ABSTRACT

AKOGLU, KEMAL. Research Based Suggestions to Overcome the Cognitive Issues in Teaching and Learning of Conditional Probability. (Under the direction of Dr. Hollylynne S. Lee).

Statistics and probability has an abstract, uncertain nature. Where conclusions in mathematical problems are mostly certain, in probability it is only possible to have ‘probabilistic’ conclusions that approximate the theoretical probability in order to make decisions. Because of its uncertain nature there have been serious cognitive issues in teaching and learning of probability. Conditional probability is specifically difficult for students to learn and it is also difficult for teachers to teach. With the help of simulation tools where students can participate in the process of building probability models and collecting data generated from those models, there are great opportunities to improve teaching and learning of conditional probability. In this thesis, research-based misconceptions and suggestions for overcoming these misconceptions are discussed. Using research-based suggestions and previous frameworks, a new framework about creating conditional probability tasks (i.e. simulation tools) has been constructed for others to use. Two *TinkerPlots* tasks are presented as examples in which task content and problems are aligned with the framework. Consequently the reflection on research issues and actual classroom uses are discussed and further questions about the subject is discussed.

Research Based Suggestions to Overcome the Cognitive Issues in Teaching and Learning of Conditional Probability

by
Kemal Akoglu

A thesis submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the degree of
Master of Science

Mathematics Education

Raleigh, North Carolina

2014

APPROVED BY:

Dr. Karen Hollebrands

Dr. Molly Fenn

Dr. Hollylynne S. Lee

BIOGRAPHY

Kemal Akoglu was born and raised in Antakya, Turkey. He graduated from Ankara University with a BS' degree in Mathematics in 2008. Kemal taught mathematics for middle school and high school students until he moved to the U.S in 2009.

Kemal has been interested in sociology and literature since his high school years. On his meeting with the sociology professor, Dr. William M. Wentworth, Kemal enrolled in the *Applied Sociology* master's program at Clemson University. After a successful year of coursework and graduate research assistantship (GRA) job in this program, Kemal took a leave of absence from the program and moved to Raleigh, North Carolina.

In Fall 2012, with the encouragement and support of his wife Tulay, he applied the master's program in Mathematics Education at North Carolina State University (NCSU). During his master's study, he worked as a GRA for Preparing to Teach Mathematics with Technology (PTMT) research project. He accomplished his coursework at this program in Spring 2014.

His interests focus on cognitive issues about probability education and the methods to overcome those issues, which he investigated in his master's thesis. After graduating from his master's program, he is admitted into the Ph.D. program at NCSU beginning in Fall 2014 to continue his academia career in mathematics education.

ACKNOWLEDGMENTS

I cannot express enough thanks to my advisor and committee chair, Dr. Hollylynne S. Lee for her continuing support, encouragement and patience. I feel one of the luckiest graduate students in the universe because of having her as my advisor.

I would like to thank my caring and loving wife, Tulay Ayyildiz Akoglu. This thesis could not have been accomplished without her support. She always trusted me more than I trusted myself, and she always encouraged me to continue.

Finally, I want to thank my beloved mother, Ganiye Akoglu. Her support and love have always been with me. Wherever I go and whatever I do, her unique smile makes me feel safe, motivated and peaceful.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

For several centuries, probability has been one of the most important components in many research areas, such as physics, chemistry, and biology, engineering, medicine, sociology, and economics. Not only in sciences, but also in our daily life, we see the importance of probability; deciding how the weather is going to be, what stocks are likely to be more profitable, the chance of getting a certain disease, etc.

Unlike the mathematics we used to know where we 'mostly' deal with concrete problems, statistics and probability has an abstract, uncertain nature. Where our conclusions in mathematical problems are and have to be certain, in probability it is only possible to have 'probabilistic' conclusions that approximate some unknown theoretical probability in order to make decisions.

Scheaffer (2006) states the differences between statistical and mathematical reasoning. For example he points that statistical reasoning mostly includes a probabilistic thinking process where mathematics often requires deterministic thinking. He describes the deterministic thinking for which every result must have an explainable cause, in contrast, for probabilistic thinking a result may be due to many unexplainable causes, as those causes come together their resulting affect is called *chance*. Another highlighting difference is about the use of numbers according to the author; as mathematics mainly is, statistics is also about numbers, but statistics use numbers in context that are called *data*; in other words, "statistics changes numbers into information” (p. 320). Although statistics is usually taught as a part of mathematics, math and stats use different ways of reasoning.

According to Britz et al. (1996) statistical thinking has its own principles: statistical work is a system of interconnected processes; variation exists in all processes; and understanding and reducing variation are keys to success in statistics. The author describes the relation between mathematics and statistics as a strong marriage. According to Scheaffer, as in marriage, statistics and mathematics “can come together not for the purpose of becoming one but so that the strengths of each can be validated, supported and enhanced” (p. 320).

Rossman, Chance and Medina (2006) also investigate the differences between statistical and mathematical reasoning science of gaining insight from data”. They state that even though statistics broadly use math, it is a separate discipline. “Statistics is the science of gaining insight from data” (p. 323). The authors point the role of context in statistics and the difference that r0ole creates between mathematics and statistics; according to them mathematics can exist independently of context; on the other hand, context is crucial in statistics and one cannot ignore the context when analyzing data. Differently from mathematics in which we need rigorous deductive reasoning, proofs based on axioms and definitions; in statistics we need inductive reasoning and uncertain conclusions. In statistics, we are able to face some situations that are impossible to imagine in mathematics: different statisticians “often come to different but reasonable conclusions when analyzing the same data” (p. 329).

Another difference highlighted by Rossman et al. (2006) is a measurement issue. The authors state that measurement issues take a significant place in statistics. Even though measurement is also important in mathematics; however in mathematics "measurement includes getting students to learn about appropriate units to measure attributes of an object such as length, area, etc. In statistics, drawing conclusions from data depends crucially on taking valid measurements of the properties being studied" (p. 326).

The differences between mathematics and statistics could be a possible reason for the misconceptions about probability. At a certain time of their study, students confront with probability as a unit in mathematics in which they need to have / develop another way of reasoning. As it is an issue for students to learn this new reasoning it is also an issue for teachers to teach probability within mathematics curriculum (Stohl, 2005).

Even though the research about probabilistic theory arose and accumulated independently, especially in education, probability has been considered a part of statistics. Because of the essential relation between probability and data, this consideration is not surprising. However, probability has a specific characteristic puts it in a unique situation; there are several cognitive issues about probability which are confronted by both teachers and students.

Concepts in the study of probability are widely misunderstood by students and teachers (e.g., Tversky & Kahneman, 1972, 1974; Fischbein & Gazit, 1984; Steinbring, 1991; Stohl & Tarr, 2002; Shaugnessy, 2003; Batanero et al, 2005). Thus, students possess some misconceptions before, and possibly after, a formal course in probability. For example, students often believe, when they toss three coins, the probability of getting three heads is ¼, since the outcomes are 3H, 2H and 1T, 1H and 2T, and 3T. Outcomes are usually judged to be equally likely by the beginning probability student. Students feel that the probability of getting a head is greater on the 10^{th} toss after a run of 9 tails. This misconception arises because people expect even short runs of coin flips to reflect the fairness of a coin (Shaughnessy, 1977).

As in probability in general, several cognitive issues in teaching and learning of conditional probability also emerge. Research shows that those cognitive issues are also resistant to the same teaching methods that have been used. A possible reason for that can be that teaching methods themselves are ineffective to overcome those issues. Since conditional probabilities play a central role in making inferences about the uncertain world, they are worthy to be studied. According to Falk (1986), the formal definition of conditional probability is easy and has no issues. On the other hand, upon careful probing into students' ideas of conditional probabilities, some misconceptions and fallacies are uncovered.

Tarr and Jones (1997) use Hogg and Tanis' (1997) definition for conditional probability, and the background for that definition is that, in some random experiments, there is interest only in those outcomes that are elements of a subset B of the sample space S. Within that concept, Hogg and Tanis define the conditional probability of an event A, given that another event B has occurred, written P(A|B), as the probability of A considers as possible outcomes only those outcomes of the random experiment that are elements of B. They note that a special case of conditional probability occurs in a random experiment conducted in "without replacement" conditions.

Without replacement conditions put a bracket between conditional probability and independence. Consider that we are given a problem where we have three white and three black marbles in an urn and we will draw two marbles in order. If the first drawn marble was put back in the urn, the first and second events are independent events, if the marble is held after the first draw; the second event now has a different sample space (only 5 marbles) and is thus conditional upon the first. Independence is a difficult idea that is better understood in relation to specific examples - classroom lessons often include discussions about conducting an experiment in class and whether two events (describing sets of possible outcomes of the experiment) are independent. Textbooks and teachers often use phrases such as event A and event B are independent events if knowledge about whether event A has occurred provides us with no knowledge about whether event B has occurred (Kelly & Zwiers, 1986). In a very subtle way an element of time is hinted at in such a statement and it often confuses students. It is therefore important to emphasize to students that whatever the temporal relationship between two (or more) independent events, a knowledge of the occurrence (or nonoccurrence) of any of the events provides one with no knowledge of the future or past outcomes of any of the other events. By planning for instruction of these two fundamental concepts we can insure that students' understanding is built up systematically. This provides a firmer foundation upon which students can acquire a grasp of probability theory (Kelly & Zwiers, 1986).

According to Tarr and Jones (1997), little research had focused on students' thinking in conditional probability and independence. Even though their paper was published in 1997, and there have since been technological and pedagogical developments in mathematics education, we are still far away from overcoming the cognitive issues which will be explained in Chapter 2. This lack of research on students' reasoning in these two concepts (conditional probability and independence) is a matter of concern given the fact that these concepts are increasingly being identified as important ideas in probability instruction for the middle school. Reaburn (2013) states that understanding conditional probability is essential for students of inferential statistics as it is used in Null Hypothesis Tests. It is also used in Bayes' theorem, in the interpretation of medical screening tests and in quality control procedures, which have been used by different areas.

As it will be discussed in Chapter 2, most of instructional methods that used physical experiments were not helpful to overcome the cognitive issues of conditional probability. Instead, research shows that cognitive issues are often increased by age and education (Fischbein & Schnarch, 1997). That shows us a significant problem about probability education: either probabilistic misconceptions are resistant to age or there is a problem in our teaching methods. In any case, there has to be steps to be taken in order to overcome that.

Differently from past, now we have advanced technological tools to help us in those issues as will be discussed further. There is almost universal agreement that technology should play a predominant role in probability and statistics education (Shaughnessy, 1992; Pfannkuch & Ziedins, 2014).

Technology allows a great opportunity for generating a large amount of data, and manipulates or/and represents the data in various ways that is not even possible to do it within the time constraints of school instruction. Furthermore, generating large data sets allows students to experience the misunderstood concepts such as ‘law of large numbers’, or ‘independence of events’ in a meaningful way (Drier, 2000).

Statistics benefits from technological tools for representing large data and making inferences. Besides data analysis, dynamic statistics software such as TinkerPlots (v. 2.0, Konold & Miller, 2011) includes the capability to model random phenomena, which gives us a golden chance to create simulations to use for probability problems.

Purpose of Thesis

The purpose of this paper is to twofold. First, I will synthesize the research about cognitive issues in learning conditional probability and independence, and analyze research-based teaching suggestions to potentially overcome those issues. A particular focus will be examining the use of a modeling approach to teaching probability that takes advantage of computer simulation power. The second purpose is to provide a framework to guide the design of conditional probability tasks and two research-based designed simulation tasks that are hypothesized to assist learners in developing better understandings of conditional probabilities.

In Chapter 2, there are two sections. The first section of the literature review is a review of literature on students’ probabilistic reasoning in conditional probability specifically. And the second section is a review of literature on modeling and simulations. This review of literature provides a background for my development of helpful tasks/methods’ characteristics to overcome those issues.

In Chapter 3, I discuss the characteristics of conditional probability tasks to be helpful in overcoming cognitive issues in teaching and learning of conditional probability. I also provide two *TinkerPlots* tasks, which are constructed to be used in high school probability classes. Chapter 4 includes an overall discussion of the implications and future directions.

CHAPTER 2

REVIEW OF LITERATURE

As stated in Chapter 1, the subject of this research is cognitive issues about conditional probability and the role of technology for overcoming these issues. Hence, Chapter 2 will be focused on the literature behind those topics. In the first part of this chapter, research about teaching and learning of conditional probability and research based methods used to teach conditional probability will be discussed. It will be pointed out that there are difficulties to both teach and learn conditional probability appropriately. Research based suggestions to overcome those problems will be presented and discussed. In the second part of this chapter the main point will be the research about design and modeling of effective tools to use in conditional probability lessons. The role of technology in the efforts of easing the difficulty about conditional probability will be highlighted, as well.

Cognitive Issues about Conditional Probability

Students' understandings of conditional probability and independence

In their famous *Science* paper, which has been referenced numerous times, Tversky and Kahnemann (1974) categorize probabilistic misconceptions in three main heuristics, which they have called representativeness, availability, and adjustment & anchoring. Even though those heuristics are used to describe probability misconceptions in general, they are highly related to reasoning used with conditional probability and independence. The representativeness heuristic in particular has been used for explaining misconceptions of conditional probability and independence.

Conditional probability and independence are two key constructs incorporated to describe and predict middle school students' reasoning in probability. Based on their empirical study, Fischbein and Gazit (1984) identified two main misconceptions in students' thinking in conditional probability:

1) Students do not realize that the sample space changes in a "without replacement" situation.

2) Students find the probability of an event in a without replacement situation by comparing the number of favorable outcomes for the event before and after the first trial rather than making comparisons with the total number of outcomes.

On the other hand, Konold et al. (1991) analyzed misconceptions about independence as either positive or negative recency effects. The Gambler's Fallacy (Negative Recency Effect) is the tendency to believe that, after a run of tails, heads should be more likely to come up. The Positive Recency Effect is the tendency to believe that, after a run of tails, a tail is more likely to come up.

These misconceptions illustrate a judgmental heuristic known as "representativeness" -the belief that a sample or even a single outcome should reflect the parent population (Kahneman & Tversky, 1972). Such a belief is powerful and generally in conflict with the concept of independence. Thus, even when students seemingly exhibit an understanding of the concept of independence, the representativeness heuristic may still prevail in their problem solving (Tarr & Jones, 1997).

In some of the early work in conditional probability, Falk (1986) found three main issues about conditional probability. First, students *interpret conditionality as causality.* If students claim that conditioning the probability of an outcome of a draw on an event that occurs later is not permissible, they may be using causal reasoning. Second, *the definition of the conditioning event is often problematic.* The probability of the target event should be conditioned on the immediate event given as datum in the problem and not on some inferred event. The examples related to this insight highlight the vital role of the basic concept of the statistical experiment, the outcomes of which define our probability space. One way to promote gaining insights concerning such problems is to devise experimental models in order to explicate the exact procedure that has generated the data and then uncovers hidden assumptions. For example, the 3 card problem which is used by several researchers; we have three cards in a hat, one is blue on both sides, one is green on both sides, and one is blue on one side and green on the other. We draw one card blindly and put it on the table as it comes out. It shows a blue face up. What is the probability that the hidden side is also blue?

The third issue discussed by Falk (1986) is that *students exhibit confusion about the inverse.* Students lack discrimination between the two directions of conditional probability, P(A|B) and P(B|A). This confusion is prevalent among students and professionals at all levels. The confusion of the inverse is especially compelling with respect to the interpretation of statistical significance tests. Whenever a random sample significantly deviates from the null hypothesis, the most natural question to ask is whether that deviation could be accounted for by random fluctuations. In fact, it is the question of all questions of statistics (Falk, 1986).

In later studies that build from Falk's early work, students' tendency to confuse the inverse is often referred to as the 'time axis fallacy' or 'Falk paradox', which will be discussed further in the paper.

The cognitive issues about conditional probability and independence are not in the field of mathematics education only. There is also a concern in psychology and cognitive sciences about conditionals and conditional probability. Oaksford and Chater (2003) state that in the psychology of reasoning, the two main theoretical approaches, mental logic and mental models; the conditional is treated as truth-functional. Experimental work has concentrated on the indicative or straight conditional. For example, if Oswald didn't shoot Kennedy, then someone else did, or, if you turn the key the car starts. According to the authors, in the philosophy of language and logic, the majority view is that straight conditionals are a matter of subjective conditional probabilities. An important consequence of this view is that people, 'do not use ''if'' to express propositions, evaluable in terms of truth. And if conditionals do not have truth conditions, then they would not appear to be truth functional, as many psychological theories assume. Thus, the main component of conditional probability has been ignored in the real world, which may be a reason for many of the misconceptions analyzed in this paper.

According to Girotto and Laird (2004), when individuals rely on mental models, they tend to think that a conditional is correct only in the situation where its antecedent and consequent hold. Hence, their predictions for the probability of the conditional depend on the other possibilities they got. If they suppose that the only relevant alternative is one in which the antecedent holds and the consequent does not, then they infer an equiprobable prediction. If they consider all the alternative prior possibilities, but assume that the conditional holds only when the antecedent and consequent hold, then they infer a conjunctive estimate.

One of the most famous conditional problems in psychology, used with college students, is found in Tversky and Kahneman (1974), the taxicab problem. In this problem, a cab was involved in a hit-and-run accident at night. Two cab companies, the Green and the Blue, operate in the city. You are given the following data:

(i) 85% of the cabs in the city are Green and 15% are Blue.

(ii) A witness identified the cab as a Blue cab.

The court tested the witness' ability to identify cabs under the appropriate visibility conditions. When presented with a sample of cabs (half of which were Blue and half of which were Green) the witness made correct identifications in 80% of the cases and incorrect identifications in 20% of cases. Then the question follows as: What is the probability that the cab involved in the accident was Blue rather than Green?

Used by psychologists over the years with college students, the taxicab problem is notoriously difficult and reveals many misconceptions in the interpretation of language and information presented. The main misconception displayed by students attempting to solve this problem is called a base-rate fallacy. This misconception occurs when the indicant or diagnostic data (i.e. eyewitness account) overshadows the base-rate information (i.e. distribution of cabs in the city). Tversky and Kahneman suggested that neglecting base-rate information depended on whether or not the evidence was given a causal interpretation; that is, base-rate data with causal interpretation influence judgments, whereas the other base-rates are dominated by other causally relevant data.

A possible reason triggering the cognitive issues about conditional probability may be the wording we use in process of teaching and assessing the topic. Some problems seem to be caused by formal wording or algebraic notation, although there is a surprising amount of difficulty even when the conditional probability is expressed in terms of simple percentages (Pollatsek et al., 1987). Research shows that those cognitive difficulties occur even with intelligent and highly educated people such as physicians, engineers, etc. An example of wording causing a misconception is: The probability of A given B could be confused with the probability of B given A, the joint probability of A and B, or even the joint frequency of A and B. In fact, one can find similar examples in daily experiences.

According to Pollatsek et al. (1987), another possible source of error is interference from causal reasoning. People may confuse conditionality with causality both because conditional probabilities are used in inferring causal relationships and because statements about both conditional probability and causality employ keywords like 'if' or 'given that'. In cases in which there is only one causal relationship (e.g. A causes B but not reverse), there will be two conditional probabilities, $p(A|B)$ and $p(B|A)$. Both conditional probabilities may be viewed as representing the same causal relationship and hence thought of as equal. Errors may also follow from the belief that causal relationships should be stronger than diagnostic ones (i.e. $p(\text{effect} \mid \text{cause}) > p(\text{cause} \mid \text{effect})$). The following two pairs of contrasting conditional statements from Pollatsek et al. (1987) are examples.

1. Which of the two events is more probable?

(a) That a girl has blue eyes if her mother has blue eyes

(b) That a mother has blue eyes if her daughter has blue eyes

(c) The two events are equally probable

2. In which prediction would you have the greatest confidence?

(a) Predicting that a person who has a fever is sick

(b) Predicting that a person who is sick has a fever

(c) Equal confidence in both predictions (p. 258)

It is documented by Watson and Kelly (2007) that the two main confusions for these types of problems above might be: contextual knowledge that implies a causal relationship for the events, and the syntax involved in answering the question. Kelly and Zwiers (1986) provided instructor suggestions that can be used to equip students with an intuitive, comprehensive understanding of the basic concepts ‘independence’ and ‘mutually exclusive’.

As we know, events are mutually exclusive when the occurrence of one event rules out the possibility of the occurrence of the other events of concern. The outcomes, for instance, on the toss of a single die are 1, 2, 3, 4, 5 or 6. The outcomes are all mutually exclusive because when a die is tossed and a number appears up, all the other numbers cannot appear. On the other hand, the events "eating scrambled eggs for breakfast" and "eating cereal for breakfast" are not mutually exclusive since it is possible that one has both for breakfast.

Students have several difficulties with the distinction between independent and dependent events. The first parallels the problem that arises with mutually exclusive events, namely, determining when events in the real world are independent or dependent. Once again, we rarely help students to bridge the gap between the fuzzy distinctions (natural) and the rigid distinctions, which are made in mathematics.

An issue that sometimes arises with students concerns is whether events can ever really be independent. For example, some writers (e.g., Capra, 1975) take a holistic approach to the universe in which the universe is considered to be a web of relationships where all things communicate intimately with one another and everything is shared. Statisticians sometimes suggest similar things, adding to student confusion. Hays (1981), for example, states that "There is surely nothing on earth that is completely independent of anything else" (p. 293). A student hearing such a statement would probably be confused. The student might reason that if nothing is completely independent of anything else, then how can we apply probability formulas that assume independence of events? The answer is that application of such formulas does not need to assume that the events are completely independent of each other, only that any relationship might be neglected. For example, everybody in the universe has some interactive gravitational attraction with every other body, but the gravitational attraction between a human being and star light years away is so minute that it can be considered non-existent. As Kelly and Zwiers (1987) cited from Underwood (1957): “The length of an astronomer's toenails isn't related to phases of the moon … and a pygmy tribe in New Guinea has little influence on the alcoholic consumption of a truck drive in Brooklyn” (p. 6).

However, I think that the term ‘negligible’ is somewhat dangerous for classroom uses. Since we already deal with a world of uncertainties where we have to transform all our mathematical knowledge into an abstract, non-deterministic field, it does not seem right to me to convince students that something is “negligible”. This makes conditional probability and independence much more difficult to reason.

One of the most difficult misconceptions about conditional probability and independence is the time axis fallacy. Reaburn (2013), in an empirical study about students’ misconceptions while entering university, uses ‘the urn problem’, which was originally described by R. Falk (1986), and also used by Fischbein and Schnarch (1997). In the problem, an urn was described that contained two white and two black balls. The first question asked for the probability of taking a white ball from the urn given that one white ball had already been removed. The second question then asked for the probability that the first ball was white, given that it is known that the second ball was white. The most common reasoning used in the second question was that the second ball could not affect the outcome of the first ball; therefore the probability was one half. This is a result of misplaced causal reasoning, and it is called time axis fallacy (or Falk’s Paradox).

As stated above, learning and teaching of conditional probability and independence have been facing serious cognitive obstacles and because of the subjects’ importance in physical and social science applications, further research is needed to analyze them. There have been, however, suggestions to overcome those issues. In next section, I will review some of those research-based suggestions.

Suggestions for overcoming cognitive issues with conditional probability and independence

In their main work on children's understanding of chance, Piaget and Inhelder (1948) concluded that, during the intuitive period (before the age of 6), children are not able to distinguish clearly between chance and necessary phenomena. That distinction appears during the concrete-operational period together with elementary forms of probabilistic estimations. According to Fischbein and Gazit (1984), in the teaching process, such intuitive attitudes must not be ignored. If they are found to be correct, they help the learner to acquire and integrate the corresponding scientific concepts. If they are not objectively acceptable, they must be eliminated and adequate intuitive representations must be developed.

If a teaching approach does not attend to possible intuitive biases, experiences may continue to mislead the learner despite the conceptual structures that may have been taught. Thus, in such teaching programs or educational tasks, one should consider the possible intuitive biases such as the problems with uncertainty, predictability, randomness, etc. Fischbein and Gazit analyze the effects of a teaching program in probability devised for 5th, 6th and 7th grade students. The program consisted of 12 lessons intended to teach the following notions: (a) certain, possible and impossible events; (b) comparison of chances; probability as a measure of likelihood of events; (c) the procedure for calculating the probability of an event, when the possible outcomes are equally likely; (d) the concept of relative frequency and its relation with that of probability; and (e) simple and compound events. The lessons based on that teaching program included practical activities such as extracting marbles, rolling dice, counting outcomes in defined situations. The relation between *a priori* calculated probabilities and empirically obtained frequencies was particularly emphasized in the article. From their study, the implication for teaching was that the implementation of a systematic program on probability might be carried out without particular difficulties, possibly starting from grade six and certainly starting from grade seven. Some intuitive obstacles have been identified but we assume that an improved version of the teaching program may overcome the respective difficulties. One may suppose that a course on probability (including practical activities) might have a positive beneficial effect on the children's prejudices and misconceptions with regard to sequences of events in uncertain situations. According to the authors, an important didactical problem arises: It seems that, with respect to proportional reasoning, the lessons had a rather negative effect. It may seem a surprising finding for a new reader of the subject; nevertheless, it is not the only example for seeing a negative effect despite time and education.

Fischbein and Schnarch (1997) find in their comprehensive empirical study that most of the misconceptions known about probability are either stable or increasing by age and education. Fischbein & Gazit (1984) think that the difficulty could be overcome if special care is devoted to this problem. They suggest that special exercises have to be devised in which students will be confronted with proportional computations and the respective implications for probability estimations.

According to Tarr and Lannin (2005), instructional tasks related to conditional probability and independence should elicit particular student conceptions and misconceptions, enabling students to reflect on the validity of their intuitions, and providing teachers' access to student thinking. Engaging students in carefully designed tasks allows the teacher to formally and informally assess student thinking and inform instructional decision-making. Tasks should be set in contexts that are familiar to students (middle school level students in their study) and promote small group and whole-class discussions. The authors state that instruction that is informed by research-based knowledge of students' thinking in conditional probability and independence can foster a coordinated understanding of both concepts. Therefore, it is vital for a researcher who is interested in this topic to be connected to the research in cognitive psychology. The authors state that a primary objective of instruction must be developing the idea that the sample space changes in without-replacement situations. Additionally, according to the authors, an instructional program must help students to consider the composition of the sample space in relation to the total number of outcomes; that is, although the number of elements of a target color may remain the same after sampling without-replacement, it is critical to consider how the entire sample space has been modified by the conditioning event. They conclude that the educational environment can lead to deeper understanding of key ideas by encouraging discussion and reflection among students regarding possible misconceptions. Similarly, other researchers, according to the article, document the importance of providing students with opportunities to collaborate on probability problems to enable them to overcome initial misconceptions and negotiate shared meanings.

Castro (1998) compared the impact of two different instructional orientations: 1) an environment that focused on eliciting student thinking and encouraging reflection on probabilistic ideas (referred to "conceptual change"), and 2) traditional instruction that centered on a clear, linear presentation of mathematical ideas without considering student conceptions and misconceptions. Castro found that misconceptions in conditional probability and independence were more resilient among those receiving "traditional instruction" than in classes that focused on "conceptual change". For example, students experiencing "traditional instruction" were more likely to retain representativeness strategies than students experiencing instruction that confronted misconceptions.

Falk (1986) suggests that one way to reduce the risk of confusion would be to dispense with such short-cut terms, which are unconditionally worded. Instead, we should strictly adhere to the symbolic language of conditional probabilities. Another didactic device is to present the problem's data in a two-dimensional frequency table, so that the two orthogonal directions for the computation of the two inverse conditional probabilities will be conspicuous.

Using references from Rittle-Johnson, Siegal and Alibali (2002), Reaburn (2013) notes the following for educators: "Procedural knowledge is defined as the 'ability to execute action sequences to solve problems' in contrast to conceptual knowledge defined 'as implicit or explicit understanding of the principles that govern a domain and the interrelation between units of knowledge in a domain'. This conceptual knowledge is flexible and not tied to specific problem types and is therefore generalizable. These two ideas are related to the ideas of surface learning, where the content of a text maybe just reproduced, and deep learning, where the student can relate parts of the material to the whole integrate it with existing knowledge and apply it in real world situations."

There are task and teaching program characteristics suggested by research, according to one of the most recent studies on the cognitive issues about conditional probability (Huerta, 2014). Nevertheless, there is very little information about conditional probability problems and we do not know what problems students have to solve, what characteristics problems possess that might have an influence on students' behavior and which make them so difficult to solve. Moreover, we have not yet considered in which direction, teaching models should be addressed so that they could help students to become competent and probabilistically literate in the actual uses of the probability. And it is known that, according to Huerta, research on probability problem solving should involve three elements, individually and in relation to each other: problems, students solving these problems, and teachers teaching students to solve them. The author describes the conditional probability problems she uses in her empirical study as this: "A probability problem is a conditional probability problem if in its formulation at least one of the quantities explicitly mentioned in the problem could be interpreted as a conditional probability" (p.614).

Continuing on Huerta's article, it is stated that no previous research has been conducted in the field of conditional probability problem solving, despite Shaughnessy's (1992) recommendation that pointed out the reality of the difficulties of teaching probability by claiming that teaching probability *is* teaching problem solving. Thus, she encourages researchers to carry out research in problem solving. Researchers may help teachers in their task of improving students' competences in solving problems of conditional probability. Thus, Huerta and her colleagues have conducted some research on conditional probability problem solving. That was carried out in order to obtain knowledge about conditional probability problems, identifying a particular family of problems -ternary problems of conditional probability- as a model for many school tasks on conditional probability. Cerdán and Huerta (1997) modeled this family of problems by means of what they called trinomial graph of the *world* of ternary problems of conditional probability. A ternary problem of conditional probability is a conditional probability problem formulated with three known quantities and one unknown quantity, to be solved for, and verifying that all probabilities in the problem are connected by ternary relationships, that is to say, additive or multiplicative relationships between the three quantities (Huerta, 2009).

According to Huerta (2009), however, because problem solving is a process, the difficulties of the problems can arise anywhere during this process. In particular, the author states that they have found difficulties when the process starts with the reading and comprehension of the problem; while students deciding to tackle the problem or not. They also found difficulties after calculations, if any, when students were trying to give an answer to the question in the problem. And, when students gave the solution of the problem by means of a quantity. Hence, not only the natural difficulty of the subject, but the very basic components we use in the problems or tasks could cause or effect the misconceptions.

Even though they are very important questions, there are no wide agreements among the researchers about when to start teaching conditional probability, and how to start introducing the subject in order to prevent the misconceptions. However, I will try to discuss some main problems I found in the methods used so far, and they may be helpful in answering those questions.

Reflection on research issues

First of all, unlike the references where it was suggested to consider the misconceptions about conditional probability and independence together, I think that independence is easier to understand for students than conditional probability. The misconceptions about independent events, such as negative and positive recency effects (i.e. the gambler's fallacy, and the hot hand belief) may be easily fixable. Either in-class experiments or computer simulations can be used to convince students that having independent events, the probability of a proceeding experiment will not be affected by previous results. Because students are able to observe that the sample space remains stable, and the experiments are still random. On the other hand, solving the misconception about conditional probability might be quite harder.

Let us consider the urn problem previously referenced in this paper. In the urn problem, I think, asking the second question was problematic, or incorrect. The reason is that students are asked to think about a future event, as "it is known that" however, the common sense tends to consider that a past event could affect a future event, but not vice versa. Hence, asking that "assume that the second draw is white…" seems as an invitation for a misconception. In my opinion, it is basically misleading to ask that question. It does not seem reasonable to force students for considering a future event as ‘given’. If we do, it is not surprising that we have unstoppable misconceptions about the urn problem. Hence, the source of misconceptions about conditional probability seems deeper. It is probably worth to question the problems we typically pose to students first and try to find more reasonable problems.

Another problem I found in empirical studies I reviewed was that analyzing a lower age student could be wrong. Even though it seems reasonable to teach the subject in lower grades as Watson and Kelly (2007) stated: “Rather than continuing to push complex topics to higher levels in the education system, perhaps they can be approached more directly at lower levels” (p. 233) .

In Tarr and Jones (1997), the authors ask a 5th grade student, Alicia, to guess the color she thinks that is more likely to appear in a spinner experiment. Her answer for the next color of spinner’s turn was blue, and Alicia says that she answered blue because it was ‘her favorite color’. Upon that explanation of her, the authors analyze Alicia’s misconception by referencing her reasoning process to availability heuristic in several paragraphs. However, I think that it cannot be enhanced as the authors did in their study. In my opinion, it could be just a 5th grade’s ‘my favorite color’ answer. The answer might have a relation to Tversky and Kahneman’s (1974) “availability” heuristic; however, it does not still prove that there is a direct relation between that non-quantitative answer of Alicia and her probabilistic reasoning. As far as I read the literature, I found that the problem with conditional probability studies could become non-effective, and even harmful in such cases.

Besides the issues in the methods researchers used, it is obvious that the subject itself contains basic cognitive problems. It should also be addressed in this paper that recent developments in technological tools could make a great contribution for the future empirical studies in the field. For example, dynamic statistical software TinkerPlots has a built-in feature called the Sampler in which one can create conditionally probability models and collect large data samples.

While the literature review has been made and research based suggestions mentioned in this paper, the next section will focus on the role of technology in modeling probability. Ultimately, the aim is to create helpful tasks for conditional probability lessons that may assist learners to overcome the cognitive issues about conditional probability.

Design and Modeling of Conditional Probability Tools

NCTM (2000) declared that “Technology is essential in teaching and learning mathematics; it influences what is taught and enhances students’ learning” (p. 24). According to Lee and Hollebrands (2006), designing technology tools can dramatically affect the students’ interaction with tools and those interactions may effect students’ mathematical problem solving. Research shows that the development of statistical software helped us to take big steps in probability education. There have been numerous empirical studies about the cognitive issues in probability and the role of technology to overcome those. For example, graphing calculators and pseudo-random number generators are well used in probability classes for students to understand better the concepts of randomness and chance. The statistical software also helped educators and students to have large sample sizes that had been a serious problem when we teach probability. On the other hand, there is little research about the suggestions to overcome specifically the cognitive obstacles about conditional probability.

Using simulations has played a significant role in the research about probability education. According to Chance and Rossman (2008) technology, and specifically simulations, can be very effective in helping students to learn statistical subjects. Before advanced computer technology became a part of the academic life, educators used to use hands-on simulation activities. For example, rolling a dice several times and collecting data by using paper and pencil. That way of teaching probability was missing a very important side of the subject; a large sample size. As one could see, it is difficult to have students understand probabilistic concepts with small sample sizes. On the other hand, it is not only students who needs to be considered while creating those technology tools; as Batanero et al (2005) stated, most of the educators have little experience with probability and share with the students various probabilistic misconceptions. Again, the lack of affordance of the technology in schools plays a part here. It seems that besides research, there has to be politic pressure to make the educational technologies reachable.

Before synthesizing the research on probability models and simulations, we need to point out the pedagogy needs to be considered here. As Stohl and Tarr (2002) based their work on the theoretical ideas that learning is a constructive process of resolving perturbations through reflection on actions and effects of actions that allows for abstraction. This process is coordinated with the social context in which meanings are negotiated through interactions with other cognizing individuals. This section of the literature review will discuss the main characteristics of designing tasks, modeling approach, embodied-design approach and semiotic approach of probability tasks.

Modeling probability and simulation tools

According to Lee and Lee (2009), students can model probabilistic situations based on their assumptions about a theoretical distribution, they can also simulate experiments to generate large amounts of data, and use the data in different ways that would be very difficult to do within the time constraints of school curriculum and instruction.

Abrahamson and Wilensky (2007) proposed a modeling approach where they highlighted the importance of the role of students in the process of designing models. According to the authors students need opportunities to re-invent some artifacts (see, touch, etc.). Only thus, according to the authors, can students appropriate these objects as thinking tools; only thus, do students develop trust in their personal mathematical agency; only thus, do students develop critical numeracy. The authors propose four key design-research constructs, which are as below:

1- Conceptual composite (mathematical concepts can be taught as sign systems, such as symbols or diagrams),
2- Learning axis (expresses an analysis of a mathematical representation to create learning tools)
3- Bridging tool (backgrounds the similarity of the idea elements, thus reminding their difference.)
4- Learning issue (that is the design challenge on understanding the concept)

For the desired design framework to implement constructivist pedagogical philosophy, it needs to satisfy some characteristics. Abrahamson and Wilensky (2007) find it important to create learning environments that capitalize on students' proto-mathematical intuitions pertaining to the target concepts; elicit students' holistic strategies, heuristics, perceptual judgments, experiential acumen, vocabulary, and previous mathematical understandings; provide materials and activities that enable students to understand the content; challenge students' strategy by showing specific situations; enable students to recognize the importance and comfort of new mathematical understanding; foster students' intuitions.

The educators' challenge is to determine the conception of probability that should be promoted in teaching and learning. According to several studies, the best direction in which to steer probability learning seems to be towards a modeling approach (Chaput et al., 2011).

Before comparing different approaches on modeling, it may be more plausible to discuss the nature of thinking about probability and models. According to Pfannkuch and Ziedins (2014), there are three interconnected ways of thinking about probability: true probability, model probability, and empirical probability. And, we could define three ways for attention to notions of a model: good model, poor model, and no model. And also, there are two interpretations of probability models: the first one is the theoretical notion of a mathematically-computed probability distribution called the classical approach; and the second one is the empirical notion of a stabilized probability distribution when some random experiment is repeated many times under the same conditions, called the frequentist approach. Batanero states that (2005), interpreting random situations in terms of probabilistic models will serve to overcome the controversy between classical, subjective and frequentist approaches.

How does the modeling process work? According to Chaput (2011), there are three stages in the modeling process, and those are:

1- Pseudo-concrete model (putting empirical observations into a working model)
2- Mathematization and formalization (translating working hypotheses into model hypotheses to design probability model).

3- Validation and interpretation in context (checking fit of a probability model to data).

It is essential for students to know why they are modeling the probability. Providing only numerical representations of real world examples make students unaware of the unique nature of probability which is full of uncertainties, differently from certain nature of mathematics in general.

In a recent paper, Pfannkuch and Ziedins (2014) state that a model is generally built for answering particular questions about a system and sometimes it could be only for understanding the behavior of that system better. According to Pfannkuch and Ziedins, a model becomes a "virtual world" where the experiments could be conducted to reflect the real world. Several researchers (Abrahamson & Wilensky, 2007; Konold & Kazak, 2008; Pfannkuch & Ziedins, 2014) highlight the importance of "model fit" as a main design construct. Model fit means the model fits to the purpose of the problem. Model probabilities are the estimates of true probabilities and they arise from modeling that we hope captures the essential behaviors of the process. Model probabilities need to satisfy certain consistency conditions to be useful for probabilists.

We could separate probability-modeling approaches in two categories: Theory-driven and data-driven. There are very strong links between the data-driven and theory-driven approaches, though. Theoretical models have often arisen as an abstraction of some aspect of a real world process, although their study may then continue long afterwards without further reference to such a motivation since the mathematical models are of interest in their own right.

Konold et al. (2011) suggest that students exposed entirely to probability situations that could be ascertained theoretically and empirically, such as the sum of two dice, coin tossing, and number of boys in a four-child family, could develop an impoverished view of probability. Hence they believe that students should experience problems for which there is no theoretical probability model or the presumed theoretical model is inadequate.

According to Pfannkuch and Ziedins (2014), teaching and learning probability from a modeling perspective is only now possible because the technology and tools are available. Also the technology we have on everyday life has opened up our minds to the possibility of operating in virtual worlds. Hence the time is ripe to consider new ways of teaching probability. Tertiary courses try to move students to a world where they work with ever richer and more complex models and move between real and virtual world systems. School students are already experiencing some of the elements of theory-driven and data-driven approaches. If they are to experience modeling, they may need to start by exploring the notion of a model as an approximation to reality, which may be fit for some purposes, but not others, and which can be improved if necessary. Teaching from a modeling perspective, moreover, would better resemble the practice of applied probabilists and should help students build a more integrated view of probability and a probabilistic way of thinking.

Abrahamson (2009) suggests the ‘embodied-design approach’ in the process of designing tools for probability education. While doing this, the author recognizes a ‘semiotic approach', which was suggested by Bakker and Hoffmann (2005). According to Abrahamson (2009), a semiotic approach appears to be a good option for the practice of design-based research. "The objects that formulate, store, convey, and mirror student expression is closely aligned with design-based researchers' analytical and pragmatic objectives, to better theorize the roles of mathematical objects in creating opportunities for mediated content learning and, to delineate principles for effective design" (p. 29). The approach Abrahamson proposes to use for the analysis of student interaction with new materials and it is an intellectual foundation for a principled design framework. In this approach, students perform semiotic acts with available media. Hence, they construct new meanings by those tools.

Prodromou and Pratt (2006) discuss two perspectives on distribution in a simulation: The modeling and data-centric perspectives. The modeling perspective pays attention to randomness and the shape of the probabilities that shape the outcomes (through some experiments). The modeling perspective reflects the reasoning process of statisticians when applying classical statistical inferences. On the other hand, the data-centric perspective on distribution pays attention to the variation and shape of data that has been collected (through a sampling process). According to Prodromou and Pratt, instead of choosing one of those perspectives, it is more plausible to combine two perspectives and they call this process 'developing a software based task to act as a window on thinking in change'. Prodromou and Pratt found that emergent "to design an environment that support students in discriminating and moving comfortably between data as a series of random outcomes at the micro-level and the shape of distribution as an emergent phenomenon at the macro-level" (p. 73). They also state that it may be possible to build an environment that allows the students to use their reasoning about causality at the same time begin to recognize its limitations in explaining local variation, and to use the parameters as causal agents. Prodromou and Pratt's pedagogical perspective is that pedagogy somehow enables students to appreciate the connection between the data-centric and modeling perspectives on distribution. In her later study, Prodromou (2012) states that the development of such pedagogy requires a research on designing tools that aim to facilitate the co-ordination of those two perspectives. The development of such pedagogy, according to Prodromou, demands appropriate pedagogical designs that facilitate the coordination of experimental probability with theoretical probability.

Abrahamson and Wilensky (2007) have considered the problem of coordinating two or more epistemologies in order to design activities that bring into close proximity the theoretical and empirical probability as elements of a conceptual composite. This allowed students to see both perspectives, thus encouraging connections to be made between them. Their approach to bridging tools inspired Prodromou to design a bridging tool, the Basketball Simulation (Prodromou & Pratt, 2006) and examine how students coordinated the two perspectives on distribution. In Prodromou's research (2012), when the students made connections from the modeling distribution to the data-centric distribution, they typically attributed intention variously to the pre-programmed nature of the computer, the characters in the software, the human modeler, or the tools within the software that trigger the random generation of data from the modeling distribution.

In my opinion, the combination of modeling and data-centric perspectives creates another problematic situation. What I understand from modeling perspective is to let students construct their own model by using a simulation tool. On the other hand, data-centric perspective intends students to focus on the shape of pre-collected data. However, I believe that, it is possible to let students have their own data, as well. Saying so, I mean the students could collect their own data and create their own model at the same time. According to Lee and Mojica (2008), teacher education efforts need to include experiences with authentic statistical inquiry, which includes the use of simulation tools, and modeling pedagogical practices that are useful to examine and discuss the data collected during the investigation. Thus, one of the questions I will try to answer in chapter 3 will be that is it possible to create such tasks where students will be able to collect their own data, create their own model by using the tool when the task will provide a chance to meet all key design constructs?

In statistics education, it is important to both model the use of technology as a problem solving tool and to take advantage of the ability to automate calculations and graphics, facilitate explorations of statistical concepts by using dynamic, interactive, visual environments, and experience first-hand the stochastic nature of statistical processes (Chance & Rossman, 2006). As in statistics education in general, educators need to make sure that they include the main characteristics of statistical investigation while using simulation tools (posing a question, collecting the data, analyzing data and interpreting the results). According to Lee and Mojica (2008), more work is needed for educators to develop conceptions of the relations between probability and statistics; it is also necessary to develop classroom practices to promote such relations. On the other hand, besides the main characteristics about statistical investigations, in the process of modeling probability tasks, Konold and Kazak (2008) suggest four characteristics in order to target four central ideas about probability that are model fit, distribution, signal & noise and law of large numbers. The authors see those four ideas as fundamental components of probability models and they list the characteristics of modeling as follows:

- *Making and testing Initial Predictions*: That is to motivate for exploration prepares to observe specifically serves to establish the purpose of understanding the activity.
- *Testing Predictions with simulated data*: Students' view the computer models they build and run them.
- *Using the sample space to explain distribution shape:* It helps to explain the data students have previously collected either from the real situation and/or from the computer simulation.
- *Seeing Signal & Noise in Empirical distribution:* To see, when chance is involved, you do not get the same thing every time. It also helps them to see in data a noisy version of their expectation.

The method used by Konold and Kazak (2008) leads students to question their initial ideas and motivates the need to develop an alternative theory. This misfit likely helps establish the more general enterprise as one of model fitting. Multiple repetitions get them to have a better sense of sample to sample variability, without their observation it wouldn't be possible to foster the sense of data as signal and noise. According to Konold and Kazak, the concepts signal-noise, sample space, LOLN (Law of Large Numbers) are not only conceptions; they are perceptions as well. Thus, the efforts for creating better education environments for probability subjects need a comprehensive psychology research, too. Indeed, it is misleading to consider conception and perception as two separate, cognitive processes, as they are highly interrelated.

According to Konold et al. (2007), teaching probability using a modeling approach has several advantages. Through modeling, students encounter and use probability in virtually the same way as practitioners do, with the purpose of better understanding some real phenomenon. Where people have strong prior conceptions, this encounter has been shown to be a very powerful technique in fostering conceptual change in domain (Drier, 2000; Konold, 1994; Pratt, 2000). Unlike the majority of probability problems students encounter, simulations generally focus on what happens in large samples, a perspective that is critical for forming or eliciting normative expectations about probabilities.

Reflection on research issues

There is no doubt that technology has increasingly been used in mathematics education research for a few decades, and the research about statistics and probability are enormously benefited from that. Before computer technologies' development, which contributed to the research about probability education, the methods to conduct tasks in classrooms (e.g. using pencil and paper) were limited. Fortunately, dynamic statistical software was invented and now we are able to use large data and simulate probability models. As discussed above, computer simulations played a huge role in the efforts of overcoming cognitive issues about conditional probability. The fundamental problems such as sample size, repeating several experiments and re-collecting those experiments in order to make the content more plausible are mostly fixed by the use of computer simulations. Nevertheless, I think that we have a problem in affordance of those tools. Even in the U.S. the actual classrooms are mostly not able to afford those technological tools. As it will further discussed in Chapter 4, there should be efforts to solve that problem.

It is not enough to have the dynamic software to overcome the cognitive issues about conditional probability that are the main issue of this research, though. As in all other mathematical and statistical units, the way we introduce the tasks is highly significant. Even though there has not been an agreement about how to design probability tasks and models, I think that it is possible to keep the commonly used task characteristics as a guide and construct a framework particularly for designing conditional probability tasks and models. Based on the literature review on the cognitive issues about conditional probability and design and modeling, in Chapter 3, I will present my framework to design good conditional probability tasks and two pre-built conditional probability tasks as examples.

CHAPTER 3

DESIGNING A FRAMEWORK AND TASKS

Considering information discussed in the previous chapter, I will further synthesize the literature to suggest several common characteristics for task questions and design principles. I will introduce simulations to help teachers and students for overcoming cognitive issues about conditional probability. The purpose of this chapter is to present a framework for those who want to design and use simulation tasks about conditional probability. The framework will then be illustrated with two simulation tasks.

The question being answered in this chapter is: *How can the important characteristics of a task for learning conditional probability with simulation tools be synthesized into a framework to guide task creation?*

Actually, that is a difficult question, because the empirical studies on designing appropriate tasks or models for overcoming the cognitive issues about conditional probability by using computer simulations are relatively recent. Because the short history of dynamic programs on probability simulations. However, we can set up a list of criteria based on the literature we have for answering this question. There is no doubt that more empirical studies will need to investigate the problem in the future.

According to Lee and Mojica (2008) a way of connecting statistics and probability could be done by processing a statistical investigation via conducting simulations or experiments. And, as Graham (1987) described in his PCAI model, all statistical investigations should have four basic principles (phases) in order to generate random data and to make inferences based on data. Those principles (phases) are: *posing a question, collecting data, analyzing data and interpreting results.* While designing a task to investigate particularly the cognitive issues about conditional probability, one has to make sure that he or she follows those phases, too. In order to create useful tasks and models for overcoming the cognitive issues about conditional probability, it is also important to consider the following questions to help define and narrow the focus of my framework:

- What are the important concepts in conditional probability to be included in the process of designing a model and creating a task?
- What is the target student level of a task that the framework can be used with?
- How will the students get informed about the tool(s) used in a task?
- Is it necessary to use hands-on activities first to investigate the task questions?
- How important is it for students to explain how the models represent the situation?
- How can the simulation motivate the purpose of the task that is overcoming the main misconceptions about conditional probability?

For the specific content focus of the framework, the difference between conditional probability and independence should be addressed. Independence of events can be considered as the opposite of conditional probability. When we say two events are independent we state that there is no such condition depending on an event that affects the other. Empirical studies (e.g. Tarr & Jones, 1997; Tarr & Lannin, 2005) investigated both concepts in their tasks. However, the link between independence and conditional probability can be problematic. For example one of the most common tasks in the research on conditional probability is the *urn problem* (Falk, 1986; Reaburn, 2013). In that problem using the situation where we continue to conduct an experiment without replacement, the students are expected to realize the sample space changes (Fischbein & Gazit, 1984), and that is the situation where students face the cognitive issues first. Repeatedly investigating the independent situations could be confusing. Hence, it is essentially important to include without replacement conditions in a good conditional probability task.

The target level of students for those tasks is an important problem. According to Watson and Kelly (2007), directly approaching the complex problem of conditional probability in lower ages may be a better decision. Even though there should be an introduction to the subject in younger ages, it is better to conduct simulation based conditional probability tasks to high school students. Because Fischbein and Schnarch (1997) showed that the evolution of probabilistic concepts by age and education is dramatically deteriorated. Thus, students need to be open to use complex tools when they try to solve complex problems. Starting to teach a problematic subject early could be a reason of keeping the subject problematic in later years. Further research should be needed for discussing this, though. Hence, the framework I present focuses on task development for high school aged students (ages 14-18).

Since the nature of probabilistic investigations has an uncertain characteristic, most of the conclusions we have can be considered as predictions based on our theoretical knowledge. The role of predictions or assumptions has an important role in probabilistic investigations (Girotto & Laird, 2004). Thus allowing students to make their own assumptions about the problems before they actually begin the modeling process could be an essential part of the task.

As Lee and Hollebrands (2006) stated, design of technology tools can dramatically affect the students' interaction with the tools and those interactions may affect students' mathematical problem solving. It is also important to show the effectiveness of the technology to students. By allowing them the chance of comparing using hands-on activities and computer-based simulations, students may see the difference simulations create. Hence in the framework I suggest, there has to be a room for hands-on activity before using computer simulations. Therefore student may realize how hands-on activities limit the sample size compare to computer-based activities (Chance & Rossman, 2008). However, the hands-on simulation can assist students in developing and understanding a model for a situation and potentially better understanding of the instantiation of that model in a computer environment.

Questions in a good conditional probability task have to be carefully chosen to examine how well a model fits to real scenarios (Pfannkuch & Ziedins, 2014). Students have to be convinced about the model fit. It could be done by giving examples from real world situations or using hands-on activities.

Teachers have to be aware of the purpose of the investigation while they are conducting the tasks. The purpose is to overcome cognitive obstacles about conditional probability with the help of computer simulations. While doing that, an important part of the framework could be teachers' questions. For example, teacher can pose a new question during the problem solving by subtracting the current sample size by one element. By doing this, the concept of causality, which is an essential actor in the study of conditional probability, could make more sense for students (Prodromou & Pratt, 2006). Moreover, one needs to make sure it is clear for students that the simulation tool is a good representation for conditional probability task. Since the real scenarios contain conditional probability take place dynamically instead of static situations. Students can notice that the simulations are a good representation for those situations where they can spontaneously see the simulation as it runs. The virtual benefits of a computer simulation has to be well emphasized (Konold et al., 2007)

Questions in the task have to be well constructed to overcome main cognitive problems about conditional probability. The main issues such as representativeness heuristic (Tversky & Kahnemann, 1974), time-axis fallacy (Falk, 1986; Fischbein & Schnarch, 1997), misconceptions of availability (Tversky & Kahnemann, 1972, 1974; Falk, 1986; Huerta, 2009) have to be addressed, since those misconceptions have not been solved without advanced technology for decades.

There needs to be one or more questions where the situation is fundamentally problematic and it is almost impossible for students to model those (Konold et al, 2011); because one of the reasons we face the difficulties is that there are situations where even imagining a model is impossible. For example, considering the difficulty of inverse conditionality (Falk, 1986), one could not create such model that the later experiment is considered as it happened. It is not an option to run a simulation by running a later step before the previous one. This is an important element to show that misconceptions in conditional probability are not only because of students' incompetence, but the subject is already problematic (Abrahamson, 2009).

After posing the questions, it is essential to make students to state their assumptions about the situation (Lee & Lee, 2009) based on the problems. Therefore, introducing the tool elaborately first and showing them a pre-built model for a relatively similar problem would be beneficial. That way, students will be able to witness how important is the modeling and how a model could represent a problem. Also, it could be a good way to let students know how important is explaining how a model represents a problematic situation (Konold & Kazak, 2008). As students watch teachers' modeling process and solving the problem with that model, they can get a sense of the effectiveness of explaining such representation in order to solve a problem.

Teachers need to ensure that they let students construct their own tools for further investigations (Konold et al, 2011). It is found that having students collect the data on their own work is better than giving them a pre-collected data (Lee & Mojica, 2008). Thus, based on the problems they are asked they to collect the data. Considering the urn problem again, students should be able to create the urn by choosing the sample, as they wish (mixer, spinner, etc.). *TinkerPlots Sampler* provides that opportunity (Konold & Kazak, 2008).

The models have to give an opportunity to students to construct new meanings, too (Abrahamson, 2009). It is also essential for a model to allow students to realize the link between theoretical and empirical probabilities. The model needs to contain features for considering it as a bridging tool (Abrahamson & Wilensky, 2007) as in the example of *Basketball Simulation* used by Prodromou and Pratt (2006).

The Framework

Summing up the elements of a good conditional probability task, the criteria presented here will be grouped into two categories; the first category is focused on task questions, and the second one is about the simulations to be used (Konold & Kazak, 2008). The framework is also organized as to what should be considered in task design during each of the four phases of a statistical investigation.

The framework is presented in Table A and then each section is elaborated and explained below.

Table 1. Framework for Designing Conditional Probability Tasks with Simulation Tools

Phase of Statistical Investigation	***Task Question Characteristics***	***Simulation Tool Characteristics***
Pose a Question	***PQ-TQ-1:*** The questions need to address the main misconceptions about conditional probability, such as representativeness problem, time axis fallacy, positive and negative recency effects, and availability heuristic. (Tversky & Kahnemann, 1974; Falk, 1986; Shaugnessy, 1992; Fischbein & Schnarch, 1997; Huerta, 2009) ***PQ-TQ-2:*** Students should be given the opportunity to make their assumptions and predictions about the problem. (Girotto & Laird, 2004; Konold & Kazak, 2008; Lee & Lee, 2009) ***PQ-TQ-3:*** The questions should motivate students to be able to ask their own questions based on the simulation, and need to get attention of students. (Abrahamson & Wilensky, 2007; Abrahamson, 2009; Konold et al, 2011)	***PQ-ST-1:*** Task scenarios should be posed that includes the use physical objects familiar to students as well as those that include unfamiliar contexts that require a modeling process. (Abrahamson, 2009; Konold et al, 2011; Pfannkuch & Ziedins, 2014) ***PQ-ST-2:*** Some questions need to represent situations where there is no possible model students can build with tools. (Falk, 1986; Abrahamson, 2009; Konold et al, 2011)
Collect Data	***CD-TQ-1:*** Students should have the opportunity to collect their own data. Thus, the questions need to be well explained and could be represented by real world scenarios. For example, task questions should suggest how to collect data and sample size. (Konold & Kazak, 2008; Pfannkuch & Ziedins, 2014)	***CD-ST-1:*** Students should have the opportunity of conducting hands-on activities before they use a technological tool and students should be able to collect the data by using the tool. (Lee & Hollebrands, 2006; Chance & Rossman, 2008; Konold & Kazak, 2008). ***CD-ST-2:*** The tool should be flexible enough to allow students to collect large samples. (Lee & Hollebrands, 2006; Konold & Kazak, 2008).

Table 1. Continued

Analyze Data	***AD-TQ-1:*** The questions should lead the students to discover how to analyze the data by informing them about each step through analysis. (Lee & Mojica, 2008; Abrahamson, 2009; Pfannkuch & Ziedins, 2014) ***AD-TQ-2:*** The students have to be given the chance of constructing new meanings, posing new questions and so, creating their own models. (Abrahamson & Wilensky, 2007; Abrahamson, 2009; Prodromou, 2012).	***AD-ST-1:*** The simulation has to provide the opportunity for analyzing results. Students should be able to use tables and graphs. (Chance & Rossman, 2006; Lee & Hollebrands, 2006; Konold & Kazak, 2008) ***AD-ST-2:*** The simulation needs to give the opportunity of creating bridging tools between theoretical and empirical probability, which stimulate the learner to construct a viable reconciliation of competing perceptions. (Abrahamson & Wilensky, 2007; Abrahamson, 2009; Pfannkuch & Ziedins, 2014).
Interpret Results	***IR-TQ-1:*** The questions need to be open for interpretations and facilitate students making connections back to the context of the problem. (Batanero, 2005).	***IR-ST-1:*** The tool affords multiple ways of summarizing data from simulations that can lead to multiple interpretations of results. (Batanero, 2005; Abrahamson, 2009; Pfannkuch & Ziedins, 2014)

The following part is a brief elaboration of each of the elements in the framework from Table A. Each characteristic is presented along with a rationale.

Phase 1: Pose a question

PQ-TQ-1: *The questions need to address the main misconceptions about conditional probability, such as representativeness problem, time axis fallacy, positive and negative recency effects, and availability heuristic. (Tversky & Kahnemann, 1974; Falk, 1986; Shaugnessy, 1992; Fischbein & Schnarch, 1997; Huerta, 2009)*

In order to target the misconceptions of students about conditional probability, it is necessary to pose questions that are constructed specifically to address those misconceptions. For example, instead of asking one single event's probability, for aiming the misconception of negative recency effect, a question could be posed by giving a certain amount of outcomes of independent events and asking the proceeding event's probability in the same model.

PQ-TQ-2: *Students should be given the opportunity to make their assumptions and predictions about the problem. (Girotto & Laird, 2004; Konold & Kazak, 2008; Lee & Lee, 2009)*

In statistics and probability, we deal with an uncertain nature. Instead of having certain answers, we find the best predictions. Asking students their initial assumptions before they solve the problem would be helpful to show this fact.

PQ-TQ-3: *The questions should motivate students to be able to ask their own questions based on the simulation, and should get their attention to the problem. (Abrahamson & Wilensky, 2007; Abrahamson, 2009; Konold et al, 2011)*

It is important to leave room for students' own curiosity. Instead of introducing a fully pre-constructed model that the tool can be used only for the task questions, leaving the simulation tool flexible to play around, thus students would be able to discover the model on their own. And that could give an opportunity for easing the cognitive obstacles, which may come from the hard concreteness of mathematical problems.

PQ-ST-1: *Task scenarios should be posed that include the use of physical objects familiar to students as well as those that include unfamiliar contexts that require a modeling process. (Abrahamson, 2009; Konold et al, 2011; Pfannkuch & Ziedins, 2014)*

Students may not feel comfortable with imagining the possible situations based on the model. Thus, it would be helpful for them to have scenarios given based on the task content before they begin work on their own. Those scenarios should have familiar and unfamiliar contexts that both require a modeling process.

PQ-ST-2: *Some questions need to represent situations where there is no possible model they can build. (Falk, 1986; Abrahamson, 2009; Konold et al, 2011)*

Models are not the ultimate cure for the cognitive problems; they cannot answer all possible questions that can be asked. Sometimes it is just impossible to model a problematic situation. For example, since time does not go backward, a question investigating time axis fallacy (Falk paradox) cannot be answered by modeling. Telling students that a future event is given and they are expected to find a previous event's probability may not be modeled.

Phase 2: Collect data

CD-TQ-1: *Students should have the opportunity to collect their own data. Thus, the questions need to be well explained and could be represented by real world scenarios. (Konold & Kazak, 2008; Pfannkuch & Ziedins, 2014)*

Apart from the rest of mathematics, in statistical and probabilistic investigations, it is important to state that students deal with data, which mostly represents the real world. That can be overcome by having students collect their own data.

CD-ST-1: *Students should have the opportunity of conducting hands-on activities before they use a technological tool and students should be able to collect the data by using the tool. (Lee & Hollebrands, 2006; Chance & Rossman, 2008; Konold & Kazak, 2008).*

It is important for students to know that simulations are not available in technological zone, but there is chance to have real world simulations as hands-on activities. However, it is also important to show how the data collection differs between two simulation methods. (i.e., in technological simulation tools, we are able to have large samples)

CD-ST-2: *The tool should be flexible enough to allow students to collect large samples (Lee & Hollebrands, 2006; Konold & Kazak, 2008).*

One of the important issues in teaching and learning of statistics and probability is having large samples to have more convincing data in order to make statistical inferences. Simulation tools should allow students to choose their sample size and be flexible to collect various amounts of data, particularly very large data sets (500 or above).

Phase 3: Analyze data

AD-TQ-1: *The questions should lead the students to discover how to analyze the data by informing them about every step they will take through their analysis. (Lee & Mojica, 2008; Abrahamson, 2009; Pfannkuch & Ziedins, 2014)*

Statistics learning process is distinctively different from mathematical learning process as it is discussed in Chapter 2. One of the reasons of that difference is the analyzing data phase of statistical investigation. Doing that without a guide could be very difficult for students who are not aware of that difference. Thus, informing students about analyzing data should be addressed in task design process.

AD-TQ-2: *The students have to be given the chance of constructing new meanings, posing new questions and so, creating own models. (Abrahamson & Wilensky, 2007; Abrahamson, 2009; Prodromou, 2012).*

Students may be taught that statistical problems have a great potential for further investigations. Having students to construct new meanings, pose new questions could open a door to create their own models.

AD-ST-1: *The simulation has to provide the opportunity for analyzing the results. Students should be able to use tables and graphs. (Chance & Rossman, 2006; Lee & Hollebrands, 2006; Konold & Kazak, 2008)*

Having students to get opportunities for working on both static and graphical analysis of results would be helpful for them to comfortably focus on cognitively problematic sides of the subject, instead of spending time on imagining or drawing the results.

AD-ST-2: *The simulation needs to give the opportunity of creating bridging tools between theoretical and empirical probability that stimulate the learner to construct a viable reconciliation of competing perceptions. (Abrahamson & Wilensky, 2007; Abrahamson, 2009; Pfannkuch & Ziedins, 2014).*

It is important for students to realize that there is a connection between theoretical probability and empirical probability. Actually seeing the simulation runs and the data is collected in a result table; theoretical probability will be more convincing.

Phase 4: Interpret results

<u>IR-TQ-1:</u> *The questions need to be open for interpretations and facilitate students making connections back to the context of the problem. (Batanero, 2005).*

The last phase of a statistical investigation is interpreting the results. Instead of asking students the probabilities of several events in which they only use their mathematical knowledge to answer, the questions need to be open for interpretations and discussions. Not only for cognitive issues about probability, but it also helps students to develop analytical thinking skills in other areas, like their scientific and social science investigations.

<u>IR-ST-1:</u> *The tool affords multiple ways of summarizing data from simulations that can lead to multiple interpretations of results. (Batanero, 2005; Abrahamson, 2009; Pfannkuch & Ziedins, 2014)*

The tool should represent more than one way of summarizing data from simulations; therefore, students would be able to interpret the results in several ways. For example, The Sampler provides several different ways to see the 'Results Table', such as scatter plot, histogram, etc.

Using TinkerPlots as a Modeling and Simulation Tool

The software TinkerPlots (Konold & Miller, 2005) integrates a probability-modeling environment with data analysis tools. Thus it allows us to design probability models to easily use in classrooms and engage students in an interactive tool (Konold & Kazak, 2008). Thus, I will use *TinkerPlots* as a simulation tool to create appropriate conditional probability tasks. At this point, I believe that this work needs a description of TinkerPlots and its features.

TinkerPlots (Konold & Miller, 2005) is a dynamic *exploratory data analysis (EDA)* and modeling software that was designed by Clifford Konold and Craig D. Miller for using by students (in grades 4 through university). It also helps teachers, education researchers and curriculum designers to use for helping students to build conceptual understandings of statistics. (Konold, 2007)

As Konold (2007) states, TinkerPlots allows students to organize data to see patterns and trends in data by visualization tools. Using TinkerPlots, students can enter their own data from other applications, or Internet. They are also able to generate data by using a sampling engine. The software comes with several multivariate data sets to make various graphs; students can construct those graphs by organizing the cases using some operations, such as *stack*, *order* and *separate*. According to Konold (2007) instead of having bar graphs, pie charts and histograms within the software, allowing students build those traditional representations by progressively organizing the data icons in a plot with the operations stack, separate and order would be a better method in order to learn. This method allows students to see the computer operations as akin to what they do when physically arranging real-world objects. Students take small steps while constructing the work and each step helps them to determine the nature of statistics. The developers of TinkerPlots state that by giving students more fundamental choices to represent the data they would develop the sense that they were making their own graphic representation rather than selecting from a set of pre-formed options. (Konold, 2007)

TinkerPlots interface is based on observations of organizing data cards on a table to make graphs to answer questions. It contains a superimposed color gradient for detecting covariation in two numeric attributes.

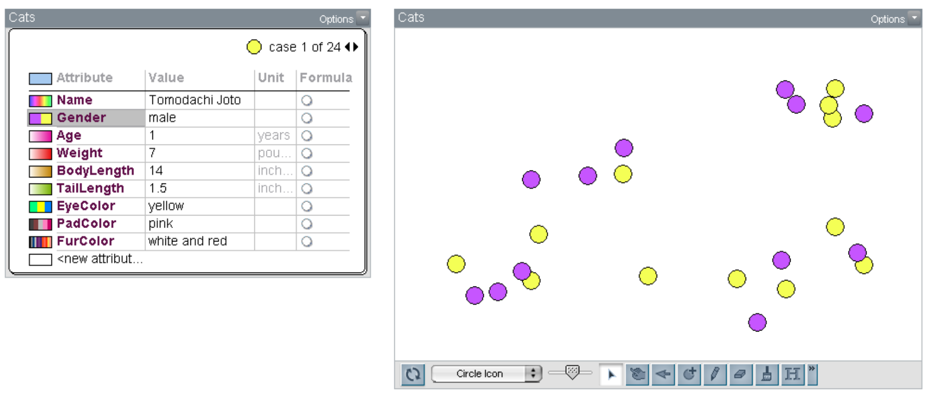

Figure 1: TinkerPlots description 1

Figure 1 is a sample of data card and plot based on a sample data (Cats.tp) within TinkerPlots. The data card on the left shows the attributes for the 1st cat in that data; by choosing the case icons on the plot or using right arrow on the up left of the data card, students could see other cases or cats' characteristics, such as name, eye color, gender, etc. The case icons are not organized in the plot and we see two colors (yellow and purple) which indicate the genders of the cats. By choosing other attributes in the data card, other specifications could be highlighted in the plot. We can organize the plot by using order, stack and separate features as seen in Figure 2 below.

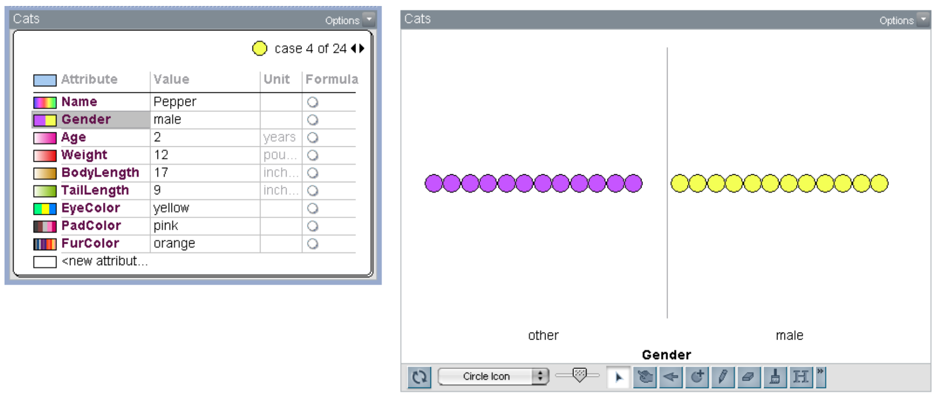

Figure 2: TinkerPlots description 2

We can also add another attribute into the plot and reorder the data. Suppose that we investigate the relation between gender and weight of the cats in that data, color changes from light red to dark red to indicate the cases from lightest to heaviest as represented in Figure 3.

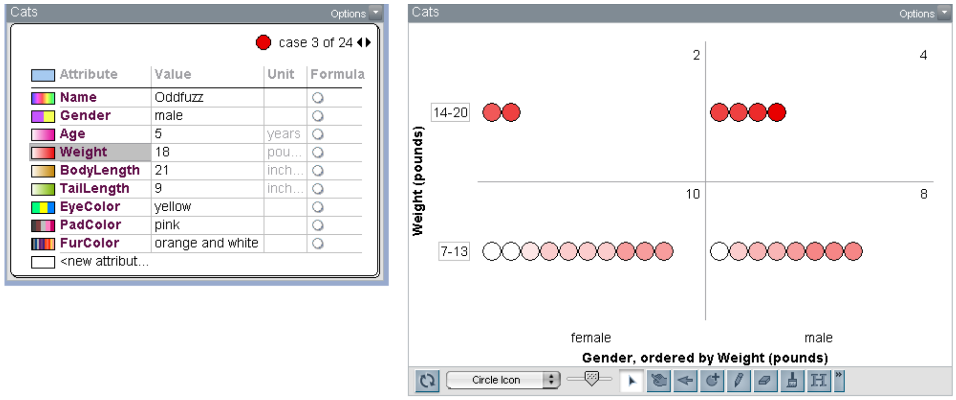

Figure 3: TinkerPlots description 3

For partitioning of data into subgroups TinkerPlots has a feature called *dividers*. Figure 4 shows the cats weights into tighter subgroups.

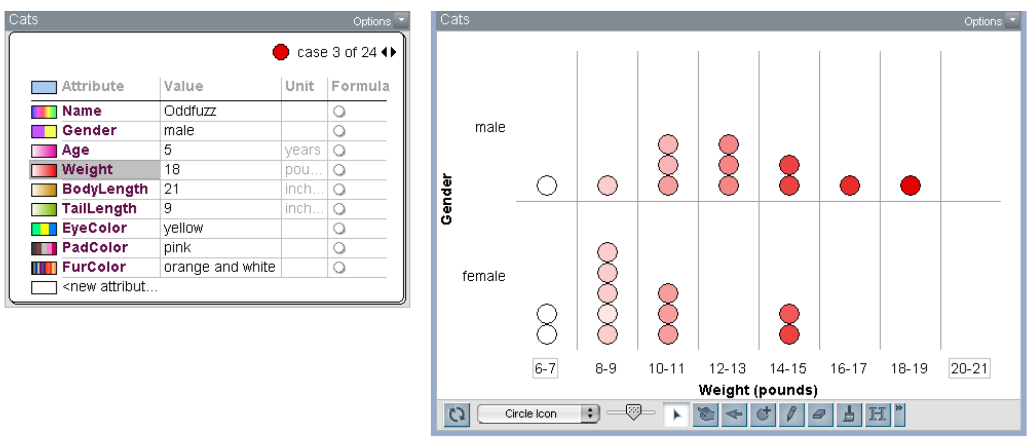

Figure 4: TinkerPlots description 4

Different from similar software such as Fathom, TinkerPlots includes hat *plot* feature, instead of box plot. Hat plot is a reformulated and generalized version of the box plot. In box plot, the whiskers are drawn along the bottom of the IQR (inter quartile range) rectangle; in hat plot the corresponding line is drawn along the bottom of the rectangle. Doing this, the designers of TinkerPlots expect that it would help to emphasize what the center rectangle is depicting the location of a central clump of the data. There are mean, median icons to point the center values in TinkerPlots. There is reference line feature to determine the precise value of case icons. Figure 5 shows how those features can be used to determine if there is a relation between gender and weight of the cats in cats.tp data sample.

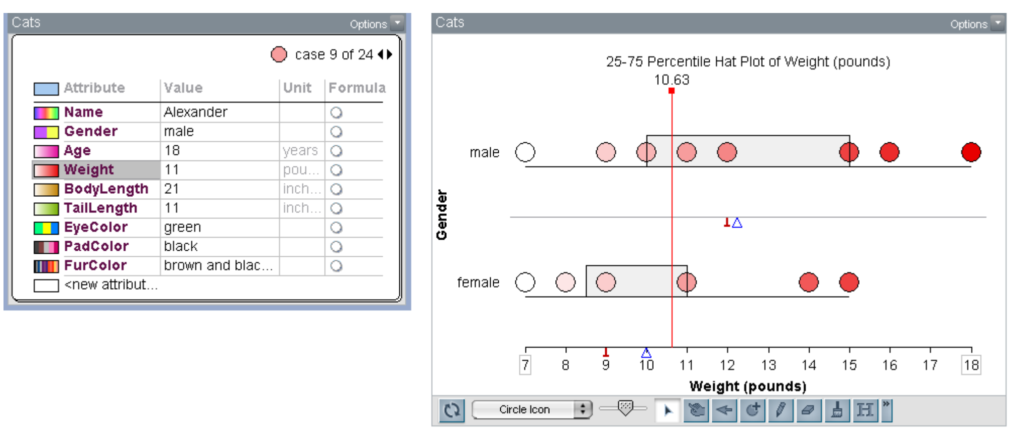

Figure 5: TinkerPlots description 5

Those features are not the only ones TinkerPlots provides as EDA software. For further information the video tutorials in the following link can be watched:

www.youtube.com/embed/gkSYlXhPOLU?list=PL439CC5309EBB1114

(KeyCurriculumPress, 2012)

In this research, my intention is using TinkerPlots as a probability-modeling tool. In TinkerPlots, a Sampler Engine was added to the dynamic statistical software TinkerPlots in a later version is a useful simulation tool for creating probability models. Let us take a look how TinkerPlots allow us to create a probability simulation by using the *Sampler* feature. We pull out a sampler into the blank page and get the interface seen in Figure 6.

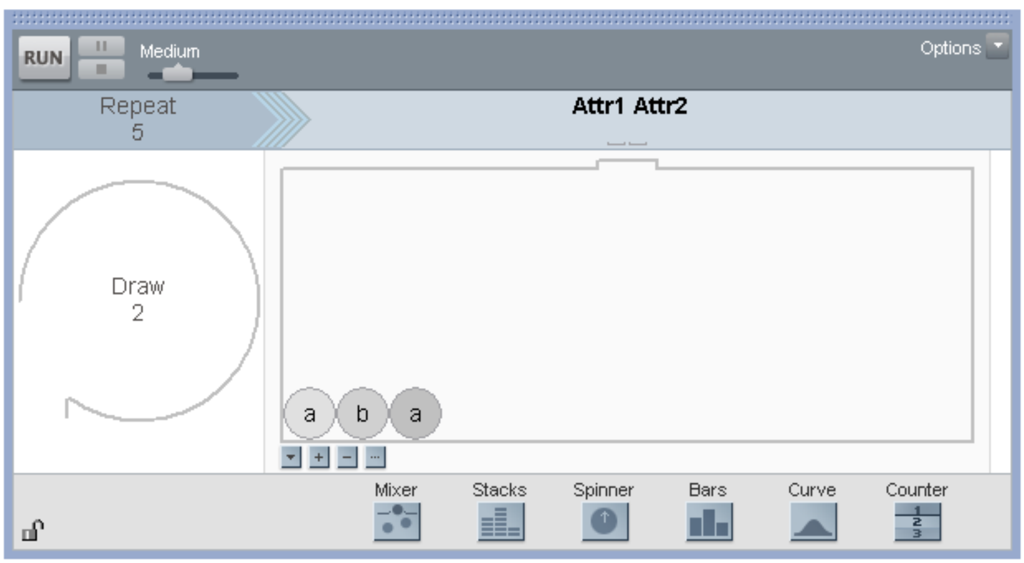

Figure 6: *TinkerPlots Sampler* 1

Assume that we want to simulate rolling two dice. We can increase the elements by plus button in the sampler until we have six elements, and change the names of elements from letters to numbers. We can change the speed of simulation from medium to slow or fast, and we are also able to change the number of repeats. After making those adjustments we just push the RUN button and watch the simulation.

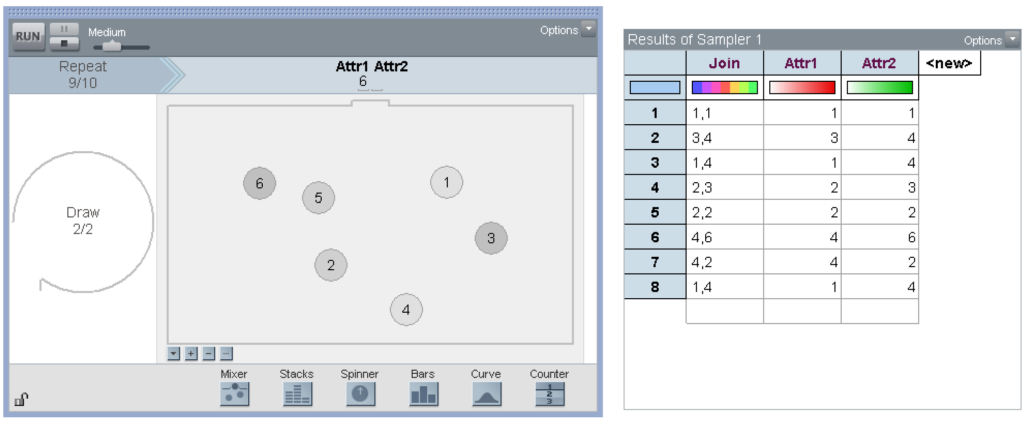

Figure 7: *TinkerPlots Sampler* 2

Now we can adjust our simulation to more resemble a rolling dice experiment. TinkerPlots allows us to change the sampler device to a spinner with equal 6 pieces, which would look more like a dice; and we can add another device next to the first one instead of drawing one device twice. We can also change the name of the attributes to "Die 1" and "Die 2", as seen in Figure 8. Suppose that we investigate a probability question about the sum of two dice, such as *"what is the probability of getting the sum of two dice greater than 9?"* Since the question includes "sum", we need to add another attribute in the result table by using the *options* menu, choosing *result attributes*, and then choosing "sum of joined values". By pulling out another plot and drag the "sum" attribute to the x-axis (or y-axis), we can see the graph for the sum of the dice. In Figure 8, it is changed the repeat number to 25; however, we are able to change it to larger numbers. As we push the RUN button in the sampler again, the result table and the “sum graph” will change, and thus we are still able to interpret the results and try to answer the question.

Figure 8: *TinkerPlots Sampler* 3

Additionally, TinkerPlots includes two important features: the ability to create new attributes with constructing formulas; and the ability to collect history. First feature allows users to create their own attributes with formulas. As an example, this important feature will be used in *Task 2: Spinodice*, later in this research. The second feature allows users to observe the collection of experiments, which could mean a difference maker in probability implementation. Suppose that you run the simulation with repeat number 100. Instead of repeating the experiment and observe those 100 repeat simulations separately, it is also possible to see all repeated experiments (history results) in a collection of history results plot. This feature will be effectively used in *Task 1: Urim & Thummim*, later in this research.

It is briefly described how *TinkerPlots Sampler* works as a probability simulation tool and how TinkerPlots Sampler can model a real world scenario. The following sections will present two conditional probability tasks as examples to use in high school classrooms.

Illustrating the Framework with Two Tasks

A framework for task design as presented in Table A can assist teachers in considering important elements to incorporate into task questions and simulation tools. The following two tasks, I believe, would be useful in teaching high school students concepts in conditional probability. Throughout each task description below, I reference different elements of the framework using the various codes as presented in Table A (e.g., *PQ-TQ-2; CD-ST-1).* Each task is also written in a complete lesson plan format and appears in Appendix A and B.

Task 1: Urim and Thummim

The origin of the story context

An article in *The Mathematics Teacher* journal, "Probability in ancient times; or shall I go down after the Philistines?" (Milton, 1989) has given a very interesting historical example about the use of probability (without using the term probability) in ancient times. The author linked her research to teaching probability by saying that nontraditional nature seldom fails to arouse the curiosity of students. Hence, finding interesting stories related to probability and telling them to students may work in teaching. The author mentions a story in *Old Testament* (actually similar stories are available in several other resources too). The story is called Urim and Thummim. The priests had a communication with a God in which they put three white and three black stones in a sacred container, when they ask 'yes-no' response type questions to The God; they randomly draw three stones without replacement from the container. If they draw all three white stones, that means God's response was a "Yes, do it!" If three stones were all black, that means The God's response was a "No, do not do it!" If the colors of stones were mixed, that meant that The God did not give a response to the question on that day, therefore the priest could repeat the same experiment another day. I believe that using such a story could be both helpful and interesting for students *(PQ-ST-1)*.

Designing the task and tool

After giving the story, students could be asked to pose some questions using this story first. What are the interesting questions they can ask? It is also important for students to discuss their assumptions and predictions as most likely and least likely based on the problem

(*PQ-TQ-2)*. Students need to be asked to state their assumptions about those questions, and then, hands-on activities could be used to try solving those questions *(PQ-TQ-3)*. Since the context of the problem uses stones in a container; students can model this situation using familiar objects of 3 black and 3 white marbles in a bag or other container (PQ-ST-1). By following the story line and drawing marbles from the bag, they can enact the context and record the results from each draw as well as the final result (Yes, No, No response and try again) (*CD-ST-1)*. As time allows, teacher try to guide students about how to analyze the results; then students collect the data. *(CD-ST-1; CD-TQ-1)*

After that, students are introduced to a pre-built model (Figure 9) and shown how to run it *(PQ-ST-1)* (The reason of choosing to introduce pre-built models will be explained later in Chapter 4). Then, they are welcome to run the simulation and observe the process. *(AD-ST-1)*

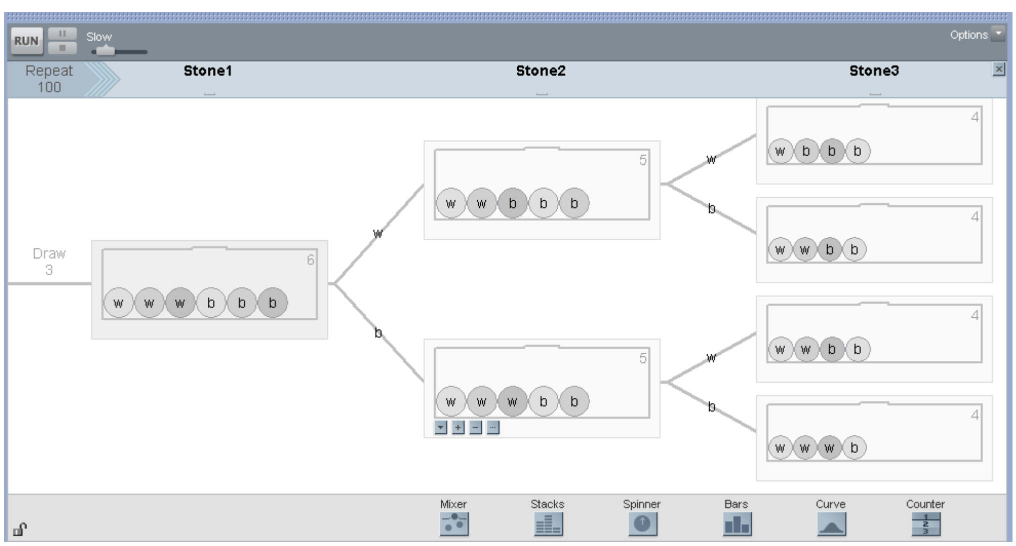

Figure 9: Urim & Thummim 1

Students could be encouraged to make some comments about the tool and compare the process with their pre-assumptions, and also hands-on activity. *(AD-TQ-2)* The similarities between hands-on activities and pre-built model will be discussed.

Possible questions may be as follow when their own work:

- *Assume that we have pre-observed experiments; after 5 days of not getting a "Do" or "Do not", response what is the probability of getting a "Do" response on day 6?*
- *What is the probability of not getting a "Do not" response after 100 days?*

More complex questions can be asked as follows:

- *Suppose that you conducted "Urim & Thummim" for 50 days by willing to get a "Do" response, but did not get a "Do" or "Do not" response yet. At that point you are offered to choose one of the options below:*
- *You can keep trying 50 days. No response after all trials will mean, "Do not". Or;*

- Suppose that instead of sacred container and stones example you are given dice to roll. You roll a dice; if you get "3" you will get "Do" response, otherwise the response is "Do not."

Which option is you going to choose and why? (IR-TQ-1; AD-TQ-1; CD-TQ-1)

Observing the students work and encouraging them to interpret the simulating process and the results *(PQ-ST-1; AD-ST-1),* teacher may propose an impossible problem as:

- Consider that it is known you will get a white stone in the second draw. How likely you get a black stone at the first draw?(PQ-ST-2)

By investigating the task, students could encounter the negative and positive recency effects, representativeness misconception, time-axis fallacy and availability heuristic. *(PQ-TQ-1; PQ-ST-2)* For example, asking students the probability of getting "Do" at sixth experiment after five "Do not" outcomes examines students' position in the gambler's fallacy; are they going to consider six trials as dependent variables and give wrong answer or are they aware of independency of the experiments?

The interactive simulation tool constructed by using the *TinkerPlots Sampler* will be useful to solve the problem in Urim & Thummim. As it was said, Figure 9 shows the sampler could be used for "Urim & Thummim" task. In all steps the mixer was used, because the experiment was reflecting the experiment where stones drawn from a sacred box. *(PQ-ST-1)* As the number of stones decrease in each step, built-in mixer misses one stone and gets ready for second draw. At the beginning the mixer has three white and three black stones. If first-drawn stone was white, in the second step the mixer contains two white, and three black stones and vice versa. Hence, students will witness without replacement condition within the simulation; the similar situation they observe during hands-on experience *when a stone is drawn, sample space is decreased by one*. *(PQ-ST-1; AD-ST-1)*

As seen in Figure 9, the sample size was chosen as 100 repeats in the simulation tool. Actually, The Sampler lets us run a simulation for millions of repeats. Since one of my goals to make sure that students see the whole process of the model, using very large sample sizes such as one million would be impossible in time manners. Thus, 100 repeat is a plausible starting number. On the other hand, using "fastest" run speed in the sampler, students would be able to collect larger data after they understand the model. *(CD-ST-2)* For example, in the Figure 10 below, the repeat number is 100,000 and the speed of the simulation is in the fastest mode.

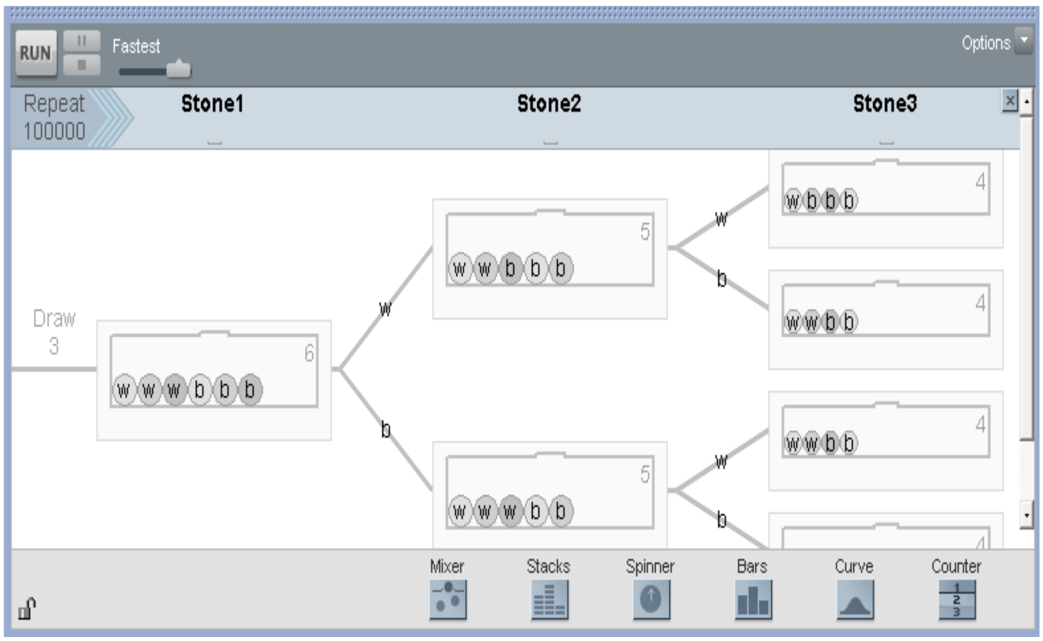

Figure 10: Urim & Thummim 2

As seen in Figure 11 below, as the simulation runs, data generated from the simulation appear in a case table labeled “Results of Sampler 1”, with results from each run appearing in a row in the table. Thus students could see each step and the result for each experiment. *(AD-ST-1; IR-ST-1)* Students should be informed about what three letters in a row mean, again. For example “w,w,w” means “yes”; “b,b,b” means a “no”, and mixed letters mean “try again later.

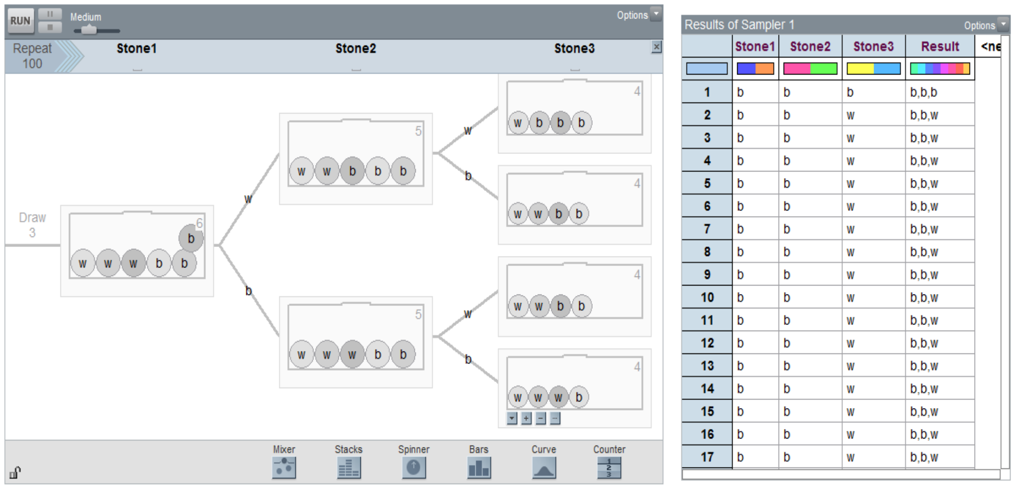

Figure 11: Urim & Thummim 3

Keeping in mind that if and only if all three stones are white the response was “yes”, if they all are black the answer was “no”, and if the color of the stones are mixed the response was “try again later”, students could use the tables and graphs to analyze the data as seen in Figure 12 Students can group the results together by splitting the plot in three bins: three blacks (b,b,b,) for “no”, three whites (w,w,w) for “yes” and the rest in one bin (b,b,w; b,w,b; b,w,w; w,b,b; w,b,w; w,w,b) for “try again later”. The percentage of interest for each result is computed. Each of these can be collected and the history of results stored in a new table. Thus, when the simulation is run again, new percentages can be computed and collected, creating data for repeated samples. These can be used to develop an empirical sampling distribution for the collected measures.

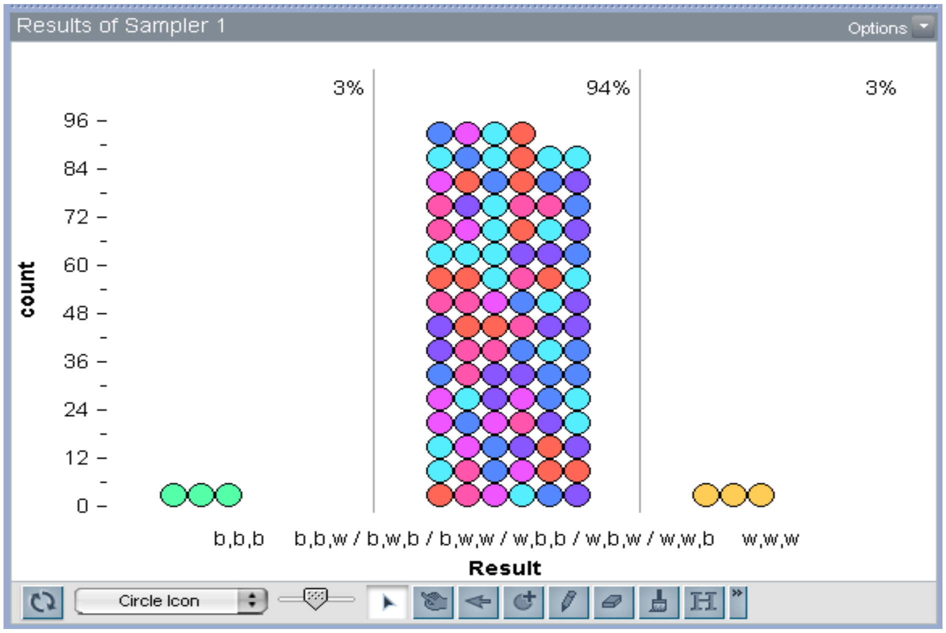

Figure 12: Urim & Thummim 4

While students are guided to use the tables and graphs based on a single “100 repeat” experiment, they would be able to compare the theoretical probability based on the problem and empirical outcomes. For example, using a question such as *“What do you find when comparing the calculated probability and the outcome you get by observing the simulation?”* would be helpful for that comparison. *(AD-ST-1; AD-ST-2)* After running the experiment several times, the subtle differences between percentages of the responses could be pointed out. The Sampler provides the opportunity of measuring the collection where students could see how the graphs based on several experiments’ results are collected for “yes”, “no”, and “try again” responses’ percentages as seen in Figures 13, 14 and 15. Students need to be well informed that each circle on those graphs represent a 100 repeat experiment, and the graph is constructed as a history of results in which 100 repeat experiments are collected. For “yes” and “no” graphs (Figure 13 and Figure 14) show a great similarity by having a right skewed distribution shape where the mean clump is around 5%. On the other hand, Figure 9 shows a left skewed shape where the mean clump is around 90%.

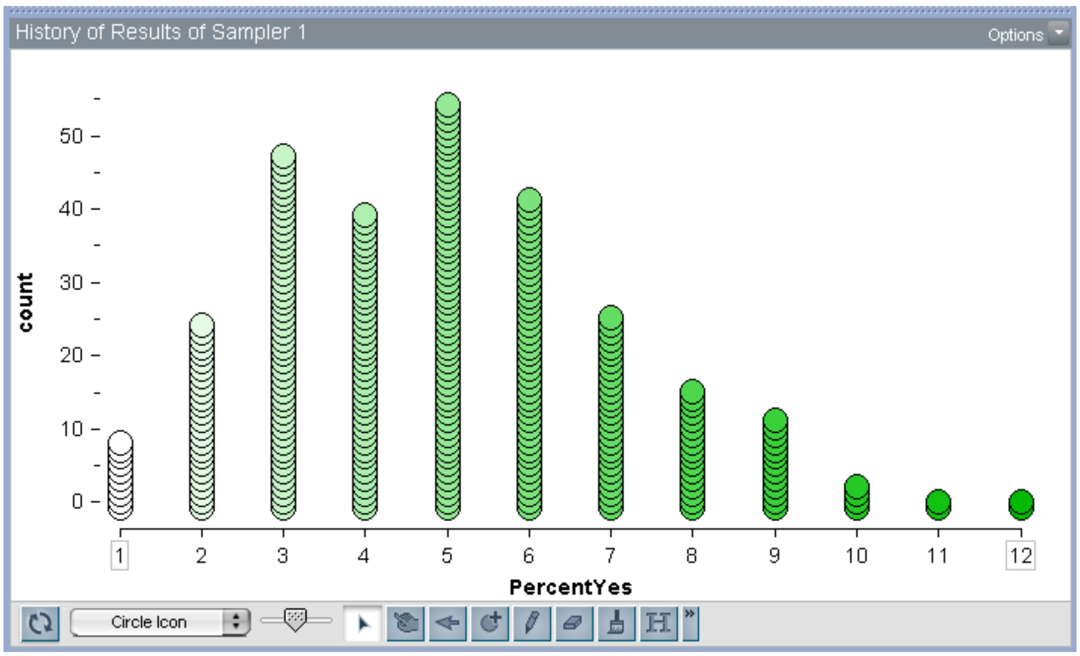

Figure 13: Urim & Thummim 5

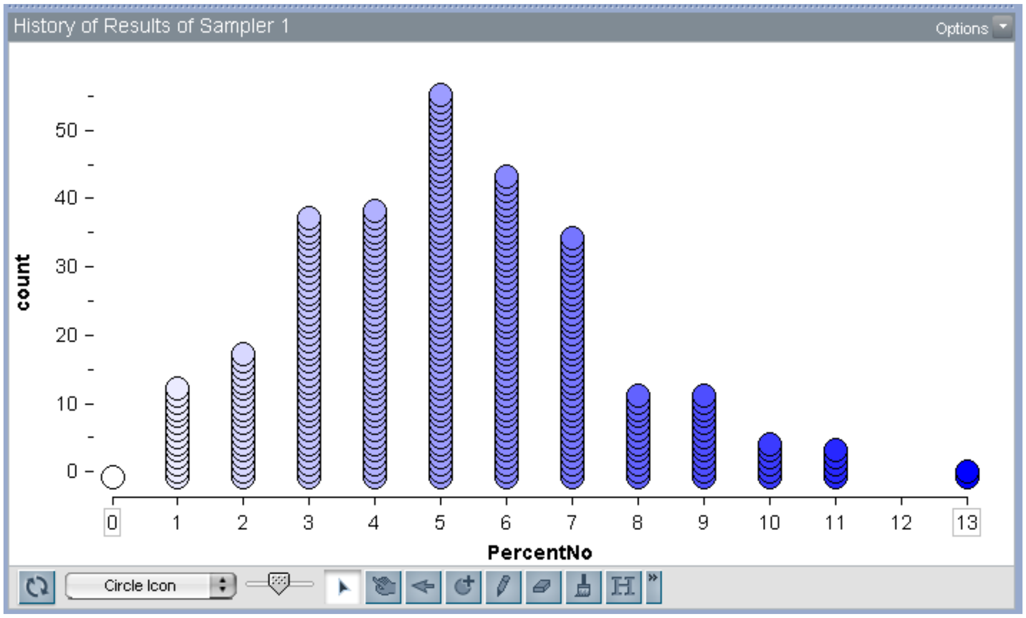

Figure 14: Urim & Thummim 6

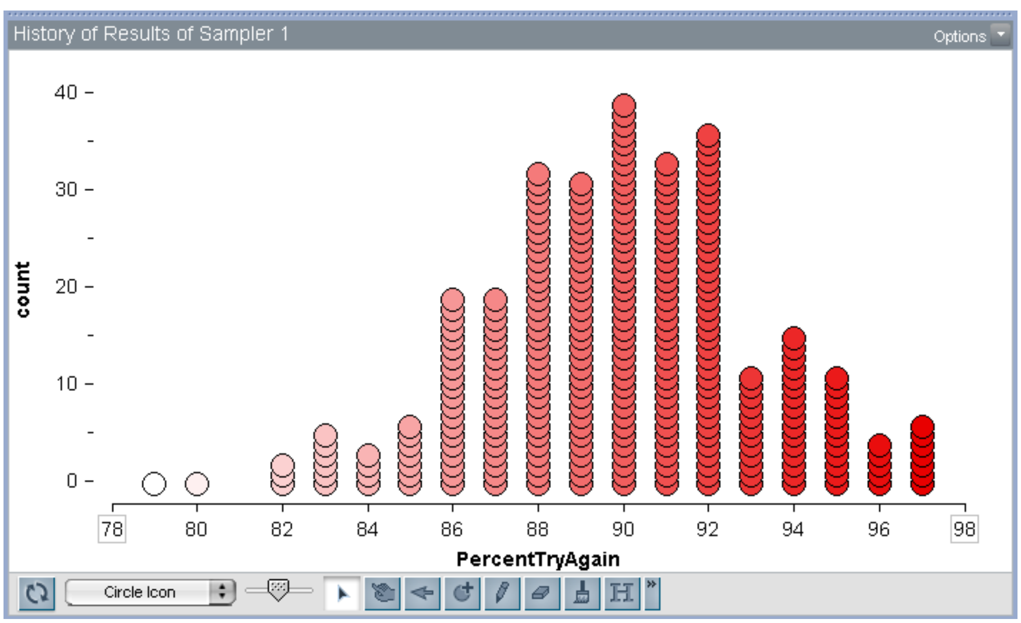

Figure 15: Urim & Thummim 7

As seen in Figure 16, all the steps explained above could be taken in a single TinkerPlots sheet, which would be a helpful feature for students to actually see the whole process. The opportunity of seeing different elements in the simulation and display of results can serve as bridging tools between empirical and theoretical probability. *(AD-ST-2)* For example "PercentYes" graph in Figure 16 reflects the fact that empirical probability is distributed around a theoretically computed probability: as 100-repeat experiments are collected, the empirical sampling distribution of "PercentYes" become more consistent with a clump of data on the theoretical mean for a yes answer (having three white stones in a row, without replacement) which is 3/6*2/5*1/4 = 1/20 = 5%. I would not expect students to compute this, instead my expectation is having them observe and interpret the simulation. *(AD-ST-2; IR-ST-1)*

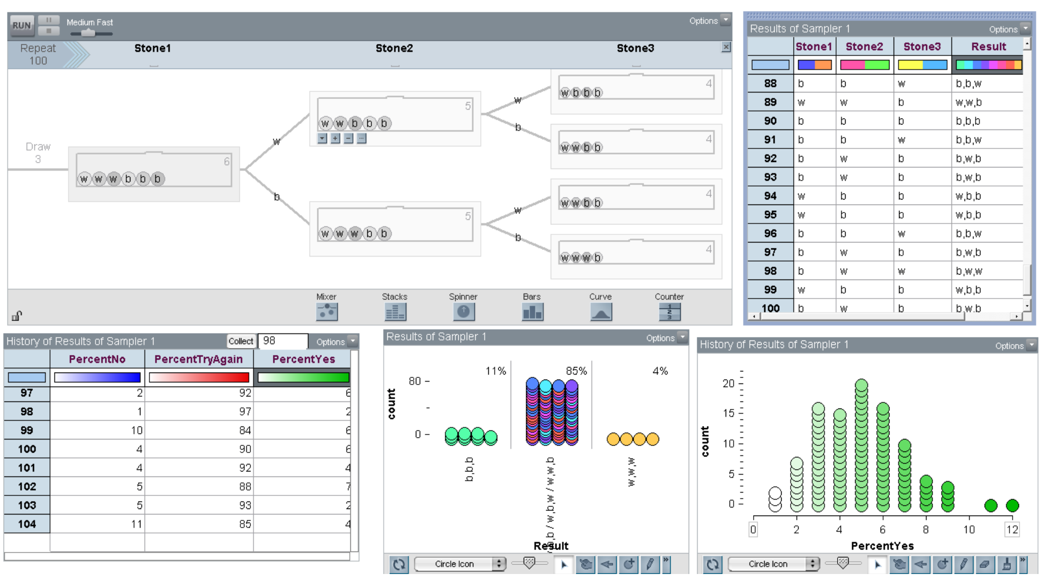

Figure 16: Urim & Thummim 8

Urim & Thummim task is intended to demonstrate the role of dependent and independent events, sample size, without replacement conditions in investigating conditional probabilities. As seen paragraphs above, the model is useful as a bridging tool between theoretical and empirical probability. Since the story content is about *decisions*, it could be considered as a good example in order to show the role of conditional probability in real world decision-making problems.

One can ask why I use a pre-built model in this task. The reason is the *TinkerPlots Sampler* is not commonly used in schools yet. Thus I think that teachers need to know how the *TinkerPlots Sampler* can be used as a modeling tool; therefore illustrating how the model is built step by step would be useful for educators.

Task2: Spinodice

The purpose of this second task is to represent a relatively more complex model and problems about conditional probability by using the *TinkerPlots Sampler*. By using well-known features of game of chances, the game *Spinodice* was tailored for this research. We need to have a spinner which is colored with equal sized red, yellow and blue parts; a single unloaded die and six boxes that have one of these prizes written: $0, $200, $300, $400, and $500. Suppose that a contestant needs to pay $100 for playing the game. As it will be explained in the game rules, *Spinodice* includes dependent and independent events, in each step *(PQ-TQ-1; PQ-ST-1)*.

More than having conditional relation between two steps, here the conditions have a continuing effect in the process. The game has a complex nature, because there are too many dependencies. Thus it will be easier to consider the probabilities of earning money in this game by building models and conducting simulations *(AD-ST-1)*.

Using a game of chance could be seen in a negative way. However, as probability researchers know, desire of winning from a game of chance (in other words, *gambling*) was an inspiring factor for the probability theory, as it will be explained in Chapter 4 again. Even though gambling is not a content we would like to use in classroom activities, it is really useful for explaining the conditional probability misconceptions. As mentioned in this thesis before, one of the well-known misconceptions about conditional probability is called ‘the gamblers fallacy’ *(PQ-ST-1)*. If it was an empirical study, this discussion might have had more room; however, the primary goal of our tasks is representing the effectiveness of modeling probabilities by using simulation tools; and at least for this study, the implementation is neglected for now.

Spinodice game rules:

The game has the conditions as explained in the following flyer:

We have a game!

There are three steps to win the big price in SPINODICE ! We have a spinner equally colored with blue, yellow and red at 1st step; then, an unloaded six faces dice at 2nd step; and finally, 5 closed boxes with some prices at 3rd step.

To play the game, as a contender you must pay $100.

1st step: You start by spinning the spinner.

a) If the result is "Red", game is over! You lost!

b) If it is blue or yellow you are going to roll a dice.

2nd step: You roll a dice.

Here you have two conditions:

(i) If the color of spinner was blue then roll the dice.

-If the result is "smaller than 6" for the dice, game is over, you lost!

-If the result is "6", you are at final step and you choose a prize box!

(ii) If the color is yellow, then roll the dice.

-If the result is equal or greater than four, then you are at final step for prize boxes; otherwise you lost!

3rd step: Choose one of five boxes.

Design of the task and tool

First, a hands-on activity can be conducted with a premade spinner, a dice and boxes in the classroom. Students spin the spinner, roll the dice, and conditionally to first two steps, open the prize boxes to observe the probability of earning money in the game *(CD-ST-1)*.

After hands-on activities, *TinkerPlots Sampler* is ready to use for modeling the game. We have two options here; either students model the game using the *TinkerPlots Sampler*, or we present a pre-built model by showing the steps and only ask the questions. I believe that choosing the first option where students build their own models is a better option, and another scenario can be explained later. However, since the tool (*TinkerPlots Sampler)* has not been used broadly, teachers have not been practicing the tool enough. Thus, I am choosing to have the scenario where teacher present each step of modeling.

Since this task has three different steps (e.g. spinning a spinner, rolling a dice, opening a box), we are able to use different representations for each step in the sampler. For example, instead of using a spinner at the 1st step we could use a mixer with three marbles in it. In this case, spinner would be chosen as a step for faithfully modeling the real game. Spinner and dice steps are supposed to be visible a contestant plays the game; however, it is important to inform students that the prize boxes step is played by seeing the boxes as sacred, hence the prizes are hidden. *(PQ-ST-2; PQ-TQ-3)*

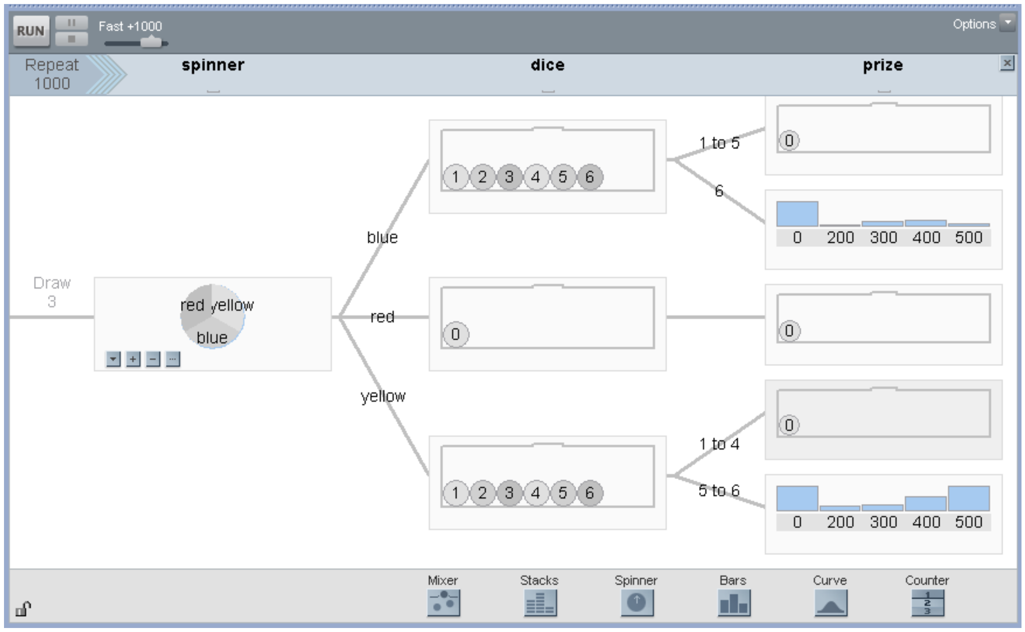

Figure 17: Spinodice 1

Task Questions:

Based on your own data you had from your experiment/experiments, use appropriate graphs and tables to answer the questions below (AD-ST-1; CD-TQ-1; PQ-ST-1):

***1)** Suppose that you have $100 and you want to buy a bicycle that is priced at $200. In order to play the game you need to pay $100.*

- Would you play this game? Why or why not?

***2)** Suppose that you landed on yellow at spinner step and you are about to roll the dice. What if another offer is made to you instead of rolling a dice?*

Offer: "Instead of rolling a dice, you could have only a "coin tossing" step and if you get head you will go to final step for prizes, if you get tail you will stop!"

Would you change your answer? Would you play this game now? Why? (Students are given another copy of the sampler to be able to alter the models in the sampler)

As the simulation runs, the results could be collected in a table and graph as shown in Figure 18 *(CD-ST-2; AD-ST-1)*.

Results of Sampler 1 Options

	PaidMo...	Join	Repeat...	RunIn...	spinner	dice	prize	EarnedMoney	Combination
701	100	yellow,4,0	701	55	yellow	4	0	no	0,4,yellow
702	100	blue,6,0	702	55	blue	6	0	no	0,6,blue
703	100	yellow,1,0	703	55	yellow	1	0	no	0,1,yellow
704	100	yellow,2,0	704	55	yellow	2	0	no	0,2,yellow
705	100	red,0,0	705	55	red	0	0	no	0,0,red
706	100	yellow,5,0	706	55	yellow	5	0	no	0,5,yellow
707	100	yellow,6,300	707	55	yellow	6	300	yes	6,300,yellow
708	100	blue,3,0	708	55	blue	3	0	no	0,3,blue
709	100	red,0,0	709	55	red	0	0	no	0,0,red
710	100	red,0,0	710	55	red	0	0	no	0,0,red
711	100	yellow,3,0	711	55	yellow	3	0	no	0,3,yellow
712	100	yellow,3,0	712	55	yellow	3	0	no	0,3,yellow
713	100	yellow,4,0	713	55	yellow	4	0	no	0,4,yellow
714	100	yellow,6,500	714	55	yellow	6	500	yes	6,500,yellow
715	100	blue,3,0	715	55	blue	3	0	no	0,3,blue
716	100	blue,4,0	716	55	blue	4	0	no	0,4,blue

Figure 18: Spinodice 2

In Figure 18, the column 'EarnedMoney' is constructed by creating a formula within the sampler for that column. That formula was as follows:

"if 'prize' > 0, then 'yes'; otherwise ' ".

Figure 19 shows how different bins represent the results.

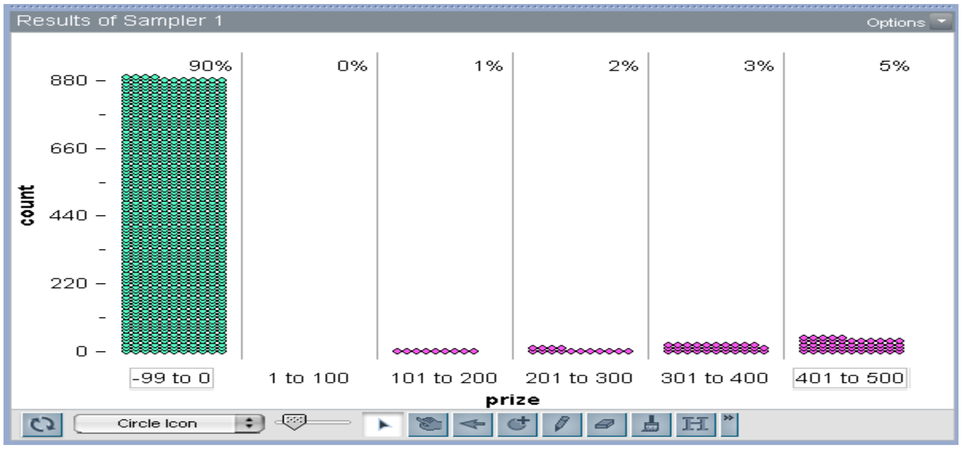

Figure 19: Spinodice 3

As seen in Figure 19, based on one simulation we run 8% (3+5=8) of the time player would win over \$300 which means the player gets at least \$200, and thus able to buy a bike. However, students need to be well informed that this simulation can be run for another 1000 trials and continue on doing these to observe whether the results change or not. Different graphs and tables can be used in order to solve the problem. *(IR-ST-1; AD-ST-1; AD-ST2)*

We can also group the bins where the player wins over \$300 in one bin. Figure 20 shows the percentage graph of 'yes' and 'no' results for EarnedMoney column. This way, students will be more comfortable in making assumptions on their decisions for first two questions about the fairness of the game. Because one of the first possible responses to the fairness question may be *"How often are we likely to win money if we repeat playing the game?"* The graph helps students in this. *(PQ-TQ-2; PQ-TQ-3; AD-ST-1)*

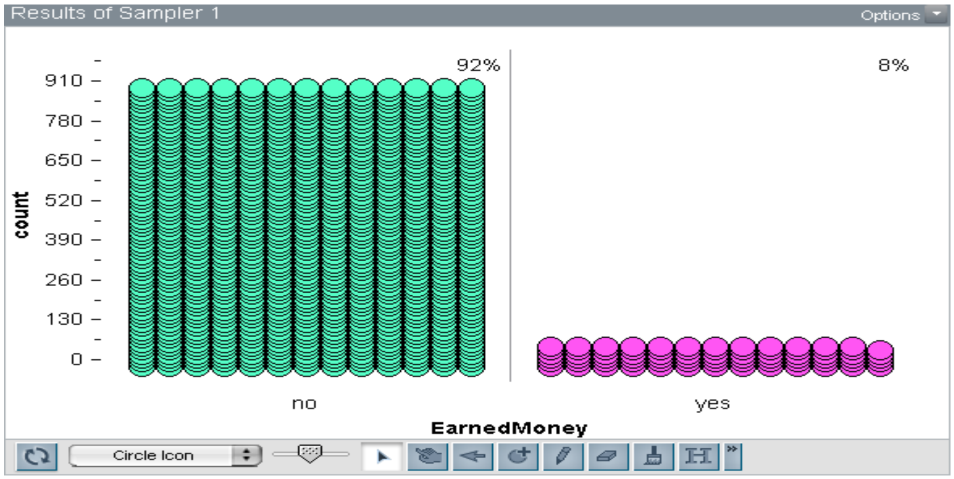

Figure 20: Spinodice 4

Then, the whole experiment with a *single trial of 1000 repeats* is shown below:

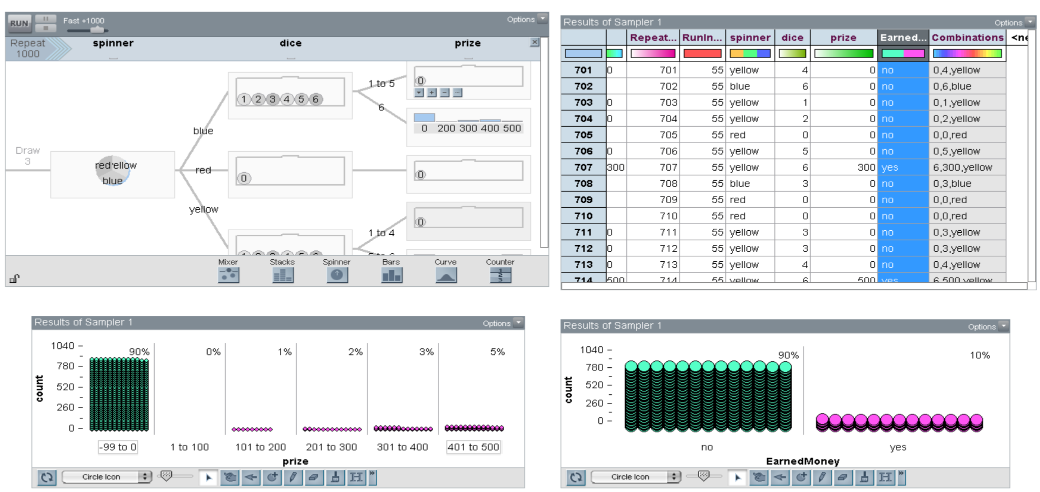

Figure 21: Spinodice 5

Even though there are 1000 repeats in above experiment, having only one trial (no matter how many repeat numbers we have) would limit us from interpreting the problem more appropriately. Hence, instead of using one single large trial, we could use *the collection of history of results* feature of the *TinkerPlots Sampler*, as seen below:

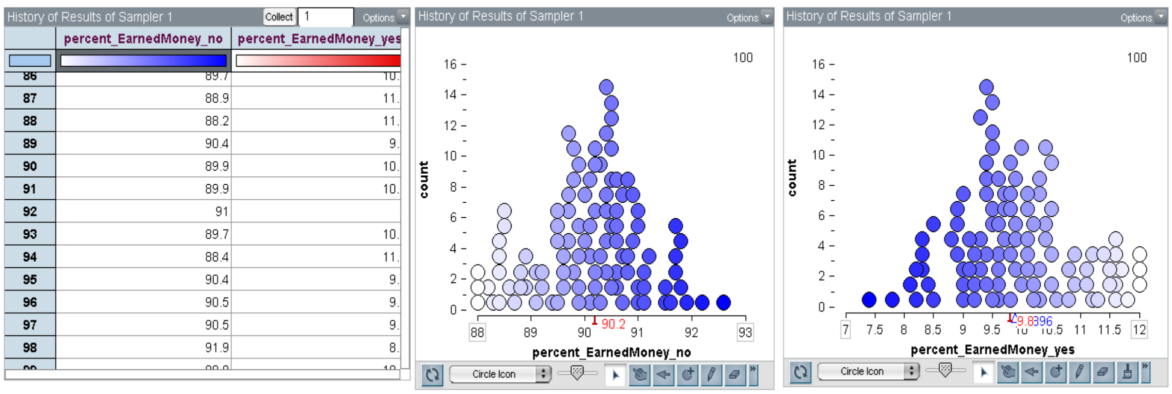

Figure 22: Spinodice 6

Using *History of Results of Sampler*, and *Collection Graph of History of Results*, students will be able to see different outcomes of trials, the overall graph of repeated trials. “percent_EarnedMoney_no” graph in Figure 22 reflects the fact that empirical probability is distributed around a theoretically computed probability: as 1000-repeat experiments are collected, the empirical sampling distribution of “percent_EarnedMoney_no” becomes more consistent with a clump of data near the theoretical mean for ‘not earned money’ answer. Similar to Task 1, my expectation is having students observe and interpret the simulation. *(AD-ST-1; AD-ST-2; IR-ST-1)* Figure 23 shows the whole screen when we decide to use collection of history of results feature:

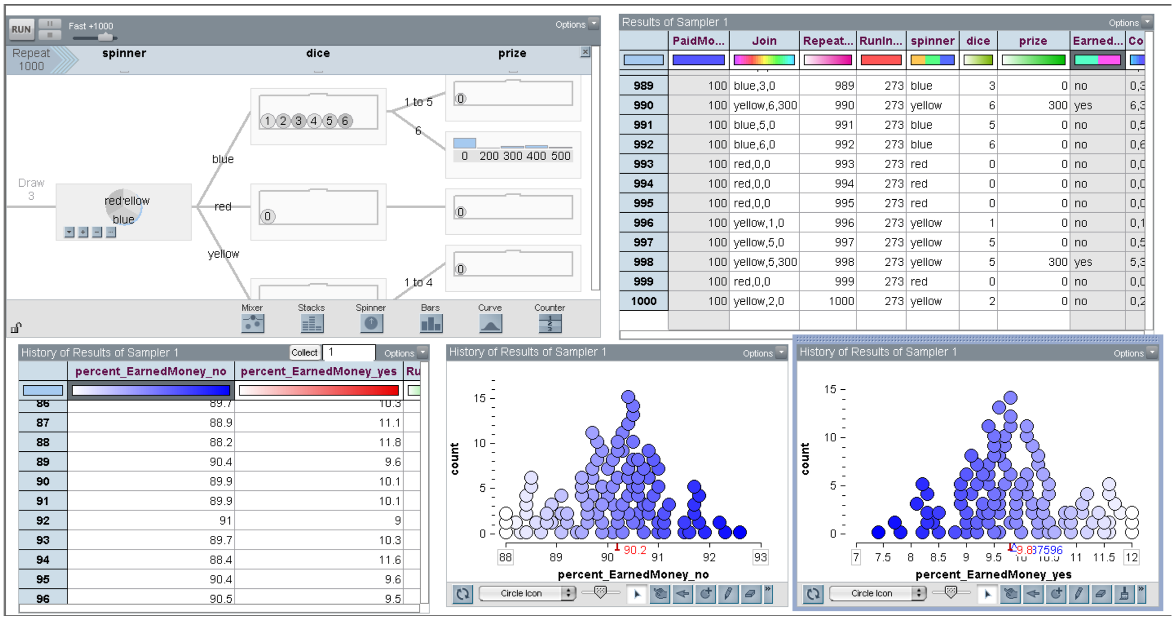

Figure 23: Spinodice 7

Two *TinkerPlots Sampler* tasks presented in Chapter 3; *Urim & Thummim* and *The Spinodice* are intended to show how TinkerPlots and the framework that is presented in Chapter 3 could be useful to develop conditional probability tasks. I believe that the framework and the simulation tool are fruitful to create lots of new useful task examples for overcoming the issues in teaching and learning of conditional probability.

CHAPTER 4

DISCUSSION AND FUTURE DIRECTIONS

In this section, there will be a brief discussion about the current status of statistics education, the framework used in this thesis, and two tasks constructed upon that framework. Also a discussion for controversial contents of the tasks, which were religious story content for the first task, and a game of chance for the second task will be in this section. Some of the limitations about the thesis will be discussed as well. And at the end, the future research about conditional probability will also be discussed.

Statistics and Mathematics: Divided or Together?

Tukey (1962) notes that "statistics is a science and no more a branch of mathematics than are physics, chemistry and economics; for if its methods fail the test of experience-not the test of logic- they are discarded" (pp. 5-6).

In the decades since Tukey's remarks, research about statistics and probability has increased enormously. In the beginning of 21st century, statistics education became more popular (Franklin et al., 2007). Since then there has been a debate for having a separate graduate program for statistics education in universities (Garfield et al., 2009). It was also important for educators to use statistics and probability in physical science and social science research. By the end of the twentieth century almost all of science had shifted to using statistical models (Salsburg, 2001). As technological tools develop the usefulness of statistical methods is increasing. The statistical applications such as exploratory data analysis

(EDA) and probability modeling are being more convenient to use with the help of technological development.

Constructing the Framework: Strengths and Limitations

First of all, a specific tool (*TinkerPlots)* heavily influences the framework introduced in this thesis. Since different tools have different features, adopting the framework for using it with another tool could require changes to the framework. It is highly recommended to carefully revise the framework in case of attempting to use it with another tool.

The framework introduced in this thesis has been constructed by using four phases of statistical investigations (*pose a question, collect data, analyze data,* and *interpret results)* and two groups of characteristics (*Task Question* and *Simulation Tool* characteristics*)*. The framework is available to use in empirical studies and is open to be developed by others. Using the framework and the tasks presented in this research were explained in Chapter 3. However, other researchers or educators could create their own tasks by using the framework's elements. Those other tasks can be conducted by allowing students to construct their own models instead of showing them the pre-built models. Pre-built models were to illustrate how possible models of conditional probability tasks could be consistent with the framework. In future empirical studies, I would not give pre-built models to students and instead let them build their own.

Do these tasks or other tasks constructed by using the framework presented in this research actually work? How can we understand if the tasks make it easier to learn conditional probability? How could one measure if students understand conditional probability better after working on the tasks built by using the framework presented here? These are important questions to answer. Because of the time constraint, I could not test my framework; instead I chose to focus on creating a wide combination of suggestions to construct a useful framework. Thus, one pitfall of this research was that I was not able to create a measuring method in order to use in possible implementations. Even though the research about conditional probability and misconceptions is relatively old, the use of computer based simulation tools is new to this field. Thus, I believe that there was a shortage of those suggestions in related literature, but I gathered those suggestions first. A further study will have to focus on that measuring method to assess whether the framework can help students better understand conditional. After all my readings on the issue, I strongly believe that that kind of research may take as much work as this thesis has taken. While this may be an exaggeration, the assessment methods used in older studies seemed problematic. For example, interview assessments were used a lot in empirical studies and as I discussed earlier in Chapter 2, that way of assessing or measuring students' understandings with post-interviews could be another reason why students continue to struggle with conditional probability. When a student answers a researcher's question without using reasoning, the assessment would be misleading. Alternatively, even though I have not deeply studied it I feel that observing students during modeling/problem solving process and building another framework for this observation process would be a better method than interviewing students after the tasks. On the other hand, the way we conduct this observation has to be studied well and as I stated, it needs a comprehensive research and a new framework. Additionally, since I plan to conduct an empirical study to test my framework and tasks later, I will focus later on the issue of assessment.

Even though all of the framework characteristics were written by commonly agreed implications for teaching conditional probability and designing simulation tools, the framework has weaknesses.

First of all, constructing a framework where it is going to be used both for a task and a simulation tool has been difficult since the idea came out. When one simulation tool is limited with the software's features, the ask content and questions can be written with limitless imagination. This could be observed in the task questions' characteristics in the framework *(PQ-TQ-1)*. I have indicated that task questions must address the misconceptions about conditional probability. However, if I choose to address "*Time Axis Fallacy*" (*Falk Paradox*) as a misconception I want students to confront, I need to ask a question where the student needs to consider a future event as given; nevertheless, it is impossible to illustrate this situation in a simulation tool.

As we know, students have individual differences in student learning and understanding. Unfortunately, the framework presented here does not clearly account for these differences. Again, for being able to do that, conducting an implementation among students seems necessary.

Another weakness of this research, possibly the most problematic part, was that the framework is not quite easy to use. Any teacher trying to use the framework in his / her classroom would need to be familiar with the literature. That could be a serious obstacle for possible implementation. This problematic situation reflects the rigor of the subject. As explained in Chapter 3, all the components of the framework were necessary and one can argue if there needs to be more. Thus, further work is needed in order to make it more comfortable for teachers to use the framework in classrooms.

Another issue was the contents I used in the tasks. One can argue about using religious story content for the first task and a game of chance (in other words *gambling*) content for the second task. I believe that I need an explanation for these selections.

Arguable Task Contents

For the first task, the idea of using the biblical *Urim&Thummim* story came out from an article I have read for the literature review of this research. *"Probability in Ancient Times; or, Shall I Go Down after Philistines?" (Milton, 1989)* suggests using the story as an example for use in conditional probability tasks; in each step of the model that story had an important detail about conditional probability. For example the sample space -which was a significant variable in investigating the cognitive issues about conditional probability - changes in each step because of drawing without replacement. Another feature of the story is allowing students to observe dependent / independent events. Moreover the story was very suitable to use in the *TinkerPlots Sampler*, and it was also easy to understand for students.

For the second task, the situation can be seen as more problematic as the schools would not agree using a gambling example in classrooms. However, my intention in using that original game of chance idea was because probability theory was discovered and developed based on the efforts of a gambler. His curiosity impressed me, as well. It is not because of an interest to gambling, but the very effective curiosity and passion for a new knowledge was impressive. In 1654, the European gambler, Chevalier de Mere posed several problems concerning gambling odds to well-known French mathematician and philosopher Blaise Pascal. Those problems initiated a series of letters between Blaise Pascal and another well-known French mathematician Pierre de Fermat. This correspondence established the basis of modern probability theory. (David, 1962) Chevalier de Mere was a clever gambler who observed a slight advantage in attempting to roll a six in four tosses of a single die and a slight disadvantage in attempting to throw two 6s in twenty-four tosses of two dice. The Chevalier de Mere noticed that the proportion of the number of tosses to the number of possible outcomes is the same for both games (4:6 for the single-die game, and 24:36 for the double-die game). Using this observation he decided that the odds of winning each game must be equal. However, from experiences with both games, he felt confident that there was an advantage for the single-die game. That leaded him to ask Blaise Pascal to explain this inconsistency. In their correspondence, Blaise Pascal and Pierre de Fermat found the basis of fundamental concepts of probability, such as sample space, probabilistic event, expected value, and binomial distribution (Maxwell, 1999). Since then, many probability researchers used gambling examples in their probability models. I think the first reason of why gambling has been used a lot in probability research was that the probability theory was came out from a gambling question. And the second and more important reason is that those games involve dependency / independency, calculating the probability of gaining, and the role of sample spaces, which are useful for probability models. An alternative content option was Genetics, which also gives the opportunity of creating tasks include dependency/independency. However, Genetics does not provide the opportunity of using without replacement situations.

On the other hand, as Konold (2007) states "Statistics is recognized as essential for good decision making in the information age of the twenty-first century, whereas mathematics has a position of strength in the school curriculum" (p. 134). I agree that the better way to help students understand probability is not asking them a "*winning of a game"* question*,* but instead asking "*making good decisions*" questions. In creating two tasks represented in this thesis, my initial goal was to create two models for explaining the role of the simulation tool in overcoming the misconceptions about conditional probability. Using the framework (Table A) constituted in this thesis, more appropriate decision making models and stories in terms of pedagogy could be used for further empirical research.

The Help of Computer Technologies

Technology gives educators a great opportunity to come up with new ideas in the efforts of overcoming the cognitive issues about conditional probability. As presented in Chapter 3, TinkerPlots has a dynamic and useful interface to create helpful tasks in order to contribute to those efforts. The way it is designed allows students to observe a conditional probability simulation as a computer game or movie. Differently from graphing calculators or other random number generators, questioning the randomization in the *TinkerPlots Sampler* is difficult; because students see the draw process or spin process as it occurs. The tool's speed and flexibility gives the opportunity for both seeing all the steps slowly and using the model for having very large samples by running the simulation in fast mode. The feature of the adding device by changing the sample size is especially useful in understanding conditional probability. As seen in *Task 1*, students are able to see how an element leaves the set and how the sample continues without that element (without replacement condition). On the other hand, in *Task 2*, the first step of Spinodice was ending when the spinner step (1st step) resulted with 'red'. However, the cells in result table were blank for the scenario when the spinner step was red. The tool does not allow us to be able to formulate blanks cells of a column in the result table. In order to include those blank cells I had to add two more empty mixers to the sampler.

As stated earlier in Chapter 3, the flexibility of having students collect one very large sample, or repeat a smaller sized trial many times and collect a statistics was important. The first approach can assist students in considering the law of large numbers, however, the second approach draws upon understanding s of sampling distributions and the central limit theorem. Both approaches show different aspects of variability, and thus both approaches are used and discussed.

It is clear that to develop such a software tool one should be familiar with the misconceptions, and suggestions for overcoming those misconceptions, about conditional probability. Focusing on each misconception separately, a better software tool could be designed. However, among several dynamic statistics software TinkerPlots was the most useful software for using in conditional probability tasks because of its sampler tool feature.

Besides having the help of technology, it is important to have teachers understand the framework and practice the simulation tool well. We can see it as the association between a maestro and his/her orchestra: If a maestro does not know the composition, then the orchestra would definitely ruin the song. Teachers have to be very well educated about conditional probability, as well as the use of the simulation tool before they use the tasks in their classroom. As Lee and Mojica (2008) indicate, “teacher education efforts need to include experiences with authentic statistical inquiry that includes use of simulation tools and modeling pedagogical practices” (p. 6).

The Future of the Research about Conditional Probability

Since the research on the role of technology for better understanding conditional probability is relatively new, there has not been enough progress yet. However, as in other fields of educational research, I believe that the enthusiasm and hard work of the researchers will be more effective to reflect benefits soon.

The research about conditional probability and simulations is moving to the direction of combining cognitive research and advanced computer technologies with mathematics education. Recently, there have been efforts on linking the mathematics education research to the research about artificial intelligence. Artificial intelligence seems like a big step in advancing the understanding of conditional probability. And, as numerous researchers have done so far, we will wonder and work more to be able to take big steps in educational research.

…

REFERENCES

Abrahamson, D. (2009). Embodied design: Constructing means for constructing meaning. *Educational Studies in Mathematics*, *70*(1), 27-47.

Abrahamson, D., & Wilensky, U. (2007). Learning axes and bridging tools in a technology-based design for statistics. *International Journal of Computers for Mathematical Learning, 12*(1), 23-55.

Bakker, A., & Hoffmann, M. H. (2005). Diagrammatic reasoning as the basis for developing concepts: A semiotic analysis of students' learning about statistical distribution. *Educational Studies in Mathematics, 60(3)*, 333-358.

Carles, M., & Huerta, M. P. (2007). Conditional probability problems and contexts. The diagnostic test context. In *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education, CERME* 5(2), 702-710.

Chance, B., Ben-Zvi, D., Garfield, J., & Medina, E. (2007). The role of technology in improving student learning of statistics. *Technology Innovations in Statistics Education, 1*(1).

David, F. N. (1962). Gods, Games and Gambling. *London: Griffin*

Falk, R. (1986). Conditional probabilities: insights and difficulties. In Proceedings of the Second International Conference on Teaching Statistics (pp. 292-297).

Fischbein, E. & Gazit, A. (1984). Does the teaching of probability improve probabilistic intuitions?: An exploratory research study. *Educational Studies in Mathematics, 15* (1), 1-24.

Franklin, C., Kader, G., Mewborn, D., Moreno, J., Peck, R., Perry, M., & Scheaffer, R. (2007). Guidelines for assessment and instruction in statistics education (GAISE) report. *Alexandria: American Statistical Association.*

Garfield, J., & Everson, M. (2009). Preparing teachers of statistics: A graduate course for future teachers. *Journal of Statistics Education*, *17*(2), 223-237.

Girotto, V., & Johnson-Laird, P. N. (2004). The probability of conditionals. *Psychologia, 47*(4), 207-222.

Huerta, M. P. (2009). On Conditional Probability Problem Solving Research-Structures and Context. *International Electronic Journal of Mathematics Education. 4*(3), 163-194.

Huerta, M. P. (2014). Researching conditional probability problem solving. In E. J. Chernoff & B. Sriraman (Eds.), *Probabilistic thinking: Presenting plural perspectives* (pp. 613-639). The Netherlands, Springer.

Ireland, S., & Watson, J. (2009). Building a connection between experimental and theoretical aspects of probability. *International Electronic Journal of Mathematics Education, 4*(3).

Johnson-Laird, P. N., & Byrne, R. J. (2002). Conditionals: A theory of meaning, pragmatics, and inference. *Psychological Review, 109*(4), 646-678.

Kelly, I. W., & Zwiers, F. W. (1986). Mutually exclusive and independence: Unravelling basic misconceptions in probability theory. *Teaching Statistics, 8,* 96-100.

Konold, C. (1996). Representing probabilities with pipe diagrams. *Mathematics Teacher, 89*(5), 378-382.

Konold, C. (2007). Designing a data analysis tool for learners. In *Thinking with data: The 33rd annual Carnegie Symposium on cognition* (pp. 267-291).

Konold, C., Harradine, A., & Kazak, S. (2007). Understanding distributions by modeling them. *International Journal of Computers for Mathematical Learning, 12*(3), 217-230.

Konold, C., & Kazak, S. (2008). Reconnecting data and chance. *Technology Innovations in Statistics Education, 2*(1).

Kulm, G., Frank, M., Campbell, P. F., & Talsma, G. (1981, April). Analysis and synthesis of mathematics problem solving processes. In *Document Besune SE 033 673 Higgins, Jon L., Ed. Research Reporting Sections, Annual Meeting of the National Council of Teachers of Mathematics (59th)* (p.25).

Lee, H. S., & Hollebrands, K. F. (2005). Students' use of technological features while solving a mathematics problem. *The Journal of Mathematical Behavior, 25*(3), 252-266.

Lee, H. S., & Hollebrands, K. F. (2008). Preparing to teach data analysis and probability with technology. *Joint ICMI/IASE study: Teaching statistics in school mathematics. Challenges for teaching and teacher education. Proceedings of the ICMI Study, 18.*

Lee, H. S., & Mojica, G. F. (2008). Examining how teachers' practices support statistical investigations. In *Proceedings of the icmi Study 18 Conference and iase 2008 Round Table Conference, Joint icmi/iase Study, Monterrey, México*..

Lee, H. S., & Lee, J. T. (2009). Reasoning about probabilistic phenomena: Lessons learned and applied in software design. *Technology Innovations in Statistics Education, 3(*2)

Lee, H. S., & Hollebrands, K. F. (2011). Characterizing and developing teachers' knowledge for teaching statistics with technology. In C. Batanero, G. Burrill & C. Reading (Eds.), *Teaching statistics in school mathematics-Challenges for teaching and teacher education* (pp. 359-369). Dordrecht, The Netherlands: Springer .

Maxwell, M. (1999). A mathematical perspective on gambling. *MIT Undergraduate Journal of Mathematics*, *1*, 123-132.

Milton, J. S. (1989). Probability in ancient times; or, shall I go down after the Philistines? *The Mathematics Teacher*, 211-213.

Oaksford, M., Chater, N., & Larkin, J. (2000). Probabilities and polarity biases in conditional inference. *Journal of Experimental Psychology: Learning, Memory, And Cognition, 26*(4), 883-899.

Oaksford, M., & Chater, N. (2003). Conditional probability and the cognitive science of conditional reasoning. *Mind & Language*, *18*(4), 359-379.

Over, D. E., & Evans, J. S. B. (2003). The probability of conditionals: The psychological evidence. *Mind & Language*, *18*(4), 340-358.

Piaget, J., & Inhelder, B. (1956). The Child's Conception of Space. *London: Routledgeand Kegan Paul. (Original work published 1948)*.

Pollatsek, A., Well, A. D., Konold, C., Hardiman, P., & Cobb, G. (1987). Understanding conditional probabilities. *Organizational Behavior and Human Decision Processes, 40*(2), 255-269.

Pratt, D. (2011). Re-connecting probability and reasoning about data in secondary school teaching. In *Proceedings of the 58th World Statistics Conference, Dublin. Retrieved on September 17th*.

Pratt, D., Jones, I., & Prodromou, T. (July, 2006). An elaboration of the design construct of phenomenalisation. In *seventh international conference on teaching statistics,* Salvador, Bahia, Brazil.

Pratt, D. (2000). Making sense of the total of two dice. *Journal for Research in Mathematics Education, 31*(5), 602-625.

Prodromou, T. (2012). Connecting experimental probability and theoretical probability. *ZDM, 44*(7), 855-868.

Prodromou, T., & Pratt, D. (2006). The role of causality in the co-ordination of two perspectives on distribution within a virtual simulation. *Statistics Education Research Journal, 5*(2), 69-88.

Reaburn, R. L. (2013). Students' understanding of conditional probability on entering university. In *36th annual conference of the Mathematics Education Research Group of Australasia (MERGA 36)* (pp. 570-577).

Rossman, A. J., Short, T. H. (1995). Conditional probability and education reform: Are they compatible. *Journal of Statistics Education, 3*(2).

Rossman, A., Chance, B., & Medina, E. (2006). Some important comparisons between statistics and mathematics, and why teachers should care. In G. F. Burrill (Ed.), *Thinking and reasoning with data and chance*: *Sixty-eighth NCTM year book* (pp. 323-334). Reston, VA: National Council of Teachers of Mathematics.

Rubin, A. (2007). Much has changed; little has changed: Revisiting the role of technology in statistics education 1992-2007. *Technology Innovations in Statistics Education, 1*(1).

Shaffer, D. W., & Kaput, J. J. (1998). Mathematics and virtual culture: An evolutionary perspective on technology and mathematics education. *Educational Studies in Mathematics, 37*(2), 97-119.

Scheaffer, R. L. (2006). Statistics and mathematics: On making a happy marriage. In G. F. Burrill (Ed.), *Thinking and reasoning with data and chance*: *Sixty-eighth NCTM year book* (pp. 309-321). Reston, VA: National Council of Teachers of Mathematics.

Stohl, H., & Tarr, J. E. (2002). Developing notions of inference using probability simulation tools. *The Journal of Mathematical Behavior, 21*(3), 319-337.

Stohl, H. (2005). Probability in teacher education and development. In Exploring Probability in School: Challenges for Teaching and Learning, Ed. G.A. Jones, pp. 345–366. New York: Springer.

Tarr, J. E., Jones, G. A. (1997). A framework for assessing middle school students' thinking in conditional probability and independence, *Mathematics Education Research Journal, 9*(1), 39-59.

Tarr, J. E., Lee, H. S., & Rider, R. (2006). When data and chance collide: Drawing inferences from empirical data. *Thinking and reasoning with data and chance,* 139-149.

Tukey, J. W. (1962). The future of data analysis. *The Annals of Mathematical Statistics*, 1-67.

Underwood, J. S., Hoadley, C., Lee, H. S., Hollebrands, K., DiGiano, C., & Renninger, K. A. (2005). IDEA: Identifying design principles in educational applets. *Educational Technology Research and Development, 53*(2), 99-112.

Tversky, A., & Kahneman, D. (1974). Judgment under uncertainty: Heuristic and biases, *Science,* 185, 1124-1131.

Tarr, J. E., & Lannin, J. K. (2005). How can teachers build notions of conditional probability and independence?. In *Exploring Probability in School* (pp. 215-238). Springer US.

Watson, J. M., & Moritz, J. B. (2002). School students' reasoning about conjunction and conditional events. *International Journal of Mathematical Education in Science and Technology, 33*(1), 59-84.

Watson, J. M., & Donne, J. M. (2009). TinkerPlots as a research tool to explore student understanding. *Technology Innovations in Statistics Education, 3*(1), 1-35.

Watson, J. M., Kelly, B. A. (2007). The development of conditional probability reasoning, *International Journal of Mathematical Education in Science and Technology,* 38:2, 213-235, doi: 10.1080/00207390601002880

APPENDICES

Appendix A

Lesson Plan for Urim & Thummim Task

(High School Mathematics)

1-Investigating Students' Understanding of Probabilistic Concepts.

2-Investigating Misconceptions of Learning of Conditional Probability.

LEARNING OBJECTIVES

Students will be able to use TinkerPlots as a technological tool.

Students will be able to investigate a conditional probability task by using a pre-designed probability sampler mini tool.

Students will be able to see how common misconceptions such as *the gambler fallacy* and *the hot-hand belief* occurs while dealing with conditional probability task.

Students will understand the importance of dependent and independent events.

Students will be able to realize the importance of probability knowledge in real world decision-making processes.

ESSENTIAL QUESTIONS:

What is a random, dependent, independent event?

How do we compute conditional probability problems?

How does probabilistic knowledge affect our everyday life?

Common Core State Mathematics Standards:

- *CCSS.Math.Content.HSS-CP.A.2 Understand that two events A and B are independent if the probability of A and B occurring together is the product of their probabilities, and use this characterization to determine if they are independent.*
- *CCSS.Math.Content.HSS-CP.A.5 Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations.*

Common Core State Mathematical Practice Standards:

- Make sense of problems and persevere in solving them.
- Use appropriate tools strategically.
- Attend to precision.
- Look for and make use of structure.

MATERIALS:

Computer with Tinkerplots software. Pencil and paper. A bag, three white and three black stones to use for Urim & Thummim hands-on experiment.

Notes to the reader:

Students have completed some units in high school probability in mathematics class and they know how to use Tinkerplots for exploratory data analysis.

Time: 90 minutes

Table 2. Lesson 1

Teacher Actions	Student Engagement
PROBLEM POSING - Say, “So far, we learned how to compute probability problems and made practices and we just saw the basic features of conditional probability.” - Ask students if they can explain what conditional probability means and give some examples. - Say, “Today, we will try to investigate a task by using a dynamic software Tinkerplots. The name of the task Urim and Thummim!” - Tell the story of Urim and Thummim, then explain the task carefully. *The Story of the Content: An article in The Mathematics Teacher journal, “Probability in Ancient Times; or Shall I Go Down After The Philistines?” (Milton, 1989) has given a very interesting historical example about the use of probability (without using the term probability) in ancient times. The author has linked her research to teaching probability by saying that nontraditional nature seldom fails to arouse the curiosity of students. Hence, finding interesting stories related to probability and telling them to students may work in teaching. The author mentions a story in Old Testament (actually similar stories are available in several other resources too.). The story is called Urim and Thummim. The priests had a communication with God in which they put three white and three black stones in a sacred container, when they ask ‘yes-no’ response type questions to The God, they randomly draw three stones without replacement from the container. If they draw all three white stones, that means God’s response was a “Yes, do it!”; if three stones were all black, that means The God’s response was a “No, do not do it!”; if the colors of stones were mixed, that meant that The God did not give a response to the question on that day, therefore the priest could repeat the same experiment another day. I believe that using such story could be both helpful and interesting for students.*	- Students respond the teacher’s question about conditional probability and try to give appropriate examples of dependent and independent events.

Table 2. Continued

	- Ask students how they could model the experiment they read in the story. Ask them to draw three stones without replacement and similar to the story, let them pretend like they make some real-life decisions based on the results. - Get them open the Tinkerplots file and first you run the simulation. Make sure that all students understand how the tool works. - Pose the sample set of questions: *1-If you draw two white stones in the first two draws, what is the probability of a getting a "do" result after the third draw?* *2- If it is given that you draw a black stone first, what is the probability of getting a "do not" result by drawing two more stones?* *3-Suppose that you conducted "Urim & Thummim" for 50 days by willing to get a "Do" response, but did not get a "Do" or "Do not" response yet. At that point you are offered to choose one of these options: - You can keep trying 50 days more and you are belonged to Urim & Thummim result. No response after all trials will mean "Do not". Or; You draw a dice; if you get "3" you will get "Do" response, otherwise the response is "Do not." Which option is you going to choose and why?* *4-Consider that it is known you will get a white stone in the second draw. How likely you get a black stone at the first draw?*	- Students read the handout and discuss the story. - Students use the bags and white/black stones to conduct the experiment in the story. - Students will open Tinkerplots in their computers and play around with the tool. They will run the simulation a few times and try to understand the tool well.
	SMALL GROUP DISCUSSIONS - As students are working in small groups, encourage students to run the experiment several times, compare their results and try to discuss the results they had. - Encourage students to talk about the dependent and independent events.	Students try to compute the probability for the 1st and 2nd questions and answer the questions. They find the 3rd and 4th questions more challenging, because instead of using the mathematical formulas they know, for those they need interpreting the scenarios.

Table 2. Continued

WHOLE CLASS DISCUSSION - Say, "I saw that most of you found the 1st question easy and some of you were OK with the 2nd question. However 3rd question seemed challenging for all of you." -Ask students, "Is there anyone answered all three questions?" -Discuss students' approaches to the questions. Find their misconceptions and visually show them why they are wrong.	- Students are confident about their first two answers; however most of them struggle in answering the 3rd and 4th questions.
SUMMARIZING AND EXTENDING -Briefly explain what they learned in this lesson. Ask their findings. -Make sure that all students are satisfied with the tool and they are able to use it. -Explain the role of technology in conditional probability education. Remind students how they were able to visually see the experiment running, and how they were able to work on large data. -Ask the students if they can pose their own questions based on the conclusions they had that day. Also, assign them to play around the software and construct new models using The Sampler.	- Understanding of dependent and independent events is important in solving conditional probability problems. - Simulation tools and models may help to overcome the cognitive issues in conditional probability.

APPENDIX B

Lesson Plan for Spinodice Task

(High School Mathematics)

1-Investigating Students' Understanding of Probabilistic Concepts.

2-Investigating Misconceptions of Learning of Conditional Probability.

LEARNING OBJECTIVES

Students will be able to use TinkerPlots as a technological tool.

Students will be able to investigate a conditional probability task by using a pre-designed probability sampler mini tool.

Students will be able to see how common misconceptions such as the time-axis fallacy (Falk paradox), representativeness and availability heuristics occur while dealing with conditional probability task.

Students will understand the importance of understanding the role of conditionals well, as well as dependent and independent events.

Students will be able to realize the importance of conditional probability knowledge in real world decision-making processes.

ESSENTIAL QUESTIONS:

What is a random, dependent, independent event?

How do we compute conditional probability problems?

How does probabilistic knowledge affect our everyday life particularly in situations that include conditional probability?

Common Core State Mathematics Standards:

- *CCSS.Math.Content.HSS-CP.A.2 Understand that two events A and B are independent if the probability of A and B occurring together is the product of their probabilities, and use this characterization to determine if they are independent.*
- *CCSS.Math.Content.HSS-CP.A.5 Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations.*
- CCSS.MATH.CONTENT.HSS.MD.B.5

  (+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values.

  - CCSS.MATH.CONTENT.HSS.MD.B.5.A

    Find the expected payoff for a game of chance. *For example, find the expected winnings from a state lottery ticket or a game at a fast-food restaurant.*
  - CCSS.MATH.CONTENT.HSS.MD.B.5.B

    Evaluate and compare strategies on the basis of expected values. *For example, compare a high-deductible versus a low-deductible automobile insurance policy using various, but reasonable, chances of having a minor or a major accident.*
- CCSS.MATH.CONTENT.HSS.MD.B.6

  (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).

Common Core State Mathematical Practice Standards:

- Make sense of problems and persevere in solving them.
- Use appropriate tools strategically.
- Attend to precision.
- Look for and make use of structure.

MATERIALS:

Computer with Tinkerplots software. Pencil and paper. A spinner with three equal parts in different colors; dice; six boxes for using in hands-on experiment.

Notes to the reader:

Students have completed some units in high school probability in mathematics class and they know how to use TinkerPlots for Exploratory Data Analysis.

Time: 90 minutes

Table 3. Lesson 2

Teacher Actions	Student Engagement
PROBLEM POSING - Say, "So far, we learned how to compute probability problems and made practices and we just saw the basic features of conditional probability." - Ask students if they can explain what conditional probability means and give some examples.	- Students respond the teacher's question about conditional probability and try to give appropriate examples of dependent and independent events.
- Say, "Today, we will try to investigate a task by using dynamic software Tinkerplots. The name of the task is *The Spinodice Game!*" - Explain the structure and rules of the game briefly and then give students handouts as shown below: *We have a game!* *There are three steps to win the big prize in SPINODICE ! We have a spinner equally colored with blue, yellow and red at 1st step; then, an unloaded six faced die at 2nd step; and finally, 6 closed boxes with money prizes at 3rd step. Those prizes are #0, $200, $300, $400, and $500 but students do not know which money in which box.* *To play the game, as a contender you must pay $100.* *1st step: You start by spinning the spinner.* *a) If the result is "Red", game is over! You lost!* *b) If it is blue or yellow you are going to roll a dice.*	- Students read the handout and discuss the game.
2nd step: You roll a dice. *Here you have two conditions:* *(i) If the color of spinner was blue then roll the dice.* *-If the result is "smaller than 6" for the dice, game is over, you lost!* *-If the result is "6", you are at final step for prize boxes!* *(ii)If the color is yellow, then roll the dice.* *-If the result is equal or greater than four, then you are at final step for prize boxes; otherwise you lost!* *3rd step: Choose one of six boxes (each includes one of the prize cards of $0, $100, $200, $300, $400, $500).*	- Students use the hands-on tools to play the game, and use paper and pencil to collect their data.

Table 3. Continued

- Ask students how they could model the experiment they read in the handout. - Let them use the premade spinner, a die and regular boxes (with money cards in them-$0, $100, $200, $300, $400, $500.) and conduct hands-on experiment first. - Ask them to spin the spinner and depending on the color they get roll the dice, and then depending on the result open of the sacred boxes. - Let them play with those tools and try to collect data whether they win or not. - Get them open the TinkerPlots file and first you run the simulation. Make sure that all students understand how the tool works. - Pose the sample set of questions with handouts or on the blackboard: *Based on your own data you had from your experiment/experiments, use appropriate graphs and tables to answer these:* ***1)*** *Suppose that you have $100 and you want to buy a bicycle which is priced at $200. In order to play the game you need to pay $100.* *- Would you play this game? Why or why not? Provide a convincing argument for your decision.* ***2)*** *Suppose that you landed on yellow on the spinner in step 1 and you are about to roll the die. What if another offer is made to you instead of finishing the game: "Instead of rolling the die, you could have a "coin tossing" step and if you get head you will go to final step for prize boxes, if you get tail you will stop!" Would you change your answer? Would you play this game now? Why?*	- Students will open TinkerPlots in their computers and play around with the tool. They will run the simulation a few times and try to understand the tool well.

Table 3. Continued

SMALL GROUP DISCUSSIONS - As students are working in small groups, encourage students to run the experiment several times, compare their results and try to discuss the results they had. - Encourage students to talk about the role of sample space, representativeness, the role of time.	- Students try to answer 1st and 2nd questions by observing the simulation as it runs. They find the 2nd question more challenging, because instead of using the mathematical formulas they know, for those they need interpreting the scenarios.
WHOLE CLASS DISCUSSION - Say, "I saw that most of you found the 1st question easy. However 2nd question seemed challenging for all of you." - Ask students, "Is there anyone answered both questions?" - Discuss students' approaches to the questions. Find their misconceptions and visually show them why they are wrong.	- Students are confident about their first answer; however most of them struggle in answering the 2nd question.
SUMMARIZING AND EXTENDING - Briefly explain what they learned in this lesson. Ask their findings. - Make sure that all students are satisfied with the tool and they are able to use it. - Explain the role of technology in conditional probability education. - Remind students how they were able to visually see the experiment running, and how they were able to work on large data. - Ask the students if they can pose their own questions based on the conclusions they had that day. Also, assign them to play around the software and construct new models using The Sampler.	- Understanding of conditionals, sample space, and representativeness are important in solving conditional probability problems. - Simulation tools and models may help to overcome the cognitive issues in conditional probability.