

Globally Convergent Algorithms for Nonsmooth Nonlinear Equations in Computational Fluid Dynamics^{*}

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Abstract

In this paper we report on a computational study in which a nonsmooth discretization of the Euler equations for flow in a nozzle is solved with splitting method which is in turn globalized with the method of pseudo-transient continuation.

Key words: Pseudo-transient continuation, nonlinear equations, steady-state solutions, global convergence, splitting methods, Euler equations, MUSCL approximation

1 Introduction

In this paper we put an approach for solving steady-state problems in compressible flow into the context of a global splitting method for a nonsmooth nonlinear equation. We use this paradigm to make a conjecture about a convergence theorem and illustrate the ideas with a numerical experiment.

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Our setting is a hybrid of two ideas, splitting methods for nonsmooth nonlinear equations [1, 2] and pseudo-transient continuation (Ψ tc) [3–5], a method for finding steady-state solutions to partial differential equations that often succeeds when methods such as line searches fail. While the theory for Ψ tc [3] requires smooth nonlinearities, equations with nonsmooth nonlinearities have been solved by using Jacobian data for related smooth problems [3, 6–8]. Even in the smooth case [9] Jacobians for simpler discretizations have been used when the nonlinearity is complicated. This is exactly the paradigm of splitting methods.

In this introductory section we begin by describing the general structure of our problem. We then give a general discussion of splitting methods and Ψ tc, presenting some known theoretical results, and pointing out the gaps between the theory and the problems we consider here.

In § 2 we describe the hybrid algorithm which we will study in the remainder of the paper. We state a conjecture about the nature of the nonsmoothness in the application and then state and prove a local convergence result based on that conjecture. A complete analysis will be done in a subsequent paper.

In § 3 we describe a sample problem, flow through a nozzle, and show how a second order discretization of the governing Euler equations leads to a nonsmooth nonlinear equation. We show how our hybrid method can be realized with an approximating smooth problem using a first order discretization. In § 4 we illustrate the ideas with a computational study.

In general terms we seek to solve a nonlinear equation of the form

$$R(U) = 0 \tag{1}$$

where R is the spatial discretization of a problem in a function space. In the case considered here, the steady-state Euler equations, a high-order discretization must incorporate a nonsmooth flux limiter in order to produce an acceptable result [10].

Because of this nonsmoothness, a conventional Newton’s method approach must be modified. Beyond that, good initial data for a steady-state problem that has complicated structural features such as shocks is difficult to find, so a nonlinear iteration that performs well for good initial data, like the splitting methods we describe in § 1.1 must be modified when the initial data are far from the solution. Even for smooth problems, a conventional strategy, such as a line search [11–13] can either stagnate at a point where the Jacobian is singular or, more seriously, converge to a non-physical solution. Ψ tc is a way to address this by directly integrating the time-dependent equation to steady state while increasing the time step to retain efficiency. While complete temporal accuracy is sacrificed with this approach, the quality of the steady state solution is unchanged.

1.1 Splitting Methods for Nonsmooth Equations

Splitting methods [1, 2] decompose R in to smooth and nonsmooth ("rough") parts

$$R(U) = R_S(U) + R_R(U), \quad (2)$$

where R_S is Lipschitz continuously differentiable. This decomposition is used in the Newton-like iteration

$$U_+ = U_c - R_S(U_c)^{-1}R(U_c), \quad (3)$$

where U_c is the current iteration and U_+ the updated iteration.

In [1], the assumptions are that

Assumption 1.1

- *There is a solution U^* .*
- *$R'_S(U^*)$ is nonsingular.*
- *R_S is Lipschitz continuously differentiable.*
- *R_R is Lipschitz continuous.*

If Assumption 1.1 holds, we can relate the splitting iteration (3) to a Newton iteration for $F(U) = R_S(U) - R_S(U^*)$. In fact, $U_+ = U_c - F'(U_c)^{-1}F(U_c) + R_S(U_c)^{-1}(R_R(U^*) - R_R(U_c))$. Therefore, using the well known convergence theory for Newton's method, we have

$$\|U_+ - U^*\| = O(\|U_c - U^*\|^2 + \gamma_R\|U_c - U^*\|). \quad (4)$$

If γ_R is small, as one assumes in [1, 14], then the iteration will converge q-linearly. Obtaining a small γ_R in practice is difficult [2, 15]. Fast multi-level methods for parabolic control problems have been based on this idea [15–17], but the splittings must be constructed carefully if the convergence in the terminal phase is to be superlinear.

Inexact Newton methods [18] use as a Newton step any s that satisfies the inexact Newton condition

$$\|R'_S(U_c)s + R(U_c)\| \leq \eta_c\|R(U_c)\|. \quad (5)$$

If, for example, the step s is computed with an iterative method for linear equations, then the forcing term η_c in (5) is simply the tolerance on the relative linear residual. So, if $U_+ = U_c + s$, where s satisfies (5) then (4) becomes

$$\|U_+ - U^*\| = O(\|U_c - U^*\|^2 + (\eta_c + \gamma_R)\|U_c - U^*\|). \quad (6)$$

1.2 Pseudo-Transient Continuation

Ψ tc is a method for computing steady-state solutions of discretized time-dependent partial differential equations. Ψ tc is effective when standard methods, such as line search methods [11–13] fail by either finding non-physical solutions or stagnating at points where the Jacobian is singular.

Let $R(U) = 0$ be the steady state equation corresponding to the time-dependent initial problem

$$\frac{dU}{dt} = -R(U), \quad U(0) = U_0. \quad (7)$$

Ψ tc begins by integrating accurately in time, closely following the transient behavior in the early stages of the iteration, until an approximate steady state is reached and then increasing the time step, sacrificing temporal accuracy in the terminal phase in favor of rapid convergence to steady state.

The equation for a Ψ tc step is

$$(\delta_c^{-1}I + J_c) s = -R(U_c), \quad \text{where } J_c \approx R'(U_c), \quad (8)$$

where δ_c is the current “time step”.

One can view (8) as a single step of a Rosenbrock method [19] if δ is fixed. Ψ tc varies the step, but not to maintain temporal accuracy, but rather to move to steady state with the largest possible steps. A common formula, which we use in the numerical experiments in § 4, for adjusting the time step is the “switched evolution relaxation” (SER) method, [20],

$$\delta_n = \min(\delta_0 \|R(U_0)\| / \|R(U_n)\|, \delta_{max}). \quad (9)$$

SER is a common approach for fluid flow problems [3, 5, 21, 22].

The convergence theory from [3] gives conditions under which Ψ tc will converge to the desired steady-state solution, which we define as $U^* = \lim_{t \rightarrow \infty} U(t)$, where U is the solution of (7). The basic assumptions are that U^* is a stable steady-state solution and that the approximate Jacobians are well conditioned. Formally:

Assumption 1.2

- R' is Lipschitz continuous and uniformly bounded.
- The initial value problem (7) has a stable steady-state solution U^* .

- $\|(I + \delta R'(U))^{-1}\| \leq (1 + \beta\delta)^{-1}$
some β , all U near U^* , $\delta > 0$.
- $R'(U^*)$ is nonsingular.

Theorem 1.1 is a summary of the convergence results from [3]. In Theorem 1.1 we use an inexact Newton method, in which the step satisfies

$$\|(\delta_n^{-1}I + R'(U_n))s_n + R(U_n)\| \leq \eta_n \|R(U_n)\|. \quad (10)$$

Theorem 1.1 *Let Assumption 1.2 hold. If δ_0 is sufficiently small, s_n satisfies (10), and $\eta_n \leq \bar{\eta} < 1$ for some $\bar{\eta}$ sufficiently small, then $\delta_n \rightarrow \infty$, $U_n \rightarrow U^*$, and the convergence in the terminal phase of the iteration is described by*

$$\|U_{n+1} - U^*\| + O((\eta_n + \delta_n^{-1})\|U_n - U^*\| + \|U_n - U^*\|^2)$$

for n sufficiently large. In particular, if $\delta_{max} = \infty$ and $\eta_n = 0$, the convergence is q -quadratic.

The analysis in [3] identifies three phases in the convergence of the Ψ tc iteration. In the early phase, δ is small and the effort is spent in resolving those transient effects that must be resolved to find the solution. At the end of the first phase, the approximation of the solution is good, and δ is increased in the second phase. In the third and final phase, δ is large and the solution is good; this combination leads to fast convergence to a highly accurate solution.

2 Hybrid Algorithm

The hybrid algorithm combines splitting and Ψ tc by simply using $J_c = R_S(U_c)$ in the equation for the step.

$$(\delta_c^{-1}I + R_S(U_c))s = -R(U_c), \quad (11)$$

and in the inexact Newton condition

$$\|(\delta_n^{-1}I + R'_S(U_n))s_n + R(U_n)\| \leq \eta_n \|R(U_n)\|. \quad (12)$$

The resulting method has been used in computational fluid dynamics for many years. In fact, it is the only practical formulation if high-order discretizations are to be used. A complex form of the hybrid was used to generate the results in [3].

While simple to formulate, there has been no formal analysis of the hybrid. This paper is a first step in that direction. We perform a computational study of a simple example with a view toward numerical verification of a convergence result. We hope to do a complete analysis in future work.

We can however, formulate a conjecture about the size of the nonsmooth component and based on that conjecture prove a local convergence result. In the application of interest here, R is the discretization of a partial differential operator. The nonsmoothness comes from methods which are required to make high-order approximations correctly resolve shocks. R_S is a low-order discretization of the same operator. For the particular methods in § 3, R_S is first-order accurate and R is second order accurate, where accuracy is measured in the l^1 norm.

This motivates our conjecture. Letting Δx denote the spatial mesh width, we assume that there is $C > 0$ such that

$$\|R'_S(U_c)^{-1}(R_R(U^*) - R_R(U_c))\| \leq C (\Delta x \|U_c - U^*\|) \quad (13)$$

for all U_c sufficiently near U^* . Here $\|\cdot\|$ will denote the l^1 norm.

We will replace Assumption 1.1 with the stronger condition,

Assumption 2.1

- *There is a solution U^* .*
- *$R'_S(U^*)$ is nonsingular.*
- *R_S is Lipschitz continuously differentiable.*
- *Equation (13) holds.*

Assumption 2.1 will allow us to prove a local convergence result. The result will show that not only does $U_n \rightarrow U^*$ but also $\delta_n \rightarrow \infty$, which will imply that the convergence rate is the same as one would expect for an implementation based on (5). This result corresponds to the terminal phase of Ψ tc as described in [3]. As is standard, we state the convergence result by examining the transition between a current iterate U_c and a new iterate U_+ .

Theorem 2.1 *Let Assumption 2.1 hold and let $0 \leq \eta_c \leq \bar{\eta} < 1$. If U_c is sufficiently near U^* , $\bar{\eta}$ and Δx are sufficiently small, and δ_c and δ_{max} are sufficiently large, then the iteration defined by (12) and (9) satisfies*

$$\|U_+ - U^*\| \leq C_1(\|U_c - U^*\|^2 + (\eta_c + \Delta x + \delta_c^{-1})\|U_c - U^*\|), \quad (14)$$

for some $C_1 > 0$. Moreover, either

$$\delta_+ = \delta_{max} \text{ or } \delta_+ > \delta_c. \quad (15)$$

Proof. Let U_c be near enough to U^* and let $\bar{\eta}$ and Δx small enough so that (5) and (6) imply that

- (14) holds for all $0 \leq \eta_c \leq \bar{\eta}$,

- $\|U_+ - U^*\| \leq \|U_c - U^*\|/2$, and
- $\|R(U_+)\| < \|R(U_c)\|$.

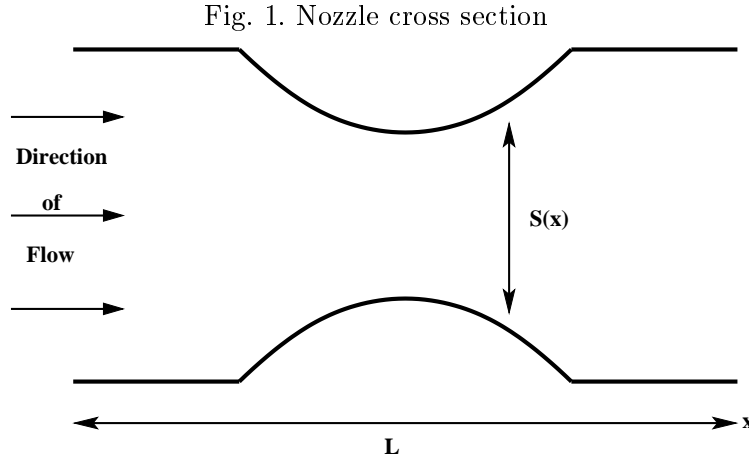
In that case, (9) implies (15). \square

Assuming that $\delta_{max} > 1/\Delta x$, which is certainly the case in realistic applications, Theorem 2.1 states that the limiting rate of convergence will be q-linear with a q-factor proportional to $\Delta x + \bar{\eta}$. In the application we describe in § 4, $\bar{\eta} = O(\Delta x)$ is roughly the error in an approximate Jacobian. We summarize these statements in a corollary.

Corollary 2.1 *Let Assumption 2.1 holds and let $0 \leq \eta_n \leq \bar{\eta} = O(\Delta x) < 1$. If U_0 is sufficiently near U^* , $\bar{\eta}$ and Δx are sufficiently small, δ_c and δ_{max} are sufficiently large, then the iteration given by (12) and (9) converges q-linearly with q-factor proportional to Δx .*

3 Euler Equations for Flow in a Nozzle

We consider flow through a nozzle of length L . The nozzle is a surface of revolution about the x axis with cross sectional area $S(x)$.



The governing equations for Quasi-One Dimensional Euler flow in conservative variables are [10]

$$\frac{\partial(\rho S)}{\partial t} + \frac{\partial(\rho u S)}{\partial x} = 0, \quad (16)$$

$$\frac{\partial(\rho u S)}{\partial t} + \frac{\partial[(\rho u^2 + p)S]}{\partial x} = p \frac{dS}{dx}, \text{ and} \quad (17)$$

$$\frac{\partial(\rho E S)}{\partial t} + \frac{\partial(\rho u H S)}{\partial x} = 0. \quad (18)$$

In (16), $\rho(t, x)$ is density, $p(t, x)$ is pressure, $u(t, x)$ is velocity, $S(x)$ is the cross-sectional area of the flow domain,

$$H = \frac{1}{\rho}(\rho E + p)$$

is stagnation enthalpy, and $E(t, x)$ is total energy. We will express the system in conservation form using the variables

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix}.$$

Let

$$F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (\rho E + p)u \end{pmatrix} \text{ and } W = \begin{pmatrix} 0 \\ p \frac{dS}{dx} \\ 0 \end{pmatrix}, \quad (19)$$

where $c = \sqrt{\gamma \frac{p}{\rho}}$ is the speed of sound, and γ is the ratio of the specific heat at constant pressure to the specific heat at constant volume. In this paper the gas is air and $\gamma = 1.4$.

The conservation form of the system is

$$SU_t + (SF)_x = W. \quad (20)$$

The steady state equation is

$$R(U) = (SF)_x - W = 0, \quad 0 < x < L, \quad (21)$$

where F and W are defined by (19).

Using the constitutive laws

$$\begin{aligned} E &= \frac{c^2}{\gamma(\gamma - 1)} + \frac{u^2}{2}, \\ \rho H &= \rho E + p, \text{ and} \\ p &= (\gamma - 1)\left(\rho E - \frac{\rho u^2}{2}\right) \quad (\text{ideal gas assumption}) \end{aligned}$$

we compute

$$\frac{\partial F}{\partial U} = \begin{bmatrix} 0 & 1 & 0 \\ -(3-\gamma)\frac{u^2}{2} & (3-\gamma)u & \gamma-1 \\ (\gamma-1)u^3 - \gamma u E & \gamma E - \frac{3}{2}(\gamma-1)u^2 & \gamma u \end{bmatrix}. \quad (22)$$

We will consider two finite volume discretizations. We divide the interval $[0, L]$ into N cells of length $\Delta x = L/N$, with cell centers x_i and cell edges $x_{i\pm\frac{1}{2}}$. All our discretizations have the form

$$R(U) = \frac{S_{i+\frac{1}{2}}F_{i+\frac{1}{2}} - S_{i-\frac{1}{2}}F_{i-\frac{1}{2}}}{\Delta x} - \begin{bmatrix} 0 \\ p_i \frac{S_{i+\frac{1}{2}} - S_{i-\frac{1}{2}}}{\Delta x} \\ 0 \end{bmatrix}. \quad (23)$$

Since U , and hence $F(U)$, is given only at the cell centers, one must define the cell-edge fluxes to fully specify the equations. The definition of the fluxes $F_{i\pm\frac{1}{2}}$ determine the order of the approximation and the smoothness of the nonlinearity.

3.1 Lax-Freidrichs Discretization

This is a first order accurate discretization. We will use it to construct R_S in the Hybrid algorithm.

The Lax-Friedrichs discretization is

$$\begin{aligned} \hat{F}_{i+\frac{1}{2}} &= \frac{1}{2}(F_{i+1} + F_i) - \frac{1}{2}|\lambda|_{i+\frac{1}{2}}(U_{i+1} - U_i), \\ \hat{F}_{i-\frac{1}{2}} &= \frac{1}{2}(F_i + F_{i-1}) - \frac{1}{2}|\lambda|_{i-\frac{1}{2}}(U_i - U_{i-1}), \end{aligned}$$

where $|\lambda| = |u| + c$.

Let

$$\hat{R}(U) = \frac{S_{i+\frac{1}{2}}\hat{F}_{i+\frac{1}{2}} - S_{i-\frac{1}{2}}\hat{F}_{i-\frac{1}{2}}}{\Delta x} - \begin{bmatrix} 0 \\ p_i \frac{S_{i+\frac{1}{2}} - S_{i-\frac{1}{2}}}{\Delta x} \\ 0 \end{bmatrix}. \quad (24)$$

Since $u \neq 0$ in this application, \widehat{F} is a smooth function of U and, when solving $R(U) = 0$, the theory in [3] is applicable.

The Jacobian $\widehat{R}'(U)$ is tridiagonal. A direct computation of $\widehat{R}'(U)$ requires additional computations of cell-edge values of λ . Instead we use the approximate Jacobian

$$J(U) = \begin{bmatrix} d_1^{(0)} & d_1^{(1)} & & & \\ d_2^{(-1)} & \cdots & \cdots & & \\ & \cdots & \cdots & d_{N-1}^{(1)} & \\ & & d_N^{(-1)} & d_N^{(0)} & \end{bmatrix}_{[3N \times 3N]} \approx \widehat{R}'(U). \quad (25)$$

In (25)

$$d_i^{(-1)} = -\frac{1}{2\Delta x} S_{i-1} \left(\left. \frac{\partial F}{\partial U} \right|_{i-1} + (|u| + c)_{i-1} I \right),$$

$$d_i^{(0)} = \frac{1}{\Delta x} S_i (|u| + c)_i I, \text{ and}$$

$$d_i^{(1)} = \frac{1}{2\Delta x} S_{i+1} \left(\left. \frac{\partial F}{\partial U} \right|_{i+1} - (|u| + c)_{i+1} I \right).$$

3.2 Roe Flux Differencing and MUSCL Extrapolation

In this section we describe a second order (in the ℓ^1 sense) method. The use of a slope-limiter is critical to this high accuracy, but is also the source of the nonsmoothness in the nonlinearity.

We define the cell-edge fluxes through right and left cell-edge values of U . Here

$$\begin{aligned} U_{i+\frac{1}{2}}^L &= U_i + \frac{1}{4} \left[(1 - \kappa) \Phi(r_{i-\frac{1}{2}}^+) (U_i - U_{i-1}) + (1 + \kappa) \Phi(r_{i+\frac{1}{2}}^-) (U_{i+1} - U_i) \right] \\ \text{and} & \\ U_{i+\frac{1}{2}}^R &= U_{i+1} - \frac{1}{4} \left[(1 - \kappa) \Phi(r_{i+\frac{3}{2}}^-) (U_{i+2} - U_{i+1}) + (1 + \kappa) \Phi(r_{i+\frac{1}{2}}^+) (U_{i+1} - U_i) \right]. \end{aligned} \quad (26)$$

In (26), r is defined by the ratio of successive differences of U :

$$r_{i-\frac{1}{2}}^+ = \frac{U_{i+2} - U_{i+1}}{U_{i+1} - U_i}, \quad r_{i+\frac{1}{2}}^- = \frac{U_i - U_{i-1}}{U_{i+1} - U_i},$$

$$r_{i+\frac{3}{2}}^- = \frac{U_{i+1} - U_i}{U_{i+2} - U_{i+1}}, \quad \text{and} \quad r_{i+\frac{1}{2}}^+ = \frac{U_{i+2} - U_{i+1}}{U_{i+1} - U_i}.$$

The nonlinear function Φ is a slope limiter. Φ is nonsmooth and must be used if a higher-order method is to perform well [10].

Two examples are the minmod limiter

$$\Phi^M(r) = \begin{cases} \min(1, r) & \text{if } r > 0 \\ 0 & \text{if } r \leq 0 \end{cases}$$

and the VanLeer limiter

$$\Phi^{VL}(r) = \frac{r + |r|}{1 + r}.$$

The cell-edge fluxes are

$$F_{i+\frac{1}{2}} = \frac{1}{2} \left(F_{i+\frac{1}{2}}^L + F_{i+\frac{1}{2}}^R - |\overline{A}|_{i+\frac{1}{2}} (U_{i+\frac{1}{2}}^R - U_{i+\frac{1}{2}}^L) \right),$$

where

$$F_{i+\frac{1}{2}}^L = F(U_{i+\frac{1}{2}}^L) \quad \text{and} \quad F_{i+\frac{1}{2}}^R = F(U_{i+\frac{1}{2}}^R).$$

The matrix $|\overline{A}|_{i+\frac{1}{2}}$ is another potential source of nonsmoothness. However in this paper the flow is entirely supersonic, and $|\overline{A}|_{i+\frac{1}{2}}$ is a smooth function of U . The construction of the Roe-averaged Jacobian \overline{A} begins by writing the Jacobian (22) in terms of ρ , u , and H to obtain

$$\frac{\partial F}{\partial U} = \begin{bmatrix} 0 & 1 & 0 \\ -(3-\gamma)\frac{u^2}{2} & (3-\gamma)u & \gamma-1 \\ (\gamma-1)u^3 - u(H + (\gamma-1)\frac{u^2}{2\rho^2}) & H + (\gamma-1)\frac{u^2}{2\rho^2} - \frac{3}{2}(\gamma-1)u^2 & \gamma u \end{bmatrix}. \quad (27)$$

We then define weighted variables,

$$R_{i+\frac{1}{2}} = \sqrt{\frac{\rho_{i+1}}{\rho_i}}, \quad \bar{\rho}_{i+\frac{1}{2}} = R_{i+\frac{1}{2}}\rho_i, \quad \bar{u}_{i+\frac{1}{2}} = \frac{R_{i+\frac{1}{2}}u_{i+1}+u_i}{R_{i+\frac{1}{2}}+1}, \quad \text{and} \quad \bar{H}_{i+\frac{1}{2}} = \frac{R_{i+\frac{1}{2}}H_{i+1}+H_i}{R_{i+\frac{1}{2}}+1}. \quad (28)$$

The Roe averaged Jacobian is

$$\bar{A}_{i+\frac{1}{2}} = \begin{bmatrix} 0 & 1 & 0 \\ -(3-\gamma)\frac{\bar{u}^2}{2} & (3-\gamma)\bar{u} & \gamma-1 \\ (\gamma-1)\bar{u}^3 - \bar{u}(\bar{H} + (\gamma-1)\frac{\bar{u}^2}{2\bar{\rho}}) \bar{H} + (\gamma-1)\frac{\bar{u}^2}{2\bar{\rho}} - \frac{3}{2}(\gamma-1)\bar{u}^2 & \gamma\bar{u} & \end{bmatrix},$$

where the variables are evaluated at cell edges.

At each cell-edge the Roe-averaged Jacobian \bar{A} has eigenvalues

$$\bar{\lambda}_{(1)} = \bar{u}, \quad \bar{\lambda}_{(2)} = \bar{u} + \bar{c}, \quad \bar{\lambda}_{(3)} = \bar{u} - \bar{c}, \quad (29)$$

and eigenvectors

$$\bar{r}^{(1)} = \begin{bmatrix} 1 \\ \bar{u} \\ \frac{\bar{u}^2}{2} \end{bmatrix}, \quad \bar{r}^{(2)} = \begin{bmatrix} 1 \\ \bar{u} + \bar{c} \\ \bar{H} + \bar{u}\bar{c} \end{bmatrix} \frac{\bar{\rho}}{2\bar{c}}, \quad \text{and} \quad \bar{r}^{(3)} = \begin{bmatrix} 1 \\ \bar{u} - \bar{c} \\ \bar{H} - \bar{u}\bar{c} \end{bmatrix} \frac{\bar{\rho}}{2\bar{c}}. \quad (30)$$

The matrix $|\bar{A}|$ is the matrix with the same eigenvectors as \bar{A} and with $\{|\bar{\lambda}_i|\}_{i=1}^3$ as the eigenvalues. These eigenvalues change sign only if parts of the flow are subsonic and other parts are supersonic, which is not the case in the example considered in this paper.

4 Numerical Results

In the computational results reported here, $L = 2$, $N = 2000$, and

$$S(x) = \begin{cases} 1 + 4(x-1)^2 & \text{if } 0.5 < x < 1.5 \\ 2 & \text{otherwise} \end{cases}$$

The boundary conditions are supersonic flow in the inlet. The initial iterate is the inlet boundary conditions at all points. To check our numerical results, we computed the exact

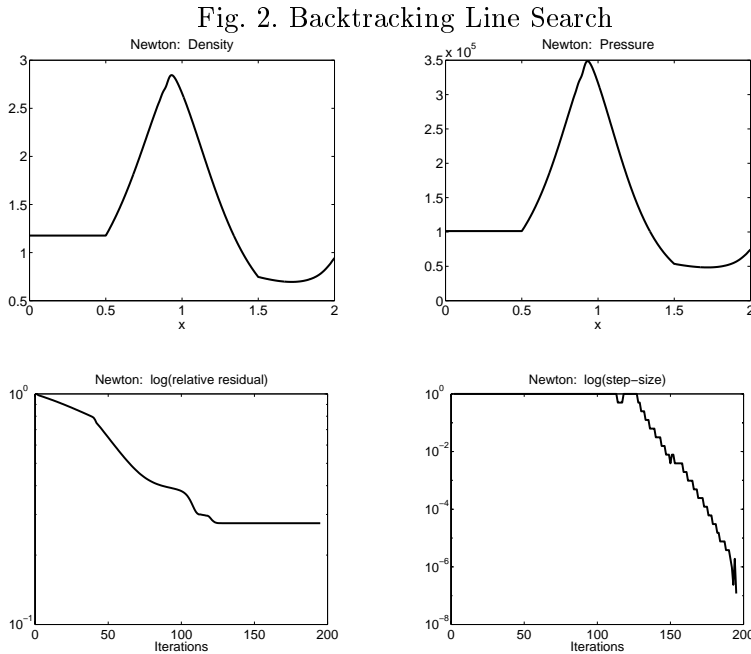
steady state solution using the isentropic gas equations along with conservation of total pressure and total temperature [23]. One of the resulting equations is nonlinear and we used Newton's method to determine its solution on each cell. The solutions from Ψtc are qualitatively and quantitatively correct according to this exact steady state solution.

We compare two methods, the hybrid and a line search approach which takes the inexact splitting Newton step $s = -J(U_c)^{-1}R(U_c)$ and then lets $U_+ = U_c + \lambda s$ where $\lambda = 2^{-m}$ where we use the smallest integer $m \geq 0$ such that

$$\|R(U_c + \lambda s)\| \leq (1 - 10^{-4}\lambda)\|R(U_c)\|. \quad (31)$$

For smooth problems there is a complete theory for this approach. Either the iteration will converge to a solution, diverge to infinity, or stagnate at a point where the Jacobian is singular [11]. For splitting Newton methods, one can prove similar results if γ_R is sufficiently small.

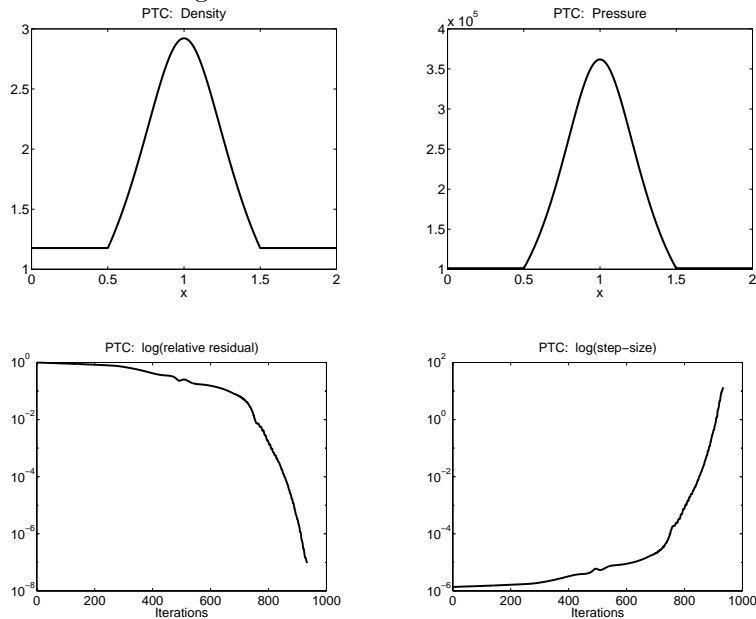
For this example, the line search fails by stagnating, as one can see both from the residual history and stepsize curves in Figure 2. The pressure and density curves are also clearly not correct, lacking the symmetry in the physical solution.



The hybrid method, where $U_+ = U_c - (\delta_c^{-1}I + J(U_c))^{-1}R(U_c)$, succeeds. The residuals in Figure 3 converge to zero rapidly in a way consistent with the prediction of Theorem 2.1 when

$$\bar{\eta} \approx \max_n \|J(U_n) - R'_S(U_n)\| = O(\Delta x).$$

Fig. 3. Pseudo-transient Continuation



5 Conclusion

We show how an approach used in the computational fluid dynamics literature for computing steady state solutions of compressible flow problems can be expressed as a splitting method. We prove a local convergence result based on an assumption about the errors in the discretizations and the nonsmooth methods developed in [1,2]. Numerical results are consistent with the predictions of the theory.

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