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ELASTIC-PLASTIC AND CREEP ANALYSES BY ASSUMED STRESS FINITE ELEMENTS

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SUMMARY

A formulation is presented of incremental finite element solutions for both initial stress and initial strain problems based on modified complementary energy principle with relaxed inter-element continuity requirement. The corresponding finite element model is the assumed stress hybrid model which has stress parameters in the interior of each element and displacements at the individual nodes as unknowns. The formulation includes an important consideration that the states of stress and strain and the beginning of each increment may not satisfy the equilibrium and compatibility equations. These imbalance and mismatch conditions all lead to correction terms for the equivalent nodal forces of the matrix equations.

The initial stress method is applied to elastic-plastic analysis of structures. In this case the stress parameters for the individual elements can be eliminated resulting to a system of equations with only nodal displacements as unknowns. Two different complementary energy principles can be formulated, in one of which the equilibrium of the final state of stress is maintained while in the other the equilibrium of the stress increments is maintained. Each of these two different formulations can be combined with different iterative schemes to be used at each incremental steps of the elastic-plastic analysis. It is also indicated clearly that for the initial stress method the state of stress at the beginning of each increment is, in general, not in equilibrium and an imbalance correction is needed. The paper presents the results of a comprehensive evaluation of various solution procedures by the initial stress method using the assumed stress hybrid element. The example used is the static response of a thick wall cylinder of elastic-perfectly plastic material under internal pressure. Solid of revolution elements with rectangular cross sections are used. For the various procedures the errors in displacements and stresses and the computer time needed for obtaining a solution under a pressure loading equal to 93.75% of the collapse load are compared. The conclusion is that the method based on the equilibrium of the stress increments is computationally more efficient than that based on the equilibrium of total stresses. It has also been found that the inclusion of equilibrium imbalance corrections is an essential step in improving the accuracy. Finally a comparison was made with results obtained by a conventional assumed-displacement elements. For this thick cylinder problem the assumed stress hybrid elements provide errors about one order less than that by the conventional elements. The accuracy of the assumed stress hybrid method is also verified by results of a thick spherical shell under internal pressure analyzed by using solid of revolution elements of quadrilateral cross-section.

The initial strain method is applied to creep analyses with steady-static creep or with transient creep under time hardening rule. In this case it is found convenient to leave the stress parameters as unknowns in the resulting system of equations, which are first order differential equations and hence are solved by many established numerical integration techniques.

1. Introduction

The assumed stress hybrid model [1][2] [3] in finite element analyses is based on element stiffness matrices, hence, similar to the conventional assumed displacement method, it consists of the nodal displacements as the unknowns in the resulting matrix equations. By means of this model, the element stiffness matrix can be derived for more general element geometry, because it is based on independently assumed stresses in the interior of each element and displacements only along the element boundaries. It has also been demonstrated that the assumed stress hybrid model usually yields more accurate solutions both for nodal displacements and for the stress distribution within each element. This model, thus, is even more attractive for elastic-plastic and creep analyses, both of which call for more accurate estimation of the stress distribution.

Systematic studies for various approaches in elastic-plastic solutions by the assumed stress hybrid model have been made [4] [5] and a procedure for analyzing creep problems by this model has also been suggested [6]. The present paper is restricted to two specific applications of the assumed stress hybrid model. The first is an elastic-plastic analysis by incremental solutions using an initial stress approach, while the second is a creep analysis under stationary or time-hardening creep behavior. The latter is, of course, also an incremental solution for which the creep strains in each time increment are considered as initial strains. In the following section the variational principle governing both initial stress and strain approaches and the corresponding finite element formulations by the assumed stress hybrid model are presented. Both problems are then illustrated by examples involving solid of revolutions under axisymmetric loading conditions.

2. Formulation

In the incremental initial stress approach which was initiated by Zienkiewicz et al [7], a finite element solution for each increment of loading is first made by assuming the material to be elastic. In the plastic region this results in an overestimation of the equivalent stress. Corrections in stress can be estimated and are introduced as initial stresses which result in an equivalent loading vector in the finite element formulation. Let $\Delta \underset{\sim}{\sigma}$ be a vector of elastic stress increments corresponding to strain increments $\Delta \underset{\sim}{\varepsilon}$ which, in turn, are expressed in terms of displacement increments $\Delta \underset{\sim}{u}$. Let $\Delta \underset{\sim}{\sigma}'$ be the actual stress increments on account of plastic deformation, then the initial stress to be considered for the next round of incremental solution is,

$$\underset{\sim}{\sigma}^{o} = \Delta \underset{\sim}{\sigma}' - \Delta \underset{\sim}{\sigma} \tag{1}$$

For the creep analysis the initial strain increments for each time increment are designated by $\Delta \underset{\sim}{\varepsilon}^{c}$.

The assumed stress hybrid model can be formulated by a modified complementary energy principle [2][3]. The stress equilibrium conditions along the interelement boundaries are introduced as conditions of constraints and the boundary displacements which are the Langrange multipliers are then introduced as additional field variables. The modified complementory energy principle can be derived from the Hellinger-Reissner principle [8]. With the contribution of initial stress and initial strain the variational functional of the Hellinger-Reissner principle takes the form,

$$\pi_R = \sum_n \Big[\int_{V_n} \{ -\tfrac{1}{2} \Delta\underset{\sim}{\sigma}^T \underset{\sim}{S} \Delta\underset{\sim}{\sigma} + (\Delta\underset{\sim}{\sigma}^T + \underset{\sim}{\sigma}^{o^T})(\underset{\sim}{D}\Delta\underset{\sim}{u}) - \Delta\underset{\sim}{\sigma}^T \Delta\underset{\sim}{\varepsilon}^c + \Delta\bar{\underset{\sim}{F}}^T \Delta\underset{\sim}{u} \} dV$$
$$+ \int_{S_{\sigma_n}} \Delta\bar{\underset{\sim}{T}}^T \Delta\underset{\sim}{u}\, dS + \int_{S_{u_n}} \Delta\underset{\sim}{T}^T (\Delta\underset{\sim}{u} - \Delta\bar{\underset{\sim}{u}})\, dS \Big] \qquad (2)$$

where

$\underset{\sim}{S}$ = elastic compliance matrix

$\underset{\sim}{D}$ = matrix of differential operators that defines the strain displacement relations

$\Delta\bar{\underset{\sim}{F}}$ = prescribed body force increments

$\Delta\underset{\sim}{T}$ = surface traction increments

V_n = volume of the n - th element

S_{u_n} = boundary of the n - th element over which displacements $\Delta\bar{\underset{\sim}{u}}$ are prescribed

S_{σ_n} = boundary of the n - th element over which tractions $\Delta\bar{\underset{\sim}{T}}$ are prescribed

Here $\sum_n$ denotes summation over all elements and the displacements $\Delta\underset{\sim}{u}$ are compatible along the inter-element boundaries.

Consider now that the stress increments $\Delta\underset{\sim}{\sigma}$ is in equilibrium, i.e.

$$\bar{\underset{\sim}{D}}\, \Delta\underset{\sim}{\sigma} + \Delta\bar{\underset{\sim}{F}} = 0 \,, \quad in\ V_n \qquad (3)$$

and $$\Delta\underset{\sim}{\sigma}^T \underset{\sim}{\nu} = \Delta\underset{\sim}{T} \quad on\ S$$

where $\bar{\underset{\sim}{D}}$ represents a matrix of differential operators and $\underset{\sim}{\nu}$ represents the directional cosine of the surface normal. Then when the divergence theorem is applied the variational functional becomes

$$\pi = -\sum_n \Big[\int_{V_n} \{ \tfrac{1}{2} \Delta\underset{\sim}{\sigma}^T \underset{\sim}{S} \Delta\underset{\sim}{\sigma} - \underset{\sim}{\sigma}^{o^T} (\underset{\sim}{D}\Delta\underset{\sim}{u}) + \Delta\underset{\sim}{\sigma}^T \Delta\underset{\sim}{\varepsilon}^c \} dV \qquad (4)$$
$$- \int_{\partial V_n} \Delta\underset{\sim}{T}^T \Delta\tilde{\underset{\sim}{u}}\, dS + \int_{S_{\sigma_n}} \Delta\bar{\underset{\sim}{T}}^T \Delta\tilde{\underset{\sim}{u}}\, dS \Big] = -\pi_{mc}$$

where ∂V_n is the entire boundary of the n - th element and the displacements along the element boundaries are now denoted by $\tilde{\underset{\sim}{u}}$. It is seen that this functional contains not only element boundary displacements but also element interior displacements $\underset{\sim}{u}$.

In the formulation of the corresponding finite element equations, it is assumed, for simplicity, that the body forces $\bar{\underset{\sim}{F}}$ are absent. Then, the stress increments and boundary traction increments for each element are expressed in terms of finite number of stress parameters $\Delta\underset{\sim}{\beta}$, i.e.

$$\Delta\underset{\sim}{\sigma} = \underset{\sim}{P}\,\Delta\underset{\sim}{\beta} \quad ; \quad \Delta\underset{\sim}{T} = \underset{\sim}{R}\,\Delta\underset{\sim}{\beta} \qquad (5)$$

The boundary displacements, the interior displacements and the strain increments are expressed in terms of nodal displacement increments $\Delta\underset{\sim}{q}$,

$$\Delta\tilde{\underset{\sim}{u}} = \underset{\sim}{L}\,\Delta\underset{\sim}{q}$$

$$\Delta \underset{\sim}{u} = \underset{\sim}{A}\, \Delta \underset{\sim}{q} \quad ; \quad \underset{\sim}{D}\, \Delta \underset{\sim}{u} = \underset{\sim}{B}\, \Delta \underset{\sim}{q} \tag{6}$$

Substituting eqs (5) and (6) into (4) gives

$$\Pi_{mc} = \sum_n \left[\tfrac{1}{2} \Delta \underset{\sim}{\beta}^T \underset{\sim}{H}\, \Delta \underset{\sim}{\beta} - \Delta \underset{\sim}{q}^T \underset{\sim}{Q}_n^{\circ} + \Delta \underset{\sim}{\beta}^T \Delta \underset{\sim}{J}^c - \Delta \underset{\sim}{\beta}^T \underset{\sim}{G}\, \Delta \underset{\sim}{q} + \Delta \underset{\sim}{q}^T \Delta \underset{\sim}{Q}_n \right] \tag{7}$$

where

$$\begin{aligned} \underset{\sim}{H} &= \int_{V_n} \underset{\sim}{P}^T \underset{\sim}{S}\, \underset{\sim}{P}\, dV \quad ; & \underset{\sim}{Q}_n^{\circ} &= \int_{V_n} \underset{\sim}{B}^T \underset{\sim}{\sigma}^{\circ}\, dV \\ \Delta \underset{\sim}{J}^c &= \int_{V_n} \underset{\sim}{P}^T \Delta \underset{\sim}{\varepsilon}^c\, dV \quad ; & \Delta \underset{\sim}{Q}_n &= \int_{S_{\sigma_n}} \underset{\sim}{R}^T \Delta \bar{\underset{\sim}{T}}\, dS \\ \underset{\sim}{G} &= \int_{\partial V_n} \underset{\sim}{R}^T \underset{\sim}{L}\, dS \end{aligned} \tag{8}$$

The vanishing of the variations of Π_{mc} with respecto to $\Delta \underset{\sim}{\beta}$ at the element level yields,

$$\Delta \underset{\sim}{\beta} = \underset{\sim}{H}^{-1} (\underset{\sim}{G}\, \Delta \underset{\sim}{q} - \Delta \underset{\sim}{J}^c) \tag{9}$$

By substituting back into eq (7), Π_{mc} becomes,

$$\Pi_{mc} = \sum_n \left[\tfrac{1}{2} \Delta \underset{\sim}{q}^T \underset{\sim}{k}_n\, \Delta \underset{\sim}{q} - \Delta \underset{\sim}{q}^T (\Delta \underset{\sim}{Q}_n + \underset{\sim}{Q}_n^{\circ} + \Delta \underset{\sim}{Q}_n^c) \right] \tag{10}$$

where

$$\begin{aligned} \underset{\sim}{k}_n &= \underset{\sim}{G}^T \underset{\sim}{H}^{-1} \underset{\sim}{G} = \text{element stiffness matrix} \\ \Delta \underset{\sim}{Q}_n^c &= \underset{\sim}{G}^T \underset{\sim}{H}^{-1} \Delta \underset{\sim}{J}^c \end{aligned} \tag{11}$$

The element matrices can be assembled into the corresponding global matrices $\underset{\sim}{K}, \Delta \underset{\sim}{Q}, \underset{\sim}{Q}^{\circ}$ and $\Delta \underset{\sim}{Q}^c$ and the variation of Π_{mc} with respect to the unrestrained nodal displacements then yields the matrix equation

$$\underset{\sim}{K}\, \Delta \underset{\sim}{q} = \Delta \underset{\sim}{Q} + \underset{\sim}{Q}^{\circ} + \Delta \underset{\sim}{Q}^c \tag{12}$$

The procedure of elastic-plastic analyses consists of giving values of $\underset{\sim}{\sigma}^{\circ}$ and evaluating $\underset{\sim}{Q}_n^{\circ}$, $\Delta \underset{\sim}{q}$ and $\Delta \underset{\sim}{\beta}$, in turn, using eqs (8), (12), and (9). Then the stress increment $\Delta \underset{\sim}{\sigma}$ can be calculated directly from eq (5). The strain increments $\Delta \underset{\sim}{\varepsilon}$ can be determined by introducing the elastic compliance matrix, and the actual stress increments $\Delta \underset{\sim}{\sigma}^{ep}$ in the plastic range is then determined for a given elastic-plastic material behaviour. Here $\Delta \underset{\sim}{\sigma}^{ep}$ will, in general, not concide with $\Delta \underset{\sim}{\sigma}'$, hence an iterative procedure is needed. Also since $\Delta \underset{\sim}{\sigma}^{ep}$ will, in general, not satisfy the stress equilibrium condition, a correction for initial equilibrium imbalance is required [9].

In the case of creep analysis, however, the incremental creep strain $\Delta \underset{\sim}{\varepsilon}^c$ is a function of the state of stress, hence the equivalent nodal forces $\Delta \underset{\sim}{Q}^c$ are functions of the stress parameters $\underset{\sim}{\beta}$. Thus eq (13) must be solved simultaneously with eq (9). For example, for stationary creep and for transient creep under time hardening rule the rates of creep strain may be expressed as

$$\dot{\underset{\sim}{\varepsilon}}^{c} = \underset{\sim}{f}\,(\underset{\sim}{\beta}\,,\,t) \tag{13}$$

The corresponding rate equations are

$$\begin{aligned} \dot{\underset{\sim}{\beta}} &= \underset{\sim}{H}^{-1}(\underset{\sim}{G}\,\dot{\underset{\sim}{q}} - \dot{\underset{\sim}{J}}^{c}) \\ \underset{\sim}{K}\,\dot{\underset{\sim}{q}} &= \dot{\underset{\sim}{Q}} + \sum_{n} \underset{\sim}{G}^{T}\underset{\sim}{H}^{-1}\dot{\underset{\sim}{J}}^{c} \end{aligned} \tag{14}$$

where

$$\dot{\underset{\sim}{J}}^{c} = \int_{V_n} \underset{\sim}{P}^{T}\underset{\sim}{f}\,(\underset{\sim}{\beta}\,,\,t)\,dV \tag{15}$$

This is a system of first-order nonlinear differential equations. With the initial conditions of $\underset{\sim}{\beta}$ and $\underset{\sim}{q}$ obtained by a linear elastic solution, eqs (14) and (15) can be solved by many numerical methods.

3. Examples

One example problem in the elastic-plastic analysis by the initial stress approach is an elastic-perfectly-plastic thick cylinder under internal pressure. This plane strain problem is solved by using a single row of five axisymmetric ring element of rectangular cross-section. An assumed stress hybrid element for rings of general quadrilateral section with 4 nodes and 8 β-parameters has been derived [5] and is used for this analysis. For a cylinder with a ratio of 2 to 1 between the outer and inner radii and an applied pressure equal to 93.75 % of the total collapse pressure, finite element solutions are performed using different loading increments and different numerical integration schemes. The effects of iteration and equilibrium imbalance corrections are also investigated. Table 1 lists the percentage errors of the radial displacement u_r at the inner surface and the stress components σ_r and σ_θ at the first element at the inner surface. Also included for comparison is the result obtained by the assumed displacement method [7] using 4-node isoparametric elements. To evaluate the efficiencies the C.P.U. time (IBM 370/155) required for each method is also listed for comparison.

These results indicate that the iterative procedure is importatnt for the accuracy of the elastic-plastic analysis and the equilibrium imbalance correction is a very inexpensive step that should be included for improving the accuracy of the solution. The computing times needed by the assumed stress hybrid method and the assumed displacement method are actually very comparable if the same number of integration stations is used. However, the assumed stress hybrid model yields much more accurate results.

The assumed stress hybrid element has also been utilized to analyze a thick sphere ($r_i/r_o = 3/5$) of elastic-perfectly-plastic material under internal pressure. In this analysis, a half of the sphere is divided into 40 quadrilateral elements, four along the radial direction and 10 along the circumferential direction. In one solution the total internal pressure is equal to 98 % of the collapse pressure and the loading increment is equal to 1/40 of the total pressure. When both iteration procedure and equilibrium imbalance corrections are applied, the percentage errors of the radial displacement u_R at the inner surface and a polar angle of 45° and, the radial and circumferential stresses σ_R and σ_θ at the centroid of an element at the inner surface and a polar angle of 49.5° are respectivly, 8.61 %, 3.48 %, and 17.60 %. The total C.P.U. time is 0.757 min. In another solution for which the loading increment is equal to 1/200 of the total pressure but the iteration procedure is not included, the corresponding errors are 7.18 %, 3.29 %, and 15.90 % respectively, and the total C.P.U. time is 0.735 min. It is seen that in this case the more efficient procedure is to use smaller loading increment and to eliminate the iteration procedure.

An example solution for the creep problems is a thickness cylinder of 0.16 inch inner radius and 0.25 inch outer radius under constant internal pressure of 365 psi. The elastic property is given by E = 2 x 10 psi and ν = 0.498 while the creep behavior is governed by stationary creep with the following relation between the rate of equivalent strain $\dot{\varepsilon}_e^c$ and the equivalent stress

$$\dot{\varepsilon}_e^c = 6.4 \times 10^{-18} \sigma_e^{4.4}$$

where the strain rate is in terms of hour^{-1} and the stress is in psi.

This is the same problem that was analized by Greenbaum and Rubinstein [10]. In the present solution one row of twelve axisymmetric ring elements of rectangular cross-section was used. Numerical integration for each element are performed by 3 x 3 Gaussian quadrature. The system of first-order nonlinear differential equation was solved by a 4th order Runge-Kutta method. The time variations of the equivalent stresses at the centroids of the inner most and outer most elements are shown in Figure I. The time scale τ is the actual time in hours. It is seen that after about one hour from the initial loading the stress distribution reaches its stationary condition. Identical solutions were obtained using different time increments in the Runge-Kutta method. The relative error of the stationary stress values is less than 0.01 %, and the total computing time up to τ = 2.0 is 3.87 sec. Reference [10] also include a result similar to Figure I. obtained by using 24 constant strain triangular elements. In that result, however, the time at which the stress distribution becomes stationary is around two hours.

4. Conclusion

The assumed stress hybrid finite element method has been derived in a rational manner for elastic-plastic analysis and for creep analysis, and its practicality has been demonstrated by several examples involving solids of revolution under axisymmetric loading conditions.

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Table I

Thick Cylinder of Elastic-Perfectly-Plastic Material

under Internal Pressure

Applied Pressure = 93.75 % of collapse pressure; load increment = total load / 60

Method	Assumed Stress Hybrid Model				Assumed Displacement Method
Iteration	No	No	Yes	Yes	Yes
Equilibrium correction	No	Yes	No	Yes	No
% error in u_r	16.39	4.61	0.87	0.35	5.00
% error in σ_r	10.09	1.56	0.47	0.23	2.74
% error in σ_θ	12.52	2.04	0.58	0.19	3.40
No. of Integration Station	4	4	4	4	9
CPU Time (Min.)	0.025	0.030	0.195	0.223	0.450

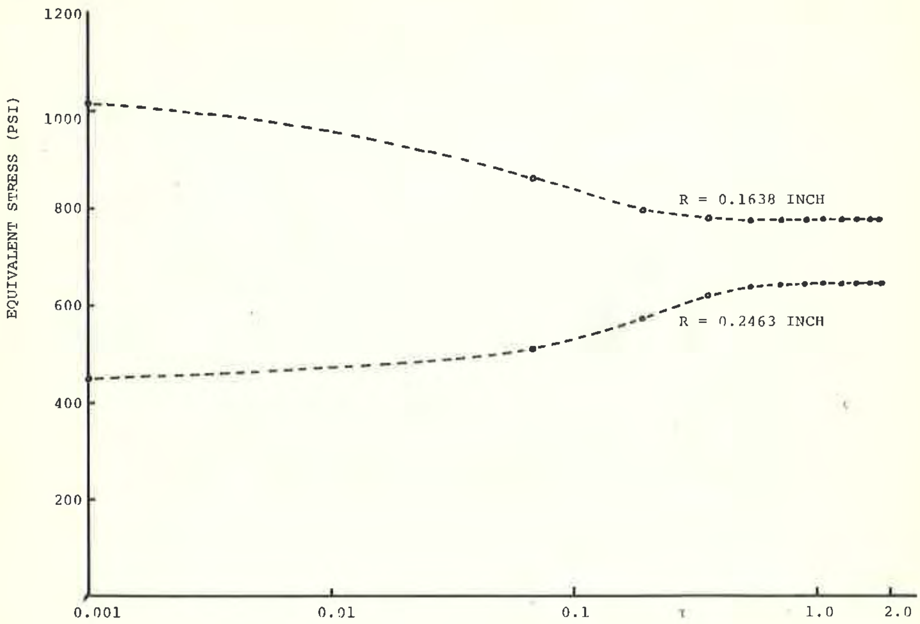

Fig. 1. Equivalent Stress vs. Parameter τ.