THE TRIANGULAR PLATE BENDING FINITE ELEMENTS FOR THE DYNAMIC RESPONSE OF THICK PLATES INCLUDING SHEAR AND ROTATORY INERTIA EFFECTS

D.T. RAMANI

Nuclear Engineering Department, Motor-Colombus Consulting Engineers Inc., Baden, Switzerland

SUMMARY

Thick flat plate-type structural components, such as end perforated plates in steam generators or lower and upper grid plates in reactor vessel, are frequently subjected to complex time-dependent random loading. These components are of great importance and their individual dynamic characteristics must be correctly predicted prior to their inclusion in an overall dynamic system subjected to transient loads. Since these plate components are extremely thick, influence of shearing deformation and rotary inertia becomes considerably significant, when studying the modes of vibration of higher frequencies for resonance and modelling technique. The finite element technique, which has proven to be a considerable success and is sometimes preferred to more exact classical methods, has been applied here to determine dynamic response of thick isotropic solid and perforated plates subjected to complex superimposed loading.

The present investigation is primarily related to development of three conforming triangular plate bending finite elements considering Mindlin's and Reissner's theories as well as an exact small deformation elasticity solution and including the effects of transverse shearing deformation and rotatory inertia on flexural motions in the response analysis of moderately thick elastic plates. The finite elements are modelled on the basis of a set of trilinear displacement functions, representing a corresponding nodal degrees of freedom system in a continuum. These functions are utilized for developing stiffness and mass matrices in each case.

The present method is consistent with Reissner's and Mindlin's theories for linear bending stress distribution and considers in-plane nodal displacements at top and bottom to vary linearly through thickness of the plate and may be assumed as; $u=u(x, y, z)$ and $v=v(x, y, z)$, whereas transverse displacement function, being independent of thickness, is expressed as $w=w(x, y)$. Average transverse shearing rotations, γ_x and γ_y, introduced in the models as linear functions and defined as: $\gamma_x=\gamma_x(x, y)$ and $\gamma_y=\gamma_y(x, y)$, are used to develop generalized strain-displacement relations of elements. This important feature for shear inclusion has an ability to specify exactly the realistic boundary conditions; however, the method is applicable to plates of uniform thickness with small deformations. The total rotations of straight lines originally normal to middle surface are assumed to remain straight but not necessarily normal to the deformed middle surface. Two finite elements developed herein have eight nodes each while the third one possesses four nodes.

While the element stiffness, $[k]^e$, for each model is developed using potential energy formulation, the element mass matrix, $[m]^e$ is determined from consistent mass concept by using same displacement field. The discretized model in each case leads to an algebraic eigenvalue problem $[[K]-\omega^2[M]]\{\Phi\}=0$, from which eigenfrequencies and mode shapes are evaluated by subspace iteration technique; while dynamic response is predicted by mode-superposition procedure ignoring damping effects.

To test the validity of present method of analysis, the proposed elements are checked against several numerical examples known from literature and an excellent agreement of the results is noticed. Also investigated is an influence of thickness on eigenfrequencies of square and circular plates. Dynamic response is studied for a clamped perforated circular plate to a certain accuracy and effectiveness. It is concluded that frequencies from Mindlin's theory model are slightly lower than those of Reissner's theory model which in turn are lower than those of classical thin plate theory.

1. Introduction

The classical flexural vibrations of thin plate and moderately thick plates are usually approximated by considering bending only and effects of shearing deformation and rotatory inertia are ignored. Thin plate theory applied to analysis of thick plates causes inaccuracies and corrections taking these effects into account may be of considerable importance in studying the modes of vibrations of higher frequencies. Importance of shear deformation and rotatory inertia was first recognized by Timoshenko [1], who considered both effects in frequencies of vibrations beams and demonstrated that these effects serve to decrease the computed frequencies due to increased inertia and flexibility of the system. An extension of plate theory to account for shearing deformation was first proposed by Reissner [2] for static deflection of moderately thick plates and number of other authors [3,4,5] have also adopted similar approach. The first presentation of consistent theory for the dynamic behaviour of plates including effects of shearing deformation and rotatory inertia was made by Uflyand [6]. However, Mindlin's [7] paper unquestionably made the most profound impact upon the subject, in which consistent set of equations relating moments and transverse shears to transverse defelction and bending rotations, was presented in the form of basic sixth-order system of partial differential equations of motion, which present great difficulty in obtaining a close-form solutions to problems involving thick plates with complex geometries.

The finite element technique has proved to be of considerable success in the application of both static and dynamic problems concerning both thin as well thick plate bending analyses. Most of the finite element dynamic analyses [8,9] in the past dealt with thin plate theory based on Kirchhoff's hypothesis in which effects of shearing deformation were neglected. However in predicting the dynamic characteristics of thick plate-type components applicable to nuclear technology, the influence of shearing deformation and rotatory inertia becomes of considerable significance. Many finite element models based on classical theory have been proposed by different authors whose account is furnished by Zienkiewicz [10]. In addition, the same reference also summarises various authors [11,12,13,14,15], who have proposed plate bending finite elements including shearing deformation for the solution of static problems. An excellent treatment has been furnished by Clough and Fellipa [9], who have prescribed a simple shear distortion mechanism in developing a quadrilateral element. Based on Reissner's theory, Pryor et.al. [20] developed a rectangular element äns using also Reissner's theory as guidelines. Ramani [16] developed trilinear displacement models in which the effects [21] of transverse shear deformation are included. A very limited development is reported in the field of finite-element applications to dynamic problems of thick plates to include the effects of shear deformation as well as rotatory inertia. Srinivas et.al. [17] have investigated Mindlin's plate model using an exact three-dimensional, small deformation theory of elasticity solution for the free-vibration analysis of simply-supported plates. Whereas Greimann and Lynn [18] have reported rectangular finite element based on Mindin's theory for free-vibration analysis of square plates. Huang [19] employed variational principle to include both effects for vibration analysis of plates, based on classical methods.

The purpose of this investigation consists in developing simple triangular element models based on the guidelines of Mindlin's and Reissner's theories as well as small deformation three-dimensional theory of elasticity solution, to demonstrate validity and convergence characteristics of these elements in predicting eigenfrequencies and dynamic response of moderately thick square and circular plates. Three different sets of displacement [22] fields to represent bending and shearing deformations are defined by simple polynomials for specified degrees of freedom proposed within three-dimensional continuum. During deformation, it is assumed that straight lines originally normal to the middle surface remain straight but not necessarily normal to the deformed surface and that transverse displacement function is independent of the thickness of plate. Excellent agreement with some of the existing exact analytical solution results is obtained to demonstrate the validity of the proposed method. The influence of transverse shear as well as rotatory inertia on eigenfrequencies of both simply-supported and clamped plates is evaluated.

2. Theoretical Formulations

The most well-known theory which includes influence of transverse shear on bending of plates was developed by Reissner [2], who assumed linear variation of bending stressses σ_{xx}, σ_{yy} and σ_{xy} and by employing three-dimensionsl equations of elasticity, obtained expressions for normal stress, σ_{zz}, as well as shear stresses, σ_{xz} and σ_{yz}. Similarly, Mindlin presented his theory, which is a direct extension of Timoshenko beam theory for two-dimensional plate vibration, in which effects of σ_{zz} are neglected. Figure 1 (a) shows an infinite simal element subjected to lateral load "q" and plate stress-resultants M_{xx}, M_{yy}, M_{xy}, Q_{xz} and Q_{yz}, which may be expressed as:

$$(M_{xx},M_{yy},M_{xy}) = \int_{-h/2}^{h/2} (\sigma_{xx},\sigma_{yy},\sigma_{xy}) Z\, dz$$
$$(Q_{xz},Q_{yz}) = \int_{-h/2}^{h/2} (\sigma_{xz},\sigma_{yz}) dz \qquad (1)$$

in which h = thickness of the plate and $\sigma_{xx},\sigma_{yy},\sigma_{xy},\sigma_{xz}$ and σ_{yz} are the stress components of three-dimensional elasticity. Using generalized Hooke's law for isotropic, homogenous, elastic material, the strains in terms of stresses may be expressed as:

$$\varepsilon_{xx} = \frac{1}{E}(\sigma_{xx}-\nu\,\sigma_{yy}-\nu\,\sigma_{zz})$$
$$\varepsilon_{yy} = \frac{1}{E}(\sigma_{yy}-\nu\,\sigma_{xx}-\nu\,\sigma_{zz})$$
$$\varepsilon_{zz} = \frac{1}{E}(\sigma_{zz}-\nu\,\sigma_{yy}-\nu\,\sigma_{xx}) \qquad (2)$$
$$\varepsilon_{xy} = \frac{1}{G}\sigma_{xy},\ \varepsilon_{yz} = \frac{1}{G}\sigma_{yz},\ \varepsilon_{xz} = \frac{1}{G}\sigma_{xz}$$

in which E is the modulus of elasticity, G is the modulus of shearing rigidity and ν is Poisson's ratio. Strains, $\varepsilon_{xx},\varepsilon_{yy},\varepsilon_{zz},\varepsilon_{xy},\varepsilon_{yz}$ and ε_{xz} may then be expressed in terms of displacement, u(x,y,z,t), v(x,y,z,t) and w(x,y,z,t) within the continuum as: $\varepsilon_{xx}=u_{,x}$, $\varepsilon_{yy}=v_{,y}$, $\varepsilon_{zz}=w_{,z}$, $\varepsilon_{xy}=(u_{,y}+v_{,x})$, $\varepsilon_{xz}=(u_{,z}+w_{,x})$, $\varepsilon_{yz}=(v_{,z}+w_{,y})$. From the preceding relations one obtains expressions for stress resultants, $M_{xx},M_{yy},M_{xy},Q_{xz}$ and Q_{yz} in terms of displacements as:

$$M_{xx} = \frac{E}{(1-\nu^2)}\int_{-h/2}^{h/2}\left(\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\right)z\, dz + \frac{\nu}{(1-\nu)}\int_{-h/2}^{h/2}\sigma_z\, z\, dz$$
$$M_{yy} = \frac{E}{(1-\nu^2)}\int_{-h/2}^{h/2}\left(\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x}\right)z\, dz + \frac{\nu}{(1-\nu)}\int_{-h/2}^{h/2}\sigma_z\, z\, dz \qquad (3)$$

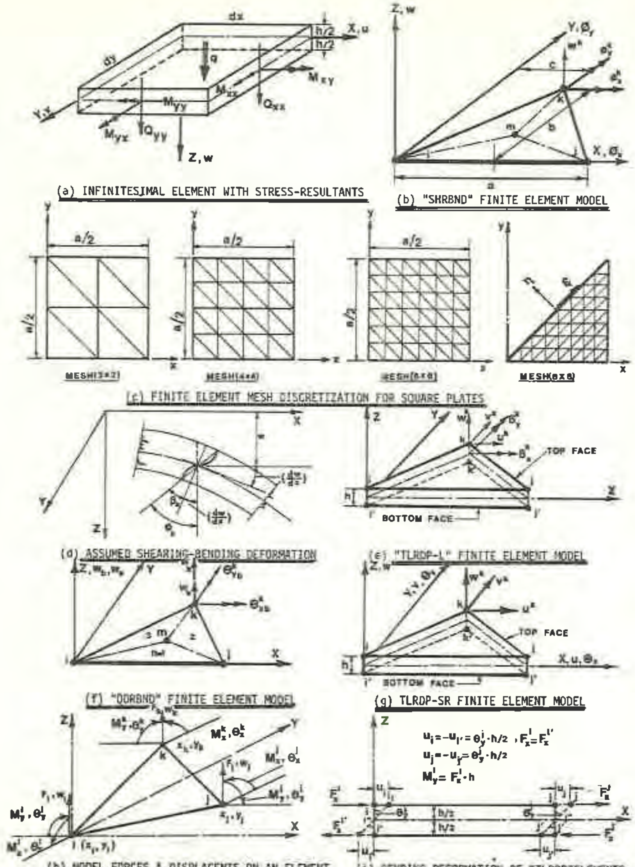

FIG: 1 FINITE ELEMENT DYNAMIC ANALYSIS OF THICK PLATES

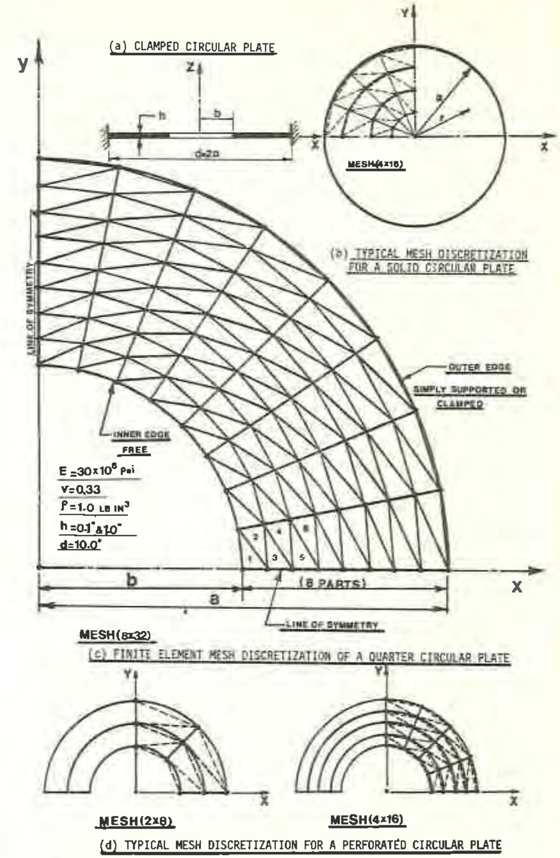

FIG: 2 FINITE ELEMENT MESH DISCRETIZATION FOR CIRCULAR PLATES

$$M_{xy} = G \int_{-h/2}^{h/2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) z \, dz$$

$$Q_{xy} = \kappa^2 G \int_{-h/2}^{h/2} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) dz \qquad (3)$$

$$Q_{yz} = \kappa^2 G \int_{-h/2}^{h/2} \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) dz \quad \text{and} \quad I_z = \int_{-h/2}^{h/2} \sigma_z \, z \, dz$$

in which constant κ^2 is introduced due to assumed variations of u, v and w. Reissner [2] assumed the constant $\kappa^2 = 5/6$ and obtained an expression for σ_{zz} from equilibrium equations to evaluate above integrals. Contrary to Reissner's assumption, Mindlin [7] neglected the integral, I_z, and assumed the constant $\kappa^2 = \pi^2/12$. Assuming the effect of integral, I_z, to be very small, the integration of above equations may be easily performed by relating u, v and w to functions of x, y and z. Mindlins and Reissner have both utilized the following relations, which physically are in agreement with the basic assumptions that plane sections

$$u(x,y,z,t) = - z\Psi_x(x,y,t), \quad v(x,y,z,t) = - z\Psi_y(x,y,t)$$
$$w(x,y,z,t) = w(x,y,t) \qquad (4)$$

remain plane and that the normal to the mid-plane before deformation is no longer normal after deformation. The equations of dynamic equilibrium for a plate element may be expressed as:

$$\left(\frac{\partial M_{xx}}{\partial x} - \frac{\partial M_{xy}}{\partial y} - Q_{xz}\right) = \frac{\rho h^3}{12} \frac{\partial^2 \Psi_x}{\partial t^2}$$
$$\left(\frac{\partial M_{yy}}{\partial y} - \frac{\partial M_{xy}}{\partial x} - Q_{yz}\right) = \frac{\rho h^3}{12} \frac{\partial^2 \Psi_y}{\partial t^2} \qquad (5)$$
$$\left(\frac{\partial Q_{xz}}{\partial x} + \frac{\partial Q_{yz}}{\partial y} + q(x,y,t)\right) = \rho h \frac{\partial^2 w}{\partial t^2}$$

in which "q" is the applied lateral load and "ρ" is the density of the material of the plate. By substituting eqs. (4) into eqs. (3) and using expressions for M_{xx}, M_{yy}, M_{xy}, Q_{xz} and Q_{yz} into eqs. (5), yields the fundamental set of equations for system as:

$$\left\{\frac{D}{2}\left[(1-\nu)\nabla^2\Psi_x + (1+\nu)\frac{\partial}{\partial x}\left(\frac{\partial \Psi_x}{\partial x} + \frac{\partial \Psi_y}{\partial y}\right)\right] - \kappa^2 Gh\left(\Psi_x - \frac{\partial w}{\partial x}\right)\right\} = \frac{\rho h^3}{12}\frac{\partial^2 \Psi_x}{\partial t^2}$$
$$\left\{\frac{D}{2}\left[(1-\nu)\nabla^2\Psi_y + (1+\nu)\frac{\partial}{\partial y}\left(\frac{\partial \Psi_x}{\partial x} + \frac{\partial \Psi_y}{\partial y}\right)\right] - \kappa^2 Gh\left(\Psi_y - \frac{\partial w}{\partial y}\right)\right\} = \frac{\rho h^3}{12}\frac{\partial^2 \Psi_y}{\partial t^2} \qquad (6)$$
$$\left\{\kappa^2 Gh\left(\nabla^2 w - \frac{\partial \Psi_x}{\partial x} - \frac{\partial \Psi_y}{\partial y}\right) + q(x,y,t)\right\} = \rho h \frac{\partial^2 w}{\partial t^2}$$

in which the terms on the right-hand sides account for the rotatory inertia of the plate element. It is abserved that above system of equations is of the sixth order with three dependent variables u, v and w. Thus, three boundary conditions are enforced along each edge.

3. Finite Element Formulation of Equations of Motion

It is apparent that equations of motion, as represented by eqs. (6), cannot be easily solved for arbitrary boundary conditions and therefore, to solve practical eigenvalue or dynamic response problems, a numerical procedure, such as finite element technique, is adopted herein. This method, which is a generalized Ritz technique, extended to dynamic problems, can be expressed by Hamilton's principle as:

$$\delta\left[\int_{t_0}^{t_1} (T_s - V_s) \, dt\right] = 0 \qquad (7)$$

in which, T_s = kinetic energy, V_s = total potential energy of the system and integration is taken over time variable t. For a holonomic system, this Hamiltonian is equivalent to Lagrange's equation of motion expressed as:

$$\left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}\right\} = 0,\ i=1,m \tag{8}$$

in which, Lagrangian $L = (T_s - V_s)$ and q_i are generalized coordinates with m = number of degrees of freedom. Considering total energy of an undamped system as made of sum of energies of individual finite elements, the quantities in eqs. (7) may be expressed as:

$$T_s = \sum_{i=1}^{n} T_i \quad , \quad V_s = \sum_{i=1}^{n} V_i \equiv \sum_{i=1}^{n} U_i + \sum_{i=1}^{n} \Omega_i \tag{9}$$

in which, U_i = strain energy of ith element, Ω_i = potential energy of applied loads on ith element and n = total number of elements. Using matrix notations, T_i, U_i and Ω_i can be expressed as:

$$T_i = \int_{A_i} \frac{\rho h}{2}\{\dot{u}\}_i^T\{u\}_i\ dA,\ U_i = \int_{A_i} \{\varepsilon\}_i^T[D]\{\varepsilon\}_i\ dA$$
$$\Omega_i = \int_{S_i} \{u\}_i^T\{P\}_i\ dS \tag{10}$$

where A_i = area of ith element, ρ = mass per unit volume of ith element, h = thickness of element and S_i = boundary of ith element on which forces, $\{P\}_i$, are specified. Expressions for time-dependent displacement field, $\{u\}_i$ and strains, $\{\varepsilon\}_i$, are usually defined in terms of time-dependent, generalized nodal displacements, $\{\delta\}_i$, from Reissner's or Mindlin's assumptions. $\{u\}$ and $\{\varepsilon\}_i$ may be expressed as:

$$\{u\}_i = [N]\{\delta\}_i \quad , \quad \{\varepsilon\}_i = [B]\{\delta\}_i \tag{11}$$

in which [N] and [B] are assumed shape-function matrix and strain-displacement matrix respectively. Substituting eqs. (11) into eqs. (10), yields the matrix relations for T_i, U_i, Ω_i as:

$$T_i = \frac{1}{2}\{\dot{\delta}\}_i^T[m]_i\{\dot{\delta}\}_i,\ U_i = \frac{1}{2}\{\delta\}_i^T[K]_i\{\delta\}_i,\ \Omega_i = \{\delta\}_i^T\{f\}_i \tag{12}$$

in which $[m]_i$, $[K]_i$ and $\{f\}_i$ are element mass matrix, stiffness matrix and nodal load matrix respectively. Matrices $[m]_i$, $[K]_i$ and $\{f\}_i$ are defined as:

$$[m]_i = \int_{A_i} \rho h[N]^T[N]dA\ , \quad [K]_i = \int_{A_i} [B]_i^T[D][B]dA$$
$$\{f\}_i = \int_{S_i} [N]^T\{P\}_i\ dS \tag{13}$$

The Langrangian function, L, for the entire undamped system may then be expressed as:

$$L = [\frac{1}{2}\{\dot{\delta}_s\}^T[M_s]\{\dot{\delta}_s\} - \frac{1}{2}\{\delta_s\}^T[K_s]\{\delta_s\} - \{\delta_s\}^T\{F_s\}] \tag{14}$$

where $\{\delta_s\}$, $[M_s]$, $[K_s]$ and $\{F_s\}$ are nodal displacement, mass, structural stiffness and nodal force matrices respectively for the entire dynamic system. Applying the Lagrangian equation of motion (eq. (8)) to the Lagrangian "L" of the given dynamic system (eq. (14)), results in the following set of m equations in matrix notations.

$$[M_s]\{\ddot{\delta}_s\} + [K_s]\{\delta_s\} + \{F_s\} = \{0\} \tag{15}$$

If the system is subjected to damping denoted by matrix, $[C_s]$, then above equation is rewritten as:

$$[M_s]\{\ddot{\delta}_s\} + [C_s]\{\dot{\delta}_s\} + [K_s]\{\delta_s\} + \{F_s\} = \{0\} \tag{16}$$

For free vibrational analysis, above equation, by letting $[C_s] = \{F_s\} = \{0\}$, reduces to

$[M_s]\{\ddot{\delta}_s\}+[K_s]\{\delta_s\}=\{0\}$. Then by assuming $\{\delta_s\}=\{\phi\}(\cos\omega t+i\sin\omega t)$, and substituting in free-vibration equation, yields the following generalized eigenvalue problem with eigenvalues, ω, and eigenvectors, ϕ, which can be evaluated by various available methods including

$$[[K_s] - \omega^2[M_s]]\{\phi\} = \{0\} \tag{17}$$

by means of sub-space iteraction technique [23,24]. For the purpose of evaluating dynamic response of the given system, mode-superposition method outlined in [24,25] has been applied herein, ignoring damping.

4. Development of Finite Element Models

Solution to the fundamental equations of motion, which were presented in the previous section, can be conveniently obtained by using finite element displacement formulations based on the guidelines of Mindlin's and Reissner's theories. Also considered is the three-dimensional linear, small deformation theory of elasticity approach in developing stress-strain relationship. The proposed method considers nodal displacements at the top and bottom faces, in a bending deformation, to vary linearly over the thickness of the plate as illustrated in Figs. 1 (e,g,i). However, nodal transverse displacement is assumed to be independent of the thickness of plate and is a function of in-plane coordinates x and y only. This condition is however relaxed in one specific model (TLRDP-SR), in which transverse displacement, w, is a function of x,y and z coordinates. These considerations and fundamental assumptions lead to the following displacement models, which are then utilized in developing element stiffness matrix, $[K]^e$, mass matrix, $[m]^e$ and nodal load matrix, $\{f\}^e$, which are then synthesized into overall system stiffness matrix, $[K_s]$, mass matrix, $[M_s]$ and nodal load matrix, $\{F_s\}$. These elements have been proposed earlier by the author [16,22] in analysing successfully static problems of thick plates including shear deformation.

4.1 TLRDP-L Element

This three-dimensional element is a flat plate triangular finite element, that incorporates a five-degree-of-freedom nodal point system. It possesses six nodes (3 each at top and bottom) as illustrated in Fig.1 (e). The displacement field within the continuum, representing nodal displacements and shearing rotations, is expressed as:

$$u = (\alpha_1+\alpha_2x+\alpha_3y)z,\quad v= (\alpha_4+\alpha_5x+\alpha_6y)z,\quad w= (\alpha_7+\alpha_8x+\alpha_9y)$$
$$\beta_x= (\alpha_{10}+\alpha_{11}x+\alpha_{12}y),\quad \beta_y = (\alpha_{13}+\alpha_{14}x+\alpha_{15}y) \tag{18}$$

in which u and v are in-plane displacements, w is transverse displacement and β_x, β_y are average transverse shearing rotations about x and y axes respectively. The generalized nodal displacements at node "i" are: $\{\delta^i\} = \{u^i,v^i,w^i,\beta_x^i,\beta_y^i\}^T$ and generalized element strain tensor may be expressed as: $\{\varepsilon\}^e=\{\varepsilon_{xx},\varepsilon_{yy},\varepsilon_{zz},\varepsilon_{xy},\varepsilon_{yz},\varepsilon_{zx}\}^T$, which is derived from exact, small deformation, three-dimensional equations of elasticity, based on the guidelines of Reissner's theory. The stress-strain relation is expressed symbolically as: $\{\sigma\}^e = [D]\{\varepsilon\}^e$, in which elemental stresses are: $\{\sigma_{xx},\sigma_{yy},\sigma_{zz},\sigma_{xy},\sigma_{yz},\sigma_{zx}\}$ and flexural rigidity matrix [D], for an isotropic, homogenous element is furnished in literature. The constant κ^2 assumed in Reissner's theory is taken to be equal to 5/6 in [D] matrix and σ_{zz}, although small in comparison to σ_{xx} and σ_{yy}, is retained in stress-strain relation. Due to anti-symmetric behavior about middle plane in bending deformation, only upper-half (or the lower half) of the element is needed in evaluating [21,22] stiffness, mass and nodal load matrices by integrating over half the volume of the element. This element may also be reduced to a thin plate finite element, designated as "TLRDP" and shown in Fig. 1(i), by assuming $\beta_x=\beta_y=0$ in the displacement field. It has been utilized to compare the eigenfrequencies and nodal response of the plate with those of exact analytical solutions, illustrated in Figs. 7,8,9,10,14.

4.2 SHRBND Element

This two-dimensional element is a flat plate triangular element, that is modelled on the guidelines of Mindlin's [22, 26] theory. It possesses four nodes and is incorporated with three degree-of-freedom nodal point system as shown in Fig. 1(b). The representation of the displacement field, (w,ϕ_x,ϕ_y), within the continuum is assumed to be a set of completely independent linear polynomials expressed as:

$$w = (\alpha_1+\alpha_2x+\alpha_3y+\alpha_4xy),\ \phi_x = (\alpha_5+\alpha_6x+\alpha_7y+\alpha_8xy) \quad (19)$$
$$\phi_y = (\alpha_9+\alpha_{10}x+\alpha_{11}y+\alpha_{12}xy)$$

where $w(x,y)$ is transverse displacement and ϕ_x and ϕ_y are total rotations of plate cross-sections about x and y axes respectively. Physically, this assumption states that plate sections remain plane and normal to the middle-plane before deformation does not remain normal after deformation. In addition, effects of normal stress, σ_{zz}, are neglected as being small. The generalized nodal displacement at node "i" are: $\{\delta^i\} = \{w^i,\phi_x^i,\phi_y^i\}^T$ and generalized strain tensor may be expressed as: $\{\varepsilon\}^e=\{\partial\phi_{x/\partial x},\partial\phi_{y/\partial y},(\partial\phi_{x/\partial y}+\partial\phi_{y/\partial x}),(\partial w/_{\partial x}-\phi_x),(\partial w/_{\partial y}-\phi_y)\}$, in which normal strain, ε_{zz}, is obviously eliminated. The stress-strain relation is symbolically expressed as: $\{\sigma\}^e=[D]\{\varepsilon\}^e$, where $\{\sigma\}^e=\{M_{xx},M_{yy},M_{xy},Q_{xz},Q_{yz}\}^T$, and flexural rigidity matrix, [D], for an isotropic, homogenous element can be easily established. To compensate for boundary condition errors, constant κ^2 in Mindlin's theory assumptions is taken to be equal to $\pi^2/12$ in [D] matrix. It may be observed that preceding assumptions satisfy the solution of eq. (6), with appropriate boundary conditions. Element stiffness mass and nodal load matrices may be conventionally derived. The stiffness matrix shows coupling effect between shearing and bending deformations, resulting in accurate evaluation of dynamic response.

4.3 QDRBND Element

In the development of this finite element, it is assumed that total deflection "w" is sum of "w_b", the deflection due to bending, ans "w_s", the deflection due to transverse shear. The cross-section rotations, ϕ_x and ϕ_y, are not independent quantities as proposed in Mindlin's theory, but functions of transverse bending deflection, w_b, as in classical theory and may be expressed as: $\phi_x=(\partial w_b/\partial x)$, $\phi_y=(\partial w_b/\partial y)$. This element possesses four node points (shown in Fig. 1-f) and each node is incorporated with four-degree-of-freedom system. The basic finite element is composed of 3 subelements and the displacement field in each is represented as:

$$(w_b)_m = (\alpha_1+\alpha_2x+\alpha_3y+\alpha_4x^2+\alpha_5xy+\alpha_6y^2)_m \quad (20)$$
$$w_s = (\beta_1+\beta_2x+\beta_3y+\beta_4xy)$$

in which $w = (w_b+w_s)$, m=1,2,3 corresponding to subelements 1,2 and 3 respectively and $(\phi_x)_m=[\partial(w_b)_m/\partial x]$, $(\phi_y)_m=[\partial(w_b)_m/\partial y]$. Mid-point node has been selected to ensure slope compatibility of w_b between adjecent elements. The generalized nodal displacement (Fig.1-f) at

node "ℓ", (ℓ = i,j,k,m) may be expressed as: $\{\delta\}^\ell = \{w_b^\ell, \phi_x^\ell, w_s^\ell, \phi_y^\ell\}$.
The element stiffness, mass and nodal load matrices for an isotropic, homogenous element are obtained by standard procedure. By using slope and deflection compatibility at the inner node [16,22], a 9x9 bending stiffness matrix, $[K_b]^e$, is obtained, which is then linearly superposed upon shear stiffness matrix, $[K_s]^e$. Constant shear is assumed through the thickness of the element by assuming κ^2=1 in shear rigidity matrix, $[D_s]$. Due to uncoupling of bending and shearing effects, this element fails to reflect the influence of shear on bending moments. However, eigenfrequencies and effect of shear on eigenfrequencies remain in excellent agreement with theoretical values.

4.4 TLRDP-SR Element

This three-dimensional finite element is similar to TLRDP-L element (see Fig.1-g) with the exception that, ε_{zz}, when evaluated in strain tensor, $\{\varepsilon\}=\{\varepsilon_{xx},\varepsilon_{yy},\varepsilon_{zz},\varepsilon_{xy},\varepsilon_{yz},\varepsilon_{zx}\}^T$, is not reduced to zero and that, in order to eliminate influence of σ_{zz} in the equations of motion, following displacement field is developed which includes the effects of rotatory inertia in both x and y directions. This element possesses six nodes (3 each at top and bottom faces) and in order to reduce the number of degrees of freedom in TLRDP-L element (discussed in 4.1), this element in incorporated with only three-degrees-of-freedom nodal point system. To introduce the coupling effect between shear and moment variables, the lateral displacement is made dependent on "z" coordinate also. The displacement field within the continuum, may be expressed as:

$$u = (\alpha_1+\alpha_2x+\alpha_3y)z \ , \ v = (\alpha_4+\alpha_5x+\alpha_6y)z$$
$$w = (\alpha_7+\alpha_8x+\alpha_9y)+z^2\bar{w}(x,y) \qquad (21)$$

in which u, v and w are displacements along x, y and z respectively. Satisfying the requirement that σ_{zz} = 0, yields following relationship for the unknown $\bar{w}$ in eq. (21) as:

$$\bar{w}(x,y) = -\frac{\nu}{2(1-\nu)}(\alpha_2+\alpha_6) \qquad (22)$$

which completely determines "w" displacement field. The terms containing z^2 in strain components, $\{\varepsilon\}^e$, are neglected as being small. Exact three-dimensional equations of elasticity [1,2] are utilized in developing strains, $\{\varepsilon\}^e$, and corresponding stresses, $\{\sigma\}^e$, which are related as: $\{\sigma\}^e=[D]\{\varepsilon\}^e$, where [D] can be easily evaluated. The shear-distribution constant κ^2 is assumed equal to 5/6 and due to anti-symmetric bending behavior about the middle plane, element stiffness, mass and nodal load matrices can be easily developed by the procedure explained previously.

5. Numerical Problems and Results

This section presents problem descriptions, which are analysed to demonstrate the validity of proposed finite element models. The use of symmetry, whenever possible, is employed and the finite element solutions are compared with known exact analytical solutions. These numerical analyses were performed with the primary objectives of investigating convergence characteristics and the criteria for different models. A comparative study of proposed models is made to demonstrate their capability in predicting the influence of transverse shear deformation and rotatory inertia. In analysing free-vibration and dynamic response problems of plates with proposed finite element models, a general-purpose three-dimensional structural analysis computer program SAP [25,26], designed to handle large-sized finite element computations, was modified to include additional options and proposed elements. Basic guidelines outlined by Clough and Bathe [24] as well as Wilson et.al. [25] and Bathe and Wilson [23] were taken into consideration for computing large eigenvalue problems with the Subspace-Iteration technique, an extended version of Ritz procedure. The objective of this technique is to calculate r eigenvectors and eigenvalues satisfying the relationsship, $[K]\{\Phi\}=[M]\{\Phi\}\omega^2$ in which the eigenvectors, Φ, from an M-orthonormal basis of the p-dimensional least dominant subspace of operators K and M and the solution is carried out by simultaneaous iteration with only q lineary independent vector, where $r<q$. For the dynamic response analysis of a relatively large-sized problem designated as: $[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{F(t)\}$, the mode-superposition technique has been employed with damping neglected in the system. In order to examine the relative performance of proposed elements, following numerical examples pertaining to simply-supported and clamped square as well as solid and perforated circular plates have been investigated. In each case, proper boundary conditions on u, v, w, w_s, β_x and β_y have been applied.

5.1 Simply Supported and Clamped Square Plates

In this investigation, the eigenvalues for simply-supported and clamped square plates with thickness-to-width ratios h/a=0.050, 0.1, 0.15, 0.20 and 0.25 are computed by using TLRDP-L, SHRBND, QDRBND and TLRDP-SR elements. Mesh sizes (see Figs.1-c) of (2x2), (4x4), (6x6) and (8x8) subdivisions are employed to determine their influence on convergence of eigenvalues λ_{11}, λ_{13} and λ_{33} of first three symmetric modes of vibrations. Exact theoretical solutions for SS thick plates and classical thin plates are furnished by Mindlin [7] and Timoshenko [27] respectively. Figures 3 and 4 illustrates relative performance of different elements for h/a=0.1 as compared to Mindlin's theory solutions (λ_{11}^M= 0.9666, λ_{13}^M = 4.30, λ_{33}^M = 7.08) for simply supported thick square plates including transverse shear deformation. Corresonding classical thin plate solutions [27] are: λ_{11}^C =1.00, λ_{13}^C=5.00 and λ_{33}^C= 9.00, which represent an upperbound limit for eigenvalues. The proposed finite element thick plate solutions in case of TLRDP-L, SHRBND and QDRBND elements are observed to yield a bit higher eigenvalues than those of Mindlin, due to effects of rotatory inertia being neglected. However, TLRDP-SR element (Shear and Rotatory inertia) yields excellent results that are in close agreement with Mindlin's solutions. In general, TLRDP-SR converges most rapidly and QDRBND converges rather slowly and in all cases convergence to λ_{11} is rather fastest as compared to eigenvalues λ_{13} and λ_{33}. Thin plate finite element TLRDP, proposed herein also furnishes identical convergent solutions to λ_{11}, λ_{13} and λ_{33}.

For clamped square plates, the finite element results as well as those of classical thin plate are presented in Figs.5,6 and 8 for thickness ratio of h/a=0.1 and first three symmetric modes. The classical theory [28] solutions for clamped plate are: λ_{11}^C = 1.82, λ_{13}^C = 6.66 and λ_{33}^C = 6.70, which represent an upper-bound limit. Thin plate finite element ,TLRDP also converges identically to classical eigenvalues λ_{11}^C, λ_{13}^C and λ_{33}^C as illustrated in Figs. 8-a,b,c. The effect of rotatory inertia and transverse shear is to lower the eigenvalues and this effect is illustrated in each case in Figs.5 and 6, for eigenvalue ratios $\lambda_{ij}/\lambda_{ij}^C$ versus thickness parameters h/a. This eigenvalue ratio, defined as the ratio of thick-plate

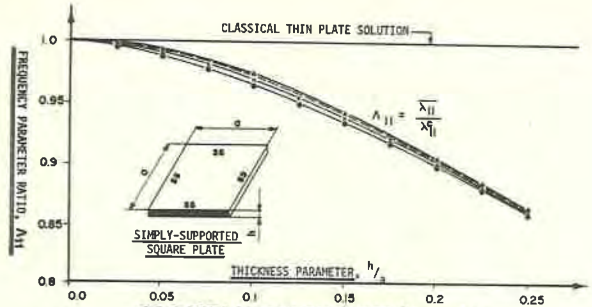

FIG: 3 EFFECT OF THICKNESS PARAMETER, h/a, ON Λ_{11}

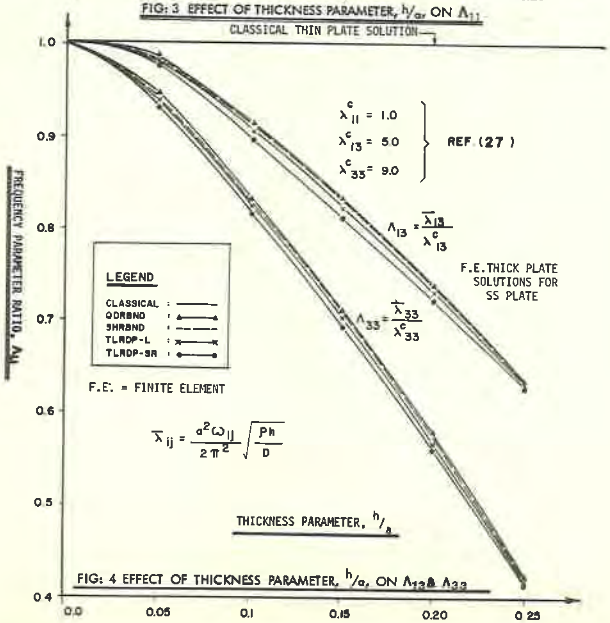

FIG: 4 EFFECT OF THICKNESS PARAMETER, h/a, ON Λ_{13} & Λ_{33}

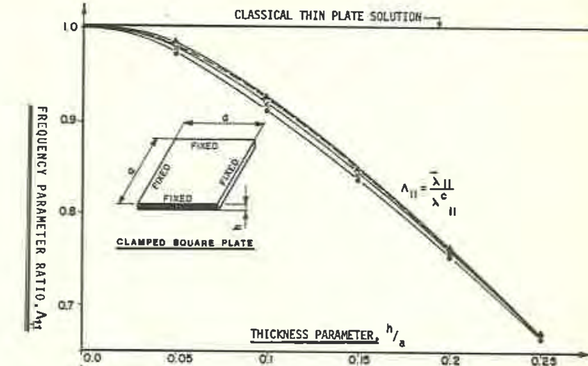

FIG: 5 EFFECT OF THICKNESS PARAMETER, h/a, ON Λ_{11}

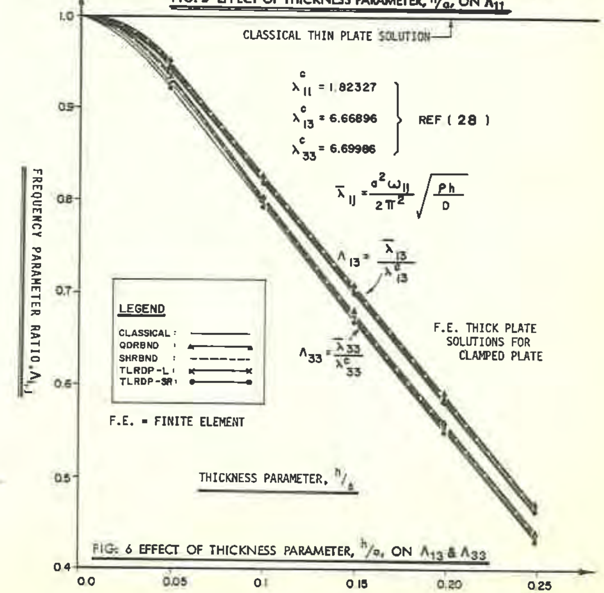

FIG: 6 EFFECT OF THICKNESS PARAMETER, h/a, ON Λ_{13} & Λ_{33}

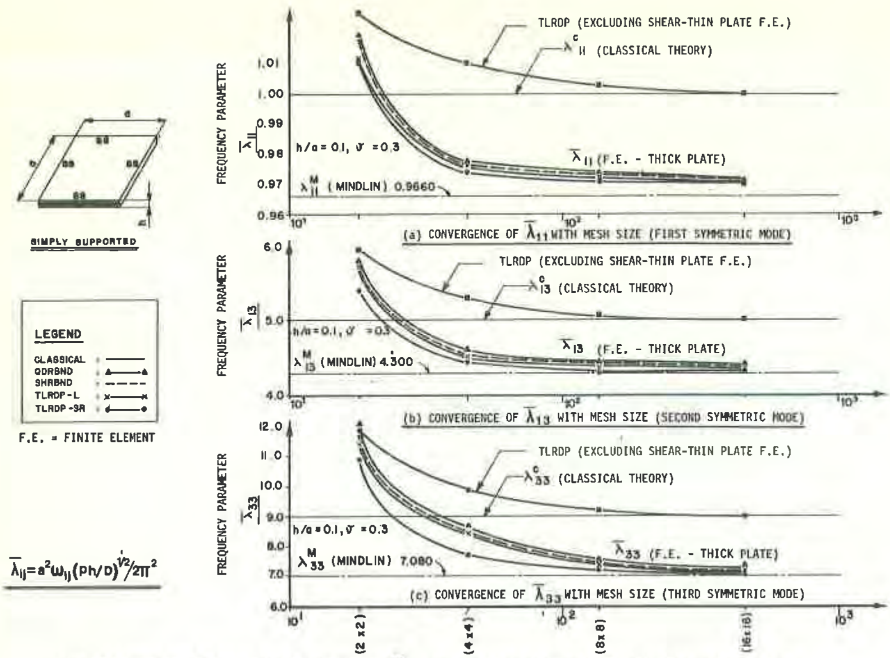

FIG: 7 CONVERGENCE OF FREQUENCY PARAMETERS, $\bar{\lambda}_{i,j}$, WITH NODAL DEGREES OF FREEDOM FOR SS-SQUARE PLATE

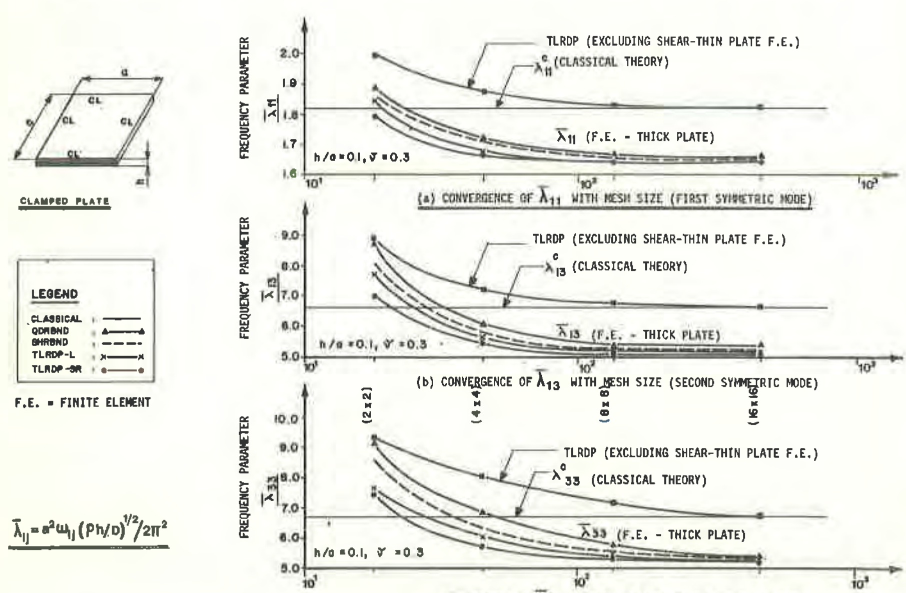

FIG: 8 CONVERGENCE OF FREQUENCY PARAMETERS, $\bar{\lambda}_{i,j}$, WITH NODAL DEGREES OF FREEDOM FOR CLAMPED SQUARE PLATE eigenvalue including shear and rotatory inertia to classical thin plate eigenvalue, is computed for each element and compared to λ_{11}, λ_{13} and λ_{33}, both for simply-supported as well as clamped plates. Figures 3,4 and 7 and 5,6 and 8 indicate considerable shear and rotatory inertia effects on free vibrations of plates. For simply-supported plates, elimination of rotatory inertia effect in finite element method, results in raising of λ_{11} by 1%, λ_{13} by 1.10% and λ_{33} by 1.15% over corresponding eigenvalues of Mindlin's theory or "TLRDP-SR" element. Similar effect for clamped plate in raising λ_{11} by 0.9%, λ_{13} by 1% and λ_{33} by 1.08% over corresponding values of TLRDP-SR element; Mindlin's theory eigenvalues for clamped plate are not available.

5.2 Simply-Supported and Clamped Circular Plates

In this investigation, the eigenvalue problems of simply-supported and clamped solid and perforated circular plates have been analysed. Figures 2(b,c,d) illustrate typical mesh discretizations using TLRDP-L, SHRBND, QDRBND and TLRDP-SR triangular finite elements with subdivisions of (1x4), (2x8), (4x16) and (8x32) on a complete circular plate. Use of triangular elements serves to illustrate their capability in approximating curved boundaries in dynamic applications. Figures 9 to 12 represent monotonic convergence of each element to non-dimensional frequency parameters, $\lambda^2_{0,0}$, $\lambda^2_{0,1}$ and $\lambda^2_{0,2}$ for the first three modes of vibration for simply-supported solid circular plates. Thickness-to-radius ratio, h/a, is taken to be equal to 0.1 and Poisson's ratio ν, was adopted as 0.33 for metallic plates. It is apparent that as the mesh representation grows finer, the parameters calculated, tend to converge and compare favorably to exact theoretical values [29] as seen in Fig. 9 (a), for the first fundamental frequency parameter, $\lambda^2_{0,0}$. Similar convergence is illustrated in Figs. 9(b,c) for frequency parameters $\lambda^2_{0,1}$ and $\lambda^2_{0,2}$, however no known exact theoretical solutions are available for these cases. Monotonic convergence with increasing mesh discretizations for clamped circular plates is illustrated in Figs. 10 (a,b,c), in which relative performance of each element is shown. First three modes, having non-dimensional frequency parameters, $\lambda^2_{0,0}$, $\lambda^2_{0,1}$ and $\lambda^2_{0,2}$ are computed using thickness parameters h/a=0.1. In each case, it is observed that TLRDP-SR element has the fastest convergence rate and QDRBND element the least. Classical thin plate solutions [30] for clamped case, are also furnished for comparison with thick plate finite element solutions. Figures 9 (a,b,c) and 10 (a,b,c) also illustrate convergence characteristics of TLRDP (thin plate element without shear inclusion) element in simulating frequency parameters, $\lambda^2_{0,0}$, $\lambda^2_{0,1}$ and $\lambda^2_{0,2}$ for both simply-supported as well as clamped plates.

Another important numerical problem, that is of vital interest to design the nuclear components, consists of a perforated thick clamped circular plate (see Fig.15) analysed to predict influence of diameter-ratio b/a, on the first fundemental frequency parameter, $\lambda_{0,0}$, using TLRDP-L (thick plate) element with mesh discretizations illustrated in Fig.2. Comparison with an exact theoretical solution [31] as well as with thin-plate (TLRDP) finite element solution, is shown in Fig. 15, indicating lowering of $\lambda_{0,0}$ due to shear inclusion. Similar plates with number of randomly spaced perforations have also been analysed by the author to evaluate fundemental frequency parameters.

5.3 Dynamic Response of a clamped circular Plate

Finally, in order to evaluate the capacity of a typical thick plate finite element in

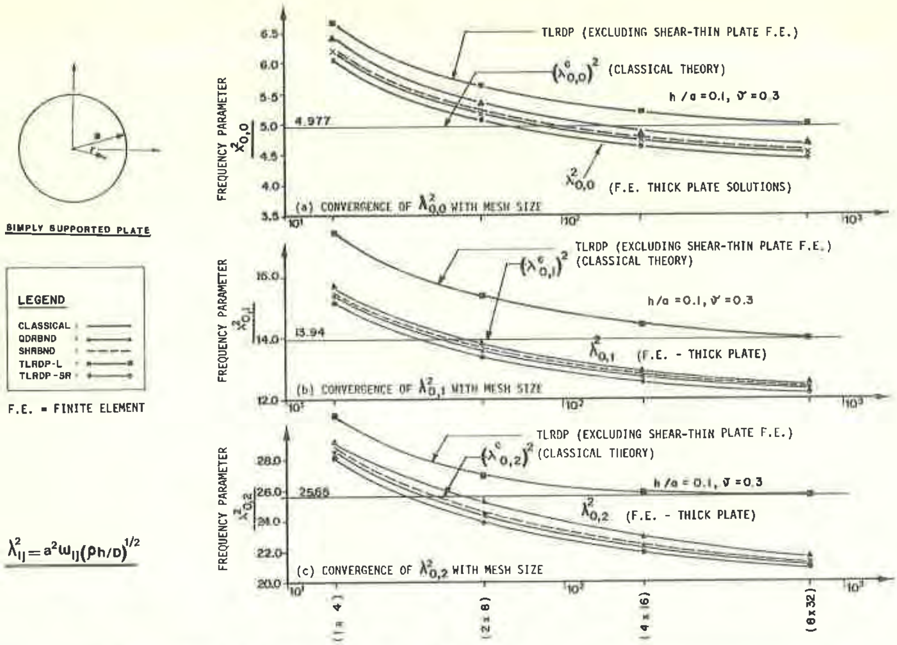

FIG: 9 CONVERGENCE OF FREQUENCY PARAMETER, $\lambda^2_{i,j}$, WITH NODAL DEGREES OF FREEDOM FOR SS CIRCULAR PLATE

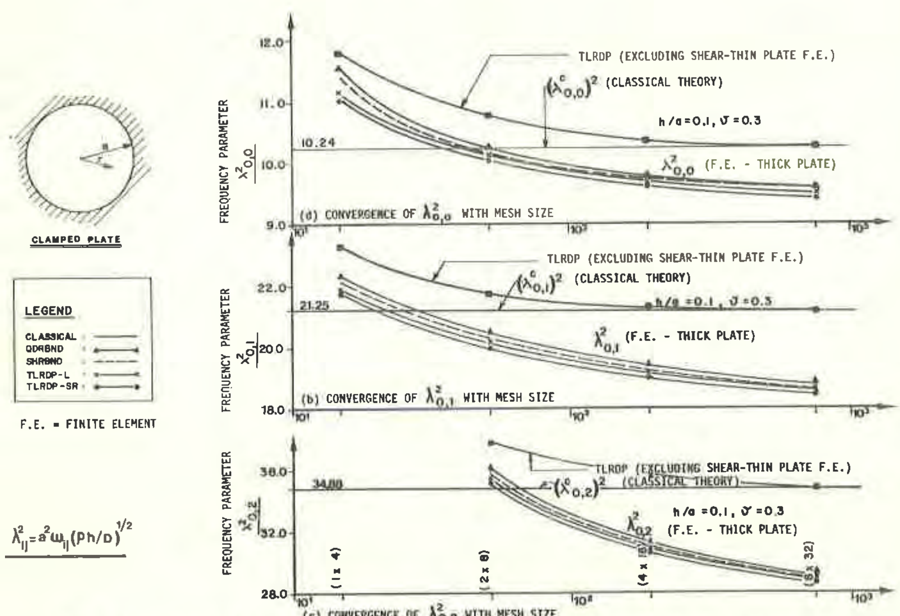

FIG: 10 CONVERGENCE OF FREQUENCY PARAMETERS, $\lambda^2_{i,j}$, WITH NODAL DEGREES OF FREEDOM FOR CLAMPED CIRCULAR PLATE

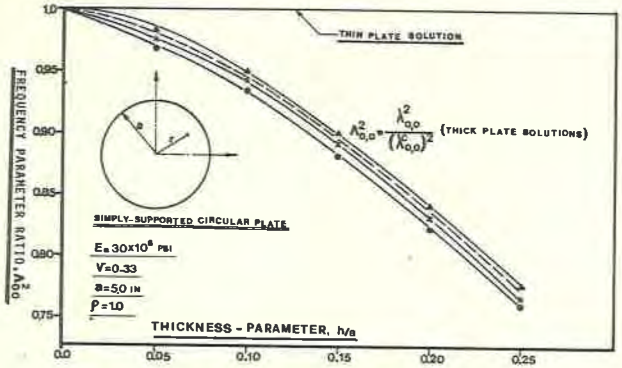

FIG:11 EFFECT OF THICKNESS PARAMETER, h/a, ON $\Lambda^2_{0,0}$

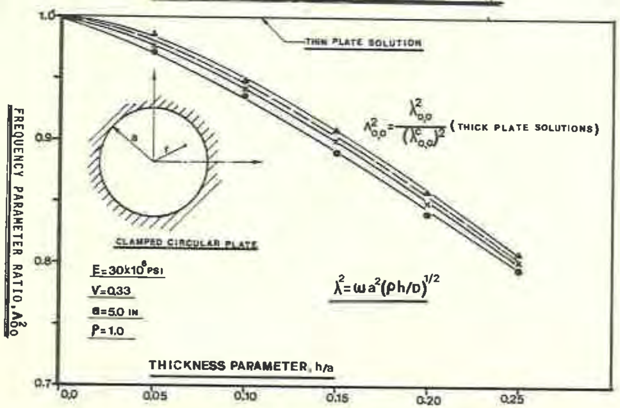

FIG: 12 EFFECT OF THICKNESS PARAMETER, h/a, ON $\Lambda^2_{0,0}$

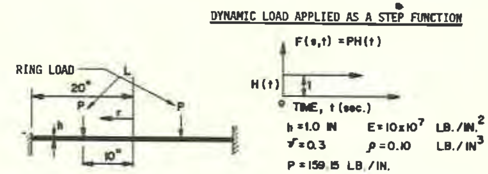

FIG: 13 CLAMPED CIRCULAR PLATE SUBJECTED TO DYNAMIC LOAD

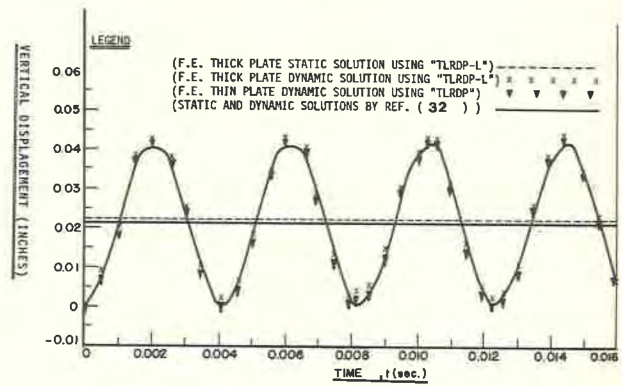

FIG: 14 VERTICAL STATIC & DYNAMIC RESPONSE OF CIRCULAR PLATE

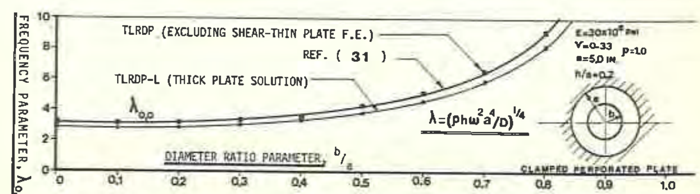

FIG: 15 EFFECT OF DIAMETER PARAMETER, b/a, ON FREQUENCY PARAMETER predicting the dynamic response of a system, a clamped circular plate subjected to a concentric ring-load applied as a step-function of time, is illustrated in Fig.14. The mesh idealization of the plate are defined in Figs.2 and the dynamic response is obtained with TLRDP-L (thick plate) finite element. Both static and dynamic load deflections are plotted and compared with the available known solutions [32] from literature. Also presented in Fig.14, is the dynamic response of the same plate using TLRDP (thin-plate) element ignoring shear and rotatory inertia effects. Excellent agreement and reliable results are obtained in each case.

6. Conclusions

The purpose of this investigation has been to develop finite element capability which can be reliably and efficiently utilized to analyse thick plate bending components in nuclear technology, by considering inclusion of transverse shear deformation and rotatory inertia in accordance with the guidelines of Reissner's and Mindlin's theories. Formulation of four compatible triangular thick plate elements is described. The proposed models are analysed and compared with respect to convergence characteristics, quality of results and accuracy as well as ease and suitability in programing computations. The use of these models in free vibration as well as in dynamic response analyses, has demonstrated that inclusion of transverse shear and rotatory inertia effects, leads to sufficient accuracy in simulating dynamic characteristics of vital nuclear components. Of special importance is the investigation into an isolated effect of rotatory inertia, whose contribution amounts to approximately 1% in lowering of fundemental frequency of plates. It is further concluded that frequencies obtained from Mindlin's theory model are slightly lower than those of Reissner's theory models, which in turn are lower than those computed from classical thin plate theory. In particular, the proposed method is well-suited for numerical analysis of thick-plate components in nuclear reactor and steam-generator.

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