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An approximate analysis method for system subjected to nonstationary random excitations

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ABSTRACT A stationary approximation method is proposed to simplify nonstationary random vibration analysis. Mean square response and first excursion probability of single-degree-of-freedom system are obtained. Next, those of the secondary system are obtained. The proposed method gives conservative value for the maximum value of the mean square response and the first excursion probability. This method gives exact value of integral of the mean square response.

1.INTRODUCTION

When system is subjected to nonstationary random vibration or system is suddenly subjected to stationary random vibration, response of the system is nonstationary random process. For the analysis of such response, the nonstationary random vibraiotn analysis method is used. However, nonstationary analysis is generally complicated and time cosuming[1],[2]. Considering above mentioned problem, stationary approximation is sometimes used for simplicity[3].

In this paper, effectiveness of a stationary approximation method is examined. A theoretical nonstationary analysis method of the mean square response of single-degree-of-freedom system is introduced. In this study, stationary approximation is defined as follows: mean square response is obtained by multiplying stationary mean square response by square of the envelope function. Mean square responses obtained by the stationary approximation are compared with those obtained by the nonstationary analysis method. The following four items are examined for various values of damping ratio and natural period. (1)the maximum value of the mean square response, (2)occurence time of the maximum value, (3)integral of the mean square response with respect to time from 0 to infinity, (4)first excursion probability.

Next, the method is applied to the secondary system, such as piping and equipment, installed on the primary system, such as building. Approximate values are compared with exact values for various values of the damping ratio, the natural period and mass ratio of the secondary system to the primary system.

It is concluded that the proposed method gives conservative value for the maximum value of the mean square response and the first excursion probability. This method gives exact value of integral of the mean square response.

2.ANALITICAL METHOD

2.1 *Analytical Model and Excitation Model*

As an analytical model, a single-degree-of-freedom system shown in Fig.1 is used. The equation of motion with respect to relative displacement z(=x-y) is

$$\ddot{z}+2\zeta\omega_n\dot{z}+\omega_n^2 z=-\ddot{y} \tag{1}$$

where ζ is the damping ratio and ω_n is the natural circular frequency. As input excitation $\ddot{y}(t)$, nonstationary white noise which is given by multiplying envelope function I(t) representing amplitude nonstationary characteristic by stationary white noise $s_y(t)$, that is,

$$\ddot{y}(t)=I(t)s_y(t) \tag{2}$$

In this study, envelope function I(t) as next equation is used.

$$I(t)=(e^{-at}-e^{-bt})/|e^{-at}-e^{-bt}|_{max} \tag{3}$$

Values of a and b are selected as 0.125 and 0.25, respectively. Fig.2 shows the envelope function.

2.2 *Mean Square Response*

The mean square reaponse of z is given by the autocorrelation function as follows.

$$R_z(t_1,t_2)=\int_{-\infty}^{\infty} G(\omega,t_1)G^*(\omega,t_2)S_0\,d\omega \tag{4}$$

where

$$G(\omega,t)=\int_0^t h(t-\xi)I(\xi)e^{i\omega\xi}\,d\xi \tag{5}$$

h(t) is unit impulse response function and $G^*(\omega,t)$ is the complex conjugate of $G(\omega,t)$. S_0 is power spectral density of stationary white noise.

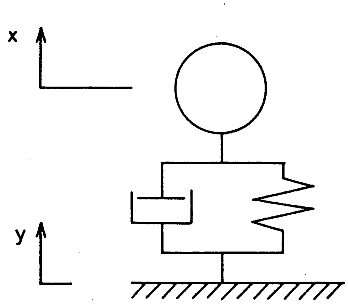

Fig.1 Analytical model of single-degree-of-freedom system

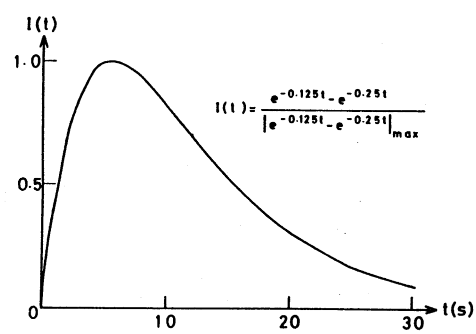

Fig.2 Envelope function

The mean square response of displacement σ_z^2 is given as:

$$\sigma_z^2(t)=R_z(t,t) \tag{6}$$

σ_z^2 is also obtained from the moment equations as:

$$\left.\begin{aligned} \frac{d\sigma_z^2}{dt} &= 2\kappa_{z\dot{z}} \\ \frac{d\sigma_{\dot{z}}^2}{dt} &= -4\zeta\omega_n\sigma_{\dot{z}}^2 - 2\omega_n^2\kappa_{z\dot{z}} + 2\pi S_0\{I(t)\}^2 \\ \frac{d\kappa_{z\dot{z}}}{dt} &= \sigma_{\dot{z}}^2 - 2\zeta\omega_n\kappa_{\dot{z}z} - \omega_n^2\sigma_z^2 \end{aligned}\right\} \tag{7}$$

where $\sigma_{\dot{z}}^2$ is the mean square response of relative velocity and $\kappa_{z\dot{z}}$ is the covariance of z and $\dot{z}$.

The integral of Eq.(4) is complicated. For simplicity, it is assumed that I(t) is independent of the integral, then Eq.(6) is expressed as follows.

$$\sigma_z^2(t)=\{I(t)\}^2\int_{-\infty}^{\infty} |H_d(\omega)|^2 S_0 d\omega \tag{8}$$

where $H_d(\omega)$ is the frequency response function for displacement response. The integrals of Eq.(8) is that for stationary random process. For stationary random process, covariance $\kappa_{z\dot{z}}$ is zero.

Integral of mean square response with respect to time from 0 to infinity is expressed as:

$$I_z=\int_0^{\infty} \sigma_z^2(t)dt \tag{9}$$

2.3 *First Excursion Probability*

First excursion probability is obtained by applying above mentioned equations. It is assumed that failure occurs at the time when $|z(t)|$ first crosses the tolerance level. In this case, the first excursion probability is obtained by the following equation.

$$P_f(t)=1-\exp\{-2\int_0^t \nu(t)dt\} \tag{10}$$

$\nu(t)$ is the expected number of crossings of the tolerance level B_D per unit time. When probability density function of input is the normal distribution, $\nu(t)$ is approximately given as:

$$\nu(t)=\frac{1}{2\pi}\frac{\sigma_{\dot{z}}}{\sigma_z}\exp\left(-\frac{B_D^2}{2\sigma_z^2}\right) \tag{11}$$

In the case where z(t) is stationary normal random process, Eq.(11) is written as:

$$\nu(t)=\frac{1}{2\pi}\omega_n\exp\left(-\frac{B_D^2}{2\sigma_z^2}\right) \tag{12}$$

The tolearance level is determined using the standard deviation of stationary response σ_{zst} as following equation.

$$BD=n\sigma_{zst} \tag{13}$$

where n is a parameter. In this paper, P_f obtained by using Eq.(11) is referred as exact value and that by using Eq.(12) is referred as approximate value.

3.RESULTS

3.1 *Mean square response*

For mean square response σ_z^2 , the following three iems are examined. (1)the maximum value, (2)occurence time of the maximum value, (3)integral of mean square response with respect to time from 0 to infinity. Table 1 shows the maximum value and occurence time of the maximum value for different values of natural period T_n ($2\pi / \omega_n$) for fixed value of damping ratio ζ =0.01. Table 2 shows the maximum value and occurence time of the maximum value for different values of ζ for fixed value of T_n =0.5s. These values are obtained for the case where power spectral density of stationary white noise is 1cm ^{2}/s ^{3}. In these tables, σ_z^2 obtained by Eq.(8) is referred as approximate value and that by Eq.(7) is referred as exact value.

From these tables, approximate value for the maximum value of mean square response is greater than exact value. Approximate value of occurence time of the maximum value is earlier than exact value. For the maximum value of the mean square response, approximate value appoaches exact value as natural period becomes shorter. For occurance time of the

Table 1 Comparison of approximate value with exact value (ζ =0.01)

T_n (s)	Maximum value (cm ^{2})		Time (s)	
	Approximate	Exact	Approximate	Exact
0.1	$6.33x10^{-3}$	$6.19x10^{-4}$	5.55	6.44
0.2	$5.07x10^{-3}$	$4.67x10^{-3}$	5.55	7.33
0.3	$1.71x10^{-2}$	$1.47x10^{-2}$	5.55	8.06
0.4	$4.05x10^{-2}$	$3.24x10^{-2}$	5.55	8.68
0.5	$7.92x10^{-2}$	$5.93x10^{-2}$	5.55	9.20

Table 2 Comparison of approximate value with exact value (T_n =0.5s)

ζ	Maximum value (cm ^{2})		Time (s)	
	Approximate	Exact	Approximate	Exact
0.01	$7.92x10^{-2}$	$5.93x10^{-2}$	5.55	9.20
0.02	$3.96x10^{-2}$	$3.53x10^{-2}$	5.55	7.71
0.05	$1.58x10^{-2}$	$1.55x10^{-2}$	5.55	6.45
0.10	$7.92x10^{-3}$	$7.87x10^{-3}$	5.55	5.98

Table 3 Comparison of approximate value with exact value

(a) ζ =0.01

T_n (s)	Integral (cm ^{2} ·s)	
	Approximate	Exact
0.1	$6.33x10^{-3}$	$6.19x10^{-4}$
0.2	$5.07x10^{-3}$	$4.67x10^{-3}$
0.3	$1.71x10^{-2}$	$1.47x10^{-2}$
0.4	$4.05x10^{-2}$	$3.24x10^{-2}$
0.5	$7.92x10^{-2}$	$5.93x10^{-2}$

(b) T_n =0.5s

ζ	Integral (cm ^{2} ·s)	
	Approximate	Exact
0.01	$8.44x10^{-1}$	$8.44x10^{-1}$
0.02	$4.22x10^{-1}$	$4.22x10^{-1}$
0.05	$1.69x10^{-1}$	$1.69x10^{-1}$
0.10	$8.44x10^{-2}$	$8.44x10^{-2}$

Table 4 Comparison of approximate value with exact value for the first excursion probability (n=2)

(a) ζ =0.01

T_n (s)	Approximate	Exact
0.1	1.00	1.00
0.2	1.00	1.00
0.3	9.97×10^{-1}	9.90×10^{-1}
0.4	9.86×10^{-1}	9.51×10^{-1}
0.5	9.68×10^{-1}	8.79×10^{-1}

(b) T_n =0.5s

ζ	Approximate	Exact
0.01	9.68×10^{-1}	8.79×10^{-1}
0.02	9.68×10^{-1}	9.45×10^{-1}
0.05	9.68×10^{-1}	9.64×10^{-1}
0.10	9.68×10^{-1}	9.67×10^{-1}

Table 5 Comparison of approximate value with exact value for the first excursion probability (n=3)

(a) ζ =0.01

T_n (s)	Approximate	Exact
0.1	6.29×10^{-1}	5.95×10^{-1}
0.2	3.91×10^{-1}	3.00×10^{-1}
0.3	2.81×10^{-1}	1.61×10^{-1}
0.4	2.20×10^{-1}	8.93×10^{-2}
0.5	1.80×10^{-1}	5.10×10^{-2}

(b) T_n =0.5s

ζ	Approximate	Exact
0.01	1.80×10^{-1}	5.10×10^{-2}
0.02	1.80×10^{-1}	1.16×10^{-1}
0.05	1.80×10^{-1}	1.66×10^{-1}
0.10	1.80×10^{-1}	1.76×10^{-1}

maximum value, difference between approximate value and exact value is larger as natural period becomes longer. And, for the maximum value, approximate value appoaches exact value as damping ratio becomes greater. For occurance time of the maximum value, difference between approximate value and exact value is larger as damping ratio becomes smaller.

Table 3(a) and (b) show values of integral of mean square response I_z . Approximate value is equal to exact value independent of damping ratio and natural period.

3.2 *First excursion probability*

When enough time passes, the first excursion probability P_f becomes constant value. Hence, constant values of P_f are obtained and examined. Results for various values of ζ and T_n are shown in Table 4 and 5 for nondimensional tolerance level n=2 and n=3, respectively. From these tables, it is found that approximate values approach exact values as damping ratio becomes greater and natural period becomes shorter. This characteristic is same as the results of Table 1 and Table 2.

4.APPLICATION TO SECONDARY SYSTEM

Next, analytical method for single-degree-of freedom system is applied to the secondary system, such as piping and equipment installed on the primary system, such as building.

4.1 *Analytical Model and mean square response*

As an analytical model of the secondary system, a two-degree-of-freedom system shown in Fig.3 is used. The upper system is the secondary system and the lower is the primary system. The equations of motion with respect to relative displacement $z_s = x_s - x_p$ and $z_p = x_p - y$ are

$$\dot{z} = Gz + f \tag{14}$$

where

$$z^T = \{z_s, z_p, \dot{z}_s, \dot{z}_p\} \tag{15}$$

$$G = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\omega_p^2 & \omega_s^2 \gamma & -2\zeta_p\omega_p & 2\zeta_s\omega_s\gamma \\ \omega_p^2 & -\omega_s^2(1+\gamma) & 2\zeta_p\omega_p & -2\zeta_s\omega_s(1+\gamma) \end{bmatrix} \tag{16}$$

$$f^T = \{0, 0, -\ddot{y}, 0\} \tag{17}$$

and ζ_s and ζ_p are the damping ratio of the secondary and the primary system, respectively and ω_s and ω_p are the natural circular frequency of the secondary and the primary system, respectively. γ is mass ratio of the secondary system to the primary system.

The second moments of response are expressed as[4]:

$$\dot{V} = GV^T + VG^T + D \tag{18}$$

$$V = \begin{bmatrix} \sigma_{zp}^2 & \kappa_{zpzs} & \kappa_{zp\dot{z}p} & \kappa_{zp\dot{z}s} \\ \kappa_{zpzs} & \sigma_{zs}^2 & \kappa_{zs\dot{z}p} & \kappa_{zs\dot{z}s} \\ \kappa_{zp\dot{z}p} & \kappa_{zs\dot{z}p} & \sigma_{\dot{z}p}^2 & \kappa_{\dot{z}p\dot{z}s} \\ \kappa_{zp\dot{z}s} & \kappa_{zs\dot{z}s} & \kappa_{\dot{z}p\dot{z}s} & \sigma_{\dot{z}s}^2 \end{bmatrix} \tag{19}$$

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2\pi S_0 \{I(t)\}^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{20}$$

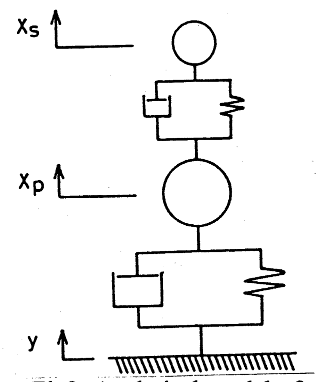

Fig3. Analytical model of the secondary system

where κ is covariance.

The second moments of response are obtained from the moment equations obtained by Eq.(19). Exact value of mean square response of relative displacement z_s, σ_{zs}^2, is obtained by Eq.(19). Approximate velue is obtained by Eq.(8).

4.2 *Results*

Table 6 shows results for differnt values of damping ratio of the secondary system ζ_s. Table 7 shows results for different values of natural period of the secondary system T_s. Table 8 shows results for different values of mass ratio of the secondary system to the primary system γ. Table 9 shows results for different values of natural period under the condition where $T_s = T_p$. From these tables, it is found that approximate value for the maximum value of mean square response is greater than exact value. Approximate value of

Table 6 Comparison of approximate value with exact value for mean square response (γ =0, ζ_p =0.05, T_s =T_p =0.5s)

ζ_s	Maximum value (cm^2)		Time (s)	
	Approximate	Exact	Approximate	Exact
0.01	6.68	4.79	5.55	11.49
0.02	2.87	2.43	5.55	9.79
0.05	$8.07x10^{-1}$	$7.57x10^{-1}$	5.55	8.18

Table 7 Comparison of approximate value with exact value for mean square response (γ =0, ζ_s =0.01, ζ_p =0.05, T_p =0.5s)

T_s (s)	Maximum value (cm^2)		Time (s)	
	Approximate	Exact	Approximate	Exact
0.25	$2.91x10^{-3}$	$2.71x10^{-3}$	5.55	6.90
0.5	6.68	4.79	5.55	11.49
1.0	1.15	$6.52x10^{-1}$	5.55	10.90

Table 8 Comparison of approximate value with exact value for mean square response (ζ_s =0.01 ζ_p =0.05, T_s =T_p =0.5s)

γ	Maximum value (cm^2)		Time (s)	
	Approximate	Exact	Approximate	Exact
0	6.68	4.79	5.55	11.49
0.01	1.13	1.08	5.55	7.78
0.05	$2.72x10^{-1}$	$2.56x10^{-1}$	5.55	7.28

Table 9 Comparison of approximate value with exact value for mean square response (γ =0, ζ_s =0.01, ζ_p =0.05, T_s =T_p)

T_s (s)	Maximum value (cm^2)		Time (s)	
	Approximate	Exact	Approximate	Exact
0.25	$8.35x10^{-1}$	$7.32x10^{-1}$	5.55	8.84
0.5	6.68	4.79	5.55	11.49
1.0	5.34x10	2.73x10	5.55	15.72

occurence time of the maximum value is ealier than exact value. Approximate value of the maximum value and occurence time of the maximum value approach exact value as damping ratio becomes greater and natural period becomes shorter. Difference between approximate value and exact value becomes smaller as mass ratio becomes greater.

Table 10 shows integral of mean square response. Approximate value of integral of mean square response is equal to exact value independent of damping ratio, natual period and mass ratio.

5.CONCLUSIONS

Mean square response and first excursion probability of the system subjected to nonstationary white noise is obtained by stationary approximation. Obtained results are summarized as follows.

(1)Approximate value for the maximum value of mean square response is greater than exact value. Approximate value of occurence time of the maximum value is earlier than exact value.

(2)The maximum value of mean square response and its occurence time obtained by stationary approximation approach those obtained by nonstationary analysis method when damping ratio becomes greater and natural period becomes shorter.

(3)The integral of mean square response obtained by stationary approximation is equal to that obtained by nonstaionary analysis independent of damping ratio and natural period.

(4)The first excursion probability obtained by stationary approximation approaches that obtained by nonstaionary analysis when damping ratio becomes greater and natural period becomes shorter.

(5)For the secondary system, approximate value for the maximum value of mean square response is greater than exact value. Approximate value of occurence time of the maximum value is earlier than exact value. Approximation method gives exact value of the integral of mean square response independent of damping ratio, natural period and mass ratio.

REFERENCES

1. Lin,Y.K. 1967. *Probabilistic Theory of Structural Dynamics.* McGraw-Hill.

2.Iyengar,R.N. and Iyengar,K.T.S.R. 1970. Probabilistic Response Analysis to Earthquakes. *Proc.of ASCE Engineering Mechanics Division* 96 June: 207-225.

3.Hasselman,T. 1972. Probabilistic Displacement Time History of a Single Degree of Freedom System. *Proc. of ASCE Engineering Mechanics Division* 98 June: 519-530.

4.Roberts,J.B. and Spanos,P.D. 1990. *Random Vibration and Statistical Linearization,* John Wiley & Sons

Table 10 Comparison of approximate value with exact value

(a) $\gamma=0$, $\zeta_p=0.05$, $T_s=T_p=0.5s$

ζ_s	Integral ($cm^2 \cdot s$)	
	Approximate	Exact
0.01	7.12x10	7.12x10
0.02	3.06x10	3.06x10
0.05	8.61	8.61

(b) $\gamma=0$, $\zeta_s=0.01$, $\zeta_p=0.05$, $T_p=0.5s$

T_s (s)	Integral ($cm^2 \cdot s$)	
	Approximate	Exact
0.25	3.10×10^{-2}	3.10×10^{-2}
0.5	7.12x10	7.12x10
1.0	1.23x10	1.23x10

(c) $\zeta_s=0.01$, $\zeta_p=0.05$, $T_s=T_p=0.5s$

γ	Integral ($cm^2 \cdot s$)	
	Approximate	Exact
0	7.12x10	7.12x10
0.01	1.20x10	1.20x10
0.05	2.90	2.90

(d) $\gamma=0$, $\zeta_s=0.01$, $\zeta_p=0.05$, $T_s=T_p$

T_s (s)	Integral ($cm^2 \cdot s$)	
	Approximate	Exact
0.25	8.90	8.90
0.5	7.12x10	7.12x10
1.0	5.70×10^{2}	5.70×10^{2}