ON THE STEADY CYCLIC STRESS ANALYSIS FOR ELASTO-PLASTIC STRUCTURES SUBJECTED TO REPEATED ACTIONS OF LOAD AND TEMPERATURE

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SUMMARY

The problem of the direct (without step-by-step analysis of all the deformation history from the very beginning of the load) determination of the steady cyclic stresses and strain rates is considered for the three-dimensional elastic-perfectly plastic continuum. The mentioned problem is formulated as a maximization (or minimization) principle for corresponding functionals. Two statements have been suggested:

I) From all the kinematically admissible cycles of plastic strain rates and corresponding residual stress rates (securing the acceptability of the total stresses in any point of the body at any time-moment) the actual steady cycle is such that the total work performed in it by the difference between total and yield stresses becomes maximum.

It is shown that the well known static (Melan) and kinematic (Koiter) shakedown (non-shakedown) theoremes directly result from the formulated principle assuming that the steady behavior of the body is perfectly elastic (or, correspondingly, that the plastic strain rates differ from zero).

II) From all the initially residual stresses fields at which the total stresses are statically possible and acceptable at any time-moment and the strain rates depending upon them are compatible in the actual steady cycle are realized the ones which ensure the minimum of the total elastic energy corresponding to the difference between the residual stresses at the beginning and at the end of the cycle.

The extreme principles suggested and proved may be practically used for the exact or approximate determination of the steady cyclic stresses and strain rates due to the given forces and temperature fields.

The mentioned principles may be extended to formulate the conditions under which the incremental collapse begins when the alternating plasticity is present.

Some examples illustrating the applications of the suggested principles are given. In all the considered cases the problems are treated either as Pontryagin's maximum principle problem, or convex or linear programming.

For structures having limited operating time the work outside the shakedown locus would be admissible in some cases. Then it is essential to know what type of the cyclic plastic deformation processes (alternating plasticity, incremental collapse or their combination) are expected. The analysis and the stress calculations made by the step-by-step procedures show the convergence of cyclic deformation processes. As the mentioned procedures are related with certain difficulties the direct determination of the strain amplitudes (in the case of alternating plasticity) and of the strain increments (when their persistent accumulation is expected) at the steady-state response of the structure would be a problem of essential interest.

The general existence and uniquiness theorems on the steady stress cycles based on the known Drucker's postulate were proved first by Frederick and Armstrong [1] . But somewhat earlier the problem of direct steady stress and strain rate calculation under the creep conditions was formulated and solved by Shorr [2, etc]. Then Ponter [3 etc.], Martin and Williams [4] proposed some approximate bounds for steady cycle charachteristics taking into consideration both plasticity and creep. The corresponding bounds were also suggested by Mroz [5] .

In this paper the problems of the stress and strain rate calculation and of the determination of the loading parameters according to the given kinematical features of the steady stress cycle are formulated on the basis of suggested general extremum principle. The latter differs from the known principles [6] as it covers the whole deformation process within the framework of one extremum problem. Such approach proves to be possible while the yield surface does not depend on the deformation history (i.e. for elastic-perfectly plastic bodies).

1. Arbitrary deformation processes

Let the acting body and surface forces $X_i(\tau)$, $p_i(\tau)$ and the temperature fields $t_i(\tau)$ at the time interval $0 \leqslant \tau \leqslant \tau_*$ be given as well as the initial state of the structure defined by the residual stress ρ_{ij}^0 (at $\tau = 0$)

$$\rho^0_{ij,j} = 0, \quad \rho^0_{ij} n_j = 0 \tag{1}$$

Then if the actual stress (consisting of elastic and self-equilibrated parts)

$$\sigma^{(e)}_{ij} + \rho^0_{ij} + \int_0^{\tau} \dot{\rho}_{ij}\, d\zeta \tag{2}$$

and the strain rate distributions at any time-moment have to be found they must satisfy the following conditions [6] :

a) The equilibrium equations -

$$\dot{\rho}_{ij} = 0, \quad \dot{\rho}_{ij} n_j = 0 \tag{3}$$

where $\dot{\rho}_{ij}$ is the residual stress rate.

b) The compatibility equations for residual strain rates including the plastic $\dot{\varepsilon}''_{ij}$ and elastic $A_{ijhk}\dot{\rho}_{hk}$ components -

$$\dot{\varepsilon}''_{ij} + A_{ijhk}\dot{\rho}_{hk} = \frac{1}{2}(\dot{u}_{i,j} + \dot{u}_{j,i}) \tag{4}$$

where $\dot{u}_i$ denotes the residual velocities and A_{ijhk} is the elastic moduli tensor.

c) The physical equations having for ideal elastic-plastic materials the form -

$$f_\alpha(\sigma^{(e)}_{ij} + \rho^0_{ij} + \int_0^\tau \dot{\rho}_{ij}\,d\zeta) \leqslant 0, \tag{5}$$

$$\dot{\varepsilon}''_{ij} = \sum_\alpha \lambda_\alpha \partial f_\alpha / \partial\sigma_{ij}, \quad \lambda_\alpha \geqslant 0 \tag{6}$$

$$\lambda_\alpha f_\alpha(\sigma^{(e)}_{ij} + \rho^0_{ij} + \int_0^\tau \dot{\rho}_{ij}\,d\zeta) = 0 \tag{7}$$

$$\lambda_\alpha \dot{f}_\alpha(\sigma^{(e)}_{ij} + \rho^0_{ij} + \int_0^\tau \dot{\rho}_{ij}\,d\zeta) = 0 \tag{8}$$

where
$$f_\alpha(\sigma_{ij}) = 0 \qquad (\alpha = 1, 2, \ldots, n) \tag{9}$$

denotes a set of regular functions defining the yield surface, λ_α - the nonnegative multiplier differing from zero only if the stresses reach the yield surface and unloading does not occur, $\sigma^{(e)}_{ij}$ - elastic stress supposed to be a known function of the loading parameters. For the further analysis it is convenient to reduce the problem to the variational one replacing the condition (7) by a specially designed functional to be minimized. Then the following extremum principle can be formulated.

1.1. In an actual deformation process defined by the given initial state and loading history such plastic strain rate fields are to be realized which satisfy conditions (3) - (5) and minimize the specific functional

$$J = \int_0^{\tau_*} d\tau \int [\sigma_{ij} - (\sigma^{(e)}_{ij} + \rho^0_{ij} + \int_0^\tau \dot{\rho}_{ij}\,d\zeta)]\dot{\varepsilon}''_{ij}\,dV \tag{10}$$

where σ_{ij} denotes the yield stresses associated with the strain rates by eq. (6).

If the displacement field includes discontinuities the righthand side of eq. (10) must be completed with the term

$$\sum_\mu \int_0^{\tau_*} d\tau \int_{S_\mu} [\sigma_{ij} - (\sigma^{(e)}_{ij} + \rho^0_{ij} + \int_0^\tau \dot{\rho}_{ij}\,d\zeta)] n_j \Delta\dot{u}_i\,dS \tag{11}$$

where $\Delta\dot{u}_i$ is the corresponding jump on the surface S_μ and n_j -the unity normal vector to the latter. For the sake of simplicity, term (11) is omitted elsewhere (except the examples below).

To prove the suggested principle one can note that according to eq.(5) the actual stresses (2) are admissible ($\sigma_{ij}^{(a)}$) and due to the Drucker's postulate

$$[\sigma_{ij} - (\sigma_{ij}^{(e)} + \rho_{ij}^{0} + \int_0^{\tau} \dot{\rho}_{ij}\, d\xi)]\dot{\varepsilon}_{ij}'' = (\sigma_{ij} - \sigma_{ij}^{(a)})\dot{\varepsilon}_{ij}'' \geqslant 0 \qquad (12)$$

where equality to zero is possible if $\sigma_{ij}^{(a)}$ and σ_{ij} (more exactly-their components essential for eq.(9)) are equal, i.e. if eq.(7) is satisfied. Hence according to nonequality (12) functional (10) is negative for all plastic strain rate fields satisfying eqs. (3) - (6) and it becomes an absolute minimum (equal to zero) only when eq. (7) holds.

The adequacy of the principle formulated as a necessary condition is now almost evident. The conditions become sufficient when in addition eq. (8) holds. But the latter being essential for exact theoretical analysis is not obligatory in practical applications when the finite strain increments are determined.

The mentioned principle permits one to obtain the necessary and sufficient conditions for the existence of any deformation process defined by the given kinematic features. For example, the principle reduces to the kinematic theorem for plastic collapse [6] assuming that the strain rates are nonconstrained at some time-moment. Similarly, assuming that the plastic strain rates are constrained at any moment one can obtain the corresponding static theorem.

2. Steady stress and strain rate cycle

Let the structure be subjected to certain forces and temperature fields changing cyclically with some period T. As the steady stress cycle is closed the following additional condition must be satisfied

$$\rho_{ij}(T) = \rho_{ij}^{0} + \int_0^{T} \dot{\rho}_{ij}\, d\tau = \rho_{ij}^{0} \qquad \left(\int_0^{T} \dot{\rho}_{ij}\, d\tau = 0 \right) \qquad (13)$$

or in the kinematic terms

$$\Delta\varepsilon_{ij}'' = \frac{1}{2}(\Delta u_{i,j} + \Delta u_{j,i}), \quad \Delta\varepsilon_{ij}'' = \int_0^{T} \dot{\varepsilon}_{ij}''\, d\tau, \quad \Delta u_i = \int_0^{T} \dot{u}_i\, d\tau \qquad (14)$$

Then principle 1.1 takes the following form.

2.1. The plastic strain rates $\dot{\varepsilon}_{ij}''$ and initial self-equilibrated stress ρ_{ij}^{0} to be realized in the actual steady cycle must satisfy eqs.(1), (3) - (6) and (13) (or equivalent eq. (14)) and minimize the functional

$$J_1 = \int_0^{T} d\tau \int (\sigma_{ij} - \sigma_{ij}^{(e)})\dot{\varepsilon}_{ij}''\, dV \qquad (15)$$

Comparing the constraints and the functionals to be minimized due to extremum principles 1.1 and 2.1 respectively one can see that the latter would be proved if the following condition holds

$$\int_0^T d\tau \int (\rho_{ij}^0 + \int_0^\tau \dot{\rho}_{ij} d\zeta)\dot{\varepsilon}_{ij}'' dv = 0 \tag{16}$$

Actually, as the self-stress state ρ_{ij}^0 does not depend on the time and the plastic strain-rate cycle is admissible due to the virtual work principle [6] we may write

$$\int_0^T d\tau \int \rho_{ij}^0 \dot{\varepsilon}_{ij}'' dv = \int \rho_{ij}^0 \Delta\varepsilon_{ij}'' dv = 0 \tag{17}$$

The second addend of the left-hand side of eq. (16) using virtual work principle and eqs. (3), (4), (13) can be transformed to the form

$$\begin{aligned} \int_0^T d\tau \int [\int_0^\tau \dot{\rho}_{ij} d\zeta]\dot{\varepsilon}_{ij}'' dv &= -\int_0^T d\tau \int A_{ijhk} [\int_0^\tau \dot{\rho}_{ij} d\zeta]\dot{\rho}_{hk} dv = \\ &= -\frac{1}{2}\int A_{ijhk}[\rho_{ij}(T)-\rho_{ij}^0][\rho_{hk}(T)-\rho_{hk}^0]dv = 0 \end{aligned} \tag{18}$$

Then eq. (16) holds and hence principle 2.1 is proved as a necessary condition. It becomes sufficient if in addition eq. (8) holds (see the corresponding remark above).

It should be noted that some other versions of the extremum principle for steady cyclic stress determination can be proposed. For example, from all fields of stresses ρ_{ij}^0 and plastic strain rates $\dot{\varepsilon}_{ij}''$ which satisfy conditions (1), (3)-(8) in the actual steady cycle are realized the ones which ensure the minimum of nonnegative functional

$$J_2 = \int_0^T d\tau \int (A_{ijhk}\dot{\rho}_{ij}\int_0^\tau \dot{\rho}_{hk} d\zeta) dv = \frac{1}{2}\int A_{ijhk}[\rho_{ij}(T)-\rho_{ij}^0][\rho_{hk}(T)-\rho_{hk}^0]dv$$

But it is practically less convenient than 2.1 due to nonlinearity of the corresponding constraints and functional.

On the basis of suggested principles 1.1, 1.2 the problems of the stress analysis (for continuous structures) can be formulated in terms of optimal control theory and solved by using the Pontriagin's maximum principle [7,8]. But in many cases the mathematical programming procedures based on descrete description of the structure are more convenient. Using the piecewise linear yield surfaces one can reduce the mentioned problems to the linear (for steady cycles when only the necessary conditions defined by principle 2.1 hold) or quadratic (for arbitrary deformation processes) programming problems.

3. Limit conditions according to the given kinematical features

Consider some special cases for principle 2.1. The simpliest case is when the plastic strain rates are equal to zero ($\dot{\varepsilon}_{ij}'' \equiv 0$) at any time-moment (the structure shakes down). Then eqs. (3), (4), (6) reduce to identities and functional (15) becomes equal to zero. According to principle 2.1 the conditions necessary for the existence of such cycle include only eq. (1) and inequality (5) which takes the form

$$f_\alpha(\sigma_{ij}^{(e)} + \rho_{ij}^{0}) \leqslant 0 \qquad (19)$$

The condition obtained would become sufficient (according to eq.(8)) if there is strict inequality. But it is not essential practically. Thus the result in fact coincides with Melan's theorem.

Similarly the kinematic inadaptation theorem established by Koiter can be inferred from principle 2.1 correspondingly assuming that the plastic strain rates unequal to zero include a common indeterminate multiplier.

A more general problem is to determine the maximum values of the loading parameters for steady stress cycle satisfying the following conditions

$$\Delta u_i^{(K)} \leqslant C^{(K)} \qquad (20)$$

where $\Delta u_i^{(K)}$ denotes the displacement increments in some points $k=1,2,\dots,n$ of the structure, $C^{(K)}$-the given values. Similarly the dual problem (to obtain the minimum values of those parameters when $\Delta u_i^{(K)} \geqslant C^{(K)}$) can be formulated. In this case the following consequence of principle 2.1.can be used.

3.1. The inequality (20) holds in the actual steady cycle if there is any strain rate field $\dot{\varepsilon}_{ij}''$ and any initial stress field ρ_{ij}^{0} simultaneously satisfying the constraints (20), (1), (3)-(6), (14) while the functional (15) attains zero value.

The problems formulated can be reduced (using some descretizing procedure) to the dual problems of convex or linear parametric programming.

A special case of essential interest is the determination of loading parameters when $\Delta u_i=0$ anywhere on the structure (it should be taken into consideration that the alternating plasticity possibly takes place). Let the variable forces and temperature fields parameters be given while the time-independent load (defined by some multiplier) is to be determined. Then the independence of the strain rates under cyclic plasticity upon the constant loads (if $\Delta u_i \equiv 0$) can be used (the corresponding proof may be omitted here). On this basis two theorems closely similar to those by Melan and Koiter (transformed to more convenient forms in [9, 10]) can be formulated. They differ from the latter only in replacing the elastic stresses $\sigma_{ij}^{(e)}$ with actual stresses $\sigma_{ij\tau}$ corresponding to the known yield limits and (only) variable forces and temperature fields.

3.2. According to the new static theorem the strain accumulation does not occur if the following inequality holds

$$\varphi(\sigma_{ij}^{0}) = \max_{\tau} f(\sigma_{ij}^{0} + \sigma_{ij\tau}) \leqslant 0 \qquad (21)$$

where σ_{ij}^{0} denotes the stresses balanced by the constant body and surface loads

$$\sigma_{ij,j}^{0} + X_i^0 = 0, \quad \sigma_{ij}^{0} n_j = p_i^0 \tag{22}$$

3.3. Respectively, due to the proposed kinematic theorem the strain accumulation (incremental collapse) will take place if the corresponding inequality is satisfied

$$\int X_i^0 \Delta u_{io} dv + \int_{S_p} p_i^0 \Delta u_{io} dS \geqslant \int \sigma_{ij*}^{0} \Delta \varepsilon_{ijo}'' dv \tag{23}$$

where σ_{ij*}^{0} is the stress reffering to the fictitious yield surface which is defined by the condition

$$\varphi(\sigma_{ij*}^{0}) = \max_{\tau} f(\sigma_{ij*}^{0} + \sigma_{ij\tau}) = 0 \tag{24}$$

(remember that the terms corresponding to the possible discontinuities of the displacement field are omitted in (23)).

The static and kinematic theorems formulated above represent the further development of the limit analysis of the elastic-perfectly plastic structures. It should be noted particularly that they form the suitable generalization of the stress superposition method appied in [9] to some simple problems of the type considered. Now all the sets of the exact and approximate methods developed for the elastic shakedown problems [9, 10] can be used to determine the general incremental collapse limit conditions due to theorems 4.2, 4.3.

4. Examples of theorems applications

Then let us consider some examples to illustrate the application of the suggested principles and theorems.

4.1. Example of steady-state cycle determination. The simple two-parametric system (Fig.1) is subjected to the constant load P and variations of the temperature (let bar 1 be cyclically heated and cooled). The plastic strain rates of the bars ($i=1,2,3$) can be presented as differences of the tension and compression rates

$$\dot{\varepsilon}_i'' = \lambda_i^{(t)} - \lambda_i^{(c)} \tag{25}$$

Then the functional (15) to be minimized takes the form

$$\int_0^T \left\{ \sum_{i=1}^{3} F_i l_i \left[(\sigma_S - \sigma_{ip} - \sigma_{it}) \lambda_i^{(t)} - (-\sigma_S - \sigma_{ip} - \sigma_{it}) \lambda_i^{(c)} \right\} d\tau \tag{26}$$

where σ_{ip}, σ_{it} denote the elastic stresses due to the force and temperature respectively, σ_S - the yield stress. In the case considered the constraints system (1), (3)-(6), (14) assumes the form

$$\sum_{i=1}^{3} \rho_i^0 F_i = 0, \qquad \sum_{i=1}^{3} \dot{\rho}_i F_i = 0 \tag{27}$$

$$(\lambda_i^{(t)} - \lambda_i^{(c)} + \frac{1}{E}\dot{\rho}_i) l_i = \underset{i}{const} \tag{28}$$

$$-\sigma_S \leqslant \sigma_{ip} + \sigma_{it} + \rho_i^0 + \int_0^{\tau} \dot{\rho}_i \, d\tau \leqslant \sigma_S \tag{29}$$

$$\lambda_i^{(t)} \geqslant 0, \qquad \lambda_i^{(c)} \geqslant 0 \tag{30}$$

$$\int_0^T l_i (\lambda_i^{(t)} - \lambda_i^{(c)}) \, d\tau = \underset{i}{const} \tag{31}$$

Thus the problem reduces to the linear programming one (the appropriate procedure is known and will not be dicussed here). By means of the symplex program one can obtain the strain amplitudes and increments corresponding to the given values of the system and loading parameters.

4.2. Example of the incremental collapse conditions determination. Let a cylindrical shell (Fig. 2a) be subjected to a time-independent ring load P and cyclic variations of the linearly distributed (across the thickness of the shell) temperature

$$t(\tau) = t_0 + \zeta t_1(\tau), \quad -t_* \leqslant t_1(\tau) \leqslant t_*, \quad \zeta = z/h, \; -1 \leqslant \zeta \leqslant 1 \tag{32}$$

The boundaries of the elastic shakedown locus for the shell were determined in [9] . The corresponding diagram ABC is presented in Fig.2b. Here the following terms are used

$$q_* = \frac{\alpha E t_*}{\sigma_S (1-\mu)}, \qquad P_0 = \frac{4\sigma_S h\sqrt{2h}}{\sqrt{R}} \tag{33}$$

The line AB corresponds to the alternating plasticity limit condition, the line BC , similarly, to the incremental collapse condition, the vertical line drown from the point C corresponds to limit equilibrium condition. Let us determine the upper bound for the parameters of the incremental collapse limit cycle at $q_* > 1$ (i.e. when alternating plasticity takes place). The actual thermal stresses and reversible plastic strains are easy to obtain due to perfect plasticity

$$\sigma_{x\tau} = \sigma_{\varphi\tau} = -\frac{\alpha E t_1(\tau)}{(1-\mu)}\zeta, \; \dot{\varepsilon}''_{x\tau} = \dot{\varepsilon}''_{\varphi\tau} = 0 \quad \text{if} \quad |\zeta| \leqslant \zeta_0 \tag{34}$$

$$\begin{aligned} \sigma_{x\tau} = \sigma_{\varphi\tau} &= -\sigma_S \, sign[\zeta t_1(\tau)], \; \int_0^{\tau} \dot{\varepsilon}''_{x\tau} dx_1 = \int_0^{\tau} \dot{\varepsilon}''_{\varphi\tau} dx_1 = \\ &= -\alpha \zeta t_1(\tau) + \frac{1-\mu}{E}\sigma_S \, sign[\zeta t_1(\tau)] \quad \text{if} \quad |\zeta| \geqslant \zeta_0 \end{aligned} \tag{35}$$

$$\zeta_0 = \frac{(1-\mu)\sigma_S}{\alpha E\,|t_1(\tau)|} = \frac{1}{q_*}\frac{t_*}{|t_1(\tau)|} \tag{36}$$

Let the adopted set of the collapse mechanisms be similar to those considered in solving a similar problem in the book [9] (see the thin lines in Fig.2a). Then the inequality (23) takes the form

$$\begin{aligned} P\Delta w_0 > 2\int_0^{L_0} dx \int_{-h}^{h} \min_{\tau}\left[(-\sigma_s - \sigma_{\varphi\tau})\left(-\frac{\Delta w_0}{R}\right)\left(1-\frac{x}{L_0}\right)\right]dz + \\ + 4\frac{\Delta w_0}{L_0}\int_{-h}^{h} \min_{\tau}\left[(\sigma_s \operatorname{sign} z - \sigma_{x\tau})z\right]dz \end{aligned} \tag{37}$$

The thermal stresses values minimizing the integrands are shown in Fig.3 by shaded lines. After simple transformations of (37) one can obtain

$$P = \frac{\sigma_S h L_0}{q_* R} + \frac{4\sigma_S h^2}{3q_*^2 L_0} \tag{38}$$

Then minimizing the load P due to the parameter L_0 results in

$$\frac{P_*}{P_0} = \frac{1}{q_*\sqrt{6q_*}} \qquad \left(L_0 = \sqrt{\frac{4Rh}{3q_*}}\right) \tag{39}$$

to which the line BD (Fig.2b) corresponds.

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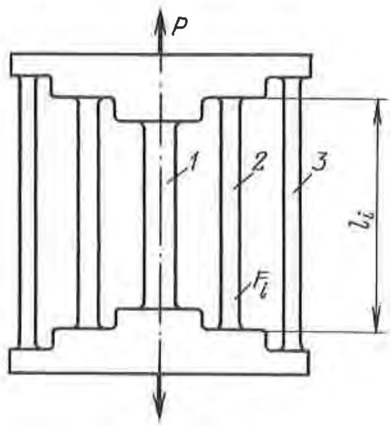

Fig. 1. Two-parametric system

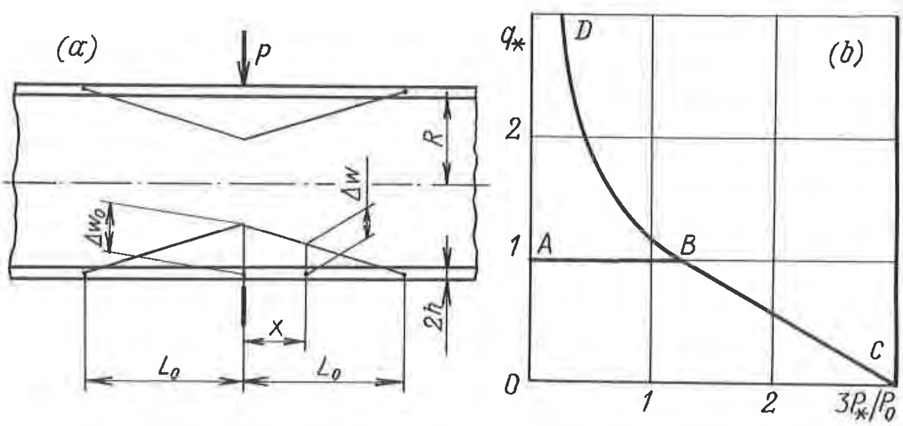

Fig.2. a) Cylindrical shell and its collapse mechanism, b) Limit conditions: ABC - shakedown, BD - incremental collapse when alternating plasticity takes place.

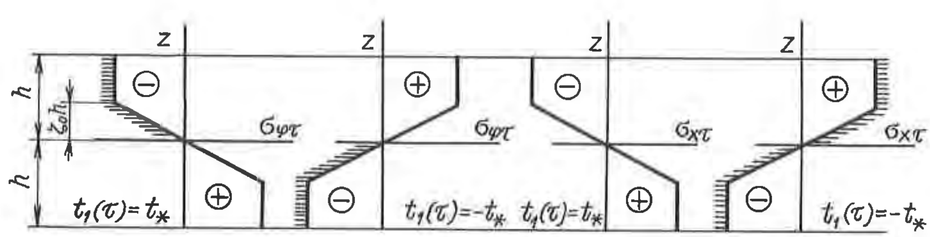

Fig.3. Thermal stresses at $q_*>1$, $P=0$ (alternating plasticity)