In [2] a representation of the group G defined over $\mathbf{k_0}$ is a rational differential algebraic group homomorphism

$$G(\mathcal{U}) \to \mathbf{GL}\,(V)\,(\mathcal{U}).$$

But such a morphism is a differential polynomial map by [2, Corollary 1, page 231] and so corresponds to a homomorphism of associated algebras. Hence, we can freely use our language of functors.

Definition 3.8 *For a linear differential algebraic group G an object $V \in \mathcal{V}$ together with a natural map of functors $r_V : G \to \mathrm{Aut}(V)$ which is a group homomorphism is called a differential G-module.*

3.4 Differential comodules

We are going to restate this in the language of comodules which we introduce now. For this we define a differential analogue of an algebraic comodule. Let A be a differential Hopf algebra.

Definition 3.9 *A finite dimensional vector space V over $\mathbf{k_0}$ is called an A-differential comodule if there is a given $\mathbf{k_0}$-linear morphism*

$$\rho : V \to V \otimes A$$

satisfying the axioms:

$$\begin{array}{ccc} V & \xrightarrow{\rho} & V \otimes A \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \mathrm{id}_V \otimes \Delta} \\ V \otimes A & \xrightarrow{\rho \otimes \mathrm{id}_A} & V \otimes A \otimes A \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{\rho} & V \otimes A \\ \downarrow{\scriptstyle \mathrm{id}_V} & & \downarrow{\scriptstyle \mathrm{id}_V \otimes \varepsilon} \\ V & \xrightarrow{\sim} & V \otimes \mathbf{k_0} \end{array}$$

together with the prolongation of ρ on $V^{(i)}$ commuting with ∂.

The definition and correctness of the prolongation are given in Theorem 3.10 and Lemma 3.11. We will show that A-differential comodules are in one-to-one correspondence with differential G-modules, where G is the functor represented by A.

3.5 Equivalent definitions of differential representations

Theorem 3.10 *Let V be a finite dimensional vector space spanned by vectors $\{v_1, \ldots, v_n\}$ over $\mathbf{k_0}$. A differential representation $\Phi : G \to \mathbf{GL}(V)$ is equivalent to V being an*

$A := \mathbf{k_0}\{G\}$-*differential comodule with respect to*

$$\rho : V \to V \otimes A.$$

In coordinates we have

$$\rho(v_j) = \sum_{i=1}^{n} v_i \otimes a_{ij}, \quad \Delta(a_{ij}) = \sum_{r=1}^{n} a_{ir} \otimes a_{rj}.$$

such that

$$\rho(\partial^p v_j) = \sum_{i=1}^{n} \sum_{q=0}^{p} \binom{p}{q} \partial^q v_i \otimes (\partial^{p-q} a_{ij})$$

gives a prolongation of ρ on $V^{(p)}$.

Proof. The representation Φ is a morphism of functors $G \to \mathrm{Aut}(V)$. According to [12, Theorem, Section 3.2] such a representation Φ of the affine group scheme G corresponds to the $\mathbf{k_0}$-linear map ρ with the following commuting diagrams:

$$\begin{array}{ccc} V & \xrightarrow{\rho} & V \otimes A \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \mathrm{id}_V \otimes \Delta} \\ V \otimes A & \xrightarrow{\rho \otimes \mathrm{id}_A} & V \otimes A \otimes A \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{\rho} & V \otimes A \\ \downarrow{\scriptstyle \mathrm{id}_V} & & \downarrow{\scriptstyle \mathrm{id}_V \otimes \varepsilon} \\ V & \xrightarrow{\sim} & V \otimes \mathbf{k_0} \end{array}$$

More precisely, the map ρ comes from the restriction of the A-linear map

$$\Phi(\mathrm{id}_A) : V \otimes A \to V \otimes A$$

to $V \otimes \mathbf{k_0} \cong V$. By [12, Corollary, Section 3.2] we have

$$\rho(v_j) = \sum_{i=1}^{n} v_i \otimes a_{ij}, \quad \Delta(a_{ij}) = \sum_{r=1}^{n} a_{ir} \otimes a_{rj}.$$

Let us demonstrate the last differential identity. We have

$$\Phi(\mathrm{id}_A) \in \mathrm{Hom}_A(V \otimes A, V \otimes A)$$

and $\Phi(\mathrm{id}_A)$ can be extended to a map $V^{(p)} \otimes A \to V^{(p)} \otimes A$ commuting with the ∂-structure. The first step gives us the following:

$$\begin{array}{ccc} v_j \otimes 1 & \xrightarrow{\partial_{V \otimes A}} & \partial v_j \otimes 1 \\ \downarrow{\scriptstyle \Phi(\mathrm{id}_A)=\rho} & & \downarrow{\scriptstyle \Phi(\mathrm{id}_A)=\rho} \\ \sum\limits_{i=1}^{n} v_i \otimes a_{ij} & \xrightarrow{\partial_{V \otimes A}} & \sum\limits_{i=1}^{n} ((\partial v_i) \otimes a_{ij} + v_i \otimes \partial a_{ij}) \end{array}$$

The formula for higher order derivatives can be obtained by induction. This, indeed, makes $V^{(p)}$ an A-comodule (see Lemma 3.11).

On the other hand, having such a $\rho : V \to V \otimes A$ one extends it by the A-linearity to $\rho_A : V \otimes A \to V \otimes A$ and then to $V^{(p)} \otimes A \to V^{(p)} \otimes A$ commuting with the ∂-structure (see Lemma 3.11 for correctness). Consider a ∂-differential $\mathbf{k_0}$-algebra B. For any $g \in \mathrm{Hom}_{\mathbf{k_0}[\partial]}(A, B)$ in $G(B)$ we have the following commutative diagram:

$$\begin{array}{ccc} V \otimes A & \xrightarrow{\Phi(\mathrm{id}_A)} & V \otimes A \\ \downarrow{\scriptstyle \mathrm{id}_V \otimes g} & & \downarrow{\scriptstyle \mathrm{id}_V \otimes g} \\ V \otimes B & \xrightarrow{\Phi(g)} & V \otimes B \end{array}$$

meaning that $\Phi(g)$ is determined by $(\mathrm{id}_V \otimes g) \circ \rho$ using B-linearity of $\Phi(g)$, where ρ is the restriction of $\rho_A := \Phi(\mathrm{id}_A)$ to V. Since $\Phi(\mathrm{id}_A)$ is made preserving the ∂-structure, the map $\Phi(g)$ does the same thing, since g is a ∂-differential algebras homomorphism $A \to B$.

Indeed, take $v = \sum c_i \partial^i v_i$, where $v_i \in V$. Let $\{e_1, \ldots, e_n\}$ be a basis of V. Then

$$\begin{aligned} \Phi(g)(v) &= (\mathrm{id}_V \otimes g) \circ \Phi(\mathrm{id}_A) \left(\sum c_i \partial^i v_i \right) = \sum c_i \cdot (\mathrm{id}_V \otimes g) \partial^i (\Phi(\mathrm{id}_A)(v_i)) = \\ &= \sum c_i \cdot (\mathrm{id}_V \otimes g) \partial^i \left(\sum_{j=1}^{n} e_j \otimes b_{ji} \right) = \\ &= \sum c_i \cdot (\mathrm{id}_V \otimes g) \left(\sum_{j=1}^{n} \sum_{r=0}^{i} \binom{i}{r} \partial^r e_j \otimes \partial^{i-r} b_{ji} \right) = \\ &= \sum c_i \left(\sum_{j=1}^{n} \sum_{r=0}^{i} \binom{i}{r} \partial^r e_j \otimes \partial^{i-r} g(b_{ji}) \right) = \\ &= \sum c_i \partial^i \left(\sum_{j=1}^{n} e_j \otimes g(b_{ji}) \right) = \sum c_i \partial^i (\Phi(g)(v_i)). \end{aligned}$$

Here we denoted $\Phi(\mathrm{id}_A)(v_i) = \sum_j e_j \otimes b_{ji}$ for some elements $b_{ij} \in A$. From this it follows that

$$\Phi(g) \in \mathrm{Hom}_{\mathbf{k_0}[\partial]} \left(\mathbf{k_0} \left\{ X_{11}, \ldots, X_{nn}, 1/\det \right\}, B \right).$$

This finally establishes a bijection between differential representations and differential comodules. ■

3.6 Prolongation of representations

Let G be a linear differential algebraic group and $\rho : G \to \mathbf{GL}(V)$ be its differential representation.

Lemma 3.11 *The action of G on each $V^{(i)}$ is algebraic, that is, $V^{(i)}$ is a $B := \mathbf{k_0}\{G\}$-comodule for all $i \geqslant 0$.*

Proof. We have a differential algebraic action on V. Let $\{e_1, \ldots, e_n\}$ be a $\mathbf{k_0}$-basis of V. For $\rho : V \to V \otimes B$ from Theorem 3.10 we have

$$\begin{aligned}
\rho(e_j) &= \sum_{i=1}^{n} e_i \otimes b_{i,j}, \\
\Delta(b_{i,j}) &= \sum_{r=1}^{n} b_{i,r} \otimes b_{r,j}, \\
\Delta(\partial^p b_{i,j}) = \partial^p(\Delta(b_{i,j})) &= \sum_{r=1}^{n} \sum_{q=0}^{p} \binom{p}{q} (\partial^q b_{i,r}) \otimes (\partial^{p-q} b_{r,j}), \\
\rho(\partial^p e_j) = \partial^p \rho(e_j) &= \sum_{i=1}^{n} \sum_{q=0}^{p} \binom{p}{q} (\partial^q e_i) \otimes (\partial^{p-q} b_{i,j}).
\end{aligned}$$

One can show by induction that if $B = (b_{i,j})_{i,j=1}^{n}$ is the "representation" matrix for $G \to \mathbf{GL}(V)$ then for a fixed number $i \geqslant 0$ we have

$$B_i := \begin{pmatrix}
B & 0 & 0 & \ldots & 0 \\
\binom{i}{1} \cdot B_t & B & 0 & \ldots & 0 \\
\binom{i}{2} \cdot B_{tt} & \binom{i-1}{1} \cdot B_t & B & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
B_{t^i} & \ldots & \ldots & \ldots & B
\end{pmatrix} = (c_{r,s})_{r,s=1}^{i \cdot n}$$

is the one for $G \to \mathbf{GL}\left(V^{(i)}\right)$, where B_{t^k} means the matrix $\left(\partial^k b_{i,j}\right)_{i,j=1}^{n}$.

It remains to show we do have a group action. Let us do this. The ordered basis of $V^{(i)}$ is

$$\{\partial^i e_1, \ldots, \partial^i e_n, \partial^{i-1} e_1, \ldots, \partial^{i-1} e_n, \ldots, e_1, \ldots, e_n\}.$$

By induction, using the fact that B_i is lower triangular we conclude that the only part of matrix we need to take care of is the set of first n columns. Let $1 \leqslant q \leqslant n$ and $n \cdot m + 1 \leqslant p \leqslant n \cdot (m+1)$. We have:

$$\begin{aligned}
\Delta(c_{p,q}) &= \Delta\left(\binom{i}{m}\partial^m b_{p-m\cdot n,q}\right) = \binom{i}{m}(\partial^m \Delta(b_{p-m\cdot n,q})) = \\
&= \binom{i}{m}\partial^m\left(\sum_{l=1}^{n} b_{p-m\cdot n,l}\otimes b_{l,q}\right) = \\
&= \sum_{l=1}^{n}\sum_{r=0}^{m}\binom{i}{m}\binom{m}{r}(\partial^r b_{p-m\cdot n,l})\otimes(\partial^{m-r} b_{l,q}) = \\
&= \sum_{l=1}^{n}\sum_{r=0}^{m}\binom{i-r}{m-r}\binom{i}{r}(\partial^r b_{p-m\cdot n,l})\otimes(\partial^{m-r} b_{l,q}),
\end{aligned}$$

because

$$\binom{i}{m}\binom{m}{r} = \frac{i!m!}{m!(i-m)!r!(m-r)!} = \frac{(i-r)!i!}{(m-r)!(i-m)!r!(i-r)!}$$

and this is exactly what we needed. ■

Chapter 4

Essential properties of differential representations

In the following we will use different equivalent definitions of differential representations of a linear differential algebraic group. Summarizing the previous sections a differential representation $V \in \mathcal{V}$ of G with algebra A can be defined in the following ways:

- by a differential morphism $G(\mathcal{U}) \to \mathbf{GL}(V)(\mathcal{U})$ (Section 3.3);
- by a natural map of functors $G \to \mathrm{Aut}(V)$ (Definition 3.8);
- by a differential A-comodule structure on V (Definition 3.9, Theorem 3.10).

Differential representations of G form a category which we denote by $\mathbf{Rep}_G$ and its objects by $\mathcal{O}b(\mathbf{Rep}_G)$.

4.1 Recovering representations

Let G be a differential algebraic group with $A := \mathbf{k_0}\{G\}$ and $G \to \mathrm{Aut}(V)$ be its faithful representation, $V \in \mathcal{V}$.

Lemma 4.1 *Every finite dimensional differential representation U of G embeds in a finite sum of copies of the regular representation.*

Proof. Denote $M = U \otimes A$. Then M is a differential comodule with

$$\mathrm{id}_U \otimes \Delta : M \to M \otimes A.$$

Since $(\mathrm{id} \otimes \Delta) \circ \rho = (\rho \otimes \mathrm{id}) \circ \rho$, the map $\rho : U \to M$ is a map of A-comodules. It is injective, because $v = (\mathrm{id} \otimes \varepsilon) \circ \rho(v)$. Finally, $M \cong A^{\dim(U)}$. ■

Proposition 4.2 *Every differential representation U of G is a subquotient of several copies of a G-module*

$$V^{(i_1)} \otimes \ldots \otimes V^{(i_k)} \otimes V^* \otimes \ldots \otimes V^*.$$

Proof. Fix a basis $\{u_1, \ldots, u_m\}$ of U. By Lemma 4.1 the representation L is an $A := \mathbf{k_0}\{G\}$-subcomodule of

$$U \otimes A = (u_1 \otimes A) \oplus \ldots \oplus (u_m \otimes A) \cong A^m.$$

Consider the canonical projections $\pi_i : A^m \to A$, which are G-equivariant maps with respect to the comultiplication $\Delta : A \to A \otimes A$. Since $U \subset A^m$, we have

$$U \subset \bigoplus_{i=1}^{m} \pi_i(U)$$

and each $\pi_i(U)$ is a G-module, because π_i is G-equivariant.

Consider the following surjection

$$\pi : B := \mathbf{k_0}\{X_{11}, \ldots, X_{nn}, 1/\det\} \to A \to 0.$$

Since $\pi_i(U)$ is a finite dimensional G-subspace of A, there exist numbers $r, s, p \in \mathbb{Z}_{\geqslant 0}$ such that $\pi_i(U)$ is contained in $\pi(L_{r,s,p})$, where

$$L_{r,s,p} := (1/\det)^r \{f(X_{ij}) \mid \deg(f) \leqslant s, \mathrm{ord}(f) \leqslant p\}.$$

There is a B-comodule structure on B given by

$$\Delta(X_{ij}) = \sum_{l=1}^{n} X_{il} \otimes X_{lj},$$
$$\Delta(\partial X_{ij}) = \sum_{l=1}^{n} ((\partial X_{il}) \otimes X_{lj} + X_{il} \otimes (\partial X_{lj}))$$

and $L_{r,s,p}$ is a B-subcomodule of B, because of

$$\Delta(X_{ij}X_{pq}) = \sum_{l,r=1}^{n} X_{il}X_{pr} \otimes X_{lj}X_{rq}$$

and Lemma 3.11. We then have that $L_{r,s,p}$ is also an $A-$subcomodule of B. Hence, each $\pi_i(L)$ is a subquotient of some $L_{r,s,p}$. Thus, we only need to show how to construct these $L_{r,s,p}$.

Fix a basis $\{v_1, \ldots, v_n\}$ of V. We have a $B-$comodule V with respect to

$$\rho(v_j) = \sum_{i=1}^{n} v_i \otimes X_{ij}.$$

For each i, $1 \leqslant i \leqslant n$, the map $\varphi_i : v_j \mapsto X_{ij}$ is $\mathbf{GL}_n$ (hence, G)-equivariant, because

$$\varphi_i(\rho_V(v_j)) = \varphi_i\left(\sum_{l=1}^{n} v_l \otimes X_{lj}\right) = \sum_{l=1}^{n} X_{il} \otimes X_{lj} = \Delta(X_{ij}) = \rho_B(\varphi_i(v_j))$$

and both ρ and Δ preserve the product rule with respect to ∂.

Consider the space of linear polynomials $L_{0,1,p}$ in the variables $\{X_{ij}\}$ and their derivatives of order up to p. An element f of such a space is of the form

$$f = \sum_{i,j=1}^{n} \sum_{q=0}^{p} c_{ij} X_{ij}^{(q)},$$

where $c_{ij} \in \mathbf{k_0}$. As it has been noticed above this space is an A-subcomodule of B. The map $(\varphi_1, \ldots, \varphi_n)$ gives an A-comodule isomorphisms between the nth power $\left(V^{(p)}\right)^n$ of the pth derivative of the original representation of G and $L_{0,1,p}$. Hence, one can construct $L_{0,1,p}$.

Consider any $s \in \mathbb{Z}_{\geqslant 2}$. The G-space $L_{0,s,p}$ is the quotient of $(L_{0,1,p})^{\otimes s}$ by the symmetric relations. So, we have all $L_{0,s,p}$. Let now $s = n = \dim_{\mathbf{k_0}} V$. Then the one-dimensional representation $\det : G \to \mathbf{k_0}$ with $g \mapsto \det(g)$ is in $L_{0,n,p}$. For $f \in \mathbf{k_0}^*$ we have

$$\det(g)(f)(x) = f(x/\det(g)) = \frac{1}{\det(g)} f(x).$$

Thus,

$$L_{r,s,p} = (\det{}^*)^{\otimes r} \otimes L_{0,s,p},$$

which is what we wanted to construct. ■

4.2 Example

We will show how Proposition 4.2 works step by step.

Example 4.3 *Consider the differential representation* $\rho : \mathbf{G_m} \to \mathbf{G_a}$ *by*

$$\mathbf{G_m} \ni y \mapsto \begin{pmatrix} 1 & \frac{\partial y}{y} \\ 0 & 1 \end{pmatrix} \in \mathbf{G_a}\,.$$

The representation ρ *corresponds to the map of algebras*

$$\rho^* : B := \mathbf{k_0}\{X_{11}, X_{12}, X_{21}, X_{22}, 1/\det\} \to A := \mathbf{k_0}\{y, 1/y\} \to 0$$

with

$$X_{11} \mapsto 1, X_{12} \mapsto y'/y,$$
$$X_{21} \mapsto 0, X_{22} \mapsto 1.$$

Take a standard basis $\{u_1, u_2\}$ *of* $\mathbf{k_0}^2$. *We then have* $\rho : U \to U \otimes A$ *given by*

$$\rho(u_1) = u_1 \otimes 1,$$
$$\rho(u_2) = u_1 \otimes (y'/y) + u_2 \otimes 1.$$

So, as an A-comodule

$$\mathbf{k_0}^2 \subset \mathrm{span}_{\mathbf{k_0}}\{u_1 \otimes 1, u_1 \otimes (y'/y)\} \oplus \mathrm{span}_{\mathbf{k_0}}\{\{u_2 \otimes 1\}\,.$$

Hence, it is sufficient to construct $\mathrm{span}_{\mathbf{k_0}}\{u_1 \otimes 1, u_1 \otimes (y'/y)\}$ *or, equivalently,*

$$W := \mathrm{span}_{\mathbf{k_0}}\left\{1, \frac{y'}{y}\right\}.$$

Consider the B-subcomodule $L_{0,1,0}$ *of linear polynomials in* $X_{11}, X_{12}, X_{21}, X_{22}$ *with coefficients in* $\mathbf{k_0}$ *which is also an A-subcomodule of B. The A-comodule W is contained in the image of* $L_{0,1,1}$ *with respect to* ρ^**. Hence, W is a subquotient of* $L_{0,1,1}$*. The A-comodule* $L_{0,1,1}$ *is constructed as follows. It is enough to get* $L_{0,1,0}$*, as* $L_{0,1,1} = \partial L_{0,1,0}$*.*

The group $\mathbf{G_m}$ *has a representation on* $V = \mathrm{span}_{\mathbf{k_0}}\{v_1, v_2\}$ *as*

$$v_1 \mapsto v_1 \otimes y,$$
$$v_2 \mapsto v_2 \otimes y.$$

Consider the two A-comodule maps

$$\varphi_1 : v_1 \mapsto X_{11},\ v_2 \mapsto X_{12},$$
$$\varphi_2 : v_1 \mapsto X_{21},\ v_2 \mapsto X_{22}.$$

The map (φ_1, φ_2) *provides an A-comodule isomorphism between* V^2 *and* $L_{0,1,0}$*. Summarizing, we need to take the 4th power of the original faithful representation of* $\mathbf{G_m}$ *on* $\mathbf{k_0}$*, compute several subquotients, and then sum up* ($\oplus$) *the result, take a subrepresentation, and differentiate to obtain the representation* ρ*.*

Chapter 5

Tannaka's theorem for linear differential algebraic groups

In this section we will show how one can recover a linear differential algebraic group knowing all its representations. For this we first prove Tannaka's theorem (Theorem 5.2). Then using this fact we reconstruct the differential Hopf algebra of functions on the group in Sections 5.3 and 5.4. Parts of the proof closely follow [10, Section 2.5]. The novelty lies in the fact that we can recover the differential structure on the Hopf algebra.

5.1 Preliminaries

Let G be a linear differential algebraic group with $A := \mathbf{k_0}\{G\}$. The category of finite dimensional differential representations (as in Definition 3.9) of G we denote by $\mathbf{Rep}_G$. Let $\omega : \mathbf{Rep}_G \to \mathcal{V}$ be the forgetful functor.

Definition 5.1 *For a ∂-differential $\mathbf{k_0}$-algebra B we define a group* $\mathrm{Aut}^{\otimes,\partial}(\omega)(B)$ *to be the set of sequences*

$$\lambda(B) = (\lambda_X \mid X \in \mathcal{O}b(\mathbf{Rep}_G)) \in \mathrm{Aut}^{\otimes,\partial}(\omega)(B)$$

such that λ_X *is a* B*-linear automorphism of* $\omega(X)\otimes B$ *for each* G*-space* X, $\omega(X) \in \mathcal{V}$, *that is,* $\lambda_X \in \mathrm{Aut}_B(\omega(X) \otimes B)$, *such that*

- *for all* X_1, X_2 *we have*

$$\lambda_{X_1 \otimes X_2} = \lambda_{X_1} \otimes \lambda_{X_2}, \tag{5.1}$$

- $\lambda_{\underline{1}}$ *is the identity map on* $\underline{1} \otimes B = B$,

- *for every* $\alpha \in \mathrm{Hom}_G(X, Y)$ *we have*

$$\lambda_Y \circ (\alpha \otimes \mathrm{id}_B) = (\alpha \otimes \mathrm{id}_B) \circ \lambda_X : X \otimes B \to Y \otimes B, \tag{5.2}$$

- *for every* X *we have*

$$\partial \circ \lambda_X = \lambda_{\partial X} \circ \partial, \tag{5.3}$$

- *the group operation* $\lambda_1(B) \cdot \lambda_2(B)$ *is defined by composition in each* $\mathrm{Aut}_B(\omega(X) \otimes B)$.

$\mathrm{Aut}^{\otimes,\partial}(\omega)$ is a functor from the category of ∂-differential $\mathbf{k_0}$-algebras to groups: $B \mapsto \mathrm{Aut}^{\otimes,\partial}(\omega)(B)$. Any $g \in G(B)$ determines an element $\lambda_g \in \mathrm{Aut}^{\otimes,\partial}(\omega)(B)$, because if $X \in \mathcal{O}b(\mathbf{Rep}_G)$ and $\Phi_B : G(B) \to \mathbf{GL}(\omega(X) \otimes B)$ then $\Phi(g)$ is a B-linear automorphism of $\omega(X) \otimes B$ and the property (5.2) is satisfied by the definition of G-equivariance of α. So, we have a morphism Φ of functors

$$G \to \mathrm{Aut}^{\otimes,\partial}(\omega), \quad g \in G(B) \mapsto \Phi(g) \in \mathrm{Aut}^{\otimes,\partial}(\omega)(B)$$

for any ∂-differential $\mathbf{k_0}$-algebra B as for any $\varphi : B_1 \to B_2$ we have the following commutative diagram:

$$\begin{array}{ccc} G(B_1) & \xrightarrow{\Phi_{B_1}} & \mathrm{Aut}^{\otimes,\partial}(\omega)(B_1) \\ \downarrow{\scriptstyle G(\varphi)} & & \downarrow{\scriptstyle \mathrm{Aut}^{\otimes,\partial}(\omega)(\varphi)} \\ G(B_2) & \xrightarrow{\Phi_{B_2}} & \mathrm{Aut}^{\otimes,\partial}(\omega)(B_2) \end{array}$$

where $G(\varphi)$ and $\mathrm{Aut}^{\otimes,\partial}(\omega)(\varphi)$ denote the morphisms

$$[\psi : A \to B_1] \mapsto [G(\varphi)(\psi) = \varphi \circ \psi : A \to B_2];$$

$$[\lambda(B_1) = (\lambda_X(B_1) : \omega(X) \otimes B_1 \to \omega(X) \otimes B_1)] \mapsto$$
$$\left[\mathrm{Aut}^{\otimes,\partial}(\omega)(\varphi)(\lambda(B_1)) = \lambda(B_2) = \left(\left(\mathrm{id}_{\omega(X)} \otimes \varphi\right) \circ \lambda_X(B_1) : \omega(X) \otimes B_2 \to \omega(X) \otimes B_2\right)\right],$$

respectively. The latter means that we take the restriction of $\lambda_X(B_1)$ to $\omega(X)$ and map it to $\omega(X) \otimes B_1$ and then apply $\mathrm{id}_{\omega(X)} \otimes \varphi$ mapping it to $\omega(X) \otimes B_2$. At the end we prolong such a map to $\omega(X) \otimes B_2$ by B_2-linearity.

5.2 The theorem

Theorem 5.2 *For a linear differential algebraic group G let $\omega : \mathbf{Rep}_G \to \mathcal{V}$ be the forgetful functor. Then*

$$G \cong \mathrm{Aut}^{\otimes,\partial}(\omega)$$

as functors.

Proof. (Following [10, Theorem 2.5.3] with *differential* modification) Since $g \in G$ determines an element of $\mathrm{Aut}^{\otimes,\partial}(\omega)$ we only need to show the converse. Let B be a ∂-differential $\mathbf{k_0}$-algebra and $(\lambda_X) \in \mathrm{Aut}^{\otimes,\partial}(\omega)(B)$. Let $V \in \mathcal{O}b(\mathbf{Rep}_G)$ not necessarily finite dimensional but locally finite. This means that every vector $v \in V$ is contained in a differential G-module $W \in \mathcal{V}$, $\dim_{\mathbf{k_0}} W < \infty$.

We will show that for given B, λ, and a locally finite differential G-module V, there exists a B-linear automorphism of $V \otimes B$ (denoted by λ_V) such that the properties (5.1), (5.2), and (5.3) are satisfied and for any $\mathcal{V} \ni W \subset V$ we have $\lambda_V|_W = \lambda_W$. For $v \in V$ we take $W \in \mathcal{O}b(\mathbf{Rep}_G)$ such that $v \in W$ and $\dim W < \infty$.

First, define $\lambda_V(v) := \lambda_W(v)$. We need to show the correctness. Let W' be another representation such that $v \in W'$. Consider the G-module $W \cap W' \ni Gv$. From (5.2)

it follows that $\lambda_{W'}(v) = \lambda_{W\cap W'}(v) = \lambda_W(v)$. Since each λ_W is invertible and linear, the map λ_V is also invertible and linear.

We are going to show now that (5.1), (5.2), and (5.3) hold for locally finite modules. For locally finite V and V' we choose $W \ni v$ and $W' \ni v'$, objects of $\mathcal{V}$. Then

$$\lambda_{V\otimes V'}(v \otimes v') = \lambda_{W\otimes W'}(v \otimes v') = \lambda_W(v) \otimes \lambda_{W'}(v') = \lambda_V(v) \otimes \lambda_{V'}(v').$$

Hence, the property (5.1) is satisfied. Also, let $\alpha \in \mathrm{Hom}_G(V, V')$. Consider $v \in V$ and W such that $v \in W$ and $\dim W < \infty$ and of the same kind $W' \supset \alpha(W)$. We obtain that

$$\lambda_{V'} \circ \alpha(v) = \lambda_{W'}\left(\alpha\big|_W(v)\right) = \alpha\big|_W \circ \lambda_W(v) = \alpha \circ \lambda_V(v).$$

We then have (5.2). Moreover, since $\partial W \subset \partial V$, we have

$$\partial \circ \lambda_V(v) = \partial \circ \lambda_W(v) = \lambda_{\partial W}(\partial v) = \lambda_{\partial V}(\partial v),$$

which implies (5.3). Thus, we may say $\lambda_V \in \mathrm{Aut}_B(V \otimes B)$.

The G-module A is locally finite via

$$\rho : A \times G \to A, \quad (\rho(g)f)(x) = f(x \cdot g)$$

by [2, Theorem, page 230], where $x, g \in G(B)$ and $f \in A = \mathbf{k_0}\{G\}$. The same is true for $A \otimes A$. Consider the multiplication map

$$m : A \otimes A \to A, \quad f \otimes h \mapsto f \cdot h$$

which is G-equivariant, because each $g \in G(B)$ is a (B-linear) algebra automorphism of $A \otimes B$. According to (5.1) and (5.2) we have

$$m \circ (\lambda_A \otimes \lambda_A) = m \circ \lambda_{A\otimes A} = \lambda_A \circ m.$$

Moreover, from (5.3) we conclude that λ_A is an ∂_t-*differential* algebraic automorphism $A \to A$. We will show that this must correspond to an element in the group. More

precisely, there exists a differential algebraic automorphism $\varphi : G(B) \to G(B)$ such that for all $f \in A$ and $g \in G(B)$ we have

$$\lambda_A(f)(g) = f(\varphi(g)).$$

We will show that this morphism φ is right multiplication by an element of $G(B)$. This will show that an element of the group corresponds to λ_A. After that we demonstrate that the algebra A can be replaced by any G-module.

For every $f \in A$ and g, $h \in G(B)$ we have $\Delta(f)(g,h) = f(gh)$. Take any $f \in A$ and $g, x, y \in G(B)$. We have

$$\Delta \circ \rho(g)(f)(x,y) = f(xyg) = (\mathrm{id}_A \otimes \rho(g)) \circ \Delta(f)(x,y).$$

Hence,

$$\Delta \circ \rho(g) = (\mathrm{id}_A \otimes \rho(g)) \circ \Delta. \tag{5.4}$$

Consider the locally finite G-module $U := A \otimes A$ via $r_U = \mathrm{id}_A \otimes \rho$. Then, by (5.4) the map Δ is G-equivariant for ρ and r_U. From (5.2) we have $\Delta \circ \lambda_A = \lambda_U \circ \Delta$. Because of (5.1) we obtain that

$$\lambda_U = \lambda_{A,\mathrm{id}_A} \otimes \lambda_{A,\rho} = \mathrm{id} \otimes \lambda_{A,\rho},$$

because $\lambda_I = \mathrm{id}$ and λ is $\mathbf{k_0}$-linear. Thus,

$$\Delta \circ \lambda_A = (\mathrm{id} \otimes \lambda_A) \circ \Delta.$$

For any $f \in A$ and g, $h \in G$ we have $\Delta \circ \lambda_A(f)(g,h) = f(\varphi(gh))$. On the other hand,

$$(\mathrm{id} \otimes \lambda_A) \circ \Delta(f)(g,h) = f(g\varphi(h)).$$

Thus,

$$\varphi(gh) = g\varphi(h).$$

Let

$$x = \varphi(e),$$

which is a *differential* algebra homomorphism $A \to B$. From this we conclude that for any $g \in G(B)$ one has $\varphi(g) = gx$. Hence, $\lambda_A = \rho(x)$. It remains to show that other automorphisms λ_V look the same (completely determined by this element x).

Consider any $V \in \mathcal{O}b(\mathbf{Rep}_G)$ with the action r_V. For any $u \in V^*$ there is a G-homomorphism

$$\varphi_u : V \to A, \quad v \mapsto \varphi_u(v), \quad \varphi_u(v)(g) := u(\rho(g) \cdot v),$$

where $v \in V$. By (5.2) and the above, we have $\rho(x) \circ \varphi_u = \lambda_A \circ \varphi_u = \varphi_u \circ \lambda_V$. Take any $g \in G(B)$ and $v \in V$. We then have

$$\rho(x)(u(r_V(g)(v))) = u(r_V(gx)(v)),$$
$$\varphi_u \circ \lambda_V(v)(g) = u(r_V(g) \circ \lambda_V(v)).$$

Thus,

$$r_V(x) = r_V(ex) = r_V(e) \circ \lambda_V = \lambda_V,$$

because the elements v, g, and u were arbitrary. ■

5.3 Recovering the algebra of regular functions on G

Similar to [10, Sections 2.5.4–2.5.8] we can recover $A = \mathbf{k_0}\{G\}$ in the following way. In addition, we show how to obtain the *differential structure* on A (see Lemma 5.6).

5.3.1 First step

We will construct the map ψ_V from "some representations" of G to the algebra A of regular differential functions on G and study main properties of ψ_V.

Recall that for $V \in \mathcal{V}$ we denote $V^{(i)} = \mathbf{k_0}[\partial]_{\leqslant i} \otimes V$ (non-commutative tensor product) and sometimes we write $V^{(0)}$ instead of V for convenience. For $V \in \mathcal{O}b(\mathbf{Rep}_G)$

and

$$v \in V,\ u \in V^*$$

we have the linear map

$$\psi_V :\ V \otimes V^* \to A,$$
$$\psi_V(v \otimes u)(g) = u(r_V(g) \cdot v).$$

Also we introduce an imbedding

$$F : V^* \to \left(V^{(1)}\right)^*, \quad F(u)(v) = u(v),\ F(u)(\partial \otimes v) = \partial(u(v)),\ v \in V.$$

Lemma 5.3 *We have the following properties:*

1. *If* $\phi \in \mathrm{Hom}_G(V, W)$ *then*

$$\psi_V \circ (\mathrm{id} \otimes \phi^*) = \psi_W \circ (\phi \otimes \mathrm{id})$$

as maps of $V \otimes W^* \to A$.

2. *We have*

$$\psi_{V \otimes W} = m \circ (\psi_V \otimes \psi_W) \circ c,$$

where $c : (V \otimes W) \otimes (V \otimes W)^* \cong (V \otimes V^*) \otimes (W \otimes W^*)$.

3. *Moreover,*

$$\partial(\psi_V(v \otimes u)) = \psi_{V^{(1)}}\left((\partial v) \otimes u\right).$$

4. *In addition,*

$$\psi_V(v \otimes w^*) = \psi_{V^{(1)}}(\partial v \otimes (\partial w)^*)$$

for all $v,\ w \in V$

5. *Finally, if for some* $l,\ 1 \leqslant l \leqslant p$, *we have* $i_l < j_l$ *then*

$$\psi_{V_1^{(q_1)} \otimes \ldots V_p^{(q_p)}}\left(\left(\partial^{i_1} v_1 \otimes \ldots \otimes \partial^{i_p} v_p\right) \otimes \left((\partial^{j_1} w_1)^* \otimes \ldots \otimes (\partial^{j_p} w_p)^*\right)\right) = 0,$$

where $v_k,\ w_k \in V_k$ *and* $\max\{i_k, j_k\} \leqslant q_k$.

Proof. For $v \in V$, $u \in V^*$, and $g \in G$ we have

$$\begin{aligned}\psi_V \circ (\mathrm{id} \otimes \phi^*)(v \otimes u)(g) &= \psi_V(v \otimes \phi^*(u))(g) = \phi^*(u)(r_V(g) \cdot v) = \\ &= u(\phi(r_V(g) \cdot v)) = u(r_W(g) \cdot \phi(v)) = \\ &= \psi_W \circ (\phi \otimes \mathrm{id})(v \otimes u)(g),\end{aligned}$$

Furthermore, consider $w \in W$ and $t \in W^*$. We then also have

$$\begin{aligned}\psi_{V\otimes W}(v \otimes w \otimes u \otimes t)(g) &= (u \otimes t)(r_{V\otimes W}(g) \cdot (v \otimes w)) = \\ &= (u \otimes t)((r_V(g) \cdot v) \otimes (r_W(g) \cdot w)) = \\ &= u(r_V(g) \cdot v) \cdot t(r_W(g) \cdot w) = \\ &= m \circ (\psi_V \otimes \psi_W)((v \otimes u) \otimes (w \otimes t))(g)) = \\ &= m \circ (\psi_V \otimes \psi_W) \circ c(v \otimes w \otimes u \otimes t)(g).\end{aligned}$$

Let $\{e_1, \ldots, e_n\}$ be a basis of V with the dual basis $\{f_1, \ldots, f_n\}$ and take $f \in \mathbf{k_0}$. We have:

$$\begin{aligned}\partial(\psi_V(f \cdot e_i \otimes f_j))(g) &= \partial(f \cdot f_j(r_V(g) \cdot e_i)) = \partial\left(f \cdot g_{ij}^V\right) = \partial(f) \cdot g_{ij}^V + f \cdot \partial\left(g_{ij}^V\right) = \\ &= \partial(f) \cdot g_{ij}^V + f \cdot \partial\left(g_{ij}^V\right) = \\ &= \partial(f) \cdot f_j(r_V(g) \cdot e_i) + f \cdot \partial(f_j(r_V(g)e_i)) = \\ &= \partial(f) \cdot \psi_{V^{(1)}}(e_i \otimes f_j)(g) + f \cdot \psi_{V^{(1)}}((\partial e_i) \otimes f_j)(g) = \\ &= \psi_{V^{(1)}}\left(\partial(f \cdot e_i) \otimes f_j\right)(g).\end{aligned}$$

Also,

$$\begin{aligned}\psi_{V^{(1)}}(\partial(f \cdot e_i) \otimes (\partial e_j)^*)(g) &= (\partial e_j)^*(r_{V^{(1)}}(g)\partial(f \cdot e_i)) = (\partial e_j)^*(\partial(r_V(g)(f \cdot e_i))) = \\ &= (\partial e_j)^*\left(\sum_{k=1}^{n} \partial(f \cdot g_{ik}) \cdot e_k + (f \cdot g_{ik}) \cdot \partial e_k\right) = \\ &= f \cdot g_{ij} = \psi_V(f \cdot e_i \otimes f_j)\end{aligned}$$

for all i and j, $1 \leqslant i, j \leqslant n$. The last statement can be shown similarly, for instance:

$$\psi_{V_1^{(1)} \otimes V_2}((v_1 \otimes v_2) \otimes ((\partial w_1)^* \otimes w_2^*))(g) = (\partial w_1)^*(r_{V_1}(g) \cdot v_1) \cdot w_2^*(r_{V_2}(g) \cdot v_2) = \\ = 0 \cdot w_2^*(r_{V_2}(g) \cdot v_2) = 0.$$

■

5.3.2 Second step

Here, we will construct a differential algebra $\mathcal{A}$ (that will be our candidate for A) using representations of G as objects of $\mathcal{V}$ together with morphisms between them and not using any other information from G.

Let

$$\mathcal{F} = \bigoplus_{V \in \mathcal{O}b(\mathbf{Rep}_G)} V \otimes V^*.$$

So, the canonical injections

$$i_V : V \otimes V^* \to \mathcal{F}$$

are defined. Consider the subspace $\mathcal{R}$ of $\mathcal{F}$ spanned by

$$\left\{ \left(i_V(\mathrm{id} \otimes \phi^*) - i_W(\phi \otimes \mathrm{id})\right)(z) \;\middle|\; V,\ W \in \mathcal{O}b(\mathbf{Rep}_G),\ \phi \in \mathrm{Hom}_G(V, W), z \in V \otimes W^* \right\},$$

$$\left\{ i_V(v \otimes w^*) - i_{V^{(1)}}(\partial v \otimes (\partial w)^*) \mid v,\ w \in V,\ V \in \mathcal{O}b(\mathbf{Rep}_G) \right\},$$

and

$$\Big\{ i_{V_1^{(q_1)} \otimes \ldots V_p^{(q_p)}} \left(\left(\partial^{i_1} v_1 \otimes \ldots \otimes \partial^{i_p} v_p\right) \otimes \left((\partial^{j_1} w_1)^* \otimes \ldots \otimes (\partial^{j_p} w_p)^*\right) \right) \;\Big| \\ \max_l (j_l - i_l) > 0,\ v_k,\ w_k \in V_k,\ \max\{i_k, j_k\} \leqslant q_k,\ V_k \in \mathcal{O}b(\mathbf{Rep}_G),\ 1 \leqslant k \leqslant p \Big\}.$$

We now put $\mathcal{A} = \mathcal{F}/\mathcal{R}$. For $v \in V$ and $u \in V^*$ we denote by

$$a_V(v \otimes u)$$

the image in $\mathcal{A}$ of

$$i_V(v \otimes u).$$

So, for any $\phi \in \mathrm{Hom}_G(V, W)$ we have

$$a_V(v \otimes \phi^*(u)) = a_W(\phi(v) \otimes u).$$

Also,

$$a_V(v \otimes w^*) = a_{V^{(1)}}(\partial v \otimes (\partial w)^*) \tag{5.5}$$

for all v, $w \in V$, and

$$a_{V_1^{(q_1)} \otimes \ldots V_p^{(q_p)}}\left(\left(\partial^{i_1} v_1 \otimes \ldots \otimes \partial^{i_p} v_p\right) \otimes \left((\partial^{j_1} w_1)^* \otimes \ldots \otimes (\partial^{j_p} w_p)^*\right)\right) = 0 \tag{5.6}$$

if for some l, $1 \leqslant l \leqslant p$, we have $i_l < j_l$.

Take $v \in V$, $w \in W$, $u \in V^*$, $t \in W^*$. Let also $\{v_i\}$ be a basis of V and $\{u_i\}$ be its dual. Introduce the following operations on $\mathcal{A}$:

$$m(a_V(v \otimes u), a_W(w \otimes t)) = a_{V \otimes W}((v \otimes w) \otimes (u \otimes t)), \tag{5.7}$$

$$\partial(a_V(v \otimes u)) = a_{V^{(1)}}(\partial v \otimes u), \tag{5.8}$$

$$\tilde{\Delta}(a_V(v \otimes u)) = \sum_j a_V(v_j \otimes u) \otimes a_V(v \otimes u_j), \tag{5.9}$$

$$\tilde{S}(a_V(v \otimes u)) = a_{V^*}(u \otimes v). \tag{5.10}$$

Proposition 5.4 *The* $\mathbf{k_0}$*-vector space* $\mathcal{A}$ *contains a non-zero vector.*

Proof. Let $\mathcal{C}$ be the category of differential representations of the trival differential group $G_e = \{e\}$. Note that the algebra of G_e is just the differential field $\mathbf{k_0}$. Take $a_V(v_1 \otimes v_2^*)$ and $a_W(0 \otimes w^*)$ with $v_1 \otimes v_2^* \neq 0$ and linearly independent v_1 and v_2. Consider the linear map:

$$\phi : V \to W, \quad v_2 \mapsto w_2,\ c \cdot v_2 \neq v \mapsto 0.$$

We have

$$\phi^*(w_2^*)(c_1 v_1 + c_2 v_2 + cv) = w_2^*(c_2 w_2) = c_2.$$

Hence, $\phi^*(w_2^*) = v_2^*$. We then have

$$a_V(v_1 \otimes v_2^*) = a_V(v_1 \otimes \phi^*(w_2^*)) = a_W(0 \otimes w_2^*) = 0.$$

Let now $0 \neq v \in V$ and $0 \neq w \in W$. Consider the linear map

$$\phi : V \to W, \quad v \mapsto w.$$

We have:

$$\phi^*(w^*)(av + bu) = w^*(aw) = a.$$

Hence, $\phi^*(w^*) = v^*$. We then have

$$a_V(v \otimes v^*) = a_V\left(v \otimes \phi^*(w^*)\right) = a_W(w \otimes w^*).$$

Thus, if $\underline{1} = \mathrm{span}_{\mathbf{k_0}}\{e\}$ then

$$\mathcal{A}_{G_e} = \mathrm{span}_{\mathbf{k_0}}\left\{a_{\underline{1}}(e \otimes e^*)\right\}.$$

Let us show that $a_{\underline{1}}(e \otimes e^*) \neq 0$ in $\mathcal{A}_{G_e}$. Since there is a surjective linear map $\mathcal{A} \to \mathcal{A}_{G_e}$, this will imply that $\mathcal{A}$ contains a non-zero vector. Indeed, we have:

$$a_{\underline{1}}(e \otimes e^*) = a_{\partial\underline{1}}(\partial e \otimes (\partial e)^*)$$

and

$$a_{\underline{1}}\left(\left(\partial^{i_1} e \otimes \ldots \partial^{i_p} e\right) \otimes \left((\partial^{j_1} e)^* \otimes \ldots \otimes (\partial^{j_p} e)^*\right)\right) = 0.$$

Therefore, $\mathcal{A}_{G_e}$ is just the quotient of $\mathcal{F}$ by the subspace generated by the relations coming from $\phi : V \to W$. And the latter subspace is not the whole $\mathcal{F}$ as it has been shown above. ■

Lemma 5.5 *The definition of m is correct and provides a structure of a commutative associative algebra on the $\mathbf{k_0}$-vector space $\mathcal{A}$ and the identity element is given by the trivial representation $\underline{1}$.*

Proof. Consider morphisms $\phi_1 : V \to X$ and $\phi_2 : W \to Y$ as G-vector spaces and vectors

$$v \in V,\ u \in V^*,\ w \in W,\ t \in W^*,\ x \in X^*,\ y \in Y^*$$

such that $\phi_1^*(x) = u$ and $\phi_2^*(y) = t$. We then have:

$$\begin{aligned} a_V(v \otimes u) \cdot a_W(w \otimes t) =& a_V(v \otimes \phi_1^*(x)) \cdot a_W(w \otimes \phi_2^*(y)) = \\ &= a_X(\phi_1(v) \otimes x) \cdot a_Y(\phi_2(w) \otimes y) = \\ &= a_{X\otimes Y}((\phi_1(v) \otimes \phi_2(w)) \otimes (x \otimes y)) = \\ &= a_{X\otimes Y}((\phi_1 \otimes \phi_2)(v \otimes w) \otimes (x \otimes y)) = \\ &= a_{V\otimes W}((v \otimes w) \otimes (u \otimes t)). \end{aligned}$$

Moreover, for $x,\ y \in V$ and $z,\ w \in W$ using formulas (5.5) and (5.6) we have:

$$\begin{aligned} a_V(x \otimes y^*) \cdot a_W(z \otimes w^*) &= a_{V\otimes W}((x \otimes z) \otimes (y^* \otimes w^*)) = \\ &= a_{(V\otimes W)^{(1)}}(\partial(x \otimes z) \otimes (\partial(y \otimes w))^*) = \\ &= a_{V^{(1)}\otimes W^{(1)}}\left((\partial x \otimes z + x \otimes \partial z) \otimes \frac{(\partial y \otimes w + y \otimes \partial w)^*}{2}\right) = \\ &= 1/2 \cdot a_{V^{(1)}\otimes W^{(1)}}((\partial x \otimes z) \otimes ((\partial y)^* \otimes w^*)) + \\ &+ 1/2 \cdot a_{V^{(1)}\otimes W^{(1)}}(x \otimes \partial z) \otimes (y^* \otimes (\partial w)^*)) = \\ &= 1/2 \cdot a_{V^{(1)}}(\partial x \otimes (\partial y)^*) \cdot a_{W^{(1)}}(z \otimes w^*) + \\ &+ 1/2 \cdot a_{V^{(1)}}(x \otimes y^*) \cdot a_{W^{(1)}}(\partial z \otimes (\partial w)^*) = \\ &= 1/2 \cdot a_{V^{(1)}}(\partial x \otimes (\partial y)^*) \cdot a_{W^{(1)}}(\partial z \otimes (\partial w)^*) + \\ &+ 1/2 \cdot a_{V^{(1)}}(\partial x \otimes (\partial y)^*) \cdot a_{W^{(1)}}(\partial z \otimes (\partial w)^*) = \\ &= a_{V^{(1)}\otimes W^{(1)}}((\partial x \otimes \partial z) \otimes ((\partial y)^* \otimes (\partial w)^*)) = \\ &= a_{V^{(1)}}(\partial x \otimes (\partial y)^*) \cdot a_{W^{(1)}}(\partial z \otimes (\partial w)^*). \end{aligned}$$

Indeed, we have used the morphism:

$$\begin{aligned} &\psi : (V \otimes W)^{(1)} \to V^{(1)} \otimes W^{(1)}, \\ &\psi : 1 \otimes x \otimes z \mapsto (1 \otimes x) \otimes (1 \otimes z), \\ &\psi : \partial \otimes x \otimes z \mapsto (\partial x) \otimes z + x \otimes (\partial z). \end{aligned}$$

Its dual

$$\psi^* : \left(V^{(1)} \otimes W^{(1)}\right)^* \to \left((V \otimes W)^{(1)}\right)^*$$

maps

$$\frac{(\partial y \otimes w + y \otimes \partial w)^*}{2} \mapsto (\partial(y \otimes w))^*.$$

Indeed,

$$\psi^* \left(\frac{(\partial y \otimes w + y \otimes \partial w)^*}{2} \right) (x \otimes z) = \left(\frac{(\partial y \otimes w + y \otimes \partial w)^*}{2} \right) (x \otimes z) =$$
$$= 0 = (\partial(y \otimes w))^*(x \otimes z),$$
$$\psi^* \left(\frac{(\partial y \otimes w + y \otimes \partial w)^*}{2} \right) (\partial(x \otimes z)) = \left(\frac{(\partial y \otimes w + y \otimes \partial w)^*}{2} \right)$$
$$((\partial x) \otimes z + x \otimes (\partial z)) =$$
$$= \frac{(\partial y)^*(\partial x) \cdot w(z) + y(x) \cdot (\partial w)^*(\partial z)}{2} =$$
$$= y^*(x) \cdot w^*(z) = (\partial(y \otimes w))^*(\partial(x \otimes z)).$$

We now prove that the multiplication is associative and commutative. Consider the morphism

$$\phi \in \mathrm{Hom}(V \otimes W, W \otimes V), \quad v \otimes w \mapsto w \otimes v.$$

We then have

$$a_V(v \otimes u) \cdot a_W(w \otimes t) = a_{V \otimes W}((v \otimes w) \otimes (u \otimes t)) = a_{W \otimes V}((w \otimes v) \otimes (t \otimes u)) =$$
$$= a_W(w \otimes t) \cdot a_V(v \otimes u).$$

So, the multiplication is commutative.

For $X \in \mathcal{O}b(\mathbf{Rep}_G)$ and $x \in X$, $y \in X^*$ we also have

$$(a_V(v \otimes u) \cdot a_W(w \otimes t)) \cdot a_X(x \otimes y) = a_{V \otimes W}((v \otimes w) \otimes (u \otimes t)) \cdot a_X(x \otimes y) =$$
$$= a_{(V \otimes W) \otimes X}((v \otimes w) \otimes x) \otimes ((t \otimes u) \otimes y)) =$$
$$= a_{V \otimes (W \otimes X)}((v \otimes (w \otimes x)) \otimes (t \otimes (u \otimes y))) =$$
$$= a_V(v \otimes t) \cdot a_{W \otimes X}((w \otimes x) \otimes (u \otimes y)) =$$
$$= a_V(v \otimes u) \cdot (a_W(w \otimes t) \cdot a_X(x \otimes y)).$$

We have shown that the multiplication is associative.

Let $\underline{1}$ be the trivial representation of the group G and $0 \neq e \in \underline{1}$, $f \in \underline{1}^*$, $f(e) = 1$. We have the morphism

$$\varphi \in \mathrm{Hom}(V, V \otimes \underline{1}), \quad v \mapsto v \otimes e.$$

Then

$$a_V(v \otimes u) \cdot a_{\underline{1}}(e \otimes f) = a_{V \otimes \underline{1}}((v \otimes e) \otimes (u \otimes f)) = a_{V \otimes \underline{1}}(\varphi(v) \otimes (u \otimes f)) =$$
$$= a_V(v \otimes \varphi^*(u \otimes f)) = a_V(v \otimes u).$$

Thus, $\mathcal{A}$ is a commutative associative algebra with unity. ■

Our main goal is to recover the *differential* Hopf algebra A. We give a differential structure on $\mathcal{A}$ and then show that this structure corresponds to the one of A. Let us describe the *intuition* behind the construction we are going to present.

We are recovering a subgroup of $\mathbf{GL}_n$. Let us denote the corresponding matrix coordinate functions by y_{ij}. We must be able to:

1. differentiate these functions y_{ij} obtaining $y'_{ij}, \ldots, y^{(p)}_{ij}, \ldots$;

2. multiply the results of this differentiation and stay in the algebra.

For bases $v_1, \ldots, v_n$ and $u_1, \ldots, u_n$ of V and V^*, respectively, the coordinate functions mapping $G \to \mathbf{k_0}$ are given by

$$y_{ij}(g) = \psi_V(v_i \otimes u_j)(g) = u_j(r_V(g) \cdot v_i),$$

where $g \in G$. The candidates for these functions in $\mathcal{A}$ are, certainly, $a_V(v_i \otimes u_j)$. Our correspondence between $\mathcal{A}$ and A must preserve differentiation. So, y'_{ij} corresponds to $\partial(a_V(v_i \otimes u_j))$ which we still need to define.

Moreover, such a definition must leave us in the same category and satisfy the product rule for differentiation. We also notice that since $y'_{ij}, \ldots, y^{(p)}_{ij}, \ldots$ and $y^{(q)}_{ij} \cdot y^{(r)}_{kl}$ are monomials, we should preserve this property for $\partial^q\left(a_V(v_i \otimes u_j)\right) \cdot \partial^r\left(a_V(v_k \otimes u_l)\right)$. These ideas are implemented in Lemma 5.6 and Theorem 5.8.

Lemma 5.6 *The natural differential structure on $\mathcal{A}$ introduced in formula* (5.8) *makes it a differential* $\mathbf{k_0}$*-algebra.*

Proof. First of all, recall that

$$\partial\left(a_V(v\otimes u)\right) = a_{V^{(1)}}((\partial v)\otimes u).$$

Recall that we let $u(\partial v) = \partial(u(v))$ for any $v \in V$ and $u \in V^*$. We need to show its correctness with respect to the morphisms:

$$\begin{aligned}\partial(a_W(\phi(v)\otimes u)) = a_{W^{(1)}}(\partial(\phi(v))\otimes u) = a_{W^{(1)}}(\phi(\partial v)\otimes u) = \\ = a_{V^{(1)}}(\partial v\otimes\phi^*(u)) = \partial(a_V(v\otimes\phi^*(u)))\end{aligned}$$

for a morphism $\phi : V \to W$ that we natrually prolong to a morphism $\phi : V^{(1)} \to W^{(1)}$ mapping $\partial v \mapsto \partial(\phi(v))$. We also check:

$$\begin{aligned}\partial\left(a_{V^{(1)}}(\partial v\otimes(\partial w)^*)\right) = a_{\left(V^{(1)}\right)^{(1)}}(\partial(\partial v)\otimes(\partial w)^*) = a_{V^{(1)}}(\partial v\otimes w^*) = \\ = \partial(a_V(v\otimes w^*)).\end{aligned}$$

for v, $w \in V$. Hence, the differentiation is correct. We need to show the product rule. We have

$$\begin{aligned}\partial\left(a_V(v\otimes u)\cdot a_W(w\otimes t)\right) &= \partial\left(a_{V\otimes W}((v\otimes w)\otimes(u\otimes t))\right) = \\ &= a_{(V\otimes W)^{(1)}}(\partial(v\otimes w)\otimes(u\otimes t)) = \\ &= a_{V^{(1)}\otimes W^{(1)}}((\partial v\otimes w)\otimes(u\otimes t)+(v\otimes\partial w)\otimes(u\otimes t)) = \\ &= a_{V^{(1)}}(\partial v\otimes u)\cdot a_W(w\otimes t)+a_V(v\otimes u)\cdot a_{W^{(1)}}(\partial w\otimes t) = \\ &= \left(\partial a_V(v\otimes u)\right)\cdot a_W(w\otimes t)+a_V(v\otimes u)\cdot\left(\partial a_W(w\otimes t)\right),\end{aligned}$$

where we use the following morphism

$$(V\otimes W)^{(1)} \to V^{(1)}\otimes W^{(1)}, \quad \partial\otimes v\otimes w \mapsto \partial v\otimes w + v\otimes\partial w.$$

■

5.3.3 Third step

Now, we can show that the differential algebra $\mathcal{A}$ we have constructed is what we were looking for.

Lemma 5.7 *Let $\mathcal{C}$ be a rigid abelian tensor category with a tensor $\mathbf{k_0}$-linear functor $\omega : \mathcal{C} \to \mathcal{V}$ then*

$$\mathrm{End}^{\otimes}(\omega) = \mathrm{Aut}^{\otimes}(\omega).$$

Proof. We expand the proof that appears in [4, Proposition 1.13]. Let $\lambda : \omega \to \omega$. For each $X \in \mathcal{C}$ there exists a morphism $t_X : \omega(X)^* \to \omega(X)^*$ such that the following diagram is commutative:

$$\begin{array}{ccc} \omega(X^*) & \xrightarrow{\lambda_{X^*}} & \omega(X^*) \\ {\scriptstyle\varphi}\downarrow{\scriptstyle\cong} & & {\scriptstyle\varphi}\downarrow{\scriptstyle\cong} \\ \omega(X)^* & \xrightarrow{t_X} & \omega(X)^* \end{array}$$

The category $\mathcal{V}$ is rigid. So, for all U, $V \in \mathcal{V}$ we have $\underline{\mathrm{Hom}}(U, V) \cong \underline{\mathrm{Hom}}(V^*, U^*)$. We then let $\mu_X := (t_X)^* : \omega(X) \to \omega(X)$. For any $f : X \to Y$ the following diagram commutes:

$$\begin{array}{ccccccc} Y & \longrightarrow & Y^* & \xrightarrow{\omega} & \omega(Y^*) & \xrightarrow{\lambda_{Y^*}} & \omega(Y^*) \\ \uparrow{\scriptstyle f} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle\omega(f^*)} & & \downarrow{\scriptstyle\omega(f^*)} \\ X & \longrightarrow & X^* & \xrightarrow{\omega} & \omega(X^*) & \xrightarrow{\lambda_{X^*}} & \omega(X^*) \end{array}$$

Gathering all commutative diagrams together we obtain that $\mu = (\mu_X) \in \mathrm{End}^{\otimes}(\omega)$. We now show that $\mu = \lambda^{-1}$. We have

$$\begin{array}{ccccc} X^* \otimes X & \xrightarrow{\omega} & \omega(X^* \otimes X) & \xrightarrow{\lambda_{X^* \otimes X}} & \omega(X^* \otimes X) \\ \downarrow{\scriptstyle \mathrm{ev}_X} & & \downarrow{\scriptstyle\omega(\mathrm{ev}_X)} & & \downarrow{\scriptstyle\omega(\mathrm{ev}_X)} \\ \underline{1} & \xrightarrow{\omega} & \omega(\underline{1}) & \xrightarrow{\mathrm{id}} & \omega(\underline{1}) \end{array}$$

The evaluation morphism takes $f \otimes x \in \omega(X)^* \otimes \omega(X)$ and evaluates providing $f(x) \in \omega(\underline{1})$ as its output. Take any $y \in \omega(X)^*$ and $x \in \omega(X)$. Then

$$\begin{aligned}(y, \mu_X \circ \lambda_X(x)) = (t_X(y), \lambda_X(x)) = (\varphi \circ \lambda_{X^*} \circ \varphi^{-1}(y), \lambda_X(x)) = \\ = \mathrm{ev}_{\omega(X)} \circ (\varphi \otimes \mathrm{id}) \circ (\lambda_{X^*} \otimes \lambda_X)(\varphi^{-1}(y), x) = \\ = \mathrm{ev}_{\omega(X)} \circ (\varphi \otimes \mathrm{id}) \circ \lambda_{X^* \otimes X}(\varphi^{-1}(y), x) = (y, x)\end{aligned}$$

and as a result λ_X is injective. Thus, λ_X is also surjective and μ_X is its inverse. We can show this differently:

$$(y, \lambda_X \circ \mu_X(x)) = \mathrm{ev}_{\omega(X)} \circ (\varphi \otimes \mathrm{id}) \circ \mu_{X^* \otimes X}(\varphi^{-1}(y), x) = (y, x)$$

as $\mu \in \mathrm{End}^{\otimes}(\omega)(R)$. ■

The proof of the following result that we give differs from the similar one in [10]. Our goal was to provide a correct differential structure. Also, if one follows our proof in a non-differential case one finds that it does *not* depend on $\mathrm{char}\,\mathbf{k_0}$. For this commutative case the change (in comparison to [10]) that we make is at the end of the proof where we take the "generic point".

Theorem 5.8 *We have*

1. *the algebra $\mathcal{A}$ is a finitely generated differential $\mathbf{k_0}$-algebra;*

2. *there is a surjective differential algebra homomorphism $\Phi : \mathcal{A} \to A$ such that*

$$\Phi \circ a_V = \psi_V$$

   for all differential G-modules V;

3. *the map Φ is a differential algebra isomorphism $\mathcal{A} \to A$.*

Proof. Let $V \in \mathcal{O}b(\mathbf{Rep}_G)$. Fix a basis $\{u_1, \ldots, u_n\}$ of V^*. Since A is locally finite and

$$\phi_{u_i} : v \mapsto u_i(r_V(\cdot) \cdot v)$$

is a G module morphism $V \to A$, there is $W \in \mathcal{O}b(\mathbf{Rep}_G)$, $W \subset A$, and $\dim W < \infty$ containing the images of ϕ_{u_i} for all i, $1 \leqslant i \leqslant n$. According to the proof of Lemma 4.1 the induced G-morphism $\phi : V \to W^n$ is injective. Hence, the map ϕ^* is surjective and for $u \in V^*$ there exist $t = (t_1, \ldots, t_n) \in W^n$ such that $u = \phi^*(t)$. We then have

$$a_V(v \otimes u) = a_V(v \otimes \phi^*((t_1, \ldots, t_n))) = a_W(\phi(v) \otimes (t_1, \ldots, t_n)).$$

Thus, the differential algebra $\mathcal{A}$ is generated by the images of the a_V for $A \supset V \in \mathcal{O}b(\mathbf{Rep}_G)$ and $\dim V < \infty$. Let V be such a G-submodule of A which also contains 1 and a finite set of generators of A as a differential $\mathbf{k_0}$-algebra. The multiplication on the algebra A defines for any $l \in \mathbb{Z}_{\geqslant 1}$ a surjective G-morphism ϕ_l from $V^{\otimes l}$ onto some $V(l) \in \mathcal{O}b(\mathbf{Rep}_G)$ with $\dim V(l) < \infty$. We then have $V(l) \subset V(l+1)$, because $1 \in V$, and $A = \bigcup_{l \geqslant 1} V(l)$.

Consider any $\mathcal{O}b(\mathbf{Rep}_G) \ni W \subset A$ with $\dim W < \infty$. There exists $l \in \mathbb{Z}_{\geqslant 1}$ such that $W \subset V(l)$. Since ϕ_l is surjective, we have

$$\operatorname{Im} a_W \subset \operatorname{Im} a_{V(l)} \subset \operatorname{Im} a_{V^{\otimes l}}.$$

Because of the multiplication structure of $\mathcal{A}$ the set $\operatorname{Im} a_{V^{\otimes l}}$ lies in the differential subalgebra generated by $\operatorname{Im} a_V$. Hence, this subalgebra is the whole $\mathcal{A}$.

The homomorphism Φ of the second statement is constructed as follows. We take an element $a_V(v \otimes u)$ and map it to $\psi_V(v, u)$ for all $V \in \mathcal{O}b(\mathbf{Rep}_G)$. Since

$$\begin{aligned}
\Phi\left(a_V(v \otimes u) \cdot a_W(w \otimes t)\right) &= \Phi\left(a_{V \otimes W}((v \otimes w) \otimes (u \otimes t))\right) = \\
&= \psi_{V \otimes W}(v \otimes w, u \otimes t) = \\
&= m \circ (\psi_V \otimes \psi_W) \circ c(v \otimes w, u \otimes t) = \\
&= m \circ (\psi_V \otimes \psi_W)\,(v \otimes u, w \otimes t) = \\
&= m\left(\Phi(a_V(v \otimes u)), \Phi(a_W(w \otimes t))\right),
\end{aligned}$$

the map Φ is a $\mathbf{k_0}$-algebra homomorphism. Let us show that it is differential. From Lemma 5.3 we have:

$$\Phi\left(\partial\left(a_V(v\otimes u)\right)\right) = \Phi\left(a_{V^{(1)}}(\partial v\otimes u\right) = \psi_{V^{(1)}}(\partial v\otimes u) = \\ = \partial\left(\psi_V(v\otimes u)\right) = \partial\left(\Phi\left(a_V(v\otimes u)\right)\right).$$

We now show the last statement. Let B be a ∂-differential $\mathbf{k_0}$-algebra. Consider a point $\xi\in \mathrm{Hom}_{\mathbf{k_0}[\partial]}(\mathcal{A},B)$ and $V\in \mathcal{O}b(\mathbf{Rep}_G)$. Fix bases $\{v_i\}$ and $\{u_j\}$ of V and V^*, respectively. There is an endomorphism λ_V of $V\otimes B$ such that

$$\langle \lambda_V(v_i), u_j\rangle = u_j(\lambda_V(v_i)) = \xi(a_V(v_i\otimes u_j)).$$

We show now that $(\lambda_V \mid V\in \mathcal{O}b(\mathbf{Rep}_G))$ satisfies the conditions of Theorem 5.2.

Let $V,W\in \mathcal{O}b(\mathbf{Rep}_G)$. Then we have

$$\begin{aligned}\langle \lambda_{V\otimes W}(v_i\otimes w_j), u_r\otimes t_l\rangle &= \xi(a_{V\otimes W}((v_i\otimes w_j)\otimes(u_r\otimes t_l))) = \\ &= \xi(a_V(v_i\otimes u_r)\cdot a_W(w_j\otimes t_l)) = \\ &= \xi(a_V(v_i\otimes u_r))\cdot\xi(a_W(w_j\otimes t_l)) = \\ &= \langle\lambda_V(v_i),u_r\rangle\cdot\langle\lambda_W(w_j),t_l\rangle = \\ &= \langle(\lambda_V\otimes\lambda_W)(v_i\otimes w_j), u_r\otimes t_l\rangle.\end{aligned}$$

Hence, $\lambda_{V\otimes W} = \lambda_V\otimes\lambda_W$. Since $a_{\underline{1}}$ is the identity in $\mathcal{A}$ and $\xi(1_{\mathcal{A}})=1$, we have $\lambda_{\underline{1}}$ is the identity. Let us show the functoriality of (λ_V). For a G-equivariant map $F: V\to W$ we have

$$\begin{aligned}\langle(\lambda_W\circ F)(v_i),t_j\rangle &= \xi(a_W(F(v_i)\otimes t_j)) = \xi(a_V(v_i\otimes F^*(t_j))) = \\ &= \langle\lambda_V(v_i),F^*(t_j)\rangle = \langle(F\circ\lambda_V)(v_i),t_j\rangle.\end{aligned}$$

Hence, $\lambda_W\circ F = F\circ\lambda_V$. Finally, since $\{v_i\}$ and $\{u_j\}$ are dual to each other, we have:

$$\begin{aligned}\langle\partial\circ\lambda_V(v_i),u_j\rangle &= \partial(u_j(\lambda_V(v_i))) = \partial\circ\xi(a_V(v_i\otimes u_j)) = \\ &= \xi(\partial a_V(v_i\otimes u_j)) = \xi(a_{\partial V}((\partial v_i)\otimes u_j)) = \langle\lambda_{\partial V}(\partial v_i),u_j\rangle.\end{aligned}$$

Moreover, let $\lambda_V(v_i) = \sum c_{ik} v_k$. Due to (5.5) we have:

$$\begin{aligned}\langle \partial \circ \lambda_V(v_i), (\partial v_j)^* \rangle &= (\partial v_j)^*(\partial(c_{ik} v_k)) = (\partial v_j)^*(\partial(c_{ik}) v_k + c_{ik} \partial v_k) = \\ &= c_{ik} = \xi(a_V(v_i \otimes v_j^*)) = \xi(a_{V^{(1)}}(\partial v_i \otimes (\partial v_j)^*)) = \\ &= \langle \lambda_{\partial V}(\partial v_i), (\partial v_j)^* \rangle.\end{aligned}$$

We then conclude that $\lambda \in \mathrm{End}^{\otimes,\partial}(\omega)$. Since the category $\mathbf{Rep}_G$ is rigid, $\lambda \in \mathrm{Aut}^{\otimes,\partial}(\omega)$ by Lemma 5.7.

Take $B = \mathcal{A}$ and the generic point $\xi = \mathrm{id}_{\mathcal{A}}$. By Theorem 5.2 there exists $x \in \mathrm{Hom}_{\mathbf{k_0}[\partial]}(A, \mathcal{A}) = G(\mathcal{A})$ such that $\lambda_V = r_V(x)$ for all $V \in \mathcal{O}b(\mathbf{Rep}_G)$. We have

$$\begin{aligned}x \circ \Phi \circ a_V(v_i \otimes u_j)) &= x \circ \psi_V(v_i \otimes u_j) = \langle r_V(x) v_i, u_j \rangle = \\ &= \langle \lambda_V(v_i), u_j \rangle = \xi(a_V(v_i \otimes u_j)).\end{aligned}$$

Hence, $x \circ \Phi = \xi = \mathrm{id}_{\mathcal{A}}$. This implies that Φ is injective. Since Φ is also surjective, we obtain that $\Phi : \mathcal{A} \to A$ is a differential algebra isomorphism. ■

5.4 Recovering Δ and S

We provide a differential Hopf algebra structure to $\mathcal{A}$. Let $V \in \mathcal{O}b(\mathbf{Rep}_G)$ and $\{v_i\}$ be its basis with the dual basis $\{u_j\}$ of V^*. Recall the $\mathbf{k_0}$-linear map

$$\tilde{\Delta} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}, \quad a_V(v \otimes u) \mapsto \sum a_V(v_i \otimes u) \otimes a_V(v \otimes u_i).$$

Lemma 5.9 *The map $\tilde{\Delta}$ is a differential algebra homomorphism and is a comultiplication.*

Proof. We first check the basis independence. Let $\{e_1, \ldots, e_n\}$ be another basis for V and $\{f_1, \ldots, f_n\}$ be its dual. Hence, there exits a matrix $C = (c_{ij}) \in \mathbf{GL}_n(\mathbf{k_0})$

such that $v_i = \sum e_j c_{ji}$. We then have

$$\sum_{i=1}^{n} a_V(v_i \otimes u) \otimes a_V(v \otimes u_i) = \sum_{i=1}^{n} a_V\left(\sum_{j=1}^{n} e_j c_{ji} \otimes u\right) \otimes a_V(v \otimes u_i) =$$
$$= \sum_{j=1}^{n} a_V\left(e_j \otimes u\right) \otimes a_V\left(v \otimes \sum_{i=1}^{n} c_{ji} u_i\right) =$$
$$= \sum_{j=1}^{n} a_V\left(e_j \otimes u\right) \otimes a_V(v \otimes f_j).$$

In addition, for $v,\ w \in V$ using (5.5) and (5.6) we have:

$$\tilde{\Delta}(a_{V^{(1)}}(\partial v \otimes (\partial w)^*)) = \sum_{i=1}^{n} a_{V^{(1)}}(e_i \otimes (\partial w)^*) \otimes a_{V^{(1)}}(\partial v \otimes e_i^*) +$$
$$+ \sum_{i=1}^{n} a_{V^{(1)}}(\partial e_i \otimes (\partial w)^*) \otimes a_{V^{(1)}}(\partial v \otimes (\partial e_i)^*) =$$
$$= \sum_{i=1}^{n} a_{V^{(1)}}(\partial e_i \otimes (\partial w)^*) \otimes a_{V^{(1)}}(\partial v \otimes (\partial e_i)^*) =$$
$$= \sum_{i=1}^{n} a_{V^{(1)}}(e_i \otimes w^*) \otimes a_{V^{(1)}}(v \otimes e_i)^*) =$$
$$= \tilde{\Delta}(a_V(v \otimes w^*)).$$

Hence, definition is correct.

We check that it is an algebra homomorphism. We have

$$\tilde{\Delta}(a_V(v \otimes u) \cdot a_W(w \otimes t)) =$$
$$= \tilde{\Delta}(a_{V \otimes W}((v \otimes w) \otimes (u \otimes t))) =$$
$$= \sum_{i,j} a_{V \otimes W}((v_i \otimes w_j) \otimes (u \otimes t)) \otimes a_{V \otimes W}((v \otimes w) \otimes (u_i \otimes t_j)) =$$
$$= \sum_{i,j} (a_V(v_i \otimes u) \otimes a_V(v \otimes u_i)) \cdot (a_W(w_j \otimes t) \otimes a_W(w \otimes t_j)) =$$
$$= \left(\sum_{i} a_V(v_i \otimes u) \otimes a_V(v \otimes u_i)\right) \cdot \left(\sum_{j} a_W(w_j \otimes t) \otimes a_W(w \otimes t_j)\right) =$$
$$= \tilde{\Delta}(a_V(v \otimes u)) \cdot \tilde{\Delta}(a_W(w \otimes t)).$$

Moreover,

$$\begin{aligned}\partial \circ \tilde{\Delta}(a_V(v \otimes u)) = &\sum (a_{V^{(1)}}(\partial v_i \otimes u) \otimes a_V(v \otimes u_i) + \\ &+ a_V(v_i \otimes u) \otimes a_{V^{(1)}}(\partial v \otimes u_i)) = \\ = &\sum (a_{V^{(1)}}(\partial v_i \otimes u) \otimes a_{V^{(1)}}(\partial v \otimes (\partial v_i)^*) + \\ &+ a_{V^{(1)}}(v_i \otimes u) \otimes a_{V^{(1)}}(\partial v \otimes u_i)) = \\ = &\tilde{\Delta}(\partial(a_V(v \otimes u))),\end{aligned}$$

where we are using the following identity:

$$a_V(v \otimes w^*) = a_{V^{(1)}}(\partial v \otimes (\partial w)^*), \quad v,\ w \in V$$

and imbedding

$$V \mapsto V^{(1)}, \quad v \mapsto 1 \otimes v,\ v \in V.$$

We finally show the coassociativity:

$$\begin{aligned}\left(\tilde{\Delta} \otimes \mathrm{id}\right) \circ \tilde{\Delta}(a_V(v \otimes u)) = &\sum_i \tilde{\Delta}(a_V(v_i \otimes u)) \otimes a_V(v \otimes u_i) = \\ = &\sum_i \left(\sum_j a_V(v_j \otimes u) \otimes a_V(v_i \otimes u_j)\right) \otimes a_V(v \otimes u_i) = \\ = &\sum_i a_V(v_i \otimes u) \otimes \left(\sum_j a_V(v_j \otimes u_i) \otimes a_V(v \otimes u_j)\right) = \\ = &\sum_i a_V(v_i \otimes u) \otimes \tilde{\Delta}(a_V(v \otimes u_i)) = \\ = &\left(\mathrm{id} \otimes \tilde{\Delta}\right) \circ \tilde{\Delta}(a_V(v \otimes u)).\end{aligned}$$

■

Denote a basis of the trivial representation $\underline{1}$ by $\{e\}$ and the dual is $\{f\}$.

Lemma 5.10 *The map $\tilde{\varepsilon} : a_V(v \otimes u) \mapsto u(v)$ is a counit for $\mathcal{A}$ corresponding to the counit of A.*

Proof. We show that the definition is correct. For v, $w \in V$ we have:

$$\tilde{\varepsilon}(a_{V^{(1)}}(\partial v \otimes (\partial w)^*)) = (\partial w)^*(\partial v) = w(v) = \tilde{\varepsilon}(a_V(v \otimes w^*)).$$

Moreover, we have that

$$\tilde{\varepsilon}\left(a_{V_1^{(q_1)}\otimes\ldots V_p^{(q_p)}}\left(\left(\partial^{i_1}v_1 \otimes \ldots \otimes \partial^{i_p}v_p\right) \otimes \left((\partial^{j_1}w_1)^* \otimes \ldots \otimes (\partial^{j_p}w_p)^*\right)\right)\right) = 0$$

if for some l, $1 \leqslant l \leqslant p$, we have $i_l < j_l$. We demonstrate that $\tilde{\varepsilon}$ and $\tilde{\Delta}$ satisfy $m \circ (\mathrm{id}_{\mathcal{A}} \otimes \tilde{\varepsilon}) \circ \tilde{\Delta} = \mathrm{id}_{\mathcal{A}}$. We have:

$$\begin{aligned} m \circ (\mathrm{id}_{\mathcal{A}} \otimes \tilde{\varepsilon}) \circ \tilde{\Delta}(a_V(v \otimes u)) &= m \circ (\mathrm{id}_{\mathcal{A}} \otimes \tilde{\varepsilon})\left(\sum a_V(v_i \otimes u) \otimes a_V(v \otimes u_i)\right) = \\ &= \sum (a_V(v_i \otimes u) \cdot u_i(v)) = \\ &= a_V\left(\left(\sum u_i(v) \cdot v_i\right) \otimes u\right) = a_V(v \otimes u). \end{aligned}$$

In addition $\tilde{\varepsilon}$ is a differential homomorphism. Indeed,

$$\tilde{\varepsilon}\left(\partial a_V(v \otimes u)\right) = \tilde{\varepsilon}\left(a_{V^{(1)}}(\partial v \otimes u)\right) = u(\partial v) = \partial(u(v)) = \partial(\tilde{\varepsilon}(a_V(v \otimes u))).$$

Finally, we show that Φ maps $\tilde{\varepsilon}$ to ε.

$$\varepsilon \circ \Phi(a_V(v \otimes u)) = \Phi(a_V(v \otimes u))(e) = u(r_V(e) \cdot v) = u(v) = \Phi(\tilde{\varepsilon}(a_V(v \otimes u)).$$

■

Proposition 5.11 *The map $\Phi : \mathcal{A} \to A$ is a differential Hopf algebra morphism.*

Proof. We show that $\tilde{\Delta}$ is mapped to Δ. We have:

$$\begin{aligned} (\Phi \otimes \Phi) \circ \tilde{\Delta}(a_V(v \otimes u))(g_1, g_2) &= (\Phi \otimes \Phi)\left(\sum_i (a_V(v_i \otimes u) \otimes a_V(v \otimes u_i)\right)(g_1, g_2) = \\ &= \sum_i u(r_V(g_1) \cdot v_i) \cdot u_i(r_V(g_2) \cdot v); \\ \Delta \circ \Phi(a_V(v \otimes u))(g_1, g_2) &= u(r_V(g_1 \cdot g_2) \cdot v) = \\ &= u(r_V(g_1) \cdot (r_V(g_2) \cdot v)). \end{aligned}$$

Let $r_V(g_2)\cdot v=\sum c_j v_j$. Then

$$\sum_i u(r_V(g_1)\cdot v_i)\cdot u_i(r_V(g_2)\cdot v)=\sum_i u(r_V(g_1)\cdot v_i)\cdot u_i\left(\sum_j c_j v_j\right)=$$
$$=\sum_i u(r_V(g_1)\cdot v_i)\cdot c_i=$$
$$=\sum_i u(r_V(g_1)\cdot c_i v_i)=$$
$$=u(r_V(g_1)\cdot(r_V(g_2)\cdot v)).$$

■

Recall the $\mathbf{k_0}$-linear map

$$\tilde{S}:\mathcal{A}\to\mathcal{A},\quad a_V(v\otimes u)\mapsto a_{V^*}(u\otimes v).$$

Lemma 5.12 *The map $\tilde{S}$ is a differential algebra homomorphism and together with $\tilde{\Delta}$ gives a differential Hopf algebra structure on $\mathcal{A}$.*

Proof. Let $\{v_1,\dots,v_n\}$ be a basis of V and $\{v_1^*,\dots,v_n^*\}$ be its dual. Let us show correctness of $\tilde{S}$:

$$\tilde{S}\left(a_{V^{(1)}}(\partial v_i\otimes(\partial v_j)^*)\right)=a_{\left(V^{(1)}\right)^*}\left((\partial v_j)^*\otimes(\partial v_i)\right)=a_{V^*}\left(v_j^*\otimes v_i\right)=\tilde{S}(v_i\otimes v_j^*),$$

where we have employed the following morphism

$$\phi_1:\left(V^{(1)}\right)^*\to V^*,\quad v_j^*\mapsto 0,\ (\partial v_j)^*\mapsto v_j^*,$$

which commutes with the G-action. We are going to demonstrate that $\tilde{S}$ commutes with ∂ :

$$\partial\left(\tilde{S}(a_V(v_i\otimes v_j^*))\right)=\partial(a_{V^*}(v_j^*\otimes v_i))=a_{(V^*)^{(1)}}\left(\partial(v_j^*)\otimes v_i\right)=$$
$$=a_{\left(V^{(1)}\right)^*}(v_j^*\otimes\partial v_i)=\tilde{S}\left(a_{V^{(1)}}\left(\partial v_i\otimes v_j^*\right)\right)=\tilde{S}(\partial a_V(v_i\otimes v_j^*)),$$

where we use the morphism

$$\phi_2 : (V^*)^{(1)} \to \left(V^{(1)}\right)^*, \quad \partial v_j^* \mapsto v_j^*,\ v_j^* \mapsto (\partial v_j)^*$$

commuting with the G-action. Moreover, $\tilde{S}$ is an algebra homomorphism:

$$\begin{aligned}\tilde{S}(a_V(v \otimes u) \cdot a_W(w \otimes t)) &= \tilde{S}(a_{V\otimes W}((v \otimes w) \otimes (u \otimes t))) = \\ &= a_{V^*\otimes W^*}((u \otimes t) \otimes (v \otimes w)) = \\ &= a_{V^*}(u \otimes v) \cdot a_{W^*}(t \otimes w) = \\ &= \tilde{S}(a_V(v \otimes u)) \cdot \tilde{S}(a_W(w \otimes t)).\end{aligned}$$

We show that it respects comultiplication:

$$\begin{aligned}m \circ (\tilde{S} \otimes \mathrm{id}) \circ \tilde{\Delta}(a_V(v \otimes u)) &= m \circ (\tilde{S} \otimes \mathrm{id}) \left(\sum_i a_V(v_i \otimes u) \otimes a_V(v \otimes u_i) \right) = \\ &= m \left(\sum_i a_{V^*}(u \otimes v_i) \otimes a_V(v \otimes u_i) \right) = \\ &= \sum_i a_{V^*\otimes V}((u \otimes v) \otimes (v_i \otimes u_i)) = \\ &= u(v) \cdot a_I(e \otimes f) = \\ &= \tilde{\varepsilon}(a_V(v \otimes u)).\end{aligned}$$

We have denoted a basis of the trivial representation $\underline{1}$ by $\{e\}$ and the dual is $\{f\}$. We also used the morphism $V^* \otimes V \to \underline{1}$ mapping $u \otimes v$ to $u(v)$. ■

Chapter 6

Examples

In this chapter we will demonstrate how the construction works on example that should the difference between the differential and commutative cases.

Here is what happens when there are no differentiations, that is, the differentiation is identically zero.

Example 6.1 *Let C be the field of constants of $\mathbf{k_0}$. Let G the multiplicative group C^*, that is, $\mathbf{G_m}(C)$. Let $V = \mathrm{span}\{v\}$ be the usual faithful representation of G. We note that for any g in G we have*

$$v \mapsto r_{V^{(1)}}(g) \cdot v = g \cdot v,$$
$$\partial v \mapsto r_{V^{(1)}}(g) \cdot \partial v = \partial(r_V(g) \cdot v) = g \cdot \partial v,$$

because $\partial g = 0$ as $g \in C^*$. We then have that

$$V^{(1)} = \mathrm{span}\{v\} \oplus \mathrm{span}\{\partial v\}$$

as representations of G. We then conclude that $\mathbf{Rep}_G$ is generated by V as a rigid abelian tensor category. Therefore, $a_V(v \otimes v^*)$ generates the whole algebra $\mathcal{A}$ using multiplication, addition, and the coinverse only. We show that the powers of $a_V(v \otimes v^*)$ are linearly independent as different tensor powers of V are not isomorphic. Finally,

$$\partial(a_V(v \otimes v^*)) = a_{V^{(1)}}(\partial v \otimes v^*) = a_V(0 \otimes v^*),$$

where we used the morphism $\phi : V^{(1)} \to V$ *mapping* $\partial v \mapsto 0$ *and* $v \mapsto v$. *Thus, we have shown that*

$$\mathcal{A} \cong C\left[x, x^{-1}\right] \cong C\left\{x, x^{-1}\right\}/[x],$$

where $\partial x = 0$, *which is the coordinate ring of* G.

Consider the differential case now.

Example 6.2 *Let* $G = \mathbf{G_m}$ *and* $\mathbf{k_0}$ *be a differentially closed field. We again generate the category* $\mathbf{Rep}_G$ *using the faithful representation* $V = \mathrm{span}\{v\}$ *but now we need differentiation. Since the category* $\mathbf{Rep}_G$ *is generated by* $a_V(v \otimes v^*)$ *and all its derivatives using multiplication, addition, and coinverse, it remains to note that* $a_V(v \otimes v^*)$ *together with all its powers and derivatives and with* $a_{V^*}(v^* \otimes v) = (a_V(v \otimes v^*))^{-1}$ *are linearly independent.*

Indeed, let

$$\sum_j \alpha_j \left(a_V(v \otimes v^*)\right)^{i_j} + \sum_{t,p} \beta_{t,p} \left(\partial^{k_t} a_V(v \otimes v^*)\right)^{q_p} = 0.$$

Since the field $\mathbf{k_0}$ *is differentially closed, there exists and element* $f \in \mathbf{k_0}$ *such that*

$$\sum_j \alpha_j f^{i_j} + \sum_t \beta_{t,p} \left(\partial^{k_t} f\right)^{q_p} \neq 0$$

if there is a non-zero coefficient. It remains to apply the differential homomorphism $\mathcal{A} \to \mathbf{k_0}$ *which first maps* $v \mapsto f \cdot v$ *and then evaluates* v^* *on the result. Contradiction. Thus,*

$$\mathcal{A} \cong \mathbf{k_0}\left\{x, x^{-1}\right\},$$

which is the coordinate ring of the linear differential algebraic group $\mathbf{G_m}$.

Chapter 7

General definition of a differential Tannakian category

Let us now generalize the results of the previous chapters by unsing the category theory approach and listing all essential properties of the category of representation of a (pro-)linear differential group that we really use to recover the group.

7.1 Definition

Definition 7.1 *A neutral differential Tannakian category $\mathcal{C}$ over a differential field $\mathbf{k_0}$ with derivation ∂_t is a*

1. *rigid*
2. *abelian*
3. *tensor*

category such that $\mathrm{End}(\underline{1})$ *is the field* $\mathbf{k_0}$ *supplied with an*

1. *exact*
2. *faithful*

3. $\mathbf{k_0}$*-linear*

4. tensor

functor $\omega : \mathcal{C} \to \mathcal{V}_{\mathbf{k_0}}$ *(see Definition 3.1) with an operation*

$$\partial^p : X \mapsto \partial^p X,$$

for each $p \in \mathbb{Z}_{\geqslant 1}$ *commuting with* ω*, and inclusion*

$$i : \partial^{p-1} X \to \partial^p X, \quad \partial^{p-1} X \ni x \mapsto x \in \partial^p X$$

and differentiation

$$\partial : X \to \partial X, \quad X \ni x \mapsto \partial x \in \partial X$$

maps satisfying the following commutative diagrams:

$$\begin{array}{ccc} \partial^{p-1} X & \xrightarrow{\omega} & \omega(X)^{(p-1)} \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ \partial^p X & \xrightarrow{\omega} & \omega(X)^{(p)} \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\omega} & \omega(X) \\ \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} \\ \partial X & \xrightarrow{\omega} & \omega(X)^{(1)} \end{array}$$

Moreover, we require that the differential structure respects tensor products and duals:

$$\begin{array}{ccc} \partial(X \otimes Y) & \xrightarrow{\omega} & (\omega(X) \otimes \omega(Y))^{(1)} \\ \downarrow & & \downarrow \\ \partial X \otimes Y \oplus X \otimes \partial Y & \xrightarrow{\omega} & \omega(X)^{(1)} \otimes \omega(Y) \oplus \omega(X) \otimes \omega(Y)^{(1)} \end{array} \tag{7.1}$$

and

$$\begin{array}{ccc} \partial(X^*) & \xrightarrow{\omega} & (\omega(X)^*)^{(1)} \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \omega(\phi)} \\ (\partial X)^* & \xrightarrow{\omega} & \left(\omega(X)^{(1)}\right)^* \end{array} \tag{7.2}$$

for each choice of basis $\{v_1, \ldots, v_n\}$ *of* $\omega(X)$ *with the dual* $\{v_1^*, \ldots, v_n^*\}$ *and*

$$\omega(\phi) : \partial(v_j^*) \mapsto v_j^*, \ v_j^* \mapsto (\partial v_j)^*.$$

Our main goal is to prove the following theorem.

Theorem 7.2 *For a neutral differential Tannakian category $\mathcal{C}$:*

- *the functor* $\mathrm{Aut}^{\otimes,\partial}(\omega)$ *from* $\mathrm{Alg_k}(\partial)$ *to* $\{\mathrm{Groups}\}$ *is representable by a differential Hopf algebra* A,
- *the functor* ω *defines an equivalence of tensor categories* $\mathcal{C} \to \mathbf{Rep}_G$, *with* G *being the affine differential group scheme represented by* A.

We start with developing a technique for this. We partially follow [4, pages 130–137] modifying their definitions and proofs to give the correct result in the differential case. We assume the differential field $\mathbf{k_0}$ to be fixed and denote $\mathcal{V}_{\mathbf{k_0}}$ by $\mathcal{V}$.

7.2 Tensor product $\mathcal{C} \otimes \mathcal{V}$ and its properties

Let now $(\mathcal{C}, \omega)$ be a differential neutral Tannakian category. Let us fix some notation. Consider objects $V \in \mathcal{O}b(\mathcal{V})$ and $X \in \mathcal{O}b(\mathcal{C})$. Here we will supply some additional to [4, pages 131–132] proofs.

7.2.1 Formal definition

Definition 7.3 *We define*

$$V \otimes X = \{[(X^n)_\alpha, \phi_{\beta,\alpha}] \mid \alpha : (\mathbf{k_0}^n) \cong V\},$$

where $(X^n)_\alpha := X \oplus \ldots \oplus X$ *and* $\phi_{\beta,\alpha} : (X^n)_\alpha \to (X^n)_\beta$ *is defined by* $\beta^{-1} \circ \alpha$ *acting as an element of* $\mathbf{GL}_n(\mathbf{k_0})$.

Lemma 7.4 $\phi_{\gamma,\beta} \circ \phi_{\beta,\alpha} = \phi_{\gamma,\alpha}$.

Proof. $\phi_{\gamma,\beta} \circ \phi_{\beta,\alpha} = \gamma^{-1} \circ \beta \circ \beta^{-1} \circ \alpha = \gamma^{-1} \circ \alpha = \phi_{\gamma,\alpha}$. ■

Definition 7.5 *For an object* $T \in \mathcal{O}b(\mathcal{C})$ *we say that* $\Phi \in \mathrm{Hom}(V \otimes X, T)$ *if* $\Phi = \{\varphi_\alpha : (X^n)_\alpha \to T\}$ *making possible diagrams with* $\phi_{\beta,\gamma}$ *commutative. Also,* $\Phi \in$

$\mathrm{Hom}(T, V \otimes X)$ *if* $\Phi = \{(\varphi_1, \ldots, \varphi_n)_\alpha\}$, $\varphi_i : T \to X$ *making possible diagrams with* $\phi_{\beta,\gamma}$ *commutative.*

Lemma 7.6 *For* $V \in \mathcal{O}b(\mathcal{V})$ *there is a canonical* $\mathbf{k_0}$*-linear map*

$$V \to \mathrm{Hom}(X, V \otimes X), \quad v \mapsto ((\varphi^1, \ldots, \varphi^n)_\alpha : X \to X_\alpha),$$
$$(\varphi^i)_\alpha(x) = w^i \cdot x,$$

where $\alpha^{-1}(v) = w,\ w = (w^1, \ldots, w^n) \in {\mathbf{k_0}}^n$.

Proof. We have:

$$\phi_{\beta,\alpha}(\alpha^{-1}(v) \cdot x) = (\beta^{-1} \circ \alpha)(\alpha^{-1}(v) \cdot x) = \beta^{-1}(v) \cdot x = \varphi_\beta(x).$$

■

7.2.2 General properties

We will also use tensor notation for elements of $V \otimes X$. Let us explain what is meant by $v \otimes x \in V \otimes X$. The object $V \otimes X$ consist of many copies of X^n. A choice α of each copy is a choice of basis of V. The map α^{-1} gives the coordinates of a vector of V w.r.t. this basis. So, $v \otimes x$ really means $\alpha^{-1}(v) \cdot x \in (X^n)_\alpha$ (multiplication in each coordinate).

Let us show an important isomorphism.

Lemma 7.7 *We have obtained a functorial isomorphism*

$$\mathrm{Hom}(V \otimes X, T) \cong \mathrm{Hom}\left(V, \mathrm{Hom}(X, T)\right),$$

where T *is an object in* $\mathcal{C}$.

Proof. Let $\Phi : V \otimes X \to T$. Consider an arbitrary element $v \in V$. By Lemma 7.6 there exists a morphism $\psi_v : X \to V \otimes X$. Hence, we define $\Psi_\Phi : V \to \mathrm{Hom}(X, T)$ by $v \mapsto \Phi \circ \psi_v$.

Let us give the inverse to this map. Take $\Psi \in \mathrm{Hom}\,(V, \mathrm{Hom}(X,T))$. For each $v \in V$ we get the morphism $\Psi(v) : X \to T$. We then define

$$\Phi_\Psi : V \otimes X \to T, \quad v \otimes x \mapsto \Psi(v)(x), \quad x \in X$$

We have

$$\Phi_{\Psi_\Phi} : V \otimes X \to T,$$
$$v \otimes x \mapsto \Psi_\Phi(v)(x) = \Phi \circ \psi_v(x) = \Phi\left(\left(\alpha^{-1}(v) \cdot x\right)_\alpha\right) = \Phi(v \otimes x),$$

for $v \in V$ and $x \in X$, because α runs through all isomorphisms ${\mathbf{k_0}}^n \cong V$. Also,

$$\Psi_{\Phi_\Psi} : V \to \mathrm{Hom}(X,T),$$
$$(v,x) \mapsto \Phi_\Psi \circ \psi_v(x) = \Phi_\Psi((\alpha^{-1}(v) \cdot x)_\alpha) = \Phi_\Psi(v \otimes x) = \Psi(v)(x).$$

Thus, we have a bijection between objects.

We now check that this isomorphism is functorial. Let $f \in \mathrm{Hom}(T_1, T_2)$, where $T_1, T_2 \in \mathcal{O}b(\mathcal{C})$. Then the following diagram is commutative

$$\begin{array}{ccccc} T_1 & \xrightarrow{\mathrm{Hom}(V\otimes X,\cdot)} & \mathrm{Hom}(V\otimes X, T_1) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathbf{k_0}}(V, \mathrm{Hom}(X,T_1)) \\ \downarrow f & & \downarrow \mathrm{Hom}(V\otimes X,\cdot)(f) & & \downarrow \mathrm{Hom}(V,\mathrm{Hom}(X,\cdot))(f) \\ T_2 & \xrightarrow{\mathrm{Hom}(V\otimes X,\cdot)} & \mathrm{Hom}(V\otimes X, T_2) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathbf{k_0}}(V, \mathrm{Hom}(X,T_2)) \end{array}$$

because

$$(\Phi, v) \mapsto \Phi \circ \psi_v \mapsto f \circ \Phi \circ \psi_v,$$
$$(\Phi, v) \mapsto (f \circ \Phi, v) \mapsto f \circ \Phi \circ \psi_v.$$

■

Lemma 7.8 *Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor. Then $F(V \otimes X) = V \otimes F(X)$.*

Proof. We have

$$F(V \otimes X) = \{[(F(X)^n)_\alpha, F(\phi_{\beta,\alpha})] \mid \alpha : {\mathbf{k_0}}^n \cong V\},$$

where $(F(X)^n)_\alpha := F(X) \oplus \ldots \oplus F(X)$ and $F(\phi_{\beta,\alpha}) : (F(X)^n)_\alpha \to (F(X)^n)_\beta$ is defined by $\beta^{-1} \circ \alpha$. ■

7.2.3 Restriction to $\mathcal{V} \otimes \mathcal{V}$

It turns out that the definition of $\otimes$ is not new at all and works fine when both objects come from the category $\mathcal{V}$.

Lemma 7.9 *Let $\mathcal{C} = \mathcal{V}$ then $V \otimes X \cong V \otimes_{\mathcal{V}} X$ (the tensor product in $\mathcal{V}$).*

Proof. Recall that

$$V \otimes X = \{[(X^n)_\alpha, \phi_{\beta,\alpha}] \mid \alpha : \mathbf{k_0}^n \cong V\},$$

where $(X^n)_\alpha := X \oplus \ldots \oplus X$ and $\phi_{\beta,\alpha} : (X^n)_\alpha \to (X^n)_\beta$ is given by $\beta^{-1} \circ \alpha$.

By Lemma 7.6 for each $v \in V$ there is a $\mathbf{k_0}$-linear map $\varphi_v : X \to V \otimes X$. Fix now $x \in X$. Then the map $v \mapsto \varphi_v(x) \in V \otimes X$ is also $\mathbf{k_0}$-linear, because α^{-1} is a $\mathbf{k_0}$-linear isomorphism. Thus, we have a bilinear map $\Phi : V \times X \to V \otimes X$. Let now $T \in \mathcal{O}b(\mathcal{V})$ with a $\mathbf{k_0}$-bilinear map $\Psi : V \times X \to T$.

Hence, for each $v \in V$ we have the $\mathbf{k_0}$-linear morphism $\Psi(v, \cdot) : X \to T$. This correspondence defines the morphism $\psi : V \to \mathrm{Hom}(X, T)$. By Lemma 7.7 this ψ corresponds to a morphism $\phi : V \otimes X \to T$ making the diagram commutative, because $v \otimes x \mapsto \psi(v)(x)$.

It remains to show that for $T = V \otimes_{\mathcal{V}} X$ the map ϕ has an inverse. We know that $V \otimes_{\mathcal{V}} X$ satisfies the universal property and $V \otimes X = [(X^n)_\alpha]$. The map $\phi : V \otimes X \to T$ is given by

$$\varphi_\alpha : (X^n)_\alpha \to T$$

with all commutative diagrams involving $\phi_{\alpha,\beta}$. Take n ordered copies of X and map them into X^n. By composing these canonical inclusions with φ_α we obtain a collection of maps $\varphi^i_\alpha : X \to T$. Hence, for each α there is an inverse

$$(\psi_1, \ldots, \psi_n)_\alpha : V \otimes_{\mathcal{V}} X \to (X^n)_\alpha$$

such that they altogether satisfy the commutative diagrams with $\phi_{\alpha,\beta}$. Thus, we obtain that $V \otimes X \cong V \otimes_{\mathcal{V}} X$. ■

Definition 7.10 $\underline{\mathrm{Hom}}(V, X) := V^* \otimes X$.

7.3 Differential comodules

Let A be a differential algebra over the field $\mathbf{k_0}$. Assume that A is supplied with the following operations:

- differential algebra homomorphism $m : A \otimes A \to A$ is the multiplication map on A,
- differential algebra homomorphism $\Delta : A \to A \otimes A$ which is a comultiplication,
- differential algebra homomorphism $\varepsilon : A \to \mathbf{k_0}$ which is a counit,
- differential algebra homomorphism $S : A \to A$ which is a coinverse.

We also assume that these maps satisfy commutative diagrams (3.1) (see also [2, page 225]). Recall that such a commutative associative differential $\mathbf{k_0}$-algebra A with the unity and operations m, Δ, S, and ε is called a *differential Hopf algebra*.

Definition 7.11 *Assume that A is just a vector space over $\mathbf{k_0}$ with a linear operation ∂ which extends the derivation on $\mathbf{k_0}$ equipped with Δ (not necessarily an algebra homomorphism) and ε commuting with ∂. In this case A is called a differential coalgebra. When, in addition, A has S, it is called a bialgebra.*

Recall that if A is a differential coalgebra with Δ and ε then for a comodule V over A the $\mathbf{k_0}$ space $\mathbf{k_0}[\partial]_{\leqslant i} \otimes V = V^{(i)}$ has a natural A-comodule structure:

$$\rho(f \otimes v) := f \otimes \rho(v) \tag{7.3}$$

for $f \in \mathbf{k_0}[\partial]_{\leqslant i}$ and $v \in V$.

We denote the category of comodules over a differential coalgebra A by $\mathbf{CoDiff}(\partial)_A$ with the induced differentiation (7.3). For a differential Hopf algebra A the functor

$$G : \mathrm{Alg_k}(\partial) \to \{\mathrm{Groups}\}, \quad R \mapsto \mathrm{Hom}(A, R)$$

is called an *affine differential algebraic group scheme* generated by A (see Section 3.3). In this case $V \in \mathcal{O}b(\mathbf{CoDiff}(\partial)_A)$ is called a *differential representation* of G (Definition 3.9, Theorem 3.10) and the category $\mathbf{CoDiff}(\partial)_A$ is denoted by $\mathbf{Rep}_G$.

7.4 Recovering a bialgebra over $\mathbf{k_0}$

Remark 7.12 *Let $\mathcal{C} = \mathbf{Rep}_G$ for a linear differential algebraic group G and ω is the forgetful functor. This is one of main applications of the theory we are developing. As we have shown in the previous chapters, one can explicitly recover the group G knowing $\mathcal{C}$.*

Let X be an object of $\mathcal{C}$ and $\{\{X\}\}$ (respectively, $\mathcal{C}_X$) be the full abelian (respectively, full abelian tensor) subcategory of $\mathcal{C}$ generated by X (respectively, containing X and closed under ∂^p for $p \in \mathbb{Z}_{\geqslant 0}$). We will also use the notation $V^{(i)}$ for $\partial^i V$, where V is an object of $\mathcal{C}_X$.

Denote by B_X the object P (or P') contained in $V^* \otimes \omega(V)$ and constructed like in [4, Lemma 2.12] for the category $\{\{X\}\}$. Its image $\omega(B_X)$ stabilizes all $\omega(Y)$, where Y is an object of $\{\{X\}\}$. Denote $P_X = (B_X)^*$. We note that $A_X := \omega(P_X)$ is a finite dimensional vector space over $\mathbf{k_0}$.

One can construct P_X directly, but this construction is informal, since it contains infinite direct sums that might not exist in the category $\mathcal{C}$. Consider

$$F_X = \bigoplus_{V \in \mathcal{O}b(\{\{X\}\})} V \otimes \omega(V)^*.$$

For an object V of $\{\{X\}\}$ we have the canonical injections:

$$i_V : V \otimes \omega(V)^* \to F_X.$$

Consider the following subobject R_X of F_X generated by the following set

$$\left\{ (i_V(\mathrm{id} \otimes \phi^*) - i_W(\phi \otimes \mathrm{id}))(z) \;\middle|\; V,\, W \in \mathcal{O}b(\{\{X\}\}),\ \phi \in \mathrm{Hom}(V, W),\, z \in V \otimes \omega(W)^* \right\}.$$

We let

$$P_X = F_X / R_X.$$

We now put

$$A_X = \omega(P_X),$$

which is a $\mathbf{k_0}$-vector space (see Lemma 7.8). Let V be an object of $\{\{X\}\}$. For

$$v \in \omega(V),\ u \in \omega(V)^*$$

we denote by

$$a_V\left(v \otimes u\right)$$

the image in A_X of

$$i_V\left(v \otimes u\right).$$

So, for any $\phi \in \mathrm{Hom}(V, W)$ we have

$$a_V\left(v \otimes \omega(\phi)^*(u)\right) = a_W(\omega(\phi)(v) \otimes u).$$

for all

$$v \in \omega(V),\ u \in \omega(W)^*.$$

Let us define a comultiplication on A_X. Let $\{v_i\}$ be a basis of $\omega(V)$ with the dual basis $\{u_j\}$ of $\omega(V)^*$. We let

$$\Delta : a_V(v \otimes u) \mapsto \sum a_V(v_i \otimes u) \otimes a_V(v \otimes u_i). \tag{7.4}$$

The counit is defined in the following way:

$$\varepsilon : a_V(v \otimes u) \mapsto u(v). \tag{7.5}$$

The coinverse is defined in the following way:

$$S : a_V(v \otimes u) \mapsto a_{V^*}(u \otimes v). \tag{7.6}$$

Proposition 7.13 *With the operations* (7.4), (7.5), *and* (7.6) *the finite dimensional* $\mathbf{k_0}$*-vector space* A_X *is a bialgebra.*

Proof. This follows directly from Lemmas 5.9, 5.10, and 5.12. ■

7.5 Equivalence of categories

Lemma 7.14 *The restriction* $\omega|_{\{\{X\}\}} : \{\{X\}\} \to \mathcal{V}$ *factors through* $\mathbf{Comod}_{A_X}$.

Proof. Let Y be an object of $\{\{X\}\}$. We introduce an A_X-comodule structure on $\omega(Y)$. Let R be a finite dimensional vector space over $\mathbf{k_0}$. Consider $\xi \in \mathrm{Hom}(A_X, R)$. Let $v \in \omega(Y)$ and $u \in \omega(Y)^*$. There is an endomorphism λ_Y of $\omega(Y) \otimes R$ such that

$$\langle \lambda_Y(v), u \rangle = u(\lambda_Y(v)) = \xi(a_Y(v \otimes u)).$$

Put $R = A_X$ and $\xi = \mathrm{id}_{A_X}$. We obtain an A_X-linear map

$$\lambda_Y : \omega(Y) \otimes A_X \to \omega(Y) \otimes A_X.$$

Composing λ_Y with the imbedding

$$\omega(Y) \to \omega(Y) \otimes A_X, \quad v \mapsto v \otimes a_{\underline{1}}(e \otimes f),$$

where $\{e\}$ is a basis of $\omega(\underline{1})$ and $f(e) = 1$, we provide an A_X-comodule structure

$$\omega(Y) \to \omega(Y) \otimes A_X$$

on $\omega(Y)$. This implies that we have defined a functor $\{\{X\}\} \to \mathbf{Comod}_{A_X}$. The forgetful functor $\mathbf{Comod}_{A_X} \to \mathcal{V}$ closes the commutative diagram. ■

Proposition 7.15 *Let* $\omega : \mathcal{C} \to \mathcal{V}$. *Then* ω *defines an equivalence of categories* $\{\{X\}\} \to \mathbf{Comod}_{A_X}$ *carrying* $\omega|_{\{\{X\}\}}$ *into the forgetful functor.*

Proof. For an object Y of $\{\{X\}\}$ it is shown in Lemma 7.14 how to put an A_X-comodule structure on $\omega(Y)$. We first demonstrate that the induced functor

$$\omega|_{\{\{X\}\}} : \{\{X\}\} \to \mathbf{Comod}_{A_X}$$

is *essentially surjective*. For a differential comodule V over A_X consider the object $P_X \otimes V$. It has a subobject S_X generated by the kernel of the morphism

$$(\varepsilon \circ \omega) \otimes \mathrm{id}_V : P_X \otimes V \to \mathbf{k_0} \otimes V \cong V.$$

We then have

$$\omega\left((P_X \otimes V)/S_X\right) = (\omega(P_X) \otimes V)/\omega(S_X) = (A_X \otimes V)/\omega(S_X) \cong V.$$

Denote $P_X \otimes V/S_X = V_X$. Now, V_X consists of the collection (Y_α) of objects Y_α in $\mathcal{C}_X$ with isomorphisms $\phi_{\beta,\alpha} : Y_\alpha \to Y_\beta$. For any α we then have $\omega(Y_\alpha) \cong V$. Hence, $\omega|_{\{\{X\}\}}$ is essentially surjective.

We show now that $\omega|_{\{\{X\}\}}$ is *full*. For $X_1, X_2 \in \mathcal{O}b(\mathcal{C}_X)$ consider the comodules V_1 and V_2 corresponding to X_1 and X_2, respectively. Let $\varphi \in \mathrm{Hom}(V_1, V_2)$. There is a map

$$\psi_\varphi := \mathrm{id} \otimes \varphi : P_X \otimes V_1 \to P_X \otimes V_2$$

inducing a map

$$(V_1)_X \to (V_2)_X.$$

By the definition of P_X we have surjections

$$\mathrm{id} \otimes \mathrm{ev} : P_X \otimes \omega(X_i) \to X_i$$

given by

$$(v \otimes u) \otimes w \mapsto u(w) \cdot v.$$

that induce isomorphisms

$$(V_i)_X \to X_i.$$

Again, $P_X \otimes V_i$ are the collections (Y_α^i) and, hence, this ψ induces a map $Y_\alpha^1 \to Y_\alpha^2$. So, we have a morphism $X_1 \to X_2$. Finally, since ω is *faithful*, we have the required equivalence of categories $\{\{X\}\} \to \mathbf{Comod}_{A_X}$. ■

7.6 Result for the whole category

Definition 7.16 *For a ∂-differential $\mathbf{k_0}$-algebra R we define a group $\mathrm{Aut}^{\otimes,\partial}(\omega)(R)$ to be the set of sequences*

$$\lambda(R) = (\lambda_X \mid X \in \mathcal{O}b(\mathcal{C})) \in \mathrm{Aut}^{\otimes,\partial}(\omega)(R)$$

such that λ_X *is a* R*-linear automorphism of* $\omega(X) \otimes R$ *for each object* X, $\omega(X) \in \mathcal{O}b(\mathcal{V})$, *that is,* $\lambda_X \in \mathrm{Aut}_R(\omega(X) \otimes R)$, *such that*

- *for all* X_1, X_2 *we have*

$$\lambda_{X_1 \otimes X_2} = \lambda_{X_1} \otimes \lambda_{X_2}, \tag{7.7}$$

- $\lambda_{\underline{1}}$ *is the identity map on* $\underline{1} \otimes R = R$,

- *for every* $\alpha \in \mathrm{Hom}(X, Y)$ *we have*

$$\lambda_Y \circ (\alpha \otimes \mathrm{id}_R) = (\alpha \otimes \mathrm{id}_R) \circ \lambda_X : X \otimes R \to Y \otimes R, \tag{7.8}$$

- *for every* X *we have*

$$\partial \circ \lambda_X = \lambda_{\partial X} \circ \partial, \tag{7.9}$$

- *the group operation* $\lambda_1(R) \cdot \lambda_2(R)$ *is defined by composition in each set* $\mathrm{Aut}_R(\omega(X) \otimes R)$.

For $X \in \mathcal{O}b(\mathcal{C})$ we let

$$\mathcal{A}_X = \varinjlim_{Y \in \mathcal{O}b(\mathcal{C}_X)} A_Y.$$

Cosider the subspace $\mathcal{R}_X$ of $\mathcal{A}_X$ genrated by the following sets:

$$\left\{ a_V(v \otimes w^*) - a_{V^{(1)}}(\partial v \otimes (\partial w)^*) \mid v \in V,\ w \in \omega(V),\ V \in \mathcal{O}b(\mathcal{C}_X) \right\},$$

and

$$\Big\{ a_{V_1^{(q_1)} \otimes \ldots V_p^{(q_p)}} \left(\left(\partial^{i_1} v_1 \otimes \ldots \otimes \partial^{i_p} v_p \right) \otimes \left((\partial^{j_1} w_1)^* \otimes \ldots \otimes (\partial^{j_p} w_p)^* \right) \right) \ \Big| \\ \max_l (j_l - i_l) > 0,\ v_k \in V_k,\ w_k \in \omega(V_k),\ \max\{i_k, j_k\} \leqslant q_k,\ V_k \in \mathcal{O}b(\mathcal{C}_X),\ 1 \leqslant k \leqslant p \Big\}.$$

In $\mathcal{A}_X / \mathcal{R}_X$ we then have:

$$a_V(v \otimes w^*) = a_{V^{(1)}}(\partial v \otimes (\partial w)^*) \tag{7.10}$$

for all v, $w \in \omega(V)$, and

$$a_{V_1^{(q_1)} \otimes \ldots V_p^{(q_p)}} \left(\left(\partial^{i_1} v_1 \otimes \ldots \otimes \partial^{i_p} v_p \right) \otimes \left((\partial^{j_1} w_1)^* \otimes \ldots \otimes (\partial^{j_p} w_p)^* \right) \right) = 0 \qquad (7.11)$$

if for some l, $1 \leqslant l \leqslant p$, we have $i_l < j_l$. We **finally** put

$$A = \varinjlim_{X \in \mathcal{O}b(\mathcal{C})} \mathcal{A}_X / \mathcal{R}_X.$$

This is a coalgebra due to the previous considerations. We are going to make it a *differential* Hopf algebra using ∂ and $\otimes$.

For this take $v \in \omega(V)$, $w \in \omega(W)$, $u \in \omega(V)^*$, $t \in \omega(W)^*$ and let

$$a_V(v \otimes u) \cdot a_W(w \otimes t) = a_{V \otimes W}((v \otimes w)) \otimes (u \otimes t)). \qquad (7.12)$$

Introduce a *differential* structure on A :

$$\partial \left(a_V(v \otimes u) \right) = a_{V^{(1)}}((\partial v) \otimes u), \qquad (7.13)$$

which can be extended to the whole A as a derivation. Note that we extend here $u \in V^*$ to an element of $\left(V^{(1)}\right)^*$ by $u(\partial w) = \partial(u(w))$.

Proposition 7.17 *With the operations* (7.12), (7.13), (7.4), (7.5), *and* (7.6) *the* $\mathbf{k_0}$*-vector space A is a direct limit of finitely generated commutative associative differential Hopf algebras with the unity.*

Proof. All the statements follow from Proposition 7.13 and Lemmas 5.5, 5.6, 5.9, 5.10, and 5.12 except for finitely generated, because our construction of $\mathcal{A}_X/\mathcal{R}_X$ and operations on it are the same as in Chapter 5. The differential algebra

$$\mathcal{A}_X/\mathcal{R}_X$$

is generated by the elements $a_X(v \otimes u)$. The statement now follows as $\omega(X) \otimes \omega(X)^*$ is a finite dimensional $\mathbf{k_0}$-vector space. ■

Proposition 7.18 *For each object X of the category $\mathcal{C}$ the identification of $\mathcal{C}_X$ with $\mathbf{CoDiff}(\partial)_{A_X}$ provides an equivalence of categories $\mathcal{C} \to \mathbf{CoDiff}(\partial)_A$ carrying ω into the forgetful functor from $\mathbf{CoDiff}(\partial)_A$ to $\mathcal{V}$.*

Proof. Follows from Proposition 7.15. Indeed, for subcategories $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \mathcal{C}$ there is a restriction map $\varphi_{1,2} : \underline{\mathrm{End}}\,(\omega|_{\mathcal{C}_2}) \to \underline{\mathrm{End}}(\omega|_{\mathcal{C}_1})$. Hence, we have the induced map $\varphi_{1,2}{}^* : (\underline{\mathrm{End}}(\omega|_{\mathcal{C}_1}))^* \to (\underline{\mathrm{End}}(\omega|_{\mathcal{C}_2}))^*$. Moreover, $(\mathcal{C}, \omega) = \varinjlim (\mathcal{C}_X, \omega|_{\mathcal{C}_X})$ with respect to the injective system of inclusion maps. ■

Corollary 7.19 *The group G_X, defined by $G_X(R) = \mathrm{Hom}_{\mathbf{k_0}[\partial]-alg}(A_X, R)$ for each differential algebra R, is a linear differential algebraic group.*

Proof. By Proposition 7.17 the differential Hopf algebra A_X is finitely generated in the differential sense. Thus, according to the proof of [1, Proposition 12, page 914] it is a coordinate ring of a linear differential algebraic group. ■

Corollary 7.20 *For the linear differential algebraic group G_X defined above we have*

$$\mathrm{Aut}^{\otimes,\partial}\,(\omega|_{\mathcal{C}_X}) \cong G_X.$$

Proof. Follows from Proposition 7.15 and Theorem 5.2. ■

7.7 Main theorem

Theorem 7.21 *Let $(\mathcal{C}, \omega)$ be a neutral differential Tannakian category. Then*

$$(\mathcal{C}, \omega) \cong \mathbf{Rep}_G$$

for the differential group scheme

$$G = \mathrm{Aut}^{\otimes,\partial}(\omega),$$

which is

1. *represented by the commutative associative differential Hopf algebra* $A = \varinjlim A_X$;

2. *a pro-linear differential algebraic group.*

Proof. The first statement is contained in Proposition 7.18. Property (1) follows from Corollary 7.20 by taking limits. According to Corollary 7.19 for each object X of $\mathcal{C}$ the group G_X is a linear differential algebraic group. Property (2) now follows. ■

Chapter 8

Applications

We will show how one can apply the theory developed above to parametric linear differential equations. The theory of parameterized linear differential equations considers equations of the form

$$\frac{\partial Y}{\partial x} = A(t,x)Y$$

where A is an $n \times n$ matrix whose entries are functions of x and of a parameter t and was developed in detail in [3]. Formally one considers a differential field k of characteristic zero with commuting derivations $\Delta = \{\partial_t, \partial_x\}$ and we assume that $\mathbf{k_0} = \{c \in \mathbf{k} \mid \partial_x(c) = 0\}$, the constants with respect to ∂_x, forms a differentially closed ∂_t-differential field[1]. Given a linear differential equation $\partial_x Y = AY$, where A is an $n \times n$ matrix with entries in $\mathbf{k}$, there exists a Δ-differential field extension K of $\mathbf{k}$ having the same ∂_x-constants as $\mathbf{k}$ and where K is generated (as a Δ-differential field) over $\mathbf{k}$ by the entries of an $n \times n$ invertible matrix Z satisfying $\partial_x Z = AZ$ (Theorem 3.5, [3]). This field is called the *parameterized Picard-Vessiot extension* associated to the equation and is unique up to differential $\mathbf{k}$-isomorphism. The group of $\mathbf{k}$-automorphisms of $\mathbf{k}$ commuting with the derivations in Δ is called the *parameterized*

[1] In [3], larger sets of derivations are considered but, for simplicity we shall consider one parametric derivation and one principal derivation. The subscripts t and x are a convenient way of distinguishing these but we do not assume that we are dealing with functions of variables t and x.

Picard-Vessiot group of the equation and can be shown to be a linear differential algebraic group. The goal of this section is to define this group using the theory developed in the previous sections.

8.1 Preliminaries

Let $\mathbf{k}, \Delta, \mathbf{k_0}$ be as above with $\mathbf{k_0}$ differentially closed. The field of constants of $\mathbf{k_0}$ w.r.t. the differential operator ∂_t is denoted by $\mathbf{k_0}^{\{\partial_t\}}$. A ∂_x-differential $\mathbf{k}$-module M is a finite dimensional $\mathbf{k}$-vector space with a linear operator ∂_x satisfying the product rule:

$$\partial_x(rm) = (\partial_x r)m + r\partial_x m$$

for all $r \in \mathbf{k}$ and $m \in M$. Starting from Section 8.2 we will also use the notation $M^{(0)}$ for such a module and M.

Let $\{e_1, \ldots, e_n\}$ be a $\mathbf{k}$-basis of M. Define the elements $a_{ij} \in \mathbf{k}$ by

$$\partial_x e_i = -\sum_{j=1}^{n} a_{ji} e_j,$$

where $1 \leqslant i \leqslant n$. Let $u = a_1 e_1 + \ldots + a_n e_n$. Then $\partial_x u = 0$ iff

$$\sum_{i=1}^{n} \partial_x(a_i) e_i - \sum_{i=1}^{n}\sum_{j=1}^{n} a_j a_{ij} e_i = 0$$

or, equivalently,

$$\partial_x \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = A \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix},$$

where $A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix}$.

We are going to define a formal object associated with a parametric linear differential equation, e.g.,

$$\partial_x y = \frac{t}{x} y$$

which is going to be equivalent to the equation in some sense and will allow us to recover the parametric differential Galois group G from getting first the category of all finite dimensional differential rational representations of G and then taking all coordinate functions of representations. We obtain the algebra $A := \mathbf{k_0}\{G\}$ of differential algebraic functions on G, then recover derivation ∂ on A, comultiplication $\Delta : A \to A \otimes A$, e.t.c.

The construction where $M^{(i)}$ is the ∂_x-module with the matrix obtained by the ith prolongation (parallel to the differential-difference construction of [5])

$$A_i = \begin{pmatrix} A & 0 & 0 & \dots & 0 \\ A_t & A & 0 & \dots & 0 \\ A_{tt} & 2A_t & A & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{t^i} & \dots & \dots & \dots & A \end{pmatrix}$$

of $A_0 = A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$ does look like an adequate one. In the next section we will see how to treat all these $M^{(i)}$, $i \geqslant 0$.

8.2 The category $\mathcal{M}$

We first give a coordinate free definition and then introduce a canonical basis and use this constructive approach to develop our theory.

Definition 8.1 *We let*

1. *an object M of $\mathcal{M}$ be an object of $\mathcal{V}$ together with a differential module structure given by ∂_x which commutes with ∂,*

2. *morphisms between objects of $\mathcal{M}$ be those which commute with the action of ∂_x :*

$$\mathrm{Hom}(M,N) = \mathrm{Hom}_{\mathbf{k}[\partial_x]}(M,N)$$

for all objects M and N of $\mathcal{M}$,

3. *subobjects, $\otimes$, $\oplus$, and * are as in the category of differential modules [11, Section 2.2].*

Remark 8.2 *Recall that*

$$M^{(i)} = \mathbf{k}[\partial]_{\leqslant i} \otimes M^{(0)},$$

where $M^{(0)}$ is another notation for M.

8.3 The equations-modules correspondence

Proposition 8.3 *Consider a parametrized differential equation $\partial_x Y = A(x,t)Y$ with $A \in \mathbf{gl_n}_n(\mathbf{k})$. Then in the sequence of differential modules starting with*

$$M^{(0)} = \mathrm{span}_k\{e_1, \ldots, e_n\}$$

given by A and continuing by induction for

$$M^{(i)} := \mathrm{span}_k\{\partial^i e_1, \ldots, \partial^i e_n, \partial^{i-1} e_1, \ldots, \partial^{i-1} e_n, \ldots, e_1, \ldots, e_n\}$$

the operation ∂ commutes with ∂_x.

Proof. Defined in such a way ∂ commutes with ∂_x and the images of bases vectors at each step are linearly independent and generate the whole module. The latter

statement follows immediately. We check the former one by induction. Fix a natural number $i \geqslant 0$. The action of ∂_x on $M^{(i)}$ and $M^{(i+1)}$ is given by the matrices

$$A_i = \begin{pmatrix} A & 0 & 0 & \dots & 0 \\ A_t & A & 0 & \dots & 0 \\ A_{tt} & 2A_t & A & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{t^i} & \binom{i}{1} A_{t^{i-1}} & \binom{i}{2} A_{t^{i-2}} & \dots & A \end{pmatrix},$$

$$A_{i+1} = \begin{pmatrix} A & 0 & 0 & \dots & 0 \\ A_t & A & 0 & \dots & 0 \\ A_{tt} & 2A_t & A & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{t^{i+1}} & \binom{i+1}{1} A_{t^i} & \binom{i+1}{2} A_{t^{i-1}} & \dots & A \end{pmatrix},$$

respectively. Hence, for any l, $1 \leqslant l \leqslant n$ and m, $0 \leqslant m \leqslant i$, we have

$$[\partial_x, \partial](\partial^m e_l) = (\partial_x \partial - \partial \partial_x)(\partial^m e_l) = \partial_x(\partial^{m+1} e_l) - \partial^{m+1}(\partial_x e_l) =$$
$$= -\sum_{q=0}^{m+1} \sum_{r=1}^{n} \binom{m+1}{q} (\partial_t^q a_{rl})(\partial^{m+1-q} e_r) + \partial^{m+1} \left(\sum_{r=1}^{n} a_{rl} e_r \right) = 0.$$

■

And vice versa:

Lemma 8.4 *For an object M the matrix A_{i+1} corresponding to the action of ∂_x on $M^{(i+1)}$ with respect to the ordered basis*

$$\{\partial^{i+1} e_1, \dots, \partial^{i+1} e_n, \dots, e_1, \dots, e_n\}$$

is of the form

$$\tilde{A}_{i+1} = \begin{pmatrix} A & 0 & 0 & \dots & 0 \\ \binom{i+1}{1} A_t & A & 0 & \dots & 0 \\ \binom{i+1}{2} A_{tt} & \binom{i}{1} A_t & A & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{t^{i+1}} & \binom{i}{i} A_{t^i} & A_{t^{i-1}} & \dots & A \end{pmatrix},$$

where

$$A_0 = A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

is the matrix for ∂_x in the module $M^{(0)}$.

Proof. For $i = 1$ the basis is $\{\partial e_1, \dots, \partial e_n, e_1, \dots, e_n\}$. For k, $1 \leqslant k \leqslant n$, we have

$$\partial_x(\partial e_k) = \partial(\partial_x(e_k)) = \partial\left(-\sum_{p=1}^{n} a_{pk} e_p\right) = -\sum_{p=1}^{n} a_{pk}\partial e_p - \sum_{p=1}^{n}(\partial_t a_{pk}) e_p,$$

$$\partial_x(e_k) = -\sum_{p=1}^{n} a_{pk} e_p.$$

Assume the result for $i = m$. We have $\{\partial^{m+1} e_1, \dots, \partial^{m+1} e_n, \dots, e_1, \dots, e_n\}$ as the ordered basis of $M^{(m+1)}$. For each l, $1 \leqslant l \leqslant n$, we have

$$\begin{aligned}
\partial_x(\partial^{m+1} e_l) &= \partial(\partial_x(\partial^m e_l)) = \\
&= \partial\left(-\sum_{q=0}^{m}\sum_{r=1}^{n}\binom{m}{q}(\partial_t^q a_{rl})\partial^{m-q} e_r\right) = \\
&= -\sum_{q=0}^{m}\sum_{r=1}^{n}\binom{m}{q}(\partial_t^{q+1} a_{rl})\partial^{m-q} e_r - \sum_{q=0}^{m}\sum_{r=1}^{n}\binom{m}{q}(\partial_t^q a_{rl})\partial(\partial^{m-q} e_r) = \\
&= -\sum_{q=0}^{m}\sum_{r=1}^{n}\binom{m}{q}(\partial_t^{q+1} a_{rl})\partial^{m+1-(q+1)} e_r - \sum_{q=0}^{m}\sum_{r=1}^{n}\binom{m}{q}(\partial_t^q a_{rl})\partial^{m+1-q} e_r = \\
&= -\sum_{q=1}^{m+1}\sum_{r=1}^{n}\binom{m}{q-1}(\partial_t^q a_{rl})\partial^{m+1-q} e_r - \sum_{q=0}^{m}\sum_{r=1}^{n}\binom{m}{q}(\partial_t^q a_{rl})\partial^{m+1-q} e_r = \\
&= -\sum_{q=0}^{m+1}\sum_{r=1}^{n}\binom{m+1}{q}(\partial_t^q a_{rl})\partial^{m+1-q} e_r,
\end{aligned}$$

because $\binom{m}{q-1} + \binom{m}{q} = \binom{m+1}{q}$. ■

Proposition 8.5 *For an object M of $\mathcal{M}$ there exists an ordered basis of each $M^{(i)}$ such that the matrix A_i corresponding to the action of ∂_x w.r.t. this basis is of the*

form

$$A_i = \begin{pmatrix} A & 0 & 0 & \dots & 0 \\ A_t & A & 0 & \dots & 0 \\ A_{tt} & \binom{2}{1} \cdot A_t & A & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{t^i} & \binom{i}{1} A_{t^{i-1}} & \binom{i}{2} A_{t^{i-2}} & \dots & A \end{pmatrix},$$

where

$$A_0 = A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

is the matrix for ∂_x in the module M.

Proof. According to Lemma 8.4 for each $M^{(i)}$ there exists an ordered basis

$$\{\partial^i e_1, \dots, \partial^i e_n, \dots, e_1, \dots, e_n\}$$

such that the matrix corresponding to ∂_x is

$$\tilde{A}_i = \begin{pmatrix} A & 0 & 0 & \dots & 0 \\ \binom{i}{1} \cdot A_t & A & 0 & \dots & 0 \\ \binom{i}{2} \cdot A_{tt} & \binom{i-1}{1} \cdot A_t & A & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{t^i} & \dots & \dots & \dots & A \end{pmatrix}.$$

Denote this basis by $\{f_1, \dots, f_{(i+1)\cdot n}\}$. Note that if we change a basis:

$$\begin{pmatrix} g_1 \\ \vdots \\ g_{(i+1)\cdot n} \end{pmatrix} = C \cdot \begin{pmatrix} f_1 \\ \vdots \\ f_{(i+1)\cdot n} \end{pmatrix}$$

and all entries of the matrix C are constants then the corresponding matrix A_i must be replaced by $(C^T)^{-1} A_i C^T$. Consider the basis $\{g_1, \dots, g_{(i+1)\cdot n}\}$ given by the following

change-of-basis matrix

$$C := \begin{pmatrix} E & 0 & 0 & \dots & 0 \\ E & \binom{i}{1} \cdot E & 0 & \dots & 0 \\ E & \binom{i-1}{1} \cdot E & \binom{i}{2} \cdot E & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ E & 2 \cdot E & 3 \cdot E & \dots & 0 \\ 0 & E & E & \dots & E \end{pmatrix}^T,$$

where E is the $n \times n$ identity matrix. Indeed, for l, $1 \leqslant l \leqslant n$ we have

$$\begin{aligned} \partial_x g_l &= \partial_x (f_l + f_{l+n} + \ldots + f_{l+(i-1)\cdot n}) = \\ &= \partial_x (\partial^{i-1} e_l + \partial^{i-2} e_l + \ldots + e_l) = \\ &= \sum_{p=0}^{i-1} \left(- \sum_{q=p}^{i} \sum_{k=1}^{n} \binom{i-p}{q-p} (\partial_t^{q-p} a_{kl}) \partial^{i-1-q} e_k \right) = \\ &= - \sum_{p=0}^{i} \sum_{k=1}^{n} \left(\sum_{q=0}^{p} \binom{i}{q} (\partial_t^q a_{kl}) \right) \partial^{i-1-p} e_k = \\ &= - \sum_{p=0}^{i} \sum_{k=1}^{n} (\partial_t^p a_{kl}) \left(\sum_{q=p}^{i} \binom{i-(q-p)}{p} \partial^{i-1-q} e_k \right) = \\ &= - \sum_{p=0}^{i} \sum_{k=1}^{n} (\partial_t^p a_{kl}) g_{k+p\cdot n} \end{aligned}$$

and for all k, $1 \leqslant k \leqslant n$, we have

$$\begin{aligned}
\partial_x(g_{k+p\cdot n}) &= \partial_x\left(\sum_{q=p}^{i}\binom{i-(q-p)}{p} f_{k+q\cdot n}\right) = \\
&= \partial_x\left(\sum_{q=p}^{i}\binom{i-(q-p)}{p}\partial^{i-1-q}e_k\right) = \\
&= \sum_{q=p}^{i}\binom{i-(q-p)}{p}\partial_x\partial^{i-1-q}e_k = \\
&= -\sum_{q=p}^{i}\binom{i-(q-p)}{p}\sum_{l=1}^{n}(\partial_t^{q-p}a_{lk})\partial^{i-1-q}e_l = \\
&= -\sum_{r=0}^{i-p}\sum_{l=1}^{n}\binom{p+r}{r}(\partial_t^r a_{lk})\left(\sum_{s=p+r}^{i}\frac{\binom{i-s+p+r}{p}\binom{i-p+p+r-s}{r}}{\binom{p+r}{r}}\partial^{i-1-s}e_l\right) = \\
&= -\sum_{r=0}^{i-p}\binom{p+r}{r}\sum_{l=1}^{n}(\partial_t^r a_{lk})\left(\sum_{s=p+r}^{i}\binom{i-s+p+r}{p+r}\partial^{i-1-s}e_l\right) = \\
&= -\sum_{r=0}^{i-p}\binom{p+r}{r}\sum_{l=1}^{n}(\partial_t^r a_{lk})g_{l+(p+r)\cdot n},
\end{aligned}$$

because

$$\begin{aligned}
\frac{\binom{i-s+p+r}{p}\binom{i-p+p+r-s}{r}}{\binom{p+r}{r}} &= \frac{(i-s+p+r)!(i+r-s)!r!p!}{p!(i-s+r)!r!(i-s)!(p+r)!} = \\
&= \frac{(i-s+p+r)!}{(i-s)!(p+r)!} = \binom{i-s+p+r}{p+r}.
\end{aligned}$$

■

8.4 Covariant solution space

Fix an object M of $\mathcal{M}$. Let K be a parametric Picard-Vessiot extension of $\mathbf{k}$ for the equation

$$\partial_x Y = AY,$$

where A is the matrix corresponding to the $\mathbf{k}$-finite dimensional $k[\partial_x]$-module M. Then the covariant solution space is

$$V = \ker(\partial_x, K \otimes M),$$

which is a vector space over the field $\mathbf{k_0}$. We let

$$V^{(i)} = \ker\left(\partial_x, K \otimes M^{(i)}\right),$$

which is an $(i+1)\cdot \dim M^{(0)} = (i+1)\cdot n$-dimensional vector space over $\mathbf{k_0}$. Let $Y \in \mathbf{GL}_n(K_0)$ be a solution matrix for $V^{(0)}$. Then, the columns of the matrix

$$Y_i := \begin{pmatrix} Y & 0 & 0 & \dots & 0 \\ Y_t & Y & 0 & \dots & 0 \\ Y_{tt} & Y_t & Y & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Y_{t^{i-1}} & Y_{t^{i-2}} & Y_{t^{i-3}} & \dots & 0 \\ Y_{t^i} & Y_{t^{i-1}} & Y_{t^{i-2}} & \dots & Y \end{pmatrix}$$

form a $\mathbf{k_0}$-basis for $V^{(i)}$, since $Y_i \in \mathbf{GL}_{n\cdot(i+1)}(K)$. Hence, for each M we have a map sending $M^{(i)} \mapsto V^{(i)}$ and $V^{(i)}$ can be identified with the vector space $\mathbf{k_0}[\partial]_{\leqslant i} \otimes V^{(0)}$. Moreover, (7.1) and (7.2) are satisfied. So, there is a map $S : M \mapsto S(M)$ from the category $\mathcal{M}$ to the category $\mathcal{V}$.

8.5 Some examples

Example 8.6 *Consider a matrix* $A \in \mathbf{M}_n\left(\mathbf{k_0}^{\{\partial_t\}}\right)$ *and the differential equation*

$$\partial_x Y = AY.$$

Then, V_i *is the* ∂_x*-differential module with the matrix*

$$A_i = \operatorname{diag}(A, \dots, A)$$

and the $(i+1)$th solution matrix is

$$Y_i = \mathrm{diag}(Y, \ldots, Y).$$

So, we do not bring anything new to the usual differential Galois theory.

Example 8.7 *Consider the differential equation*

$$\partial_x y = \frac{t}{x} y.$$

Then,

$$\begin{aligned}
A_0 &= \left(\tfrac{t}{x}\right), \\
A_1 &= \begin{pmatrix} \frac{t}{x} & 0 \\ \frac{1}{x} & \frac{t}{x} \end{pmatrix}, \\
&\vdots \\
A_i &= \begin{pmatrix} \frac{t}{x} & 0 & 0 & 0 & \ldots & 0 \\ \frac{1}{x} & \frac{t}{x} & 0 & 0 & \ldots & 0 \\ 0 & \frac{2}{x} & \frac{t}{x} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & \frac{t}{x} \end{pmatrix}, \\
&\vdots
\end{aligned}$$

and the solution matrices are

$$
\begin{aligned}
Y_0 &= \begin{pmatrix} x^t \end{pmatrix}, \\
Y_1 &= \begin{pmatrix} x^t & 0 \\ x^t \log x & x^t \end{pmatrix}, \\
&\vdots \\
Y_i &= \begin{pmatrix} x^t & 0 & 0 & 0 & \dots & 0 \\ x^t \log x & x^t & 0 & 0 & \dots & 0 \\ x^t (\log x)^2 & x^t \log x & x^t & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x^t (\log x)^i & x^t (\log x)^{i-1} & x^t (\log x)^{i-2} & x^t (\log x)^{i-3} & \dots & x^t \end{pmatrix}, \\
&\vdots
\end{aligned}
$$

8.6 Fiber functor

Proposition 8.8 *The map S is an exact $\mathbf{k_0}$-linear faithful tensor covariant functor commuting with ∂.*

Proof. We have:

- Consider a short exact sequence

$$0 \longrightarrow L \xrightarrow{\varphi_1} M \xrightarrow{\varphi_2} N \longrightarrow 0$$

  Since $\mathbf{k}$ is a field, the sequence

$$0 \longrightarrow K \otimes_k L \xrightarrow{\mathrm{id}_K \otimes \varphi_1} K \otimes_k M \xrightarrow{\mathrm{id}_K \otimes \varphi_2} K \otimes_k N \longrightarrow 0 \tag{8.1}$$

is also exact. We have

$$\begin{aligned}\dim_K K\otimes M-\dim_K K\otimes L&=\dim_K K\otimes N,\\ \dim_{\mathbf{k_0}}\ker\left(\partial_x,K\otimes M\right)&=\dim_K K\otimes M,\\ \dim_{\mathbf{k_0}}\ker\left(\partial_x,K\otimes L\right)&=\dim_K K\otimes L,\\ \dim_{\mathbf{k_0}}\ker\left(\partial_x,K\otimes N\right)&=\dim_K K\otimes N.\end{aligned}$$

Consider the following short sequence:

$$\begin{aligned}0\longrightarrow\ \ker\left(\partial_x,K\otimes_k L\right)\ \longrightarrow\ \ker\left(\partial_x,K\otimes_k M\right)\ \longrightarrow\\ \longrightarrow\ \ker\left(\partial_x,K\otimes_k N\right)\ \longrightarrow\qquad 0\end{aligned}\tag{8.2}$$

The first and second elements of this sequence are subsets of the corresponding elements of the sequence (8.1) and, hence, sequence (8.2) is exact at those elements. Since $\ker(\partial_x,K\otimes M)/\ker(\partial_x,K\otimes L)$ is isomorphic to a subspace of $\ker(\partial_x,K\otimes N)$ and according to the dimensional equalities we have

$$\dim_{\mathbf{k_0}}\ker\left(\partial_x,K\otimes M\right)/\ker\left(\partial_x,K\otimes L\right)=\dim_{\mathbf{k_0}}\ker\left(\partial_x,K\otimes N\right),$$

the sequence (8.2) is exact at the third element.

- We have $\varphi=a\varphi_1+b\varphi_2$ commutes with ∂_x for all $\varphi_i\in\mathrm{Hom}(M,N)$ and $a,b\in\mathbf{k_0}$. Hence, φ is mapped to $aS(\varphi_1)+bS(\varphi_2)$. Thus, S is $\mathbf{k_0}$-linear.

- A non-zero module provides a non-zero solution.

- For modules M and N we only need to show that

$$\ker\left(\partial_x,K\otimes M\right)\otimes\ker\left(\partial_x,K\otimes N\right)=\ker\left(\partial_x,K\otimes M\otimes N\right),$$

but this follows as in the usual differential case by taking K-bases $\{e_i\}$ and $\{f_j\}$ of $K\otimes M$ and $K\otimes N$, respectively, with $\partial_x(e_i)=\partial_x(f_j)=0$.

- Let $\varphi\in\mathrm{Hom}(M,N)$. Then it is mapped to the morphism $\mathrm{id}\otimes\varphi$ of the spaces $\ker(\partial_x,K\otimes M)$ and $\ker(\partial_x,K\otimes N)$.

■

Corollary 8.9 *The category $\mathcal{M}$ together with the functor S is a neutral differential Tannakian category (Definition 7.1).*

Corollary 8.10 *For a parametric linear differential equation*

$$\partial_x Y = A(x,t)Y$$

a parametric differential Galois group G is the functor $\mathrm{Aut}^{\otimes,\partial}(S)$, *where S is the functor associated with $M \in \mathcal{M}$, where the action of ∂_x defined by the matrix $A \in \mathbf{M}_n(\mathbf{k})$.*

Proof. From Theorem 7.21 and Corollary 8.9 it follows that the category $\mathcal{C}_M$ generated by the module M and all its derivatives $M^{(i)}$ and the dual M^* using the operations $\oplus$, $\otimes$, and subquotient together with the functor S form a neutral differential Tannakian category. We say that $\mathcal{C}_M$ is *differentially generated* by M. Moreover, $\mathcal{C}_M$ is equivalent to the category of representations of the linear differential algebraic group $\mathrm{Aut}^{\otimes,\partial}(S)$. It remains to show that for the parametric Galois group G its category of representations is equivalent to $\mathcal{C}_M$.

Similar to [11, Theorem 2.33], in order to get this equivalence we notice the following. Consider a parametrized Picard-Vessiot extension K of $\mathbf{k}$ corresponding to the equation $\partial_x Y = AY$. This field K is generated over $\mathbf{k}$ by a fundamental set of solutions of the differential equation and G acts on the $\mathbf{k_0}-$linear space generated by these solutions. So, G acts on K and commutes with ∂_x. Hence, if for $f \otimes m \in K \otimes M$ and $\sigma \in G$ we let

$$\sigma(f \otimes m) = \sigma(f) \otimes m,$$

then the linear differential algebraic group G acts faithfully on $S(M)$, which is an object of the category $\mathcal{V}$.

From Proposition 4.2 it follows that the category $\mathbf{Rep}_G$ is *differentially* generated by a faithful representation V of G. We take $V = S(M)$ and, hence, for every representation W there exists an object M_W in $\mathcal{C}_M$ such that $S(M_W) = W$. Hence, the map S is essentially surjective. We need to show that

$$\mathrm{Hom}(M, N) \to \mathrm{Hom}(S(M), S(N))$$

is a bijection. We have:

$$\mathrm{Hom}_{\mathbf{k}[\partial_x]}(M, N) = \ker(\partial_x, M^* \otimes N) = \mathrm{Hom}_{\mathbf{k}[\partial_x]}(\mathbf{k_0}, M^* \otimes N).$$

So, it is enough to show the bijection for $M = \mathbf{k}$. We have $S(k) = \mathbf{k_0}$, which is the trivial representation of the group G. Moreover,

$$\mathrm{Hom}_{\mathbf{k}[\partial_x]}(\mathbf{k_0}, N) = \left\{ n \in N \;\middle|\; \partial_x n = 0 \right\}$$

and

$$\mathrm{Hom}_{\mathbf{k_0}}(\mathbf{k_0}, S(N)) = \left\{ v \in \ker(\partial_x, K \otimes N) \;\middle|\; gv = v \text{ for all } g \in G \right\}.$$

But the fixed points K^G are $\mathbf{k}$ (see [3, Theorem 3.5]). Hence, if $g(f \otimes n) = f \otimes n$ for $f \in K$ and $n \in N$ for all $g \in G$ then $f \in \mathbf{k}$, and

$$(K \otimes N)^G = N.$$

Thus, we have shown a bijection between $\mathrm{Hom}(M, N)$ and $\mathrm{Hom}(S(M), S(N))$. ■

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