# Approximate Creep Analysis for Cyclic Change of Loading and Temperature

A.S.L. Chan, C.B. Spiliopoulos

*Aeronautics Department, Imperial College of Science & Technology, Prince Consort Road, London SW7 2BY, U.K.*

Summary

The paper describes a method for estimating the long term effects on structures under cyclic changes of loading and temperature. For loads less than a certain critical value (shakedown limit), it is shown that the stress in the structure will, after a sufficient number of cycles, asymtote to a cyclic stationary state, consisting of an elastic part in response to the cyclic loading, plus a constant system of self-equilibrating residual stresses, corresponding to which the creep energy dissipation over an entire cycle of loading is a maximum, hence providing an upper bound to the total work per cycle after the structure has shakedown. Instead of following the exact time history to reach this state, this solution is obtained by a successive approximation procedure starting with any admissible residual stress distribution as an initial trial value, and correcting it by the effect of the creep and plastic strains accumulated over an entire cycle, which are in general not compatible and require a self-equilibrating stress system to give an elastic strain so as to make the total strain satisfy compatibility. This is repeated until the steady state is reached if and when no further correction is necessary. If the incremental procedure does not converge to such a state, then the level of loading is outside the shakedown limit. The nonlinear problem at every approximation step may be solved by any convenient method. The procedure is an artificial device for obtaining the solution with no correspondence to the real event, and thus can be accelerated by a suitable choice of initial starting value for residual stress. It gives the upper-bound cyclic stationary solution for a given amplitude of cyclic loading, and may be used to find the shakedown limit if so desired.

## 1. Introduction

A great deal of effort has already been spent on the assessment of the long term effects on structures under cyclic loading and temperature. It is rightly recognised that an exact evaluation of the structural response at every instant of time would be unnecessary as well as uneconomical, and the purpose could be served by simpler estimations of the bounds on the deformation or dissipated energy. This paper discussed a new procedure for the calculation of an upper bound solution, established by following the usual energy principles as given in references [1,2], but modified to include the effect of plastic deformation and to remove some of the unnecessary restrictions, to be applied with the finite element method. It stipulates that, for loading below a certain shakedown limit, the structure will settle down to a cyclic stationary state after a sufficient number of cycles, with a constant system of plastic strain and residual stress, and will respond to the cyclic loading only elastically. The creep strain rate can then be determined from the creep law and then the amount of deformation incurred over cycle estimated.

The problem is thereby reduced to that of finding the residual stress sytem corresponding to the cyclic stationary state, which must of course satisfy the usual equilibrium and compatibility requirements. This is done by a series of successive approximations, starting from any admissible initial residual stress system, and correcting it by the plastic and creep strains accumulated over an entire cycle, during which the residual stress system is maintained constant. Thus, the procedure resembles a sequence of psuedo-time steps, solving a boundary value problem at discrete points of time corresponding to the end of a loading when no force is applied and only the residual stress sytem exists. If the applied load is below the shakedown limit, the calculation will converge and the residual stress will remain unchanged. This will not take place if the applied load exceeds the shakedown limit.

Although the calculation has the appearance of a time stepping procedure, it does not represent the real event since it may start with any initial admissible residual stress system. The real time sequence is approached if initially the residual stress is taken to be zero, but convergence to the cyclic stationary state will be slow. On the other hand, if the residual stress system is taken to be that due to a constant application of the maximum load inducing only creep strains, (with no plastic strain taken into account), then convergence to the required solution can be extremely fast.

The nonlinear boundary value problem at every step of the successive approximation sequence can be solved iteratively by the Newton-Raphson method. The solution of the somewhat simpler problem at the beginning to obtain the initial trial values for the residual stresses may be treated in the same way, or by any other convenient method which would achieve rapid convergence.

## 2. Formulation of the Upper Bound

Consider a structure subjected to a periodic loading $R(t)$ at time $t$ with a cycling time $\tau$ so that $R(t+\tau) = R(t)$, inducing a strain $\varepsilon(t)$ in response. The total rate of strain $\varepsilon$ may be considered as the sum of

$$\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_p + \dot{\varepsilon}_c = \dot{\varepsilon}_e + \dot{\gamma} \tag{1}$$

where $\dot{\varepsilon}_e$, $\dot{\varepsilon}_p$, and $\dot{\varepsilon}_c$ are the elastic, plastic and creep strain rate components, related to the stress $\sigma$ by Hooke's law, the Prandtl-Reuss plastic flow rule and the viscous energy dissipation function respectively. $\dot{\gamma}$ is the inelastic strain rate, being the sum of $\dot{\varepsilon}_p + \dot{\varepsilon}_c$. Here, all stresses and strains are defined as column matrices. These constitutive laws are of the form

$$\dot{\varepsilon}_e = C\,\dot{\sigma} \tag{2}$$

$$\left.\begin{aligned} \dot{\varepsilon}_p &= \lambda\left(\frac{\partial f}{\partial \sigma}\right)^t \text{ for } f(\sigma) = 0 \\ &= 0 \qquad\quad \text{ for } f(\sigma) < 0 \end{aligned}\right\} \tag{3}$$

where $f(\sigma) = 0$ defines the yield surface for the material, assumed to be perfectly plastic. The creep energy dissipation function is defined as

$$\dot{D}(\sigma) = \sigma^t\,\dot{\sigma}_c = \phi(\sigma)\ g(t) \tag{4}$$

where $\phi(\sigma)$ is a homogeneous function of $\sigma$ of degree $n+1$, ($n$ is the creep index in Norton's power law), and $g(t)$ is a function of time, so that

$$\dot{\varepsilon}_c = \frac{1}{n+1}\left(\frac{\partial \phi}{\partial \sigma}\right)^t g(t) \tag{5}$$

(note that since $\phi = \frac{\partial \phi}{\partial \sigma}\,d\sigma$ where $d\sigma$ is a column, $\frac{\partial \phi}{\partial \sigma}$ is a row matrix to give a scalar $d\phi$).

For stable materials, the Drucker's postulate shows that the surface $\phi$ is convex, in much the same way as the plastic yield surface $f$ is convex.

For two states of stresses $\sigma$ and $\bar{\sigma}$ on the $\phi$ surface, the convexity condition gives

$$\frac{\partial \phi}{\partial \sigma}\,(\bar{\sigma} - \sigma) \leqslant 0 \tag{6}$$

where $\frac{\partial \phi}{\partial \sigma}$ is the outward normal gradient to the surface at $\sigma$ (Fig. 1). The identity

$$0 \equiv \phi\,(\bar{\sigma}) - \phi(\sigma) \tag{7}$$

may be added to eq.(6) and using the definitions in eqs.(4 & 5), it becomes

$$(n+1)\ \dot{\varepsilon}_c^{\,t}\,(\bar{\sigma} - \sigma) \leqslant \bar{\sigma}^t\,\dot{\bar{\varepsilon}}_c - \sigma^t\,\dot{\varepsilon}_c \tag{8}$$

rearranging and replacing $\dot{\varepsilon}_c$ by $\dot{\gamma} - \dot{\varepsilon}_p$, eq.(8) becomes

$$\begin{aligned} \sigma\,\dot{\gamma} \leqslant\ & (n+1)(\sigma - \bar{\sigma})^t\dot{\gamma} + (n+1)(\bar{\sigma} - \sigma)^t\dot{\varepsilon}_p \\ & + \sigma^t\,\dot{\varepsilon}_p + \bar{\sigma}^t\,\dot{\bar{\varepsilon}}_c \end{aligned} \tag{9}$$

The total inelastic work over one complete cycle of loading in the structure is then

$$\int_V \int_0^T \sigma^t\,\dot{\gamma}\,dt\,dV \leqslant \int_V \int_0^T \left[\text{right hand side of (9)}\right] dt\,dV \tag{10}$$

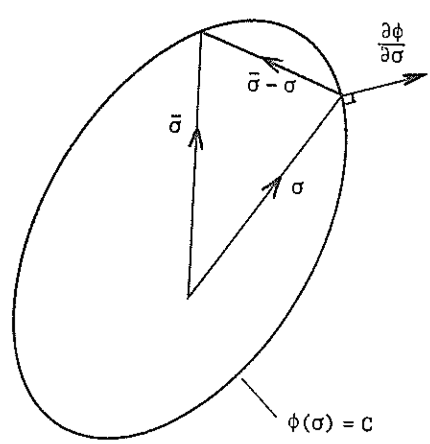

FIG. 1.

The stress state $\sigma$ may be considered to be the combination of an elastic stress distribution $\sigma_e(t)$ in equilibrium with the cyclic applied loading, together with a residual self-equilibrating stess system $\rho(t)$, such that their combination

$$\sigma(t) = \sigma_e(t) + \rho(t) \tag{11}$$

gives the compatible strain rate $\dot{\varepsilon}$

The stress $\bar{\sigma}$ may be arbitrarily defined to be the combination of the elastic response $\sigma_e(t)$ to the applied load and a constant (in time) system of residual stress $\bar{\rho}$

$$\bar{\sigma}(t) = \sigma_e(t) + \bar{\rho} \tag{12}$$

where $\bar{\rho}$ is such that $\bar{\sigma}$ is within the plastic yield limits

$$f(\bar{\sigma}) \leqslant 0 \tag{13}$$

and has the same creep energy dissipation rate as $\sigma$, as stated in eq.(7).

The difference $\sigma - \bar{\sigma} = \rho - \bar{\rho}$ is a self-equilibrating stress sytem corresponding to zero external applied load. The rate of virtual work done by this stress system on the compatible strain rate $\dot{\varepsilon}$ is zero. Hence

$$(\sigma - \bar{\sigma})^t \dot{\gamma} = -(\sigma - \bar{\sigma})^t \dot{\varepsilon}_e \tag{14}$$

When this is integrated over one complete cycle of loading

$$\int_0^\tau (\sigma - \bar{\sigma})^t \dot{\gamma}\, dt = 0 \tag{15}$$

since the elastic strain $\varepsilon_e$ is completely recovered after a cycle. The first term on the right hand side of eq.(10) therefore vanishes.

Applying the same kind of reasoning with respect to the convex yield surface $f$, as that for $\phi$ in establishing the inequality (6), (e.g. consider Fig. 1 as an $f$ surface instead of a $\phi$ surface), an analogous inequality

$$(\bar{\sigma} - \sigma)^t \dot{\varepsilon}_p \leqslant 0 \tag{16}$$

can also be established.

Hence the inequality eq.(10) is strengthened if the second term on the right hand side is ignored. The elastic strain rate $\dot{\varepsilon}_e$ may also be added to $\dot{\gamma}$ on the left hand side of (10) since it has no effect when integrated over a complete cycle. The inequality (10) then becomes

$$\int_0^\tau \left(\int_V \sigma^t \dot{\varepsilon}\, dV\right) dt \leqslant \int_0^\tau \left(\int_V \bar{\sigma}^t \dot{\bar{\varepsilon}}_c\, dV\right) dt + \int_0^\tau \int_V \sigma^t \dot{\varepsilon}_p\, dV\, dt \tag{17}$$

Thus the total creep dissipation energy over a cycle of the cyclic stationary solution $\bar{\sigma}$, together with the total plastic dissipation energy over the cycles, provides an upper bound to the total work per cycle. For a cyclic loading which would eventually allow the structure to shakedown, the plastic strain rate $\dot{\varepsilon}_p = 0$ and hence the plastic dissipation energy will disappear, then the total work will be bounded by the creep dissipation energy of the upper bound solution only.

It can readily be seen that when no plastic deformation is involved, all terms involving $\dot{\varepsilon}_p$ will disappear, but the conclusion reached above remains completely valid. Furthermore, if any thermal strain is present, due to, say, a cyclic change of temperature $T(t)$, it may be included as part of the elastic strain since it will return to the original value after a complete cycle of loading and the above formulation is once again unchanged.

It has been shown, reference [3], that if the stress system $\bar{\sigma}$ exists, the actual stress system $\sigma$ will approach this state after a sufficient number of loading cycles. $\bar{\sigma}$ is known as the cyclic stationary state in which the structure has shakedown after some initial plastic deformation, and is now responding to the applied cyclic loading purely elastically, together with a constant system of residual stresses. The creep dissipation energy of the cyclic stationary stress system also provides an upper bound to the total work per cycle of loading after shakedown has taken place and $\dot{\varepsilon}_p$ has vanished.

## 3. Calculation Procedure

To obtain this assymtotic solution for $\sigma$, which would provide also an upper bound on the creep energy dissipated over a cycle of loading, it is only necessary to find the self-equilibrating stress system $\bar{\rho}$, constant in time, which when added to the elastic response $\sigma_e(t)$, will give the required cyclic stationary solution $\bar{\sigma}(t)$, providing that it does not exceed the yield limit anywhere, and gives the same creep energy dissipation rate as the actual stress system $\sigma$, (i.e. conditions in eqs.(7) and (13) are satisfied).

This solution may be obtained quite artificially, starting with any admissible self-equilibrating stress system $\bar{\rho}_o(x, y, z)$, and adding it to the elastic response $\sigma_e(t)$ to obtain

$$\bar{\sigma}_o(t) = \sigma_e(t) + \bar{\rho}_o \tag{18}$$

as an initial guess for $\bar{\sigma}$. A creep strain rate $\dot{\varepsilon}_c$ corresponding to $\bar{\sigma}_o$ can now be calculated from the creep law in eq.(5) which would automatically ensure that the condition required in eq.(7) is satisfied. Likewise, a plastic strain rate $\dot{\varepsilon}_p$ can be calculated by the plastic flow law defined in eq.(3).

These rates may be integrated over a loading cycle of time $\tau$ to give

$$\Delta\varepsilon_c = \int_o^\tau \dot{\varepsilon}_c \, dt \tag{19}$$

$$\Delta\varepsilon_p = \int_o^\tau \dot{\varepsilon}_p \, dt \tag{20}$$

In general, these strain increments will not be compatible, and will require a correction to the self-equilibrating stress system $\Delta\bar{\rho}$, which will give rise to an elastic strain increment by eq.(2)

$$\Delta\varepsilon_e = C\,\Delta\bar{\rho} \tag{21}$$

such that the total strain increment

$$\Delta\varepsilon = \Delta\varepsilon_e + \Delta\varepsilon_c + \Delta\varepsilon_p \tag{22}$$

are compatible. The compatibility requirement on $\Delta\varepsilon$, and the equilibrium requirement on

$\Delta\bar{\rho}$ , define as usual a boundary value problem, in this case nonlinear, which can be solved by any of the available methods. The correction $\Delta\rho$ must be such that $\bar{\sigma}_o + \Delta\bar{\rho}$ does not exceed the yield limit in the iterative solution. For loads $R(t)$ below a certain shake-down limit, here modified somewhat due to the presence of the creep strain, repetition of this successive approximation procedure will converge to give the required $\bar{\rho}$ when $\Delta\bar{\rho}$ becomes sufficiently small, and the resulting stress system $\bar{\sigma}$ will satisfy the yield criterion defined in condition (13).

This procedure is merely an artificial device for finding the correct cyclic stationary solution $\bar{\sigma}$ which gives the upper bound to the energy dissipation over a cycle. It does not correspond with any real event. In the case when $\bar{\rho}_o$ is taken to be zero, or the true residual stress system if the structure has been loaded before, and the cycle time $\tau$ is short, so that the residual stress over a cycle is approximately constant, then the steps of the calculation will approximately approach to the real events which take place over the time sequence, with the stress states determined at discrete points of time coinciding with the ends of the loading cycles. However, practical experience found the convergence rate for such a calculation to be slow. Convergence may be accelerated considerably if $\bar{\rho}_o$ is taken to be the residual stress given by the application of a constant maximum load, inducing only creep strains without any plastic deformation.

The nonlinear problem involved in the initial solution, as well as the solution at every subsequent successive approximation step, may be solved by, say, the Newton-Raphson iterative method, or any other iterative method that would give rapid convergence.

## 4. Example

In this preliminary report, the simple problem of the bending of a beam by a moment varying linearly over a time cycle $\tau = 10$ hours is used as an example, (Fig. 2). The beam has a cross-section 10 mm square, made of a perfectly plastic material with a Young's modulus $E = 1.8 \times 10^5$ N/mm$^2$, a yield stress $Y = 130$ N/mm$^2$, and a creep strain rate given by $\dot{\varepsilon}_c = k\,\sigma^n$ , where $k = 0.817 \times 10^{-18}$ in SI units (with length in mm) and $n = 7$. The end section of the beam is assumed to be constrained in such a way that plane cross-sections remain plane.

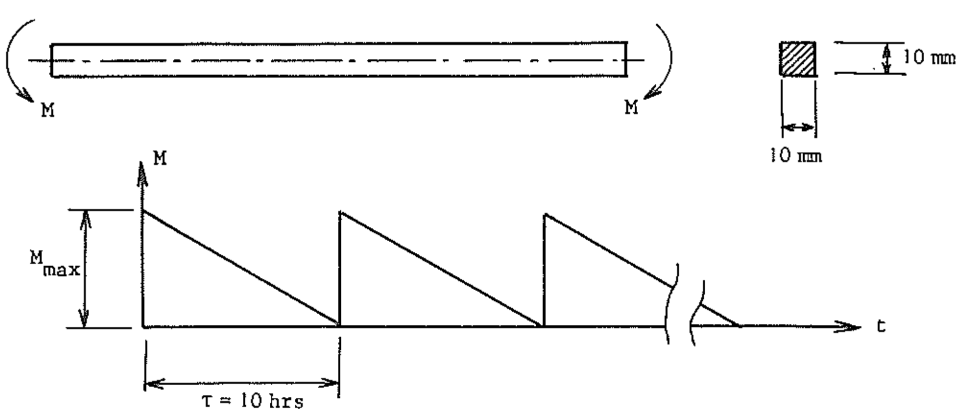


FIG. 2.

For this beam, the moment at the elastic limit $M_e = 21.67$ N-m, and the fully plastic moment $M_p = 32.5$ N-m, which is also the plastic shakedown limit if there is no creep. The elastic response $\sigma(t)$ is simply a linear stress distribution over the cross-section proportional to $M(t)$ in magnitude. The residual stress distribution $\bar{\rho}$ at the cyclic stationary state corresponding to a $M_{max} = 30$ N-m is shown on Fig. 3. This value of $M_{max}$ is actually larger than the modified shakedown limit of $\frac{n}{n+1} M_p = \frac{7}{8} \times 32.5 = 28.44$ stipulated in reference [1].

The initial guess $\bar{\rho}_o$ is taken to be that due to the effect of creep alone for a constant application of $M_{max}$ shown in Fig. 3, the calculation requires only 8 corrective steps to converge to the required steady state $\bar{\rho}$, with about 3 iterations for each step, totaling 25 iterations for the entire calculation, including the initial solution for $\bar{\rho}_o$. On the other hand, if $\bar{\rho}_o$ is taken to be zero, the calculation actually requires the subdivision of the cycle time $\tau$ into smaller time intervals before the "stiff" equation would yield a solution. The total iteration count for this approaches 130 before convergence takes place. The advantage of the former choice of $\bar{\rho}_o$ is apparent.

More complicated examples requiring finite element idealisations for solution are being prepared at the time of writing, reference [4]

## References


[1] PONTER, A.R.S., "Deformation, Displacement, and Work Bounds for Structures in a State of Creep and subject to Variable Loading", Jnl. Appl. Mechanics, 39, pp.953-958 (1972).

[2] LECKIE, F.A., PONTER, A.R.S., "Deformation Bounds for Bodies which Creep in the Plastic Range", Jnl. Appl. Mechanics, 37, pp.426-430 (1970).

[3] FREDERICK, C.O., ARMSTRONG, P.J., "Convergent Internal Stresses and Steady Cyclic States of Stress", Jnl. of Strain Analysis, 1, (2), (1966).

[4] SPILIOPOULOS, K., "Cyclic Loading Inelastic Analysis of Structures", Ph.D. thesis in preparation, Aeronautics Department, Imperial College, (University of London).

RESIDUAL STRESSES FOR RECTANGULAR BEAM CROSS-SECTION

FIG.3.