Abstract

COOK, WILLIAM JEFFREY. Affine Lie Algebras, Vertex Operator Algebras and Combinatorial Identities. (Under the direction of Kailash C. Misra and Haisheng Li)

Affine Lie algebra representations have many connections with different areas of mathematics and physics. One such connection in mathematics is with number theory and in particular combinatorial identities. In this thesis, we study affine Lie algebra representation theory and obtain new families of combinatorial identities of Rogers-Ramanujan type.

It is well known that when $\tilde{\mathfrak{g}}$ is an untwisted affine Lie algebra and k is a positive integer, the integrable highest weight $\tilde{\mathfrak{g}}$-module $L(k\Lambda_0)$ has the structure of a vertex operator algebra. Using this structure, we will obtain recurrence relations for the characters of all integrable highest-weight modules of $\tilde{\mathfrak{g}}$. In the case when $\tilde{\mathfrak{g}}$ is of (ADE)-type and $k = 1$, we solve the recurrence relations and obtain the full characters of the adjoint module $L(\Lambda_0)$. Then, taking the principal specialization, we obtain new families of multisum identities of Rogers-Ramanujan type.

AFFINE LIE ALGEBRAS, VERTEX OPERATOR ALGEBRAS AND COMBINATORIAL IDENTITIES

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Biography

William J. Cook (Bill) was born on February 12, 1980 in Johnson City, NY. He spent most of his childhood in Vestal, NY and moved with his family to Banner Elk, NC in 1993. He was homeschooled from the fifth grade until graduation. Being homeschooled allowed him to start attending college classes at Lees McRae College in Banner Elk in 1996. Bill entered Bob Jones University in Greenville, SC as a sophomore in 1997 and graduated in December 1999 with a B.S. in Mathematics and minor in Computer Science. He then entered NC State University in January 2000 and completed his M.S. in Mathematics in December 2001. He graduated in May 2005 with a Ph.D. in Mathematics and minor in Computer Science.

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And we know that all things work together for good to them that love God, to them who are called according to his purpose.

Romans 8:28 (KJV)

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In highschool, when Mom's calculus got too rusty, Dad helped me get into the second semester calculus class taught by Dr. John Alsup at Lees MacRae College. And even though it wasn't quite time to start college yet and Lees MacRae isn't cheap, Dad still found the money to allow me to go. The following year, when I took introductory real analysis from Dr. Alsup, I almost gave up on math. Without Dr. Alsup's patience, kindness, and encouragement, I would not have made it here today.

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> *...of making many books there is no end; and much study is a weariness of the flesh.*
>
> *Let us hear the conclusion of the whole matter: Fear God, and keep his commandments: for this is the whole duty of man.*

Ecclesiastes 12:12b-13 (KJV)

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Chapter 1

Introduction

In 1968, Victor Kac [15] and Robert Moody [26] independently discovered a new class of Lie algebras called Kac-Moody algebras, which are infinite dimensional analogs of finite dimensional semisimple Lie algebras. The study of these algebras has led to a deep theory with important applications in physics and other areas of mathematics.

A special type of Kac-Moody algebras called affine Lie algebras have particularly rich and beautiful structures. Their representation theory has many interesting applications in mathematics and physics, particularly in number theory. In 1978, Lepowsky and Milne [19] showed that the product sides of the famous Rogers-Ramanujan identities appear in the principal specialization of certain characters of the irreducible representations of the simplest affine Lie algebra $A_1^{(1)}$. In [21], Lepowsky and Wilson proved the Rogers-Ramanujan identities using Lie theoretic methods. These methods relied heavily upon their vertex operator constructions of the basic representations of $A_1^{(1)}$ [20]. This unexpected connection to number theory along with other unanticipated connections to different areas of physics (such as conformal field theory and string theory) attracted many researchers to the representation theory of affine Lie algebras and ultimately led to the discovery of new combinatorial identities (see for example [3] and [7]) and new algebraic structures such as vertex (operator) algebras (see for example [2], [8], [10], [17], and [18]).

In [11], Frenkel and Zhu showed that certain affine Lie algebra modules ($L(k\Lambda_0)$) have the structure of vertex operator algebras. Recently, Capparelli, Lepowsky and Milas ([4] and [5]) used the vertex operator algebra structure for $A_1^{(1)}$-modules $L(k\Lambda_0)$ and the intertwining operators to obtain Rogers-Ramanujan recursions and Rogers-Selberg recursions. Motivated by this work, we started our investigation to obtain recurrence relations for the characters of $L(k\Lambda_0)$ for any untwisted affine Lie algebra. In what follows, viewing $L(k\Lambda_0)$ (k a positive integer) as a vertex operator algebra and using certain results from [22], [23], [24], and [25], we derive recurrence relations for the full character of a level k integrable highest-weight untwisted affine Lie algebra module $L(\Lambda)$ where $L(\Lambda)$ is viewed as an irreducible vertex operator algebra module for the vertex operator algebra $L(k\Lambda_0)$.

For affine Lie algebras of (ADE)-type when $k = 1$, we solve these recurrence relations and obtain formulas for the characters of the affine Lie algebra module $L(\Lambda_0)$. The formulas we obtain were conjectured in [6]. In [12] and [13], Georgiev proved (the homogeneous specialization of) these formulas for the affine Lie algebra

$A_\ell^{(1)}$ using a different approach.

Starting with our formulas for the full character of $L(\Lambda_0)$, we then take the principal specialization of this character and compare it with previously known principal character formulas (see [16]). Thus, we obtain new families of multisum identities of Rogers-Ramanujan type.

In Chapter 2, we review the definitions of Lie algebras and their representations along with the definition of the universal enveloping algebra of a Lie algebra and the statement of the Poincaré-Birkhoff-Witt Theorem. A detailed introduction to these topics can be found in [14]. Next we define finite dimensional semisimple Lie algebras and affine Lie algebras using generalized Cartan matrices, generators and relations. We then define the notion of integrable highest weight modules of affine Lie algebras and we construct Verma modules using the previously discussed universal enveloping algebras. In the last section of this chapter, we construct untwisted affine Lie algebras using central extensions of loop algebras over their underlying finite dimensional simple Lie algebras. An excellent introduction to affine (Kac-Moody) Lie algebras and their representation theory can be found in [16].

In Chapter 3, we review the definition of vertex (operator) algebras and their representations. Then, we take the Verma modules constructed in chapter 2 and show that, in addition to their affine Lie algebra module structure, they also have the structure of a vertex algebra. In fact, some of these modules (for example $L(k\Lambda_0)$, k a positive integer) have the structure of a vertex operator algebra. In the following section, we discuss the lattice construction of the basic module $L(\Lambda_0)$ which is needed in chapter 4. The final section of chapter 3 is devoted to reviewing certain results from Haisheng Li's work. We obtain automorphisms of irreducible $L(k\Lambda_0)$-modules. These automorphisms are essential to obtain our recurrence relations in chapter 4.

The final chapter, Chapter 4, contains the main results. We begin by defining full, homogeneous, and principal characters of an irreducible $L(k\Lambda_0)$-module. The relationship between the full character and homogeneous character is obvious. Proposition 4.3 reveals the relationship between the full character and the principal character. In the next section, we state our main result, Theorem 4.5,

$$\chi_W(x_1, ..., x_\ell; q) = (x_i q)^{\frac{2}{\langle \alpha_i, \alpha_i \rangle} k} \chi_W(x_1 q^{a_{1i}}, ..., x_\ell q^{a_{\ell i}}; q) \tag{1.1}$$

for $1 \leq i \leq \ell$ where W is an irreducible $L(k\Lambda_0)$-module (k a positive integer) and ℓ is the rank of the underlying finite dimensional simple Lie algebra. Using equation (1.1), we obtain recurrence relations for coefficients of $x_1^{n_1} ... x_\ell^{n_\ell}$. In the case where the underlying finite dimensional simple Lie algebra is simply laced (i.e. (ADE)-type) and $k = 1$, these recurrence relations simplify nicely and we are able to solve them and write a formula for any coefficient in terms of one initial condition. Using the lattice construction of $L(\Lambda_0)$, we obtain the necessary initial condition and obtain a formula for the full character of the basic module $L(\Lambda_0)$. Finally, using Proposition 4.5, we obtain a formula for the principal character of $L(\Lambda_0)$. But a product formula for this character is already known (see [16]). Thus equating our multisum character formulas with the known product formulas we obtain several families of multisum identities of Rogers-Ramanujan type. Theorem 4.12 gives a list of the identities. In the final section, as an illustration, we state several special cases of these identities.

Appendix A contains a list of the Dynkin diagrams and Cartan matrices of the finite dimensional simple Lie algebras, and the Dynkin diagrams and generalized Cartan matrices of the untwisted affine Lie algebras.

Chapter 2

Affine (Kac-Moody) Lie Algebras

In this chapter, we will give the definitions of Lie algebras and their representations (over the complex numbers, $\mathbb{C}$). Then we will define the special class of Lie algebras called affine (Kac-Moody) Lie algebras and an important class of representations associated with affine Lie algebras called highest-weight representations. For more details about Lie algebras (especially finite dimensional theory), see [14]. A detailed introduction to Kac-Moody algebras can be found in [16].

2.1 Lie Algebras and Their Representations

Definition 2.1. *A* Lie algebra *is a vector space (over a field $\mathbb{F}$) $\mathfrak{g}$ equipped with a bilinear multiplication,*

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \tag{2.1}$$

(called a bracket*), satisfying the following axioms.*

Skew-Symmetry *For each* $g \in \mathfrak{g}$*,* $[g, g] = 0$ *(when* $\mathbb{F}$ *is characteristic 0, this is equivalent to* $\forall\, g, h \in \mathfrak{g}$*,* $[g, h] = -[h, g]$*).*

Jacobi Identity *For each* $f, g, h \in \mathfrak{g}$*,*

$$[[f, g], h] + [[g, h], f] + [[h, f], g] = 0. \tag{2.2}$$

A linear map, $\varphi : \mathfrak{g} \to \mathfrak{g}'$*, between two Lie algebras is called a* homomorphism *if it preserves brackets (i.e.* $\varphi([g, h]) = [\varphi(g), \varphi(h)]$ *for each* $g, h \in \mathfrak{g}$*).*

Given a Lie algebra $\mathfrak{g}$, define a linear map $\mathrm{ad}(g) : \mathfrak{g} \to \mathfrak{g}$ for each $g \in \mathfrak{g}$ by:

$$\mathrm{ad}(g)(x) = [g, x] \qquad\qquad x \in \mathfrak{g}. \tag{2.3}$$

The map, ad, is called the *adjoint* map. We can rewrite the Jacobi identity (using skew-symmetry) and get the following equivalent identity ($g, x, y \in \mathfrak{g}$):

$$\mathrm{ad}(g)([x, y]) = [\mathrm{ad}(g)(x), y] + [x, \mathrm{ad}(g)(y)]. \tag{2.4}$$

In other words, for each $g \in \mathfrak{g}$, $\mathrm{ad}(g)$ is a *derivation.*

Example 2.2. *Let* $\mathcal{A}$ *be an associative algebra over* $\mathbb{F}$*. We can make* $\mathcal{A}$ *into a Lie* *algebra by giving it the* commutator bracket

$$[a, b] = ab - ba \qquad \text{for all } \ a, b \in \mathcal{A}. \tag{2.5}$$

Example 2.3. *Let V be a vector space over $\mathbb{F}$. The set of all linear endomorphisms,* $\operatorname{End}(V)$*, is an associative algebra where its multiplication is function composition. Let $\mathfrak{gl}(V)$ denote the Lie algebra obtained when* $\operatorname{End}(V)$ *is given the commutator bracket.*

Definition 2.4. *Let $\mathfrak{g}$ be a Lie algebra and V a vector space with a bilinear map $\cdot : \mathfrak{g} \times V \to V$ denoted $(g, v) \mapsto g \cdot v$ such that $[g, h] \cdot v = g \cdot (h \cdot v) - h \cdot (g \cdot v)$ for every $g, h \in \mathfrak{g}$ and $v \in V$. Then V is called a* $\mathfrak{g}$-module.

For each $g \in \mathfrak{g}$, we have the following linear map: $\varphi(g)(v) = g \cdot v$ $(v \in V)$. It is easy to see that $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a homomorphism (called a representation *of $\mathfrak{g}$ on V).*

Conversely, given a representation $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$, V becomes a $\mathfrak{g}$-module with action: $g \cdot v = \varphi(g)(v)$ $(g \in \mathfrak{g}$ and $v \in V)$.

Example 2.5. *Notice that* $\operatorname{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ *is a (Lie algebra) homomorphism, so* ad *is called the* adjoint representation.

2.2 Universal Enveloping Algebras

Universal enveloping algebras are indispensable tools for studying Lie algebras and their representations. They will allow us to construct large classes of representations. In this section, we will state the definition of the universal enveloping algebra and one very important theorem.

Definition 2.6. *Let $\mathfrak{g}$ be a Lie algebra, $U(\mathfrak{g})$ an associative algebra and $j : \mathfrak{g} \to U(\mathfrak{g})$ a homomorphism (of Lie algebras). The pair, $U(\mathfrak{g})$ and j, is a* universal enveloping algebra *of $\mathfrak{g}$ if given any other pair, say A paired with $k : \mathfrak{g} \to A$, we have that k factors through j in a unique way (i.e. there exists a unique associative algebra homomorphism $\varphi : U(\mathfrak{g}) \to A$ such that $k = \varphi \circ j$).*

It can be shown [14] that, for each Lie algebra $\mathfrak{g}$, there exists an universal enveloping algebra $U(\mathfrak{g})$ and moreover, it is unique (up to isomorphism). The following theorem is commonly referred to as the Poincaré-Birkhoff-Witt Theorem (or the PBW Theorem for short).

Theorem 2.7. *(Corollary C on page 92 of [14]) Let $\mathfrak{g}$ be a Lie algebra with ordered basis $\{x_\alpha \,|\, \alpha \in \Omega\}$. Let $U(\mathfrak{g})$ paired with $j : \mathfrak{g} \to U(\mathfrak{g})$ be its universal enveloping algebra. Then $\{j(x_{\alpha_1})...j(x_{\alpha_n}) \,|\, n \geq 0,\ \alpha_1 \leq \alpha_2 \leq ... \leq \alpha_n,\ \alpha_i \in \Omega\}$ is a basis for $U(\mathfrak{g})$.*

In particular, $j : \mathfrak{g} \to U(\mathfrak{g})$ is a monomorphism.

Note that if $y, z \in \mathfrak{g}$, $[y, z] = 0$, then y and z commute in the associative algebra

$U(\mathfrak{g})$. Thus (by the PBW theorem), we conclude that:

Corollary 2.8. *Suppose that* $\mathfrak{g}$ *is the direct sum of* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *as Lie algebras (i.e.* $g_1 + g_2, h_1 + h_2 \in \mathfrak{g}_1 \oplus \mathfrak{g}_2 = \mathfrak{g}$ *then* $[g_1 + g_2, h_1 + h_2] = [g_1, h_1] + [g_2, h_2]$*). Then we have the following natural isomorphism of associative algebras:* $U(\mathfrak{g}) \cong U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2)$*.*

2.3 Affine Lie Algebras

Affine Lie algebras are a generalization of finite dimensional semisimple Lie algebras. We begin this section by defining generalized Cartan matrices and then define affine Lie algebras using generators and relations.

Definition 2.9. *Let* $C = (a_{ij})$ *be an* $n \times n$ *matrix with integer entries. We call* C *a* generalized Cartan matrix (GCM) *if the following properties hold.*

C1 *For all* i*,* $a_{ii} = 2$*.*

C2 *For all* $i \neq j$*,* $a_{ij} \leq 0$*.*

C3 *For all* i *and* j*,* $a_{ij} = 0$ *if and only if* $a_{ji} = 0$*.*

If in addition to properties **C1-C3***, we have that:*

C4 *The GCM,* C*, is a positive definite matrix.*

Then C *is called a* Cartan matrix.

Let C be an $n \times n$ GCM. Suppose there exists a permutation, σ, of the indices $\{1, 2, ..., n\}$ such that

$$C' = (a_{\sigma(i),\sigma(j)}) = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \tag{2.6}$$

then C and C' are called *decomposable*, and C' is the direct sum of C_1 and C_2. It is easy to see that C_1 and C_2 must be GCMs themselves. If no such decomposition exists, C is called *indecomposable*.

Theorem 2.10. *(Theorem 4.3 on page 48 of [16]) Let C be an $n \times n$ indecomposable GCM. Then C belongs to exactly one of the following categories:*

Finite Type *There exists a column vector $\theta \in \mathbb{Z}^n$ of positive integers such that $C\theta$ is a column vector of positive integers. Moreover, C is positive definite.*

Affine Type *There exists a column vector $\theta \in \mathbb{Z}^n$ of positive integers such that $C\theta = 0$. Moreover, C is positive semi-definite and* $\operatorname{rank}(C) = n - 1$.

Indefinite Type *There exists a column vector $\theta \in \mathbb{Z}^n$ of positive integers such that $C\theta$ is a column vector of negative integers.*

In [16], all finite and affine type indecomposable GCMs are classified using Dynkin diagrams. For a list of finite type Cartan matrices, see Appendix A. A special subclass of affine GCMs (type 1 affine GCMs associated with untwisted affine Lie algebras) can also be found in the appendix.

Definition 2.11. *Let* $C = (a_{ij})_{1 \leq i,j \leq \ell}$ *be an* $\ell \times \ell$ *Cartan matrix (i.e. a direct sum of finite type GCMs). Then* C *is nondegenerate (since it is positive definite). Thus* $\text{rank}(C) = \ell$. *We say that* $(\mathfrak{h}, \Pi, \Pi^\vee)$ *is a* realization *of* C *if:*

- *The vector space,* $\mathfrak{h}$*, has dimension* ℓ.
- *The set,* $\Pi = \{\alpha_1, ..., \alpha_\ell\}$*, is a basis for the dual space* $\mathfrak{h}^*$.
- *The set,* $\Pi^\vee = \{H_1, ..., H_\ell\}$*, is a basis for* $\mathfrak{h}$.
- *For all* $1 \leq i, j \leq \ell$*, we have that* $\alpha_i(H_j) = a_{ji}$.

Let $\mathfrak{g}(C)$ *be the Lie algebra generated by* $\{E_1, ..., E_\ell\} \cup \{F_1, ..., F_\ell\} \cup \mathfrak{h}$ *with the following relations among the generators.*

- $[E_i, F_j] = \delta_{ij} H_i$ *for* $1 \leq i, j \leq \ell$
- $[H, H'] = 0$ *for all* $H, H' \in \mathfrak{h}$ *(i.e.* $\mathfrak{h}$ *is an abelian subalgebra.)*
- $[H, E_i] = \alpha_i(H) E_i$ *for all* $H \in \mathfrak{h}$ *and* $1 \leq i \leq \ell$
- $[H, F_i] = -\alpha_i(H) F_i$ *for all* $H \in \mathfrak{h}$ *and* $1 \leq i \leq \ell$
- $(\text{ad}(E_i))^{(1-a_{ij})}(E_j)$ *for* $1 \leq i \neq j \leq \ell$
- $(\text{ad}(F_i))^{(1-a_{ij})}(F_j)$ *for* $1 \leq i \neq j \leq \ell$

Then $\mathfrak{g}(C)$ *is a finite dimensional semisimple Lie algebra (see e.g. [14] Serre's Theorem on page 99). This semisimple Lie algebra is also called the* Kac-Moody algebra *of finite type associated with the generalized Cartan matrix* C*. The* ℓ *dimensional* *vector space,* $\mathfrak{h}$*, is called the* Cartan subalgebra (CSA)*, and* $E_1, ..., E_\ell$*,* $F_1, ..., F_\ell$ *are called the* Chevalley generators*. The size of the GCM,* ℓ*, is called the* rank *of the algebra* $\mathfrak{g}(C)$*.*

Definition 2.12. *Let* $C = (a_{ij})_{0 \leq i,j \leq \ell}$ *be an* $(\ell+1) \times (\ell+1)$ *affine type GCM. Then* $\text{rank}(C) = \ell$*. We say that* $(\mathfrak{h}, \Pi, \Pi^\vee)$ *is a* realization *of* C *if:*

- *The vector space,* $\mathfrak{h}$*, has dimension* $\ell + 2$*.*
- *The subset,* $\Pi = \{\alpha_0, ..., \alpha_\ell\}$*, of* $\mathfrak{h}^*$ *is linearly independent.*
- *The subset,* $\Pi^\vee = \{h_0, ..., h_\ell\}$*, of* $\mathfrak{h}$ *is linearly independent.*
- *For all* $0 \leq i, j \leq \ell$*, we have that* $\alpha_i(h_j) = a_{ji}$*.*

Let $\mathfrak{g}(C)$ *be the Lie algebra generated by* $\{e_0, ..., e_\ell\} \cup \{f_0, ..., f_\ell\} \cup \mathfrak{h}$ *with the following relations among the generators.*

- $[e_i, f_j] = \delta_{ij} h_i$ *for* $0 \leq i, j \leq \ell$
- $[h, h'] = 0$ *for all* $h, h' \in \mathfrak{h}$ *(i.e.* $\mathfrak{h}$ *is an abelian subalgebra)*
- $[h, e_i] = \alpha_i(h) e_i$ *for all* $h \in \mathfrak{h}$ *and* $0 \leq i \leq \ell$
- $[h, f_i] = -\alpha_i(h) f_i$ *for all* $h \in \mathfrak{h}$ *and* $0 \leq i \leq \ell$
- $(\text{ad}(e_i))^{(1-a_{ij})}(e_j)$ *for* $0 \leq i \neq j \leq \ell$
- $(\text{ad}(f_i))^{(1-a_{ij})}(f_j)$ *for* $0 \leq i \neq j \leq \ell$

Then $\mathfrak{g}(C)$ *is called the* Kac-Moody algebra *associated with the generalized Cartan matrix* C *(see [16]). We also say that* $\mathfrak{g}(C)$ *is an affine Lie algebra (since* C *is an affine type GCM). This algebra is generated by* $e_0, ..., e_\ell$ *and* $f_0, ..., f_\ell$*. These elements are called the* Chevalley generators*. The size of the GCM,* $\ell + 1$*, is called the* rank *of the Kac-Moody algebra* $\mathfrak{g}(C)$*.*

Notice that we have chosen uppercase E's, F's, and H's to denote the Chevalley generators in the finite dimensional case and lowercase e's, f's, and h's to denote the Chevalley generators in the affine case. Also note that the Chevalley generators for the finite dimensional case have indices beginning with 1, whereas the Chevalley generators for the affine case have indices beginning with 0.

In both cases, $\mathfrak{h}$ is called the *Cartan subalgebra (CSA)*, elements of Π are called *simple roots* and elements of $\Pi^\vee$ are called *simple coroots*. Let

$$Q = \left\{ \sum_{\alpha_i \in \Pi} k_i \alpha_i \,|\, k_i \in \mathbb{Z} \right\} \tag{2.7}$$

denote the *root lattice* and

$$Q_+ = \left\{ \sum_{\alpha_i \in \Pi} k_i \alpha_i \,|\, k_i \in \mathbb{Z}_{\geq 0} \right\} \tag{2.8}$$

denote the *positive root lattice*. Let

$$P = \{\lambda \in \mathfrak{h}^* \,|\, \lambda(h_i) \in \mathbb{Z}, h_i \in \Pi^\vee\} \tag{2.9}$$

denote the *weight lattice* and

$$P_+ = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0}, h_i \in \Pi^\vee\} \tag{2.10}$$

denote the set of *dominant integral weights*.

Also, let

$$Q^\vee = \left\{ \sum_{h_i \in \Pi^\vee} k_i h_i \mid k_i \in \mathbb{Z} \right\} \tag{2.11}$$

denote the *coroot lattice* and let

$$P^\vee = \{h \in \mathfrak{h} \mid \alpha_i(h) \in \mathbb{Z}, \alpha_i \in \Pi\} \tag{2.12}$$

denote the *coweight lattice*.

We can partially order $\mathfrak{h}^*$ by letting $(\lambda, \mu \in \mathfrak{h}^*)$:

$$\lambda \geq \mu \text{ if } \lambda - \mu \in Q_+ \tag{2.13}$$

Given $\alpha \in \mathfrak{h}^*$, let

$$\mathfrak{g}(C)_\alpha = \{g \in \mathfrak{g}(C) \mid [h, g] = \alpha(h)g, \forall h \in \mathfrak{h}\}. \tag{2.14}$$

If $\alpha \neq 0$ and $\mathfrak{g}(C)_\alpha \neq 0$, then $\mathfrak{g}(C)_\alpha$ is a *root space* and α is a *root*. Denote the set of roots by Δ and *positive roots* by $\Delta_+ = \Delta \cap Q_+$. Note that the set of *negative roots*

$\Delta_- = \Delta \cap -Q_+ = -\Delta_+$ is disjoint from the set of positive roots. Also, $\Delta = \Delta_+ \cup \Delta_-$ (any root is either positive or negative). Finally, note that $\mathfrak{g}(C)_0 = \mathfrak{h}$. We have the following:

$$\mathfrak{g}(C) = \bigoplus_{\alpha \in Q} \mathfrak{g}(C)_\alpha = \left(\bigoplus_{\alpha \in \Delta} \mathfrak{g}(C)_\alpha \right) \oplus \mathfrak{h}. \tag{2.15}$$

When C is of finite type, let $\mathfrak{n}_+$ be the subalgebra generated by $E_1, ..., E_\ell$, and let $\mathfrak{n}_-$ be the subalgebra by $F_1, ..., F_\ell$. When C is of affine type, let $\mathfrak{n}_+$ be the subalgebra generated by $e_0, ..., e_\ell$, and let $\mathfrak{n}_-$ be the subalgebra by $f_0, ..., f_\ell$. In both cases, we have the following triangular decomposition (see [16]):

$$\mathfrak{g}(C) = \left(\bigoplus_{\alpha \in \Delta_-} \mathfrak{g}(C)_\alpha \right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_+} \mathfrak{g}(C)_\alpha \right) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \tag{2.16}$$

Let $\alpha = \sum_{\alpha_i \in \Pi} k_i \alpha_i \in Q$. Define the *height* of the root α to be

$$\mathrm{ht}(\alpha) = \sum k_i. \tag{2.17}$$

When C is a finite type GCM, since $\mathfrak{g}(C)$ is a finite dimensional simple Lie algebra, Δ is a finite set.

A generalized Cartan matrix A is said to be *symmetrizable* with *symmetrization* B if there exists an invertible diagonal matrix $D = \mathrm{diag}(\epsilon_1, \epsilon_2, ..., \epsilon_n)$ and a symmetric matrix B such that $A = DB$. It can be shown that if A is symmetrizable, then there exists a diagonal matrix D and symmetric matrix B with rational entries such that $A = DB$. Moreover, these are determined uniquely (up to a scalar multiple). Note that all finite and affine type GCMs are symmetrizable.

When C is a finite type GCM, $\mathfrak{g}(C)$ has a nondegenerate symmetric invariant bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g}(C) \times \mathfrak{g}(C) \to \mathbb{C}$ such that:

- For all $H \in \mathfrak{h}$ and $1 \leq i \leq \ell$, $\langle H_i, H \rangle = \alpha_i(H)\epsilon_i$.
- For all $\alpha, \beta \in \Delta$, $g \in \mathfrak{g}(C)_\alpha$ and $g' \in \mathfrak{g}(C)_\beta$, $\langle g, g' \rangle = 0$ whenever $\alpha + \beta \neq 0$.
- For all $1 \leq i, j \leq \ell$, $\langle E_i, F_j \rangle = \delta_{ij}\epsilon_i$.

When C is an affine type GCM, $\mathfrak{g}(C)$ has a nondegenerate symmetric invariant bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g}(C) \times \mathfrak{g}(C) \to \mathbb{C}$ such that:

- For all $h \in \mathfrak{h}$ and $0 \leq i \leq \ell$, $\langle h_i, h \rangle = \alpha_i(h)\epsilon_i$.
- For all $\alpha, \beta \in \Delta$, $g \in \mathfrak{g}(C)_\alpha$ and $g' \in \mathfrak{g}(C)_\beta$, $\langle g, g' \rangle = 0$ whenever $\alpha + \beta \neq 0$.
- For all $0 \leq i, j \leq \ell$, $\langle e_i, f_j \rangle = \delta_{ij}\epsilon_i$.

In both cases, the restriction of this form to $\mathfrak{h}$ is still nondegenerate. Thus we have an isomorphism between $\mathfrak{h}$ and $\mathfrak{h}^*$ induced by this nondegenerate form. So, we induce a nondegenerate symmetric bilinear form on $\mathfrak{h}^*$. We have that:

$$\langle \alpha_i, \alpha_j \rangle = b_{ij} = a_{ij}/\epsilon_i \qquad \text{for } \alpha_i, \alpha_j \in \Pi. \tag{2.18}$$

The roots associated with a finite type GCM can have at most two root lengths called *long* and *short* (when there is only one length all roots are called long roots).

Let us normalize our nondegenerate symmetric invariant bilinear form so that long roots have squared length 2 (that is $|\alpha|^2 = \langle \alpha, \alpha \rangle = 2$ for each long root $\alpha \in \Delta$).

When C is a finite type GCM, there is a unique long root of maximal height: $\theta \in \Delta$ ($\mathrm{ht}(\alpha) < \mathrm{ht}(\theta)$ for all $\alpha \in \Delta$, $\alpha \neq \theta$). This root is called the *highest long root*.

When C is an affine type GCM, $\mathfrak{g}(C)$ has a natural integer gradation called the principal gradation.

Definition 2.13. *Define the* principal derivation, $d_P : \mathfrak{g}(C) \to \mathfrak{g}(C)$, *as follows:*

$$d_P(e_i) = e_i, \quad d_P(f_i) = -f_i, \quad \text{and } d_P(h) = 0 \tag{2.19}$$

for all $h \in \mathfrak{h}$ and $0 \leq i \leq \ell$.

Then extend d_P from the Chevalley generators to all of $\mathfrak{g}(C)$ by using the Leibnitz rule:

$$d_P([x, y]) = [d_P(x), y] + [x, d_P(y)]. \tag{2.20}$$

Let $\mathfrak{g}(C)^P_m = \{g \in \mathfrak{g}(C) \,|\, d_P(g) = mg\}$. This gives us the *principal gradation*:

$$\mathfrak{g}(C) = \bigoplus_{m \in \mathbb{Z}} \mathfrak{g}(C)^P_m \qquad \text{and} \qquad \mathfrak{g}(C)^P_m = \bigoplus_{\alpha \in \Delta, \mathrm{ht}(\alpha) = m} \mathfrak{g}(C)_\alpha. \tag{2.21}$$

Notice that $[\mathfrak{g}(C)^P_m, \mathfrak{g}(C)^P_n] \subseteq \mathfrak{g}(C)^P_{m+n}$ for all $m, n \in \mathbb{Z}$.

2.4 Integrable Highest-Weight Representations

The collection of all modules of affine Lie algebras is much too diverse for us to study in detail, so we will restrict our attention to modules with certain "nice" properties. In particular, we will study certain integrable highest-weight modules.

Definition 2.14. *A $\mathfrak{g}(C)$-module V is* $\mathfrak{h}$-diagonalizable *if*

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda \qquad \text{where} \qquad V_\lambda = \{v \in V \mid h \cdot v = \lambda(h)v, \forall h \in \mathfrak{h}\}. \tag{2.22}$$

If $V_\lambda \neq 0$, then V_λ is called a weight space *with* weight λ. *Nonzero elements of weight spaces are called* weight vectors. *Let*

$$P(V) = \{\lambda \in \mathfrak{h}^* \mid V_\lambda \neq 0\} \tag{2.23}$$

denote the set of all weights of V.

Recall that $\mathfrak{h}^*$ is partially ordered (see equation (2.13)). If $D \subseteq \mathfrak{h}^*$ and $\mu \leq \lambda$ for each $\mu \in D$, then D is *dominated* by λ. Given a $\mathfrak{g}(C)$-module V, let

$$D(\lambda) = \{\mu \in \mathfrak{h}^* \mid \mu \leq \lambda\} \qquad \text{where} \qquad \lambda \in \mathfrak{h}^*. \tag{2.24}$$

Definition 2.15. *Let us fix an affine Lie algebra $\mathfrak{g}(C)$. The category $\mathcal{O}$ has $\mathfrak{g}(C)$-module homomorphisms as its morphisms. A $\mathfrak{g}(C)$-module V is an object of $\mathcal{O}$ if and only if:*

- *The module* V *is* $\mathfrak{h}$*-diagonalizable.*

- *There exists some* $m \geq 0$ *and* $\mu_1, ..., \mu_m \in \mathfrak{h}^*$ *such that*

$$P(V) \subseteq D(\mu_1) \cup D(\mu_2) \cup ... \cup D(\mu_m).$$

Given a $\mathfrak{g}(A)$-module V, $g \in \mathfrak{g}(C)$ is called *locally nilpotent* if for every $v \in V$ there exists some positive integer N such that

$$g^N \cdot v = g \cdot (g \cdot (...(g \cdot v)...)) = 0. \tag{2.25}$$

Definition 2.16. *Fix an affine Lie algebra* $\mathfrak{g}(C)$*. A* $\mathfrak{g}(C)$*-module* V *is called* integrable *if:*

- *The module* V *is* $\mathfrak{h}$*-diagonalizable.*

- *The Chevalley generators* $e_0, ..., e_\ell$ *and* $f_0, ..., f_\ell$ *of* $\mathfrak{g}(C)$ *are locally nilpotent.*

Definition 2.17. *Let* V *be a* $\mathfrak{g}(C)$*-module for some Kac-Moody algebra* $\mathfrak{g}(C)$*.* V *is a* highest-weight *module if there exists some* $\Lambda \in \mathfrak{h}^*$ *and* $v_\Lambda \in V$ *such that:*

- *For all* $g \in \mathfrak{n}_+$*,* $g \cdot v_\Lambda = 0$ *(i.e.* $\mathfrak{n}_+(v_\Lambda) = 0$*).*

- *For all* $h \in \mathfrak{h}$*,* $h \cdot v_\Lambda = \Lambda(h)v_\Lambda$*.*

- *For each* $v \in V$ *there exists some* $x \in U(\mathfrak{g}(C))$ *such that* $x \cdot v_\Lambda = v$ *(i.e.* $U(\mathfrak{g}(C))(v_\Lambda) = V$ *or* V *is generated by* v_Λ*).*

The weight vector v_Λ *is called the* highest-weight vector*, and* Λ *is called the* highest-weight*.*

It follows from the fact that V is generated from the highest-weight vector that:

- $V = \bigoplus_{\lambda \leq \Lambda} V_\lambda$ (V is $\mathfrak{h}$-diagonalizable.)
- $V_\Lambda = \mathbb{C}v_\Lambda$ (the highest-weight vector is unique up to scalar multiples.)
- All of the weight spaces, V_λ, are finite dimensional.

Let us now construct a universal highest-weight module with highest-weight $\Lambda \in \mathfrak{h}^*$. First, let $\mathbb{C}_\Lambda$ be a $(\mathfrak{h}+\mathfrak{n}_+)$-module with a copy of $\mathbb{C}$ as its underlying vector space. Let $\mathfrak{n}_+$ act on $\mathbb{C}_\Lambda$ trivially (i.e. $x \cdot 1 = 0$ for all $x \in \mathfrak{n}_+$), and let $h \cdot 1 = \Lambda(h)$ for all $h \in \mathfrak{h}$. Now induce a $\mathfrak{g}(C)$ module from $\mathbb{C}_\Lambda$ as follows:

$$M(\Lambda) = U(\mathfrak{g}(C)) \otimes_{U(\mathfrak{h}+\mathfrak{n}_+)} \mathbb{C}_\Lambda. \tag{2.26}$$

This $\mathfrak{g}(C)$-module is the *Verma module* with highest-weight Λ.

Now $M(\Lambda)$ is a highest-weight module. Hence, it is generated by its highest-weight vector $v_\Lambda = 1$. Thus if N is any *proper* submodule of $M(\Lambda)$, we must have that $1 \notin N$. Let N be the sum of all proper submodules of $M(\Lambda)$. Then we still have that $1 \notin N$. Therefore, N itself is the maximal proper submodule of $M(\Lambda)$. Therefore, (since N is maximal) we have that

$$L(\Lambda) = M(\Lambda)/N \tag{2.27}$$

is an irreducible (highest-weight) $\mathfrak{g}(C)$-module.

Theorem 2.18. *(Proposition 9.3 on page 148 and Lemma 10.1 on page 171 of [16]) Let V be an irreducible $\mathfrak{g}(C)$-module from category $\mathcal{O}$. Then there exists (a unique) $\Lambda \in \mathfrak{h}^*$ such that $L(\Lambda) \cong V$.*

Moreover, $L(\Lambda)$ is integrable if and only if Λ is a dominant integral weight of $\mathfrak{g}(C)$ (i.e. $\Lambda \in P_+$).

2.5 Constructing Untwisted Affine Lie Algebras

Let us now restrict our attention to a special type of Kac-Moody algebras. We noted before that the GCMs break up into three distinct classes: finite, affine, and indefinite. Of the affine Lie algebras, the untwisted affine Lie algebras are the easiest to construct (a list of these untwisted affine GCMs can be found in Appendix A). Given a finite dimensional simple Lie algebra $\mathfrak{g}$ of type X_ℓ (see the appendix for a list), we will construct the affine Lie algebra of type $X_\ell^{(1)}$. We denote this algebra by $\tilde{\mathfrak{g}}$ and call it the *affinization* of $\mathfrak{g}$.

Let $\mathfrak{g}$ be *any* Lie algebra with a nondegenerate symmetric invariant bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$. Define

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \tag{2.28}$$

to be the Lie algebra with the brackets defined as follows (on homogeneous terms):

$$[\hat{\mathfrak{g}}, c] = 0 \qquad (c \text{ is central}) \tag{2.29}$$

and

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + m\langle a, b\rangle \delta_{m+n,0}c \tag{2.30}$$

where $a, b \in \mathfrak{g}$, $m, n \in \mathbb{Z}$, and $\delta_{i,j}$ is the Kronecker delta ($\delta_{i,j}$ is 1 when $i = j$ and 0 otherwise).

Sometimes $\hat{\mathfrak{g}}$ is referred to as the *affinization* of $\mathfrak{g}$. However, to correspond with our previous definition of affine Lie algebras, we need more. Define $d : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$ by

$$d = 1 \otimes t\frac{\mathrm{d}}{\mathrm{d}t} \tag{2.31}$$

on $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ and $d(c) = 0$. The map d is a derivation of $\hat{\mathfrak{g}}$. Define $\tilde{\mathfrak{g}}$ to be the semi-direct product Lie algebra:

$$\tilde{\mathfrak{g}} = \hat{\mathfrak{g}} \rtimes \mathbb{C}d \qquad \text{where} \qquad [d, d] = 0 \text{ and } [d, g] = d(g) \text{ for } g \in \hat{\mathfrak{g}}. \tag{2.32}$$

Just as with the principal derivation (see definition 2.13), we can use the *homogeneous degree derivation*

$$d_H = -d \tag{2.33}$$

to obtain a gradation of our Lie algebra. Let

$$\tilde{\mathfrak{g}}_m^H = \{g \in \tilde{\mathfrak{g}} \,|\, d_H(g) = mg\} \qquad \text{for } m \in \mathbb{Z}. \tag{2.34}$$

Then we have that:

$$\tilde{\mathfrak{g}} = \bigoplus_{m \in \mathbb{Z}} \tilde{\mathfrak{g}}_m^H \tag{2.35}$$

where $\tilde{\mathfrak{g}}_m^H = \mathfrak{g} \otimes t^{-m}$ for $m \neq 0$ and $\tilde{\mathfrak{g}}_0^H = \mathfrak{g} + \mathbb{C}c + \mathbb{C}d$. Likewise, let $\hat{\mathfrak{g}}_m^H = \{g \in \hat{\mathfrak{g}} \,|\, d_H(g) = mg\}$ for $m \in \mathbb{Z}$. Then $\hat{\mathfrak{g}} = \bigoplus_{m \in \mathbb{Z}} \hat{\mathfrak{g}}_m^H$, $\hat{\mathfrak{g}}_m^H = \tilde{\mathfrak{g}}_m^H$ for $m \neq 0$, and $\hat{\mathfrak{g}}_0^H = \mathfrak{g} + \mathbb{C}c$.

When $\mathfrak{g}$ is a finite dimensional simple Lie algebra with Cartan matrix C of type X_ℓ, $\tilde{\mathfrak{g}}$ is an affine Lie algebra with GCM of type $X_\ell^{(1)}$ (see e.g. [16]). We say that $\tilde{\mathfrak{g}}$ is the *(untwisted) affine algebra* associated with $\mathfrak{g}$.

Let C be the GCM of $\mathfrak{g}$ where C is of type X_ℓ. Let us describe the Chevalley generators of $\tilde{\mathfrak{g}}$ in terms of the Chevalley generators of $\mathfrak{g}$. Let $\mathfrak{h}$ be a CSA of $\mathfrak{g}$ and let E_i, F_i, H_i $(1 \leq i \leq \ell)$ be the Chevalley generators of $\mathfrak{g}$.

Recall that $\dim(\mathfrak{h}) = \ell$ and H_i's are linearly independent (thus they form a basis for $\mathfrak{h}$). Let $\lambda_1, ..., \lambda_n$ be the corresponding dual basis for $\mathfrak{h}^*$ (i.e. $\lambda_i(h_j) = \delta_{ij}$). The λ_i's are called the *fundamental weights* of $\mathfrak{g}$.

Likewise, the simple roots $\alpha_1, ..., \alpha_\ell$ form a basis for $\mathfrak{h}^*$. Let $H^{(1)}, ..., H^{(\ell)}$ be the corresponding dual basis for $\mathfrak{h}$ (i.e. $\alpha_i(H^{(j)}) = \delta_{ij}$). The $H^{(i)}$'s are called the *fundamental coweights*.

Let

$$\rho^\vee = \sum_{i=1}^{\ell} H^{(i)}. \tag{2.36}$$

Notice that $\alpha_i(\rho^\vee) = \sum_{j=1}^{\ell} \alpha_i(H^{(j)}) = \sum_{j=1}^{\ell} \delta_{ij} = 1$ for all $1 \leq i \leq \ell$.

Recall that Q and P denote the root lattice and weight lattice of $\mathfrak{g}$. Notice that:

$$Q = \mathbb{Z}\alpha_1 + ... + \mathbb{Z}\alpha_\ell \tag{2.37}$$

and

$$P = \mathbb{Z}\lambda_1 + ... + \mathbb{Z}\lambda_\ell. \tag{2.38}$$

Likewise, recall that $Q^\vee$ and $P^\vee$ denote the coroot and coweight lattice of $\mathfrak{g}$. Notice that:

$$Q^\vee = \mathbb{Z}H_1 + ... + \mathbb{Z}H_\ell \tag{2.39}$$

and

$$P^\vee = \mathbb{Z}H^{(1)} + ... + \mathbb{Z}H^{(\ell)}. \tag{2.40}$$

For any $\lambda \in \mathfrak{h}^*$ and $k \in \mathbb{C}$, we can extend λ to a linear functional (k, λ) in $\tilde{\mathfrak{h}}^* = (\mathfrak{h} + \mathbb{C}c + \mathbb{C}d)^*$ as follows: let $(k, \lambda)(h) = \lambda(h)$ for all $h \in \mathfrak{h}$, $(k, \lambda)(c) = k$, and $(k, \lambda)(d) = 0$. Define $\Lambda_i = (1, \lambda_i)$ for $1 \leq i \leq \ell$ and $\Lambda_0 = (1, 0)$ (i.e. let $\lambda_0 = 0$). Using this notation, we have the following:

$$L(k, k\lambda_i) = L(k\Lambda_i) \qquad \text{for } 0 \leq i \leq \ell. \tag{2.41}$$

In particular, $L(k,0) = L(k\Lambda_0)$.

Recall that θ is the highest long root of $\mathfrak{g}$. Choose $E_\theta \in \mathfrak{g}_\theta$ (the θ root space) and $F_\theta \in \mathfrak{g}_{-\theta}$ (the $-\theta$ root space) such that $\langle E_\theta, F_\theta \rangle = 1$. Also let $H_\theta = [E_\theta, F_\theta]$. Then the following hold:

$$\begin{aligned} e_0 &= F_\theta \otimes t \\ f_0 &= E_\theta \otimes t^{-1} \\ h_0 &= [e_0, f_0] \end{aligned} \tag{2.42}$$

and for $1 \leq i \leq \ell$

$$\begin{aligned} e_i &= E_i \otimes 1 \\ f_i &= F_i \otimes 1 \\ h_i &= H_i \otimes 1, \end{aligned} \tag{2.43}$$

where $e_0, ..., e_\ell$ and $f_0, ..., f_\ell$ are the Chevalley generators of $\tilde{\mathfrak{g}}$ (we also say that they are the Chevalley generators of $\hat{\mathfrak{g}}$).

Chapter 3

Vertex Operator Algebras

For a detailed introduction to vertex operator algebras and their representations, we refer the reader to [8], [10], [17], and [18]. We will be using the notation and terminology of [18] in what follows.

Let us establish the following notation (R is a commutative ring with unit):

$$R[x] = \{\sum_{i=0}^{n} a_i x^i \,|\, n \geq 0, i = 0...n, a_i \in R\} \qquad \text{Polynomials}$$

$$R[x, x^{-1}] = \{\sum_{i=-m}^{n} a_i x^i \,|\, m, n \geq 0, i = -m...n, a_i \in R\} \qquad \text{Laurent Polynomials}$$

$$R[[x]] = \{\sum_{i=0}^{\infty} a_i x^i \,|\, a_i \in R\} \qquad \text{Formal Power Series}$$

$$R[[x, x^{-1}]] = \{\sum_{i \in \mathbb{Z}} a_i x^i \,|\, a_i \in R\} \qquad \text{Formal Laurent Series}$$

$$R((x)) = \{\sum_{i \in \mathbb{Z}} a_i x^i \,|\, a_i \in R, a_i = 0 \text{ for } i \ll 0\} \qquad \text{Lower-Truncated Laurent Series}$$

Note: $a_i = 0$ for $i \ll 0$ means that there exists a integer $N \leq 0$ such that $a_i = 0$ for all $i < N$. (That is, a_i is zero for i sufficiently negative.)

Notice that $R[[x, x^{-1}]]$ contains all of these sets. $R[[x, x^{-1}]]$ is a $R[x, x^{-1}]$-module. We can always add two series: $\sum a_i x^i + \sum b_i x^i = \sum (a_i + b_i) x^i$. We can always multiply by elements of R: $r \sum a_i x^i = \sum (r a_i) x^i$, and we can always multiply by powers of x: $x^k \sum a_i x^i = \sum a_{i-k} x^i$. However, we cannot always multiply elements of $R[[x, x^{-1}]]$ together. For example, $\sum_{i<0} x^i$ times $\sum_{i>0} x^i$ is ill defined.

Definition 3.1. *For $a(x) = \sum_{n \in \mathbb{Z}} a_n x^n \in R[[x, x^{-1}]]$ and $y \in R$, we introduce the following notation:*

$$\lim_{x \to y} a(x) = a(x)\Big|_{x=y} = \sum_{n \in \mathbb{Z}} a_n y^n = a(y). \tag{3.1}$$

That is, substitute y for x (of course this may not always exist or make sense). Let

$$\mathrm{Res}_x a(x) = a_{-1}. \tag{3.2}$$

That is, $\mathrm{Res}_x a(x)$ *gives the coefficient of x^{-1}. We call this the* residue *of $a(x)$ at x.*

Definition 3.2. *Let n be any complex number. Define*

$$\binom{n}{0} = 1 \tag{3.3}$$

and for any positive integer k, *the* binomial coefficients *are defined by*

$$\binom{n}{k} = \frac{n(n-1)...(n-k+1)}{k!}. \tag{3.4}$$

Definition 3.3. *For any real number* n,

$$(x+y)^n = \sum_{k\geq 0} \binom{n}{k} x^{n-k} y^k. \tag{3.5}$$

Notice that when $(x+y)^n = \sum_{k\geq 0}\binom{n}{k}x^{n-k}y^k = x^n(1+\frac{y}{x})^n$ *is considered as a series in two complex variables, it converges for all values of* x *and* y *such that* $|\frac{y}{x}| < 1$ *(i.e.* $|y| < |x|$*).*

Remark 3.4. *Notice that* $(x+y)^n$ *is not equal to* $(y+x)^n$ *(unless* n *is a non-negative integer). We always expand in non-negative powers of the second variable:*

$$(x+y)^n \in \mathbb{C}[[x, x^{-1}]][[y]].$$

In some books and papers (for example, see [17]), $(x+y)^n$ *is denoted by* $\iota_{x,y}(x+y)^n$ *and* $(y+x)^n$ *is denoted by* $\iota_{y,x}(x+y)^n$ *to stress which expansion is being considered.*

Let us define the "formal δ-function at $x = 1$" to be the series

$$\delta(x) = \sum_{n\in\mathbb{Z}} x^n. \tag{3.6}$$

Note that in some texts (for example, see [17]), the formal δ-function is defined to be the series (x and y are two commuting indeterminates)

$$\delta(x-y) = x^{-1}\sum_{n\in\mathbb{Z}}\left(\frac{x}{y}\right)^n,$$

which is $x^{-1}\delta\left(\frac{x}{y}\right)$ in our notation.

The formal δ-function has many interesting properties. Let us consider a few here.

Proposition 3.5. *Let x and y be commuting indeterminates.*

- *For $a(x) \in R[[x, x^{-1}]]$,*

$$\mathrm{Res}_x a(x)x^{-1}\delta\left(\frac{x}{y}\right) = a(y). \tag{3.7}$$

- *We have the following relations:*

$$x^{-1}\delta\left(\frac{x}{y}\right) = y^{-1}\delta\left(\frac{y}{x}\right) = y^{-1}\delta\left(\frac{x}{y}\right). \tag{3.8}$$

- *Let $f(x,y) \in R[[x, x^{-1}, y, y^{-1}]]$ be such that $\lim_{x\to y} f(x,y) = f(y,y)$ exists. Then, we have that*

$$f(x,y)\,\delta\left(\frac{x}{y}\right) = f(x,x)\,\delta\left(\frac{x}{y}\right) = f(y,y)\,\delta\left(\frac{x}{y}\right). \tag{3.9}$$

- *The formal δ-function is an* expansion of zero*:*

$$\delta(x) = (1-x)^{-1} - (-x+1)^{-1}. \tag{3.10}$$

So, we have that

$$x^{-1}\delta\left(\frac{x}{y}\right) = (x-y)^{-1} - (-y+x)^{-1}. \tag{3.11}$$

The derivatives of $\delta(x)$ are also expansions of zero (n is a non-negative integer):

$$\frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n}\delta(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{(n)}\delta(x) = (1-x)^{-1-n} - (-x+1)^{-1-n}. \tag{3.12}$$

- *For any non-negative integers k and n, we have that*

$$(x-y)^{k}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{(n)}\delta\left(\frac{x}{y}\right) = 0 \tag{3.13}$$

whenever $k > n$.

3.1 Vertex Operator Algebras and Their Modules

In this section, we review the definitions of vertex (operator) algebras and their modules.

Definition 3.6. *Let V be a vector space (over $\mathbb{C}$) and*

$$a(x) = \sum_{n\in\mathbb{Z}} a_n x^{-n-1} \in \mathrm{End}\,(V)[[x, x^{-1}]]. \tag{3.14}$$

Suppose that $a_n(v) = 0$ for $n \gg 0$, then $a(x)$ is a field. *That is,*

$$a(x) \in \mathrm{Hom}\,(V, V((x)))\,. \tag{3.15}$$

Definition 3.7. *Let V be a vector space (over $\mathbb{C}$) with a distinguished vector $\mathbf{1} \in V$ (the* vacuum vector*) and a linear map $Y(\cdot, x) : V \to \mathrm{End}\,(V)[[x, x^{-1}]]$,*

$$v \mapsto Y(v, x) = \sum_{n\in\mathbb{Z}} v_n x^{-n-1} \tag{3.16}$$

such that $Y(v, x)$ is a field for all $v \in V$, and the following properties hold.

Vacuum Property *For every $v \in V$,*

$$Y(\mathbf{1}, x)v = v \tag{3.17}$$

($Y(\mathbf{1}, x) = 1_V$ is the identity map).

Creation Property *For every $v \in V$, $Y(v, x)\mathbf{1} \in V[[x]]$ (that is, $v_n\mathbf{1} = 0$ for all $n \geq 0$) and*

$$\lim_{x\to 0} Y(v, x)\mathbf{1} = v_{-1}\mathbf{1} = v. \tag{3.18}$$

Jacobi Identity *For every $u, v \in V$ the following identity holds:*

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y(u, x_1)Y(v, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)Y(v, x_2)Y(u, x_1) =$$

$$x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)Y(Y(u, x_0)v, x_2). \tag{3.19}$$

Then, $(V, Y, \mathbf{1})$ *(or briefly, V) is a* vertex algebra.

We will call $Y(v, x)$ the *vertex operator* associated with v. The coefficient v_n of x^{-n-1} is called the n^{th} *mode* of v.

Definition 3.8. *Let V be a vertex algebra such that the following properties hold.*

Grading *The vector space, V, is $\mathbb{Z}$-graded (that is, $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ where $\coprod$ denotes the direct sum). If $v \in V_{(n)}$, then v has* weight *n denoted by* $\mathrm{wt}(v) = n$*. In addition, assume that*

$$\dim V_{(n)} < \infty \qquad \text{and} \qquad V_{(n)} = 0 \text{ for } n \ll 0. \tag{3.20}$$

Conformal Vector *Suppose that V has a distinguished vector $\omega \in V_{(2)}$ called the* conformal vector*. Establish the following notation:*

$$Y(\omega, x) = \sum_{n\in\mathbb{Z}} \omega_n x^{-n-1} = \sum_{n\in\mathbb{Z}} L(n)x^{-n-2} \tag{3.21}$$

(i.e. $L(n) = \omega_{n+1}$*).*

Suppose that the modes of ω *satisfy the* Virasoro *relations* $(m, n \in \mathbb{Z})$*:*

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3 - m}{12}\delta_{m+n,0}\, c_V \tag{3.22}$$

where $c_V \in \mathbb{C}$ *is called the* central charge *(or* rank*) of* V*.*

Compatible Gradation *For any homogeneous vector* $v \in V_{(n)}$*, require that:*

$$L(0)v = nv = \mathrm{wt}(v)v. \tag{3.23}$$

Translation Invariance *(Also called the* $L(-1)$-derivative property*.) For every* $v \in V$*,*

$$Y(L(-1)v, x) = \frac{\mathrm{d}}{\mathrm{d}x}Y(v, x) \tag{3.24}$$

where $\frac{\mathrm{d}}{\mathrm{d}x}Y(v, x) = \sum_{n\in\mathbb{Z}} v_n(-n-1)x^{-n-2}$ *(the formal derivative).*

Then, $(V, Y, \mathbf{1}, \omega)$ *(or briefly,* V*) is a* vertex operator algebra (VOA).

Definition 3.9. *Let* V^1 *and* V^2 *be vertex algebras. Let* $\varphi : V^1 \to V^2$ *be a linear map such that for each* $v, w \in V^1$*,*

$$Y(\varphi(v), x)\varphi(w) = \varphi(Y(v, x)w) \tag{3.25}$$

(i.e. $\varphi(v)_n\varphi(w) = \varphi(v_n w)$ for each $n \in \mathbb{Z}$). In addition, suppose that

$$\varphi(\mathbf{1}) = \mathbf{1}. \tag{3.26}$$

Then φ is a vertex algebra homomorphism*.*

If V^1 and V^2 are also vertex operator algebras and

$$\varphi(\omega) = \omega, \tag{3.27}$$

then φ is a vertex operator algebra homomorphism*.*

Any bijective homomorphism is called an isomorphism*.*

Next, we need to define vertex algebra modules and vertex operator algebra modules.

Definition 3.10. *Let V be a vertex algebra. Let W be a vector space equipped with a linear map $Y_W(\cdot, x) : V \to \mathrm{End}\,(W)[[x, x^{-1}]]$ where*

$$v \mapsto Y_W(v, x) = \sum_{n\in\mathbb{Z}} v_n x^{-n-1} \tag{3.28}$$

for each $v \in V$ such that $Y_W(v, x)$ is a field and the following properties hold.

Vacuum Property *For every $w \in W$,*

$$Y_W(\mathbf{1}, x)w = w \tag{3.29}$$

(i.e. $Y_W(\mathbf{1}, x) = 1_W$ is the identity map).

Jacobi Identity *For every $u, v \in V$ the following identity holds:*

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_W(u, x_1)Y_W(v, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y_W(v, x_2)Y_W(u, x_1) =$$

$$x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y_W(Y(u, x_0)v, x_2). \tag{3.30}$$

*Then, (W, Y_W) (or briefly, W) is a V-*module *(or more precisely, a* vertex algebra module*).*

Again, $Y_W(v, x)$ is called the vertex operator associated with v, and v_n is called the n^{th} mode of v.

Definition 3.11. *Let V be a vertex operator algebra. Let W be a (vertex algebra) V-module. Again establish the notation:*

$$Y_W(\omega, x) = \sum_{n\in\mathbb{Z}} \omega_n x^{-n-1} = \sum_{n\in\mathbb{Z}} L(n)x^{-n-2} \tag{3.31}$$

(i.e. $L(n) = \omega_{n+1}$). Suppose that W is $\mathbb{C}$-graded:

$$W = \coprod_{a\in\mathbb{C}} W_{(a)} \text{ where } W_{(a)} = \{w \in W \mid L(0)w = aw\} \tag{3.32}$$

(again, the grading is given by the operator $L(0)$). In addition, suppose that for all

$a \in \mathbb{C}$, $\dim W_{(a)} < \infty$ and $W_{(a+r)} = 0$ for all sufficiently negative real numbers r.

*Then, W is a V-*module *(or more precisely, a* vertex operator algebra module*).*

Definition 3.12. *Let V be a vertex algebra. Let W^1 and W^2 be V-modules. Let $\varphi : W^1 \to W^2$ be a linear map such that for each $v \in V$ and $w \in W^1$ we have:*

$$\varphi(Y_{W_1}(v,x)w) = Y_{W_2}(v,x)\varphi(w) \tag{3.33}$$

(i.e. $\varphi(v_n w) = v_n \varphi(w)$ for each $n \in \mathbb{Z}$). Then, φ is a vertex algebra module homomorphism.

Note that if φ is a vertex algebra module homomorphism for two vertex operator algebra modules, then φ is a vertex operator algebra module homomorphism (there is no distinction).

Also, bijective module homomorphisms are called isomorphisms.

Example 3.13. *Let V be a vertex algebra. Then it is easy to see that V itself is a V-module. In fact, if V is a vertex operator algebra, then V is a vertex operator algebra V-module. In both cases, we call V the* adjoint module.

3.2 Vertex Operator Algebra Structure for $L(k\Lambda_0)$

For what follows, let $\mathfrak{g}$ be a finite dimensional simple Lie algebra (over $\mathbb{C}$) equipped with its normalized nondegenerate symmetric invariant bilinear form, $\langle \cdot, \cdot \rangle$.

For any $\tilde{\mathfrak{g}}$ or $\hat{\mathfrak{g}}$-module, V, let $a(n)$ denote the action of $a \otimes t^n$ on V where $a \in \mathfrak{g}$ and $n \in \mathbb{Z}$. Consider the following generating function:

$$a(x) = \sum_{n \in \mathbb{Z}} a(n) x^{-n-1} \qquad a \in \mathfrak{g}. \tag{3.34}$$

We can rewrite the commutation relations (2.30) in terms of these generating functions as follows (let x and y be commuting indeterminates and $a, b \in \mathfrak{g}$):

$$[a(x), b(y)] = [a, b](y) y^{-1} \delta\left(\frac{x}{y}\right) - \langle a, b \rangle \frac{\mathrm{d}}{\mathrm{d}x} y^{-1} \delta\left(\frac{x}{y}\right) c. \tag{3.35}$$

Definition 3.14. *Let V be a vector space and $a(x), b(x), c(x) \in \operatorname{End}(V)[[x, x^{-1}]]$ be fields. Then, for $n \in \mathbb{Z}$ let:*

$$a(x)_n b(x) = \operatorname{Res}_y \left((y - x)^n a(y) b(x) - (-x + y)^n b(x) a(y) \right). \tag{3.36}$$

Note that $a(x)_n b(x) \in \operatorname{End}(V)[[x, x^{-1}]]$ is also a field. For $m, n \in \mathbb{Z}$, $a(x)_n b(x)_m c(x)$ is defined to be $a(x)_n (b(x)_m c(x))$.

Recall the definition of the $\tilde{\mathfrak{g}}$-module $L(k, 0)$ (k some complex number) from equations (2.27) and (2.41).

Theorem 3.15. *(see e.g. Theorem 6.2.11 on page 207 of [18]) Let $k \in \mathbb{C}$. Then,*

$L(k,0)$ has the structure of a vertex algebra given by the following:

$$Y(\mathbf{1},x) = 1_{L(k,0)}, \tag{3.37}$$

$$Y(a,x) = a(x) \qquad \text{for each } \ a \in \mathfrak{g}, \tag{3.38}$$

and

$$Y(a_1(n_1)a_2(n_2)...a_r(n_r)1,x) = a_1(x)_{n_1}a_2(x)_{n_2}...a_r(x)_{n_r}1_{L(k,0)} \tag{3.39}$$

for each $r \geq 0$, $a_1, a_2, ..., a_r \in \mathfrak{g}$, and $n_1, n_2, ..., n_r \in \mathbb{Z}$.

Let $\{u^{(i)}\}_{i=1}^{d}$ ($\dim(\mathfrak{g}) = d$) be a basis for $\mathfrak{g}$ such that $\langle u^{(i)}, u^{(j)} \rangle = \delta_{i,j}$ (an orthonormal basis for $\mathfrak{g}$). Let $k \in \mathbb{C}$ such that $k \neq -h^{\vee}$ where $h^{\vee}$ is the dual Coxeter number of $\mathfrak{g}$. We note here that $h^{\vee}$ is always a positive integer (for a table of dual Coxeter numbers see [16] page 80). Define the following:

$$\omega = \frac{1}{2(k+h^{\vee})}\sum_{i=1}^{d} u^{(i)}(-1)u^{(i)}(-1)\mathbf{1}. \tag{3.40}$$

Then, we have the following vertex operator:

$$Y(\omega,x) = \sum_{n\in\mathbb{Z}} \omega_n x^{-n-1} = \sum_{n\in\mathbb{Z}} L(n)x^{-n-2}. \tag{3.41}$$

Theorem 3.16. *(see e.g. Theorems 6.2.16 on page 211 and 6.2.18 on page 214 of*

[18]) Let $k \neq -h^\vee$, $a \in \mathfrak{g}$, and $m, n \in \mathbb{Z}$. Recall that $\dim(\mathfrak{g}) = d$. Then

$$[L(m), a(n)] = -na(m+n), \tag{3.42}$$

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3 - m}{12}\frac{dk}{k + h^\vee}\delta_{m+n,0}, \tag{3.43}$$

and

$$L(0)v = nv = \mathrm{wt}(v)v \qquad \text{whenever} \qquad v \in L(k,0)_{(n)}. \tag{3.44}$$

Moreover, $L(k,0)$ is a vertex operator algebra.

3.3 Lattice Construction for ADE Level 1

We will need an initial condition in order to solve certain recurrence relations in a later section. To obtain this initial condition, we will use Frenkel and Kac's [9] explicit construction of the basic representation $L(1,0)$.

Let us restrict our attention to a finite dimensional simple Lie algebra $\mathfrak{g}$ of (ADE)-type. In this case, all of the roots of $\mathfrak{g}$ have one length (i.e. all roots are long roots). Recall that we have normalized our bilinear form so that $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in \Delta$. Also, recall that $Q = \mathbb{Z}\alpha_1 + ... + \mathbb{Z}\alpha_\ell$ is the root lattice. The root lattice is a positive definite even integral lattice. That is, for all $\alpha, \beta \in Q$: $\langle \alpha, \alpha \rangle > 0$ when $\alpha \neq 0$, $\langle \alpha, \alpha \rangle \in 2\mathbb{Z}$, and $\langle \alpha, \beta \rangle \in \mathbb{Z}$.

Let $\mathfrak{h} = Q \otimes_{\mathbb{Z}} \mathbb{C}$ (thus $\dim \mathfrak{h} = \ell$). We can naturally extend the symmetric bilinear form $\langle\cdot,\cdot\rangle$ on Q to $\mathfrak{h}$ and give $\mathfrak{h}$ the trivial bracket ($[h, h'] = 0$ for all $h, h' \in \mathfrak{h}$).

Therefore, $\mathfrak{h}$ is an abelian Lie algebra with a nondegenerate symmetric invariant bilinear form. We can then affinize $\mathfrak{h}$ to $\hat{\mathfrak{h}} = (\mathfrak{h} \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}c$ with brackets given by: $[a \otimes t^m, b \otimes t^n] = \langle a, b\rangle m\delta_{m+n,0}c$ for $a, b \in \mathfrak{h}$ and $m, n \in \mathbb{Z}$. Also, let c be central. Let $\hat{\mathfrak{h}}_- = \mathfrak{h} \otimes t\mathbb{C}[t]$ and $\hat{\mathfrak{h}}_+ = \mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}]$.

Let $\mathbb{C}_1$ be a $(\hat{\mathfrak{h}}_- + \mathfrak{h} + \mathbb{C}c)$-module with underlying vector space $\mathbb{C}$, $(\hat{\mathfrak{h}}_- + \mathfrak{h})$ acting trivially and $c \cdot 1 = 1$. Let $M(1) = U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{h}}_-)} \mathbb{C}_1$. Note that $M(1) \cong U(\hat{\mathfrak{h}}_+) \cong S(\hat{\mathfrak{h}}_+)$ linearly where $S(\hat{\mathfrak{h}}_+)$ is the symmetric algebra of the vector space $\hat{\mathfrak{h}}_+$.

Let $\alpha(n)$ denote the action of $\alpha \otimes t^n$ on $M(1)$ where $\alpha \in \mathfrak{h}$ and $n \in \mathbb{Z}$. Form the vertex operator $\alpha(x) = \sum_{n\in\mathbb{Z}} \alpha(n)x^{-n-1}$. Finally, let $\{u^{(i)}\}_{i=1}^{\ell}$ be an orthonormal basis for $\mathfrak{h}$ with respect to $\langle\cdot,\cdot\rangle$.

Theorem 3.17. *(see e.g. Theorem 6.3.2 page 219 of [18]) The $\hat{h}$-module, $M(1)$, is a vertex operator algebra with the following vertex operators:*

$$Y(\mathbf{1}, x) = 1_{M(1)}, \tag{3.45}$$

$$Y(\alpha, x) = \alpha(x) \qquad \text{for all } \alpha \in \mathfrak{h}, \tag{3.46}$$

and

$$Y(\alpha_1(n_1)...\alpha_r(n_r)1, x) = \alpha_1(x)_{n_1}...\alpha_r(x)_{n_r}1_{M(1)} \tag{3.47}$$

for all $r \geq 0$, $\alpha_1, ..., \alpha_r \in \mathfrak{h}$ and $n_1, ..., n_r \in \mathbb{Z}$.

The conformal vector of $M(1)$ is given by:

$$\omega = \frac{1}{2}\sum_{i=1}^{\ell} u^{(i)}(-1)u^{(i)}(-1)\mathbf{1}. \tag{3.48}$$

Note that $M(1)$ has central charge ℓ.

Let $\mathbb{C}[Q]$ be the group algebra of the abelian group Q. Let $V_Q = M(1) \otimes \mathbb{C}[Q]$. We extend the action of $\hat{\mathfrak{h}}$ from $M(1)$ to V_Q. Let $\left(\hat{\mathfrak{h}}_- + \hat{\mathfrak{h}}_+ + \mathbb{C}c\right)$ act trivially on $\mathbb{C}[Q]$ and let $\mathfrak{h}$ act on $\mathbb{C}[Q]$ as follows:

$$h(0)e^{\alpha} = \langle h, \alpha\rangle e^{\alpha} \qquad h \in \mathfrak{h},\ \alpha \in Q. \tag{3.49}$$

Then, V_Q is an $\hat{\mathfrak{h}}$-module when viewed as a tensor product of the $\hat{\mathfrak{h}}$-modules $M(1)$ and $\mathbb{C}[Q]$. The central element c acts as the identity,

$$h(0)(v \otimes e^{\alpha}) = \langle h, \alpha\rangle (v \otimes e^{\alpha}), \tag{3.50}$$

and

$$h(n)(v \otimes e^{\alpha}) = (h(n)v) \otimes e^{\alpha} \tag{3.51}$$

for all $v \in M(1)$, $\alpha \in Q$, $h \in \mathfrak{h}$, and $0 \neq n \in \mathbb{Z}$.

Finally, for $v \in M(1)$, $h \in \mathfrak{h}$, and $\alpha \in Q$ define:

$$x^{h}(v \otimes e^{\alpha}) = x^{\langle h, \alpha\rangle}(v \otimes e^{\alpha}). \tag{3.52}$$

We wish to describe the vertex algebra structure of V_Q. Consider the following operators ($\alpha \in \mathfrak{h}$):

$$E^{\pm}(\alpha, x) = \exp\left(\sum_{n \in \pm\mathbb{Z}_{>0}} \frac{\alpha(n)}{n} x^{-n}\right). \tag{3.53}$$

Now we can define the vertex operator associated with $e^{\alpha} \in \mathbb{C}[Q]$.

Theorem 3.18. *(see e.g. Theorem 6.5.1 page 240 of [18]) The $\hat{\mathfrak{h}}$-module, V_Q, is a vertex operator algebra with vertex algebra structure given by:*

$$Y(e^{\alpha}, x) = E^{-}(-\alpha, x)E^{+}(-\alpha, x)e^{\alpha}x^{\alpha} \tag{3.54}$$

for $\alpha \in Q$. For $\alpha \in \mathfrak{h}$, we define $Y(\alpha, x) = \alpha(x)$.

The conformal vector of V_Q is

$$\omega = \frac{1}{2}\sum_{i=1}^{\ell} u^{(i)}(-1)u^{(i)}(-1)\mathbf{1} \tag{3.55}$$

with central charge ℓ.

Theorem 3.19. *(see [9] and [10]) As vertex operator algebras, $V_Q \cong L(1, 0)$.*

3.4 Isomorphisms of VOA Modules

In this section, we will review some results obtained by Haisheng Li. For this section and what follows, we assume that k is a positive integer.

First, we recall several equivalent definitions of primary fields.

Proposition 3.20. *(Corollary 4.10 on page 127 of [17]) Let V be a vertex operator algebra and $a \in V$. Then, $Y(a,x)$ is a* primary field of conformal weight $\Delta \in \mathbb{C}$ *if and only if one of the following equivalent conditions hold:*

- $L_n a = \delta_{n,0}\Delta a$ *for all non-negative integers n;*
- $[L_m, Y(a,x)] = x^m \left(x\frac{\mathrm{d}}{\mathrm{d}x} + \Delta(m+1)\right) Y(a,x)$ *for all $m \in \mathbb{Z}$;*
- $[L_m, a_n] = ((\Delta - 1)m - n)a_{m+n}$ *for all $m, n \in \mathbb{Z}$.*

Let V be a vertex operator algebra and $\alpha \in V_{(1)}$ such that $Y(\alpha, x)$ is a *primary field* of conformal weight 1 (i.e. $L(n)\alpha = \delta_{n,0}\alpha$ for all non-negative integers n). Also, suppose that for some fixed constant $\gamma \in \mathbb{C}$, we have that

$$\alpha(n)\alpha = \delta_{n,1}\gamma\mathbf{1} \qquad \text{for all } \ n \in \mathbb{Z}_{\geq 0}. \tag{3.56}$$

Note that equation (3.56) is satisfied if the modes of α satisfy the Heisenberg commutation relations:

$$[\alpha(m), \alpha(n)] = m\delta_{m+n,0}\gamma 1_V \qquad \text{for all } \ m, n \in \mathbb{Z}. \tag{3.57}$$

In terms of generating functions, this says that

$$[Y(\alpha, x), Y(\alpha, y)] = \frac{\mathrm{d}}{\mathrm{d}y} y^{-1}\delta\left(\frac{x}{y}\right)\gamma Y(\mathbf{1}, x). \tag{3.58}$$

Such a vector α is called a *free boson.* In addition to being a weight 1 primary field and satisfying (3.56), suppose that $\alpha(0)$ acts semi-simply on V with integer eigenvalues. Consider the following element of $\mathrm{End}\,(V)[[x, x^{-1}]]$:

$$\Delta(\alpha, x) = x^{\alpha(0)} \exp\left(\sum_{m=1}^{\infty} \frac{\alpha(m)}{-m}(-x)^{-m}\right). \tag{3.59}$$

Proposition 3.21. *(Proposition 5.4 on page 230 of [22]) Suppose that α satisfies the conditions above. Let W be any (irreducible) V-vertex algebra module. Define*

$$(W^{(\alpha)}, Y_\alpha(\cdot, x)) = (W, Y(\Delta(\alpha, x)\cdot, x)). \tag{3.60}$$

Then, $(W^{(\alpha)}, Y_\alpha(\cdot, x))$ is an (irreducible) V-vertex algebra module.

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra, k be a positive integer, and let α be an element of the coroot lattice. Then, by equation (3.42), $Y(\alpha, x)$ is a weight 1 primary field. From equation (2.30), we see that the modes of α satisfy the Heisenberg commutation relations and that $\alpha(0)$ acts semi-simply with integer eigenvalues. Thus, Proposition 3.21 applies. In fact, we have the following result.

Proposition 3.22. *(Proposition 2.25 on page 664 of [25]) Let $(W, Y(\cdot, x))$ be any irreducible vertex operator algebra module of $L(k, 0)$ and let $\alpha \in Q^\vee$. Then, as vertex operator algebra modules of $L(k, 0)$,*

$$(W, Y(\cdot, x)) \cong (W^{(\alpha)}, Y_\alpha(\cdot, x)) = (W, Y(\Delta(\alpha, x)\cdot, x)). \tag{3.61}$$

Next, let us record how the $\Delta(\cdot, x)$ map affects certain vertex operators of an irreducible $L(k,0)$-module W. In [23] and [25], Haisheng Li calculated how $\Delta(\cdot, x)$ changes the conformal vector, but we also need to calculate $\Delta(\cdot, x)$'s effect on elements of the CSA.

For $a \in \mathfrak{g}$, establish the following notation:

$$Y_h(a, x) = Y(\Delta(h, x)a, x) = \sum_{n\in\mathbb{Z}} a_{(h)}(n)x^{-n-1}. \tag{3.62}$$

Thus, $H_{(h)}(0)$ is the image of $H(0)$ under the isomorphism between $L(k,0)$-modules $L(k,0)$ and $L(k,0)^{(h)}$. Likewise, if ω is the conformal vector in $L(k,0)$, denote the action of the conformal vector in $L(k,0)^{(h)}$ by:

$$Y_h(\omega, x) = Y(\Delta(h, x)\omega, x) = \sum_{n\in\mathbb{Z}} \omega_{(h)}(n)x^{-n-1} = \sum_{n\in\mathbb{Z}} L_{(h)}(n)x^{-n-2}. \tag{3.63}$$

Now let us calculate explicitly what $H'_{(H)}(0)$ and $L(0)_{(H)}$ are in terms of $H'(0)$ and $L(0)$ where $H, H' \in \mathfrak{h}$.

When $m \geq 0$, we have the following (by the commutation relations from equation (2.30)):

$$[H(m), H'(-1)]\mathbf{1} = [H, H'](m-1)\mathbf{1} + m\langle H, H'\rangle\delta_{m-1,0}k\mathbf{1} = mk\langle H, H'\rangle\delta_{m,1}\mathbf{1}.$$

Using the vacuum axiom, we see that $H(m)\mathbf{1} = 0$ when $m \geq 0$. Therefore,

$$H(m)H' = H(m)H'(-1)\mathbf{1} - H'(-1)H(m)\mathbf{1} = mk\langle H, H'\rangle\delta_{m,1}\mathbf{1}.$$

Notice that $H(0)H' = \langle H, H'\rangle\delta_{0,1}k\mathbf{1} = 0$. So, we get that $x^{H(0)}H' = x^0 = 1$. First, we find that

$$\begin{aligned}\left(\sum_{m\geq 1}\frac{H(m)}{-m}(-x)^{-m}\right)H' &= \sum_{m\geq 1}\frac{\langle H, H'\rangle\delta_{m,1}k}{-m}(-x)^{-m}\mathbf{1}\\ &= \langle H, H'\rangle k\mathbf{1}x^{-1}.\end{aligned}$$

Therefore, when $r > 1$,

$$\left(\sum_{m\geq 1}\frac{H(m)}{-m}(-x)^{-m}\right)^r H' = \left(\sum_{m\geq 1}\frac{H(m)}{-m}(-x)^{-m}\right)^{r-1}\left(\langle H, H'\rangle k\mathbf{1}x^{-1}\right) = 0$$

because $H(m)\mathbf{1} = 0$ for $m \geq 1$ (vacuum axiom). Putting this together, we find that

$$\begin{aligned}\Delta(H, x)H' &= H' + \left(\sum_{m\geq 1}\frac{H(m)}{-m}(-x)^{-m}\right)H' + \frac{1}{2!}\left(\sum_{m\geq 1}\frac{H(m)}{-m}(-x)^{-m}\right)^2 H' + \dots\\ &= H' + \langle H, H'\rangle k\mathbf{1}x^{-1} + \frac{1}{2!}0 + \dots\end{aligned}$$

in $L(k, 0)[x, x^{-1}]$.

Therefore, we have the following formula for $Y_H(H', x)$:

$$\begin{aligned} Y_H(H', x) &= Y(\Delta(H, x)H', x) \\ &= Y(H' + \langle H, H'\rangle kx^{-1}\mathbf{1}, x) \\ &= Y(H', x) + \langle H, H'\rangle kx^{-1}. \end{aligned} \tag{3.64}$$

Looking at the residue of this equation, we have the following lemma.

Lemma 3.23. *Let $H, H' \in \mathfrak{h} \subseteq \mathfrak{g}$ and let W be an irreducible vertex operator algebra module of $L(k, 0)$. The image of $H'(0)$ in $W^{(H)}$ under the map, $\Delta(H, x)$, is*

$$H'_{(H)}(0) = H'(0) + \langle H, H'\rangle k. \tag{3.65}$$

Now we want to find a formula for $L_{(H)}(0)$.

When $a \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$, Theorem 3.16 tells us that $[a(m), L(n)] = ma(m+n)$ in $L(k, 0)$. Recall that $\omega = L(-2)\mathbf{1}$. Thus, we have that

$$a(m)\omega = a(m)L(-2)\mathbf{1} = L(-2)a(m)\mathbf{1} + ma(m-2)\mathbf{1}. \tag{3.66}$$

The vacuum axiom gives us $a(m)\mathbf{1} = 0$ for $m \geq 0$ and $a(-1)\mathbf{1} = a$. Thus, $a(m-2)\mathbf{1} = m\delta_{m-2,-1}a$ for $m \geq 1$. So, we get $a(m)\omega = 0 + m\delta_{m-2,-1}a = m\delta_{m,1}a$ for $m \geq 1$.

$$a(m)\omega = \delta_{m,1}a \qquad \text{for } a \in \mathfrak{g} \text{ and } m \geq 1 \tag{3.67}$$

This implies that $H(0)\omega = \delta_{0,1}H = 0$. Thus, $x^{H(0)}\omega = x^0 = 1$. First, we find that

$$\left(\sum_{m\geq 1}\frac{H(m)}{-m}(-x)^{-m}\right)\omega = \sum_{m\geq 1}\frac{\delta_{m,1}H}{-m}(-x)^{-m} = x^{-1}H. \tag{3.68}$$

Using (3.68), we have that

$$\begin{aligned}\left(\sum_{m\geq 1}\frac{H(m)}{-m}(-x)^{-m}\right)^2\omega &= \left(\sum_{m\geq 1}\frac{H(m)}{-m}(-x)^{-m}\right)(x^{-1}H)\\ &= x^{-1}\sum_{m\geq 1}\frac{\langle H,H\rangle\delta_{m,1}k}{-m}(-x)^{-m}\mathbf{1}\\ &= x^{-1}\langle H,H\rangle k\mathbf{1}x^{-1} = x^{-2}\langle H,H\rangle k\mathbf{1}.\end{aligned}$$

Therefore, when $r > 2$,

$$\left(\sum_{m\geq 1}\frac{H(m)}{-m}(-x)^{-m}\right)^r\omega = \left(\sum_{m\geq 1}\frac{H(m)}{-m}(-x)^{-m}\right)^{r-2}\left(x^{-2}\langle H,H\rangle k\mathbf{1}\right) = 0$$

(because the vacuum axiom gives us $H(m)\mathbf{1} = 0$ for $m \geq 1$). Now, putting all of this together, we get:

$$\begin{aligned}\Delta(H,x)\omega &= \omega + \left(\sum_{m\geq 1}\frac{H(m)}{-m}(-x)^{-m}\right)\omega + \frac{1}{2!}\left(\sum_{m\geq 1}\frac{H(m)}{-m}(-x)^{-m}\right)^2\omega + \dots\\ &= \omega + Hx^{-1} + \frac{1}{2!}(\langle H,H\rangle k\mathbf{1}x^{-2}) + \frac{1}{3!}0 + \dots\end{aligned}$$

in $L(k,0)[x,x^{-1}]$.

Therefore, we have the following formula for $Y_H(\omega, x)$:

$$\begin{aligned} Y(\Delta(H,x)\omega, x) &= Y(\omega + Hx^{-1} + \frac{\langle H, H\rangle k}{2}\mathbf{1}x^{-2}, x) \\ &= Y(\omega, x) + x^{-1}Y(H, x) + x^{-2}\frac{\langle H, H\rangle}{2}k. \end{aligned} \tag{3.69}$$

Thus, we have the following result.

Lemma 3.24. *(see Remark 2.23 on page 663 of [25]) Let $H \in \mathfrak{h} \subseteq \mathfrak{g}$ and let W be an irreducible vertex operator algebra module of $L(k, 0)$. The image of $L(0)$ in $W^{(H)}$ under the $\Delta(H, x)$ map is*

$$L_{(H)}(0) = L(0) + H(0) + \frac{\langle H, H\rangle}{2}k. \tag{3.70}$$

Chapter 4

Multisum Identities

In this chapter, we will find recurrence relations for the character of an irreducible $L(k,0)$-module (k a positive integer). Then, we will restrict our attention to the case when $\mathfrak{g}$ (the underlying finite dimensional simple Lie algebra) is simply laced (i.e. (ADE)-type) and $k = 1$. In this case, we will solve the recurrence relations and obtain a formula for the character of $L(1,0)$.

In the last section, we will specialize this formula to the principal picture and combine it with previously known principal character formulas to obtain a new class of multi-sum identities.

4.1 Characters of $L(k,0)$-modules

Both d_H and d_P are derivations for $\hat{\mathfrak{g}}$. We can easily extend them to any highest-weight module (such as $L(\Lambda)$) by letting $d_H(v_\Lambda) = d_P(v_\Lambda) = 0$.

Notice that for all $a \in \mathfrak{g}$ and $n \in \mathbb{Z}$, we have

$$[d_H, a \otimes t^n] = d_H(a \otimes t^n) = na \otimes t^n.$$

So, we have that $[L(0) - d_H, a(n)] = 0$ in $L(\Lambda)$. Therefore,

$$L(0) = d_H + \mu \tag{4.1}$$

for some complex number μ.

Definition 4.1. *Let $L(\Lambda)$ be an irreducible $L(k,0)$-module. We define the (full)* character *of $L(\Lambda)$ to be*

$$\chi_{L(\Lambda)}(x_1, ..., x_\ell; q) = \mathrm{tr}_{L(\Lambda)} x_1^{H^{(1)}(0)} ... x_\ell^{H^{(\ell)}(0)} q^{L(0)}, \tag{4.2}$$

the homogeneous *character to be*

$$\chi^H_{L(\Lambda)}(q) = \mathrm{tr}_{L(\Lambda)} q^{d_H} = q^{-\mu} \chi_{L(\Lambda)}(1, ..., 1; q), \tag{4.3}$$

and the principal *character to be*

$$\chi^P_{L(\Lambda)}(q) = \mathrm{tr}_{L(\Lambda)} q^{-d_P}. \tag{4.4}$$

Lemma 4.2. *We have that*

$$d_P = (\mathrm{ht}(\theta) + 1)(-d_H) + \mathrm{ad}(\rho^\vee) \tag{4.5}$$

in $\hat{\mathfrak{g}}$ *(and* $L(\Lambda)$*).*

Proof. Both d_P and $(\mathrm{ht}(\theta) + 1)(-d_H) + \mathrm{ad}(\rho^\vee)$ act on the central element c as zero. For $1 \leq i \leq \ell$, we have that

$$\begin{aligned}
\left((\mathrm{ht}(\theta) + 1)(-d_H) + \mathrm{ad}(\rho^\vee)\right)(e_i) &= (\mathrm{ht}(\theta) + 1)0 + [\rho^\vee, e_i] \\
&= \alpha_i(\rho^\vee) E_i \otimes 1 \\
&= E_i \otimes 1 = e_i = d_P(e_i)
\end{aligned}$$

and

$$\begin{aligned}
\left((\mathrm{ht}(\theta) + 1)(-d_H) + \mathrm{ad}(\rho^\vee)\right)(f_i) &= (\mathrm{ht}(\theta) + 1)0 + [\rho^\vee, f_i] \\
&= -\alpha_i(\rho^\vee) F_i \otimes 1 \\
&= -F_i \otimes 1 = -f_i = d_P(f_i).
\end{aligned}$$

We also have that

$$\begin{aligned}
\left((\mathrm{ht}(\theta)+1)(-d_H)+\mathrm{ad}(\rho^\vee)\right)(e_0) &= \left((\mathrm{ht}(\theta)+1)d+\mathrm{ad}(\rho^\vee)\right)(F_\theta\otimes t)\\
&= (\mathrm{ht}(\theta)+1)F_\theta\otimes t-\theta(\rho^\vee)F_\theta\otimes t\\
&= (\mathrm{ht}(\theta)+1)F_\theta\otimes t-\mathrm{ht}(\theta)F_\theta\otimes t\\
&= F_\theta\otimes t=e_0=d_P(e_0)
\end{aligned}$$

and

$$\begin{aligned}
\left((\mathrm{ht}(\theta)+1)(-d_H)+\mathrm{ad}(\rho^\vee)\right)(f_0) &= \left((\mathrm{ht}(\theta)+1)d+\mathrm{ad}(\rho^\vee)\right)(E_\theta\otimes t^{-1})\\
&= (\mathrm{ht}(\theta)+1)(-1)E_\theta\otimes t^{-1}+\theta(\rho^\vee)E_\theta\otimes t^{-1}\\
&= -(\mathrm{ht}(\theta)+1)E_\theta\otimes t^{-1}+\mathrm{ht}(\theta)E_\theta\otimes t^{-1}\\
&= -E_\theta\otimes t^{-1}=-f_0=d_P(f_0).
\end{aligned}$$

We have shown that the equality holds on a generating set of $\hat{\mathfrak{g}}$. Therefore, it holds for all of $\hat{\mathfrak{g}}$ (and thus on $L(\Lambda)$ also). □

Proposition 4.3. *Let $L(\Lambda)$ be an irreducible $L(k,0)$-module (k a positive integer). We obtain the principal character of $L(\Lambda)$ from the full character of $L(\Lambda)$ as follows:*

$$\chi^P_{L(\Lambda)}(q)=q^{-(\mathrm{ht}(\theta)+1)\mu}\chi_{L(\Lambda)}(q^{-1},...,q^{-1};q^{\mathrm{ht}(\theta)+1}). \tag{4.6}$$

Proof.

$$
\begin{aligned}
& q^{-(\mathrm{ht}(\theta)+1)\mu}\chi_{L(\Lambda)}(q^{-1},...,q^{-1};q^{\mathrm{ht}(\theta)+1}) \\
&= q^{-(\mathrm{ht}(\theta)+1)\mu}\mathrm{tr}_{L(\Lambda)}q^{-H^{(1)}(0)}...q^{-H^{(\ell)}(0)}q^{(\mathrm{ht}(\theta)+1)L(0)} \\
&= \mathrm{tr}_{L(\Lambda)}q^{-H^{(1)}(0)}...q^{-H^{(\ell)}(0)}q^{(\mathrm{ht}(\theta)+1)(L(0)-\mu)} \\
&= \mathrm{tr}_{L(\Lambda)}q^{-H^{(1)}(0)}...q^{-H^{(\ell)}(0)}q^{(\mathrm{ht}(\theta)+1)d_H} \\
&= \mathrm{tr}_{L(\Lambda)}q^{-(\mathrm{ht}(\theta)+1)(-d_H)-\sum_{i=1}^{\ell}H^{(i)}(0)} \\
&= \mathrm{tr}_{L(\Lambda)}q^{-((\mathrm{ht}(\theta)+1)(-d_H)+\rho^\vee(0))} = \mathrm{tr}_{L(\Lambda)}q^{-d_P} = \chi^P_{L(\Lambda)}(q)
\end{aligned}
$$

□

4.2 The Recurrence Relations

In this section, we obtain recurrence relations for the character of an irreducible $L(k,0)$-module (k a positive integer). Then, after expanding the character, we use our recurrence relations for the character to obtain recurrence relations for the coefficients of the character.

Lemma 4.4. *Let $H \in \mathfrak{h}$ such that $W \cong W^{(H)}$ as $L(k,0)$-modules (k a positive integer). Then, we have that*

$$\chi_W(x_1,...,x_\ell;q) = \mathrm{tr}_W x_1^{H^{(1)}_{(H)}(0)}...x_\ell^{H^{(\ell)}_{(H)}(0)}q^{L_{(H)}(0)}. \tag{4.7}$$

Proof. Let $\varphi : W \to W^{(H)}$ be an $L(k,0)$-module isomorphism. Suppose that $v \in W$ such that $H^{(i)}(0)v = n_i v$ for $1 \leq i \leq \ell$, $n_i \in \mathbb{Z}$. Also, suppose that $L(0)v = mv$ for some $m \in \mathbb{Z}$. Then, consider $\varphi(v) \in W^{(H)}$. We have that

$$H^{(i)}_{(H)}(0)\varphi(v) = \varphi(H^{(i)}_{(H)}(0)v) = \varphi(n_i v) = n_i \varphi(v)$$

and also

$$L_{(H)}(0)\varphi(v) = \varphi(L(0)v) = \varphi(mv) = m\varphi(v).$$

Now, φ is a bijection. Thus, for any $n_1, ..., n_\ell, m \in \mathbb{Z}$,

$$\begin{aligned} & \dim\{v \in W \mid H^{(i)}(0)v = n_i v, L(0)v = mv\} \\ = \quad & \dim\{w \in W \mid H^{(i)}_{(H)}(0)w = n_i w, L_{(H)}(0)w = mw\}. \end{aligned}$$

The lemma then follows. □

Theorem 4.5. *Let W be an irreducible $L(k,0)$-module (k a positive integer). Then, we have the following relations ($1 \leq i \leq \ell$):*

$$\chi_W(x_1, ..., x_\ell; q) = (x_i q)^{\frac{2}{\langle \alpha_i, \alpha_i \rangle} k} \chi_W(x_1 q^{a_{1i}}, ..., x_\ell q^{a_{\ell i}}; q).$$

Recall that a_{ij} denotes the i^{th} row, j^{th} column entry of the Cartan matrix of $\mathfrak{g}$.

Proof. Fix some $1 \leq i \leq \ell$. Consider the simple coroot $H_i \in Q^\vee$. By Proposition

3.22, we have that $W \cong W^{(H_i)}$ (as vertex operator algebra modules of $L(k, 0)$). Thus by Proposition 4.5, we have that

$$\chi_W(x_1, ..., x_\ell; q) = \mathrm{tr}_W x_1^{H^{(1)}_{(H_i)}(0)} ... x_\ell^{H^{(\ell)}_{(H_i)}(0)} q^{L_{(H_i)}(0)}.$$

Recall equation (3.65): $H^{(j)}_{(H_i)}(0) = H^{(j)}(0) + \langle H^{(j)}, H_i \rangle k$ and note that $\langle H^{(j)}, H_i \rangle = \frac{2}{\langle \alpha_i, \alpha_i \rangle} \delta_{i,j} k$. Therefore, we have that

$$H^{(j)}_{(H_i)}(0) = H^{(j)}(0) + \frac{2}{\langle \alpha_i, \alpha_i \rangle} \delta_{i,j} k.$$

Now, recall equation (3.70): $L_{(H_i)}(0) = L(0) + H_i(0) + \frac{\langle H_i, H_i \rangle}{2} k$. Note that $H_i = \sum_{j=1}^{\ell} a_{j,i} H^{(j)}$ and $\langle H_i, H_i \rangle = \frac{4}{\langle \alpha_i, \alpha_i \rangle}$. Therefore, we have

$$L_{(H_i)}(0) = L(0) + \sum_{j=1}^{\ell} a_{j,i} H^{(j)}(0) + \frac{2}{\langle \alpha_i, \alpha_i \rangle} k.$$

This gives us the following:

$$\begin{aligned}
&\chi_W(x_1, ..., x_\ell; q) \\
&= \mathrm{tr}_W x_1^{H^{(1)}(0) + \frac{2}{\langle \alpha_i, \alpha_i \rangle} \delta_{i,1} k} ... x_\ell^{H^{(\ell)}(0) + \frac{2}{\langle \alpha_i, \alpha_i \rangle} \delta_{i,\ell} k} q^{L(0) + \sum_{j=1}^{\ell} a_{ji} H^{(j)}(0) + \frac{2}{\langle \alpha_i, \alpha_i \rangle} k} \\
&= (x_i q)^{\frac{2}{\langle \alpha_i, \alpha_i \rangle} k} \mathrm{tr}_W (x_1 q^{a_{1i}})^{H^{(1)}(0)} ... (x_\ell q^{a_{\ell i}})^{H^{(\ell)}(0)} q^{L(0)} \\
&= (x_i q)^{\frac{2}{\langle \alpha_i, \alpha_i \rangle} k} \chi_W(x_1 q^{a_{1i}}, ..., x_\ell q^{a_{\ell i}}; q).
\end{aligned}$$

□

For $\mathbf{n} = (n_1, ..., n_\ell) \in \mathbb{Z}^\ell$, we establish the following notation:

$$\mathbf{x}^{\mathbf{n}} = x_1^{n_1} ... x_\ell^{n_\ell}. \tag{4.8}$$

Define $A(\mathbf{n}; q) \in q^\mu \mathbb{C}[[q]]$ by

$$\chi_W(x_1, ..., x_\ell; q) = \sum_{\mathbf{n} \in \mathbb{Z}^\ell} A(\mathbf{n}; q) \mathbf{x}^{\mathbf{n}} \tag{4.9}$$

(recall that $L(0) = d_h + \mu$ for some complex number μ).

Let us rewrite the recurrence relations of Theorem 4.5 in terms of these coefficients:

$$\begin{aligned}
&\chi_W(x_1, ..., x_\ell; q) \\
&= (x_i q)^{\frac{2}{\langle \alpha_i, \alpha_i \rangle} k} \chi_W(x_1 q^{a_{1i}}, ..., x_\ell q^{a_{\ell i}}; q) \\
&= (x_i q)^{\frac{2}{\langle \alpha_i, \alpha_i \rangle} k} \sum_{n_1, ..., n_\ell \in \mathbb{Z}} A(n_1, ..., n_\ell; q) (x_1 q^{a_{1i}})^{n_1} ... (x_\ell q^{a_{\ell i}})^{n_\ell} \\
&= \sum_{n_1, ..., n_\ell \in \mathbb{Z}} A(n_1, ..., n_\ell; q) q^{\frac{2}{\langle \alpha_i, \alpha_i \rangle} k + \sum_{m=1}^{\ell} a_{mi} n_m} x_1^{n_1} ... x_{i-1}^{n_{i-1}} x_i^{n_i + \frac{2}{\langle \alpha_i, \alpha_i \rangle} k} x_{i+1}^{n_{i+1}} ... x_\ell^{n_\ell}.
\end{aligned}$$

For $1 \leq i \leq \ell$, $\frac{2}{\langle \alpha_i, \alpha_i \rangle}$ is a positive integer. Thus, we can replace n_i with $n_i - \frac{2}{\langle \alpha_i, \alpha_i \rangle} k$

(since the above sum ranges over all of the integers). Therefore, we get that

$$\begin{aligned}
&\sum_{n_1,\dots,n_\ell\in\mathbb{Z}} A(n_1,...,n_\ell;q)x_1^{n_1}...x_\ell^{n_\ell} \\
&= \sum_{n_1,\dots,n_\ell\in\mathbb{Z}} A(n_1,...,n_{i-1},n_i-\frac{2}{\langle\alpha_i,\alpha_i\rangle}k,n_{i+1},...,n_\ell;q) \\
&\qquad q^{\frac{2}{\langle\alpha_i,\alpha_i\rangle}k-a_{ii}\frac{2}{\langle\alpha_i,\alpha_i\rangle}k+\sum_{m=1}^{\ell}a_{mi}n_m}x_1^{n_1}...x_\ell^{n_\ell} \\
&= \sum_{n_1,\dots,n_\ell\in\mathbb{Z}} A(n_1,...,n_{i-1},n_i-\frac{2}{\langle\alpha_i,\alpha_i\rangle}k,n_{i+1},...,n_\ell;q) \\
&\qquad q^{-\frac{2}{\langle\alpha_i,\alpha_i\rangle}k+\sum_{m=1}^{\ell}a_{mi}n_m}x_1^{n_1}...x_\ell^{n_\ell}.
\end{aligned}$$

Equating coefficients, we have the following proposition.

Proposition 4.6. *Let $\mathfrak{g}$ be any finite dimensional simple Lie algebra over $\mathbb{C}$ of rank ℓ with Cartan matrix $C=(a_{ij})$. Let k be a positive integer, let W be an irreducible $L(k,0)$-module, and let $1\leq i\leq\ell$. Then, for $n_1,...,n_\ell\in\mathbb{Z}$ we have that*

$$\begin{aligned}
&A(n_1,...,n_\ell;q) \\
&= A(n_1,...,n_{i-1},n_i-\frac{2}{\langle\alpha_i,\alpha_i\rangle}k,n_{i+1},...,n_\ell;q)q^{-\frac{2}{\langle\alpha_i,\alpha_i\rangle}k+\sum_{m=1}^{\ell}a_{mi}n_m} \qquad (4.10)
\end{aligned}$$

where $\chi_W(\mathbf{x};q)=\sum_{\mathbf{n}\in\mathbb{Z}^\ell}A(\mathbf{n};q)\mathbf{x}^{\mathbf{n}}$.

4.3 Solving The Recurrence Relations

Now, let us assume that $\mathfrak{g}$ is of (ADE)-type. Then all roots are long roots and thus $\langle \alpha_i, \alpha_i \rangle = 2$ for all $1 \leq i \leq \ell$. Therefore, $\frac{2}{\langle \alpha_i, \alpha_i \rangle} = 1$ for $1 \leq i \leq \ell$. So, our recurrence relations simplify to

$$A(n_1, \ldots, n_\ell; q) = A(n_1, \ldots, n_{i-1}, n_i - k, n_{i+1}, \ldots, n_\ell; q) q^{-k + \sum_{m=1}^{\ell} a_{mi} n_m} \tag{4.11}$$

for $n_1, \ldots, n_\ell \in \mathbb{Z}$, $1 \leq i \leq \ell$.

When $\mathfrak{g}$ is of (ADE)-type and the level $k = 1$, the recurrence relations (4.11) simplify to the following relations:

$$A(n_1, \ldots, n_\ell; q) = A(n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_\ell; q) q^{-1 + \sum_{m=1}^{\ell} a_{mi} n_m} \tag{4.12}$$

for $n_1, \ldots, n_\ell \in \mathbb{Z}$, $1 \leq i \leq \ell$.

We now solve these recurrence relations by "eliminating" one index at a time.

Lemma 4.7. *Assume $\mathfrak{g}$ is of (ADE)-type and $k = 1$. For $1 \leq i \leq \ell$ and $n_1, \ldots, n_\ell \in \mathbb{Z}$, we have*

$$A(n_1, \ldots, n_\ell; q) = A(n_1, \ldots, n_{i-1}, 0, n_{i+1}, \ldots, n_\ell; q) q^{-n_i^2 + \sum_{m=1}^{\ell} n_m a_{mi} n_i}. \tag{4.13}$$

Proof. Let $1 \leq i \leq \ell$. When $n_i = 0$, we have

$$\begin{aligned} & A(n_1, \ldots, n_{i-1}, 0, n_{i+1}, \ldots, n_\ell; q) q^{-0^2 + \sum_{m=1}^{\ell} n_m a_{mi}(0)} \\ = \quad & A(n_1, \ldots, n_{i-1}, 0, n_{i+1}, \ldots, n_\ell; q) q^0. \end{aligned}$$

Thus, the formula holds trivially for $n_i = 0$.

Assume that the formula holds for $n_i - 1$. Then, we have

$$\begin{aligned} & A(n_1, \ldots, n_\ell; q) \\ = \quad & A(n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_\ell; q) q^{-1 + \sum_{m=1}^{\ell} a_{mi} n_m} \\ = \quad & A(n_1, \ldots, n_{i-1}, 0, n_{i+1}, \ldots, n_\ell; q) \\ & \qquad q^{-(n_i - 1)^2 + \sum_{m=1}^{\ell} (n_m - \delta_{m,i}) a_{mi} (n_i - 1)} q^{-1 + \sum_{m=1}^{\ell} a_{mi} n_m} \\ = \quad & A(n_1, \ldots, n_{i-1}, 0, n_{i+1}, \ldots, n_\ell; q) \\ & \qquad q^{-n_i^2 + \sum_{m=1}^{\ell} n_m a_{mi} n_i}. \end{aligned}$$

Thus, if the formula holds for $n_i - 1$, it also holds for n_i.

Again, assume that the formula holds for n_i. This gives us the following:

$$A(n_1, \ldots, n_\ell; q) = A(n_1, \ldots, n_{i-1}, 0, n_{i+1}, \ldots, n_\ell; q) q^{-n_i^2 + \sum_{m=1}^{\ell} n_m a_{mi} n_i}. \tag{4.14}$$

The recurrences (4.12) give us

$$A(n_1,\ldots,n_\ell;q) = A(n_1,\ldots,n_{i-1},n_i-1,n_{i+1},\ldots,n_\ell;q)q^{-1+\sum_{m=1}^{\ell} a_{mi}n_m}.$$

Substituting this into (4.14), we get

$$\begin{aligned} & A(n_1,\ldots,n_{i-1},n_i-1,n_{i+1},\ldots,n_\ell;q)q^{-1+\sum_{m=1}^{\ell} a_{mi}n_m} \\ = \;& A(n_1,\ldots,n_{i-1},0,n_{i+1},\ldots,n_\ell;q)q^{-n_i^2+\sum_{m=1}^{\ell} n_m a_{mi} n_i}. \end{aligned}$$

Dividing both sides by $q^{-1+\sum_{m=1}^{\ell} a_{mi}n_m}$, the exponent of q becomes 0 on the left hand side and the exponent of q on the right hand side becomes

$$\begin{aligned} & -n_i^2 + \sum_{m=1}^{\ell} n_m a_{mi} n_i + 1 - \sum_{m=1}^{\ell} a_{mi} n_m \\ = \;& -n_i^2 + 1 + \sum_{m=1}^{\ell} n_m a_{mi}(n_i-1) \\ = \;& -n_i^2 + 1 + \sum_{m=1}^{\ell} n_m a_{mi}(n_i-1) - \sum_{m=1}^{\ell} \delta_{mi} a_{mi}(n_i-1) + \sum_{m=1}^{\ell} \delta_{mi} a_{mi}(n_i-1) \\ = \;& -n_i^2 + 1 + \sum_{m=1}^{\ell} (n_m - \delta_{mi}) a_{mi}(n_i-1) + \sum_{m=1}^{\ell} \delta_{mi} a_{mi}(n_i-1) \\ = \;& -n_i^2 + 1 + \sum_{m=1}^{\ell} (n_m - \delta_{mi}) a_{mi}(n_i-1) + a_{ii}(n_i-1) \\ = \;& -(n_i-1)^2 + 1 + \sum_{m=1}^{\ell} (n_m - \delta_{mi}) a_{mi}(n_i-1). \end{aligned}$$

Therefore, we have

$$
\begin{aligned}
&A(n_1,\dots,n_{i-1},n_i-1,n_{i+1},\dots,n_\ell;q)\\
&= A(n_1,\dots,n_{i-1},0,n_{i+1},\dots,n_\ell;q)q^{-(n_i-1)^2+1+\sum_{m=1}^{\ell}(n_m-\delta_{mi})a_{mi}(n_i-1)}.
\end{aligned}
$$

Thus, if the formula holds for n_i, it also holds for n_i-1. Therefore, by induction, the formula holds for all $n_i \in \mathbb{Z}$. □

Now we can apply Lemma 4.7 repeatedly to get

$$
\begin{aligned}
&A(n_1,\dots,n_\ell;q)\\
&= A(0,n_2,\dots,n_\ell;q)q^{-n_1^2+\sum_{m=1}^{\ell} n_m a_{m1} n_1}\\
&= A(0,0,n_3,\dots,n_\ell;q)q^{-n_1^2-n_2^2+\sum_{m=1}^{\ell} n_m a_{m1} n_1+\sum_{m=2}^{\ell} n_m a_{m2} n_2}\\
&= \dots\\
&= A(0,\dots,0;q)q^{-\sum_{m=1}^{\ell} n_m^2+\sum_{m=1}^{\ell} n_m a_{m1} n_1+\sum_{m=2}^{\ell} n_m a_{m2} n_2+\cdots+n_1 a_{\ell\ell} n_\ell}\\
&= A(0,\dots,0;q)q^{\frac{1}{2}\sum_{i=1}^{\ell}\sum_{m_i=1}^{\ell} n_{m_i} a_{m_i i} n_i}. \qquad (4.15)
\end{aligned}
$$

Next, notice that $\sum_{i=1}^{\ell}\sum_{m_i=1}^{\ell} n_{m_i} a_{m_i i} n_i = \mathbf{n}C\mathbf{n}^t$ where C is the Cartan matrix of $\mathfrak{g}$. Using (4.15), we have the following result.

Proposition 4.8. *When $\mathfrak{g}$ is of (ADE)-type, the coeffecient $A(n_1,...,n_\ell;q)$ from the* *character of $L(1,0)$ is given by the following:*

$$A(n_1,\ldots,n_\ell;q)=A(0,\ldots,0;q)q^{\frac{1}{2}\mathbf{n}C\mathbf{n}^t}. \tag{4.16}$$

4.4 Determining The Initial Condition

We have seen that in the case where $\mathfrak{g}$ is simply laced (i.e. (ADE)-type) and the level is taken to be $k=1$, we need only one initial condition. The lattice construction of $L(1,0)$ will furnish us with the necessary initial condition.

Note that we chose the conformal vector for $L(1,0)$ so that $d_H=L(0)$ ($\mu=0$). Then, recall that we defined the character of $L(1,0)$ to be $\chi(x_1,...,x_\ell;q)=\mathrm{tr}_{L(1,0)}x_1^{H^{(1)}(0)}...x_\ell^{H^{(\ell)}(0)}q^{d_H}=\mathrm{tr}_{L(1,0)}x_1^{H^{(1)}(0)}...x_\ell^{H^{(\ell)}(0)}q^{L(0)}$.

This character can be expanded in powers of x_1, ..., x_ℓ. We wish to examine $A(0,...,0;q)$ (the coefficient of $x_1^0x_2^0...x_\ell^0$). Therefore, we should compute the trace only for vectors $v\in L(1,0)$ such that $H^{(i)}(0)v=0$ for all $1\le i\le\ell$. Hence, we compute the trace over the subspace

$$W=\{v\in V_Q\,|\,H^{(i)}(0)v=0\text{ for }i=1,...,\ell\}=\{v\in V_Q\,|\,h(0)v=0\text{ for }h\in\mathfrak{h}\}. \tag{4.17}$$

But recall that $h(0)e^\alpha=\langle h,\alpha\rangle e^\alpha$ for all $h\in\mathfrak{h}$. Therefore, $h(0)e^\alpha=0$ for all $h\in\mathfrak{h}$ only when $\alpha=0$. Thus, $W=M(1)\otimes 1$.

Recall that $M(1)$ is linearly isomorphic to $S(\hat{\mathfrak{h}}_+)$. In fact, we have that

$$\operatorname{tr}_W q^{L(0)} = \prod_{i=1}^{\infty}(1-q^j)^{-\ell}. \tag{4.18}$$

Hence, we have shown the following.

Lemma 4.9. *When $\mathfrak{g}$ is (ADE)-type, if we expand out the character of $L(1,0)$,*

$$\chi_{L(1,0)}(q) = \sum_{(n_1,\ldots,n_\ell)\in\mathbb{Z}^\ell} A(n_1,\ldots,n_\ell;q)x_1^{n_1}\ldots x_\ell^{n_\ell}, \tag{4.19}$$

then

$$A(0,\ldots,0;q) = \prod_{i=1}^{\infty}(1-q^j)^{-\ell}. \tag{4.20}$$

Using (4.16) and (4.20), we obtain the following:

$$A(n_1,\ldots,n_\ell;q) = \prod_{j\geq 1}(1-q^j)^{-\ell}q^{\frac{1}{2}\mathbf{n}C\mathbf{n}^t}. \tag{4.21}$$

Therefore, we have the following character formula.

Theorem 4.10. *Assume that $\mathfrak{g}$ is of (ADE)-type with Cartan matrix $C = (a_{ij})$. Then,*

$$\chi_{L(1,0)}(x_1,\ldots,x_\ell;q) = \prod_{j\geq 1}(1-q^j)^{-\ell}\sum_{\mathbf{n}\in\mathbb{Z}^\ell} q^{\frac{1}{2}\mathbf{n}C\mathbf{n}^t}\mathbf{x}^{\mathbf{n}}. \tag{4.22}$$

Furthermore,

$$\chi^H_{L(1,0)}(q) = \chi_{L(1,0)}(1,\ldots,1;q) = \prod_{j\geq 1}(1-q^j)^{-\ell}\sum_{\mathbf{n}\in\mathbb{Z}^\ell} q^{\frac{1}{2}\mathbf{n}C\mathbf{n}^t}. \tag{4.23}$$

4.5 Principal Characters and Multisum Identities

In this section, we use two different expressions of the principal characters of the basic representations to obtain certain interesting multisum identities of Rogers-Ramanujan type.

Corollary 4.11. *Assume that $\mathfrak{g}$ is of (ADE)-type. Then, the principal character for $L(1,0)$ can be written as*

$$\chi^P_{L(1,0)}(q) = \prod_{j\geq 1}(1-q^{(\mathrm{ht}(\theta)+1)j})^{-\ell}\sum_{\mathbf{n}\in\mathbb{Z}^\ell} q^{\frac{\mathrm{ht}(\theta)+1}{2}\mathbf{n}C\mathbf{n}^t-\sum_{i=1}^{\ell} n_i} \tag{4.24}$$

where C is the Cartan matrix of $\mathfrak{g}$ and $\mathrm{ht}(\theta)$ is the height of the highest long root.

Proof. By Theorem 4.10, we have

$$\chi_{L(1,0)}(x_1,\ldots,x_\ell;q) = \prod_{j\geq 1}(1-q^j)^{-\ell}\sum_{\mathbf{n}\in\mathbb{Z}^\ell} q^{\frac{1}{2}\mathbf{n}C\mathbf{n}^t}\mathbf{x}^{\mathbf{n}}.$$

Then,

$$\chi_{L(1,0)}(q^{-1},\ldots,q^{-1};q^{\mathrm{ht}(\theta)+1}) = \prod_{j\geq 1}(1-q^{(\mathrm{ht}(\theta)+1)j})^{-\ell}\sum_{\mathbf{n}\in\mathbb{Z}^\ell} q^{\frac{\mathrm{ht}(\theta)+1}{2}\mathbf{n}C\mathbf{n}^t} q^{-n_1}\cdots q^{-n_\ell}.$$

Note that the $L(0)$-weight of the highest weight subspace of $L(1,0)$ is 0. Now the Corollary follows from Proposition 4.3. □

Next, we consider each of the different types of $\mathfrak{g}$ (i.e. A, D, and E) separately.

First, consider $\mathfrak{g}$ of type A_ℓ. Recall (see e.g. [16]) that the principal character of $L(1,0)$ is given by

$$\chi^P_{L(1,0)}(q) = \prod_{j\geq 1} \frac{(1-q^{(\ell+1)j})}{(1-q^j)} \tag{4.25}$$

and $\mathrm{ht}(\theta) = \ell$. Therefore, we have that

$$\prod_{j\geq 1} \frac{(1-q^{(\ell+1)j})}{(1-q^j)} = \prod_{j\geq 1}(1-q^{(l+1)j})^{-\ell} \sum_{\mathbf{n}\in\mathbb{Z}^\ell} q^{\frac{\ell+1}{2}\mathbf{n}C\mathbf{n}^t - \sum_{i=1}^{\ell} n_i} \tag{4.26}$$

where C is the Cartan matrix of type A_ℓ. Multiply both sides of (4.26) by $\prod_{j\geq 1}(1-q^{(l+1)j})^\ell$ to get

$$\prod_{j\geq 1} \frac{(1-q^{(\ell+1)j})^{(\ell+1)}}{(1-q^j)} = \sum_{\mathbf{n}\in\mathbb{Z}^\ell} q^{\frac{\ell+1}{2}\mathbf{n}C\mathbf{n}^t - \sum_{i=1}^{\ell} n_i}. \tag{4.27}$$

If $\mathfrak{g}$ is of type D_ℓ, we know (see e.g. [16]) that the principal character of $L(1,0)$ is given by

$$\chi^P_{L(1,0)}(q) = \prod_{j\geq 1}(1-q^{2j-1})^{-1}(1-q^{(\ell-1)(2j-1)})^{-1} \tag{4.28}$$

and $\mathrm{ht}(\theta) = 2\ell-3$. Therefore, we have that

$$\prod_{j\geq 1}(1-q^{2j-1})^{-1}(1-q^{(\ell-1)(2j-1)})^{-1} = \prod_{j\geq 1}(1-q^{2(\ell-1)j})^{-\ell} \sum_{\mathbf{n}\in\mathbb{Z}^\ell} q^{(\ell-1)\mathbf{n}C\mathbf{n}^t - \sum_{i=1}^{\ell} n_i} \tag{4.29}$$

where C is the Cartan matrix of type D_ℓ. Multiply both sides of (4.29) by $\prod_{j\geq 1}(1-q^{2(l-1)j})^\ell$ to get

$$\prod_{j\geq 1}\frac{(1-q^{2(\ell-1)j})^\ell}{(1-q^{2j-1})(1-q^{(\ell-1)(2j-1)})} = \sum_{\mathbf{n}\in\mathbb{Z}^\ell} q^{(\ell-1)\mathbf{n}C\mathbf{n}^t-\sum_{i=1}^{\ell} n_i}. \tag{4.30}$$

To simplify the following formulas, we recall the *Euler product function* which is given by

$$\varphi(q) = \prod_{j\geq 1}(1-q^j). \tag{4.31}$$

If $\mathfrak{g}$ is of type E_6, then the principal character of $L(1,0)$ is given by (see e.g. [16])

$$\chi^P_{L(1,0)}(q) = \prod_{j\equiv\pm1,\pm4,\pm5\,(\mathrm{mod}12)}(1-q^j)^{-1} \tag{4.32}$$

and $\mathrm{ht}(\theta)=11$. Therefore, we have that

$$\prod_{j\equiv\pm1,\pm4,\pm5\,(\mathrm{mod}12)}(1-q^j)^{-1} = \varphi(q^{12})^{-6}\sum_{\mathbf{n}\in\mathbb{Z}^6} q^{6\mathbf{n}C\mathbf{n}^t-\sum_{i=1}^{6} n_i} \tag{4.33}$$

where C is the Cartan matrix of type E_6. Multiply both sides of (4.33) by $\varphi(q^{12})^6$ to get

$$\varphi(q^{12})^6\prod_{j\equiv\pm1,\pm4,\pm5\,(\mathrm{mod}12)}(1-q^j)^{-1} = \sum_{\mathbf{n}\in\mathbb{Z}^6} q^{6\mathbf{n}C\mathbf{n}^t-\sum_{i=1}^{6} n_i}. \tag{4.34}$$

If $\mathfrak{g}$ is of type E_7, then the principal character of $L(1,0)$ is given by (see e.g. [16])

$$\chi^P_{L(1,0)}(q) = \prod_{j\equiv\pm1,\pm5,\pm7,9\,(\mathrm{mod}18)} (1-q^j)^{-1} \tag{4.35}$$

and $\mathrm{ht}(\theta) = 17$. Therefore, we have that

$$\prod_{j\equiv\pm1,\pm5,\pm7,9\,(\mathrm{mod}18)} (1-q^j)^{-1} = \varphi(q^{18})^{-7} \sum_{\mathbf{n}\in\mathbb{Z}^7} q^{9\mathbf{n}C\mathbf{n}^t - \sum_{i=1}^{7} n_i} \tag{4.36}$$

where C is the Cartan matrix of type E_7. Multiply both sides of (4.36) by $\varphi(q^{18})^7$ to get

$$\varphi(q^{18})^7 \prod_{j\equiv\pm1,\pm5,\pm7,9\,(\mathrm{mod}18)} (1-q^j)^{-1} = \sum_{\mathbf{n}\in\mathbb{Z}^7} q^{9\mathbf{n}C\mathbf{n}^t - \sum_{i=1}^{7} n_i}. \tag{4.37}$$

If $\mathfrak{g}$ is of type E_8, then the principal character of $L(1,0)$ is given by (see e.g. [16])

$$\chi^P_{L(1,0)}(q) = \prod_{j\equiv\pm1,\pm7,\pm11,\pm13\,(\mathrm{mod}30)} (1-q^j)^{-1} \tag{4.38}$$

and $\mathrm{ht}(\theta) = 29$. Therefore, we have that

$$\prod_{j\equiv\pm1,\pm7,\pm11,\pm13\,(\mathrm{mod}30)} (1-q^j)^{-1} = \varphi(q^{30})^{-8} \sum_{\mathbf{n}\in\mathbb{Z}^8} q^{15\mathbf{n}C\mathbf{n}^t - \sum_{i=1}^{8} n_i} \tag{4.39}$$

where C is the Cartan matrix of type E_8. Multiply both sides of 4.39 by $\varphi(q^{30})^8$ to get

$$\varphi(q^{30})^8 \prod_{j\equiv\pm1,\pm7,\pm11,\pm13\,(\mathrm{mod}30)} (1-q^j)^{-1} = \sum_{\mathbf{n}\in\mathbb{Z}^8} q^{15\mathbf{n}C\mathbf{n}^t - \sum_{i=1}^{8} n_i}. \tag{4.40}$$

To summarize we have:

Theorem 4.12. *If $C = (a_{ij})$ is the Cartan matrix of type A_ℓ, then*

$$\prod_{j\geq 1} \frac{(1-q^{(\ell+1)j})^{(\ell+1)}}{(1-q^j)} = \sum_{\mathbf{n}\in\mathbb{Z}^\ell} q^{\frac{\ell+1}{2}\mathbf{n}C\mathbf{n}^t - \sum_{i=1}^{\ell} n_i}. \tag{4.41}$$

If C is the Cartan matrix of type D_ℓ, then

$$\prod_{j\geq 1} \frac{(1-q^{2(\ell-1)j})^{\ell}}{(1-q^{2j-1})(1-q^{(\ell-1)(2j-1)})} = \sum_{\mathbf{n}\in\mathbb{Z}^\ell} q^{(\ell-1)\mathbf{n}C\mathbf{n}^t - \sum_{i=1}^{\ell} n_i}. \tag{4.42}$$

If C is the Cartan matrix of type E_6, then

$$\varphi(q^{12})^6 \prod_{j\equiv\pm1,\pm4,\pm5 \pmod{12}} (1-q^j)^{-1} = \sum_{\mathbf{n}\in\mathbb{Z}^6} q^{6\mathbf{n}C\mathbf{n}^t - \sum_{i=1}^{6} n_i}. \tag{4.43}$$

If C is the Cartan matrix of type E_7, then

$$\varphi(q^{18})^7 \prod_{j\equiv\pm1,\pm5,\pm7,9 \pmod{18}} (1-q^j)^{-1} = \sum_{\mathbf{n}\in\mathbb{Z}^7} q^{9\mathbf{n}C\mathbf{n}^t - \sum_{i=1}^{7} n_i}. \tag{4.44}$$

If C is the Cartan matrix of type E_8, then

$$\varphi(q^{30})^8 \prod_{j\equiv\pm1,\pm7,\pm11,\pm13 \pmod{30}} (1-q^j)^{-1} = \sum_{\mathbf{n}\in\mathbb{Z}^8} q^{15\mathbf{n}C\mathbf{n}^t - \sum_{i=1}^{8} n_i}. \tag{4.45}$$

4.6 Multisum Examples

Example 4.13. *The multisum identity obtained from the principal character of* $L(\Lambda_0)$ *when* $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ *(i.e. type* A_1*) is an easy consequence of Theorem 2.8 on page 21 in [1]. The theorem states that for* $z \neq 0$ *and* $|q| < 1$*, we have that*

$$\sum_{n\in\mathbb{Z}} z^n q^{n^2} = \prod_{n=0}^{\infty}(1-q^{2n+2})(1+zq^{2n+1})(1+z^{-1}q^{2n+1}). \tag{4.46}$$

Let us specialize $z \to q^{-1}$ *and* $q \to q^2$*. Thus, we have*

$$\begin{aligned}\sum_{n\in\mathbb{Z}} q^{2n^2-n} &= \prod_{j=0}^{\infty}(1-q^{4j+4})(1+q^{-1}q^{4j+2})(1+q^{1}q^{4n+2}) \\ &= \prod_{j=1}^{\infty}(1-q^{4j})(1+q^{4j-3})(1+q^{4j-1}).\end{aligned}$$

Notice that $4j-3$ *combined with* $4j-1$ *for all* $j \geq 1$ *gives us all odd positive integers. Thus, we have*

$$\begin{aligned}\sum_{n\in\mathbb{Z}} q^{2n^2-n} &= \prod_{j=1}^{\infty}(1-q^{4j})(1+q^{2j-1}) \\ &= \prod_{j=1}^{\infty}\left[(1-q^{2j})(1+q^{2j})\right](1+q^{2j-1}).\end{aligned}$$

Next, notice that $2j$ together with $2j-1$ for $j \geq 1$ gives us all positive integers. Thus, we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}} q^{2n^2-n} &= \prod_{j=1}^{\infty}(1-q^{2j})(1+q^j) \\ &= \prod_{j=1}^{\infty}(1-q^{2j})(1+q^j)\frac{(1-q^j)}{(1-q^j)} \\ &= \prod_{j=1}^{\infty}\frac{(1-q^{2j})(1+q^j)(1-q^j)}{(1-q^j)} \\ &= \prod_{j=1}^{\infty}\frac{(1-q^{2j})^2}{(1-q^j)}.\end{aligned}$$

Therefore, we have found that

$$\sum_{n\in\mathbb{Z}} q^{2n^2-n} = \prod_{j=1}^{\infty}\frac{(1-q^{2j})^2}{(1-q^j)}. \tag{4.47}$$

This is precisely (4.41) when $\ell = 1$.

Example 4.14. *Let us write out the identity for $\mathfrak{sl}_{\ell+1}$ (i.e. type A_ℓ) explicitly.*

First, let us look at $\mathfrak{sl}_3$. Referring to Appendix A, we see that the Cartan matrix of $\mathfrak{sl}_3$ is:

$$C = (a_{ij})_{1\leq i,j\leq 3} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{4.48}$$

Thus, we have that

$$(n_1, n_2)\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = 2n_1^2 - 2n_1n_2 + 2n_2^2 = 2(n_1^2 - n_1n_2 + n_2^2).$$

So, when $\ell = 2$ equation (4.41) becomes

$$\prod_{j\geq 1} \frac{(1-q^{3j})^3}{(1-q^j)} = \sum_{n_1,n_2\in\mathbb{Z}} q^{3(n_1^2-n_1n_2+n_2^2)-(n_1+n_2)}. \tag{4.49}$$

In general, for type A_ℓ, we have that

$$\prod_{j\geq 1} \frac{(1-q^{(\ell+1)j})^{(\ell+1)}}{(1-q^j)} = \sum_{n_1,\ldots,n_\ell\in\mathbb{Z}} q^{(\ell+1)\sum_{i=1}^{\ell} n_i^2-(\ell+1)\sum_{i=1}^{\ell-1} n_in_{i+1}-\sum_{i=1}^{\ell} n_i}. \tag{4.50}$$

Example 4.15. *Finally, let us consider the identity obtained from the character of the basic representation, $L(\Lambda_0)$, with underlying finite dimensional simple Lie algebra $\mathfrak{g} = \mathfrak{so}_8(\mathbb{C})$ (i.e. type D_4).*

Referring to Appendix A, we see that the type D_4 Cartan matrix is

$$C = (a_{ij})_{1\leq i,j\leq 4} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}. \tag{4.51}$$

Therefore, we have for all $\mathbf{n} \in \mathbb{Z}^4$,

$$\mathbf{n}C\mathbf{n}^t = 2(n_1^2 + n_2^2 + n_3^2 + n_4^2 - n_1n_2 - n_2n_3 - n_2n_4). \tag{4.52}$$

So, when $\ell = 4$, *equation (4.42) becomes*

$$\prod_{j\geq 1} \frac{(1-q^{6j})^4}{(1-q^{(2j-1)})(1-q^{(6j-3)})} =$$

$$\sum_{n_1,n_2,n_3,n_4\in\mathbb{Z}} q^{6(n_1^2+n_2^2+n_3^2+n_4^2-n_1n_2-n_2n_3-n_2n_4)-(n_1+n_2+n_3+n_4)}. \tag{4.53}$$

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Appendix A

Finite and Affine Type Matrices

A.1 Finite Type Matrices

Type A_ℓ $(\ell \geq 1)$

$$(a_{i,j})_{1 \leq i,j \leq \ell} = \begin{pmatrix} 2 & -1 & 0 & . & 0 \\ -1 & 2 & -1 & . & . \\ 0 & -1 & . & . & 0 \\ . & . & . & 2 & -1 \\ 0 & . & 0 & -1 & 2 \end{pmatrix}$$

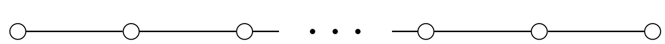

Figure A.1: Type A_ℓ Dynkin diagram

Figure A.2: Type B_ℓ Dynkin diagram

Type B_ℓ $(\ell \geq 2)$

$$(a_{i,j})_{1\leq i,j\leq \ell} = \begin{pmatrix} 2 & -1 & . & 0 & 0 \\ -1 & . & . & . & . \\ . & . & 2 & -1 & 0 \\ 0 & . & -1 & 2 & -2 \\ 0 & . & 0 & -1 & 2 \end{pmatrix}$$

Figure A.3: Type C_ℓ Dynkin diagram

Type C_ℓ $(\ell \geq 3)$

$$(a_{i,j})_{1\leq i,j\leq \ell} = \begin{pmatrix} 2 & -1 & . & 0 & 0 \\ -1 & . & . & . & . \\ . & . & 2 & -1 & 0 \\ 0 & . & -1 & 2 & -1 \\ 0 & . & 0 & -2 & 2 \end{pmatrix}$$

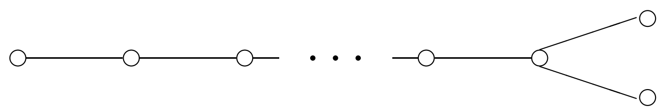

Figure A.4: Type D_ℓ Dynkin diagram

Type D_ℓ ($\ell \geq 4$)

$$(a_{i,j})_{1\leq i,j\leq \ell} = \begin{pmatrix} 2 & -1 & . & 0 & 0 \\ -1 & . & . & . & . \\ . & . & 2 & -1 & -1 \\ 0 & . & -1 & 2 & 0 \\ 0 & . & -1 & 0 & 2 \end{pmatrix}$$

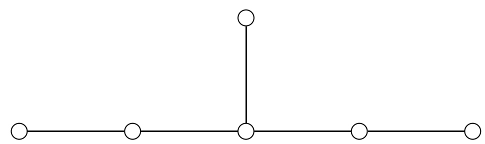

Figure A.5: Type E_6 Dynkin diagram

Type E_6

$$(a_{i,j})_{1\le i,j\le 6} = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

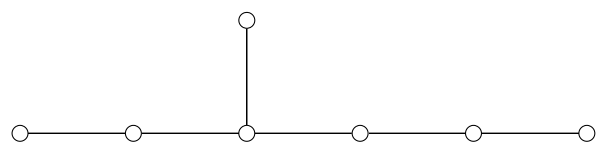

Figure A.6: Type E_7 Dynkin diagram

Type E_7

$$(a_{i,j})_{1\leq i,j\leq 7} = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

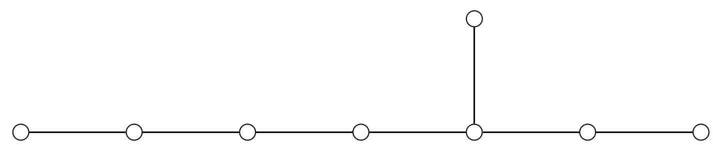

Figure A.7: Type E_8 Dynkin diagram

Type E_8

$$(a_{i,j})_{1\leq i,j\leq 8} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

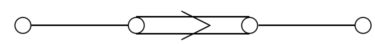

Figure A.8: Type F_4 Dynkin diagram

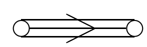

Figure A.9: Type G_2 Dynkin diagram

Type F_4 $$(a_{i,j})_{1\leq i,j\leq 4} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

Type G_2 $$(a_{i,j})_{1\leq i,j\leq 2} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

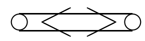

Figure A.10: Type $A_1^{(1)}$ Dynkin diagram

A.2 Affine (Type 1) Matrices

This section contains a list of generalized Cartan matrices and Dynkin diagrams associated with affine Lie algebras of type 1 (the untwisted affine algebras). Notice that after deleting the first row and the first column of a GCM of type $X_\ell^{(1)}$, one is left with a Cartan matrix of type X_ℓ.

Type $A_1^{(1)}$

$$(a_{i,j})_{1\le i,j\le 2} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

Type $A_\ell^{(1)}$ $(\ell \geq 2)$

$$(a_{i,j})_{0\le i,j\le \ell} = \begin{pmatrix} 2 & -1 & 0 & 0 & . & 0 & -1 \\ -1 & 2 & -1 & 0 & . & 0 & 0 \\ 0 & -1 & 2 & -1 & . & . & . \\ 0 & 0 & -1 & . & . & 0 & 0 \\ . & . & . & . & 2 & -1 & 0 \\ 0 & 0 & . & 0 & -1 & 2 & -1 \\ -1 & 0 & . & 0 & 0 & -1 & 2 \end{pmatrix}$$

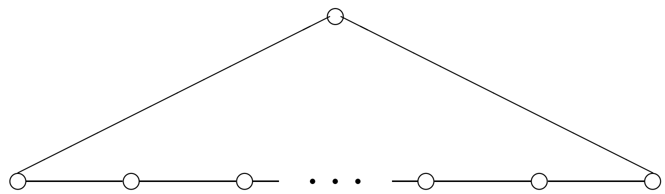

Figure A.11: Type $A_\ell^{(1)}$ Dynkin diagram

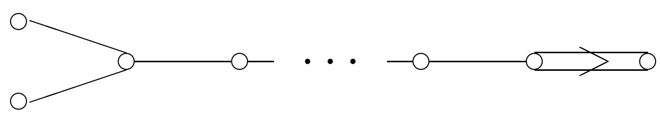

Figure A.12: Type $B_\ell^{(1)}$ Dynkin diagram

Type $B_\ell^{(1)}$ $(\ell \geq 2)$

$$(a_{i,j})_{0\leq i,j\leq \ell} = \begin{pmatrix} 2 & 0 & -1 & 0 & . & 0 & 0 \\ 0 & 2 & -1 & 0 & . & 0 & 0 \\ -1 & -1 & 2 & -1 & . & . & . \\ 0 & 0 & -1 & . & . & 0 & 0 \\ . & . & . & . & 2 & -1 & 0 \\ 0 & 0 & . & 0 & -1 & 2 & -2 \\ 0 & 0 & . & 0 & 0 & -1 & 2 \end{pmatrix}$$

Figure A.13: Type $C_\ell^{(1)}$ Dynkin diagram

Type $C_\ell^{(1)}$ $(\ell \geq 3)$

$$(a_{i,j})_{0\leq i,j\leq \ell} = \begin{pmatrix} 2 & -2 & 0 & 0 & . & 0 & 0 \\ -1 & 2 & -1 & 0 & . & 0 & 0 \\ 0 & -1 & 2 & -1 & . & . & . \\ 0 & 0 & -1 & . & . & 0 & 0 \\ . & . & . & . & 2 & -1 & 0 \\ 0 & 0 & . & 0 & -1 & 2 & -1 \\ 0 & 0 & . & 0 & 0 & -2 & 2 \end{pmatrix}$$

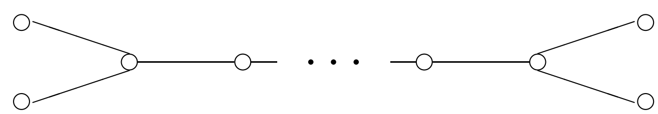

Figure A.14: Type $D_\ell^{(1)}$ Dynkin diagram

Type $D_\ell^{(1)}$ $(\ell \geq 4)$

$$(a_{i,j})_{0\leq i,j\leq \ell} = \begin{pmatrix} 2 & 0 & -1 & 0 & . & 0 & 0 \\ 0 & 2 & -1 & 0 & . & 0 & 0 \\ -1 & -1 & 2 & -1 & . & . & . \\ 0 & 0 & -1 & . & . & 0 & 0 \\ . & . & . & . & 2 & -1 & -1 \\ 0 & 0 & . & 0 & -1 & 2 & 0 \\ 0 & 0 & . & 0 & -1 & 0 & 2 \end{pmatrix}$$

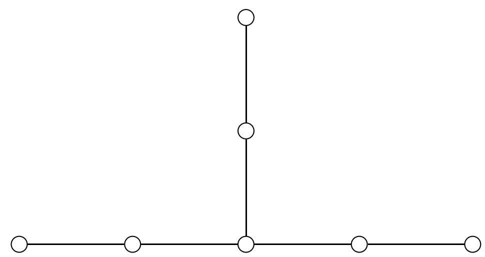

Figure A.15: Type $E_6^{(1)}$ Dynkin diagram

Type $E_6^{(1)}$

$$(a_{i,j})_{0\leq i,j\leq 6} = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

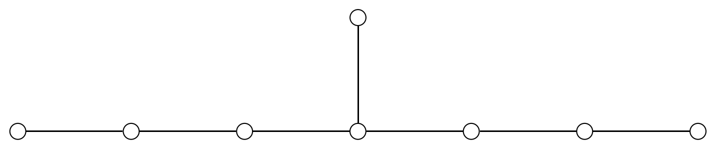

Figure A.16: Type $E_7^{(1)}$ Dynkin diagram

Type $E_7^{(1)}$

$$(a_{i,j})_{0\leq i,j\leq 7} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

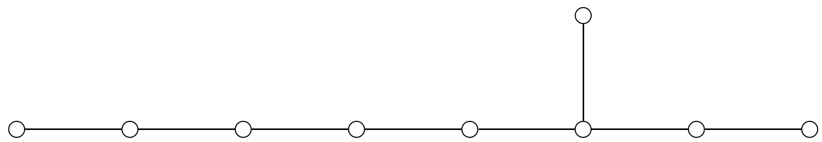

Figure A.17: Type $E_8^{(1)}$ Dynkin diagram

Type $E_8^{(1)}$

$$(a_{i,j})_{0\le i,j\le 8} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

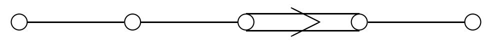

Figure A.18: Type $F_4^{(1)}$ Dynkin diagram

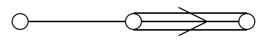

Figure A.19: Type $G_2^{(1)}$ Dynkin diagram

Type $F_4^{(1)}$

$$(a_{i,j})_{0\leq i,j\leq 4} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -2 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Type $G_2^{(1)}$

$$(a_{i,j})_{0\leq i,j\leq 2} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}$$