A CHARACTERIZATION OF THE GRAPH

OF THE T_m ASSOCIATION SCHEME

by

Thomas A. Dowling

University of North Carolina

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Abstract

The graph G of the T_m association scheme is a graph on $\binom{n}{m}$ vertices, where $2 \leq m \leq n/2$, such that the vertices may be identified with unordered m-plets $(a_1, a_2, \ldots a_m)$ from among the n symbols $1, 2, \ldots, n$ and two vertices of G are adjacent if and only if the corresponding m-plets have exactly m-1 common symbols. If $d(x,y)$ denotes the distance between two vertices x and y and $\Delta(x,y)$ denotes the number of vertices adjacent to both x and y, then G is a connected graph possessing the following properties: (a_1) The number of vertices in G is $\binom{n}{m}$; (a_2) G is regular of valence $m(n-m)$; (a_3) $\Delta(x,y) = n-2$ if $d(x,y) = 1$; (a_4) $\Delta(x,y) = 4$ if $d(x,y) = 2$. In this paper it is shown that, conversely, if G is a connected graph and (a_1)-(a_4) hold for some $m > 2$ and $n > 2m(m-1) + 4$, then G is necessarily the graph of the T_m association scheme. This result generalizes previously known results for the cases m = 2, 3 corresponding to the triangular and tetrahedral schemes.

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DEPARTMENT OF STATISTICS
UNIVERSITY OF NORTH CAROLINA
Chapel Hill, N. C.

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1. Introduction. An m-class association scheme, called the T_m scheme, was introduced in [7] as a generalization of the well-known triangular scheme with two associate classes. The T_m scheme may be defined as follows: Let m and n be positive integers satisfying $2 \leq m \leq n/2$, and suppose we are given $v = \binom{n}{m}$ treatments $\varphi_1, \varphi_2, \ldots, \varphi_v$. To each of the v unordered m-plets $(a_1, a_2, \ldots, a_m)$ of distinct symbols from among $1, 2, \ldots, n$, we associate a treatment φ in some one-to-one manner. Two treatments φ and φ' are said to be u-th associates of each other if and only if the corresponding m-plets $(a_1, a_2, \ldots, a_m)$ and $(a_1', a_2', \ldots, a_m')$ contain exactly m-u symbols in common. The triangular association scheme clearly corresponds to the case m = 2.

The parameters of the T_m scheme are given in [7] as follows: The number of treatments is

$$v = \binom{n}{m}. \tag{1}$$

The number of u-th associates of each treatment is

$$n_u = \binom{m}{u}\binom{n-m}{u}, \quad u = 0, 1, \ldots, m. \tag{2}$$

For any two treatments φ and φ' which are t-th associates, the number of treatments which are s-th associates of φ and at the same time u-th associates of φ' is

$$p_{su}^{t} = \sum_{i=0}^{m-t} \binom{m-t}{i} \binom{t}{m-s-i} \binom{t}{m-u-i} \binom{n-m-t}{s+u-m+i}, \quad u, s, t = 0, 1, \ldots, m, \tag{3}$$

where as usual we take $\binom{x}{y} = 0$ if $x < y$ or $y < 0$.

Given any association scheme, the graph of the scheme is constructed by identifying the v treatments with vertices, and joining two vertices by an edge if and only if the corresponding treatments are first associates.

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Thus two vertices of the graph G of a T_m scheme are adjacent if and only if the corresponding m-plets have exactly m-1 symbols in common. More generally, one can easily show that two vertices of G are at distance u from each other if and only if the corresponding m-plets have exactly m-u symbols in common, i.e., if and only if the corresponding treatments are u-th associates.

Using (1), (2), and (3) we have in particular for a T_m scheme $v = \binom{n}{m}$, $n_1 = m(n-m)$, $p_{11}^1 = n-2$, $p_{11}^2 = 4$. If for any finite, connected graph we denote by $d(x,y)$ the distance between the vertices x and y, and by $\Delta(x,y)$ the number of vertices adjacent to both x and y, then the above parameters imply that the graph G of a T_m scheme possesses the following properties:

(a_1) The number of vertices in G is $\binom{n}{m}$;

(a_2) G is regular of valence $m(n-m)$;

(a_3) $\Delta(x,y) = n-2$ if $d(x,y) = 1$;

(a_4) $\Delta(x,y) = 4$ if $d(x,y) = 2$.

In the case $m = 2$, corresponding to the triangular association scheme, Connor [2] showed (using somewhat different terminology) that if $n > 8$ the triangular graph is characterized by properties (a_1)-(a_4). Shrikhande [8], Li-Chien [5,6], and Hoffman [3,4] completed Connor's work by showing that the same result holds if $n < 8$, but that if $n = 8$ then there exist graphs satisfying (a_1)-(a_4) which are not triangular. Recently Bose and Laskar [1] have investigated the graph of the T_3 scheme, which they called a tetrahedral graph, and were able to show that a connected graph satisfying (a_1)-(a_4) for $m = 3$ is tetrahedral if $n > 16$. In this paper we generalize these results to an arbitrary $m \geq 2$. The main theorem states that if G is a connected graph satisfying (a_1)-(a_4) for some integers $m \geq 2$ and $n > 2m(m-1) + 4$, then G is the graph of a T_m scheme. Equivalently, we can say that the T_m scheme is characterized by the four parameters v, n_1, p_{11}^1, p_{11}^2 when $n > 2m(m-1) + 4$. Little is known at present concerning the uniqueness of the T_m scheme when $m \geq 3$ and $n \leq 2m(m-1) + 4$.

<u>2. Characterization of the T_m graph.</u> We shall consider only finite, undirected graphs G with no loops and no multiple edges. For completeness it is convenient to recall here several definitions concerning such graphs.

A chain $C = (x_1, x_2, \ldots, x_t)$ is a sequence of $t \geq 2$ distinct vertices of G, not necessarily all distinct, such that any two consecutive vertices in the sequence are adjacent. Thus the pairs $(x_1,x_2), (x_2,x_3), \ldots, (x_{t-1},x_t)$ are edges of G. The number of edges $t-1$ is the length of C, and C is said to join x_1 and x_t.

G is a connected graph if there exists a chain joining x and y for every pair of distinct vertices x,y. For a connected graph the distance $d(x,y)$ is defined to be the length of the shortest chain joining x and y. Clearly $d(x,y) = 1$ if and only if x and y are adjacent vertices. By convention we take $d(x,x) = 0$.

The valence $v(x)$ of a vertex x is the number of vertices in G adjacent to x. If there exists an integer n_1 such that $v(x) = n_1$ for every vertex x in G, then G is said to be regular of valence n_1.

For any two distinct vertices x,y of G, we denote by $\Delta(x,y)$ the number of vertices adjacent to both x and y. Obviously $\Delta(x,y) = 0$ if $d(x,y) > 2$. If x and y are adjacent vertices, then $\Delta(x,y)$ is called the edge-degree of the edge (x,y). A regular graph G for which all edges have the same edge-degree Δ is said to be edge-regular with edge-degree Δ.

A clique K is a set of mutually adjacent vertices in G. K is said to be complete if there does not exist a vertex $x \notin K$ such that $x \cup K$ is a clique.

We now quote a result due to Bose and Laskar [1] which is of fundamental importance in the sequel. Consider an edge-regular graph G with valence $r(k-1)$ and edge-degree $(k-2) + \alpha$ such that $\Delta(x,y) \leq 1 + \beta$ for every pair of non-adjacent vertices x,y, where r,k,α,β are fixed integers satisfying $r \geq 1$, $k \geq 2$, $\alpha \geq 0, \beta \geq 0$, and $r\beta - 2\alpha \geq 0$. A grand clique K in G is defined to be a complete clique satisfying $|K| \geq k - (r-1)\alpha$. Then we have the following theorem [1] for G:

Theorem. If $k > \max[p(r,\alpha,\beta), \rho(r,\alpha,\beta)]$, where

$$p(r,\alpha,\beta) = 1 + \frac{1}{2}(r+1)(r\beta-2\alpha),$$
$$\rho(r,\alpha,\beta) = 1 + \beta + (2r-1)\alpha,$$

then

(i) each vertex of G is contained in exactly r grand cliques;

(ii) each pair of adjacent vertices is contained in exactly one grand clique.

In the remainder of this paper we shall denote by G a connected graph possessing properties (a_1)-(a_4) of Section 1 for some integers $m \geq 3$ and $n > 2m(m-1) + 4$.

Lemma 1. Each vertex of G is contained in exactly m grand cliques, and each pair of adjacent vertices is contained in exactly one grand clique. If K is a grand clique, then $|K| \geq n-m(m-1)$.

Proof. If we set $r = m$, $k = n-m+1$, $\alpha = m-1$, $\beta = 3$, then G satisfies the conditions of the previous theorem. In this case we have

$$p(r\ \alpha,\beta) = 1 + \frac{1}{2}\ (m+1)(m+2),$$
$$\rho(r,\alpha,\beta) = 4 + \quad (2m-1)(m-1).$$

Thus

$$\rho(r,\alpha,\beta) - p(r,\alpha,\beta) = \frac{3}{2}\ (m-1)(m-2) \geq 0.$$

Hence the theorem holds for G if

$$n-m+1 \ > \ 4 + (2m-1)(m-1),$$

i.e., if

$$n \ > \ 2m(m-1) + 4,$$

a condition which we have assumed to be satisfied.

By definition we have for any grand clique K in G,

$$|K| \ \geq \ k - (r-1)\alpha$$
$$= \ n - m(m-1).$$

It is clear from Lemma 1 that any two grand cliques in G can intersect in at most one vertex. Also if x is any vertex and y is a vertex adjacent to x, then y is contained in one and only one of the m grand cliques containing x. Since there are $m(n-m)$ vertices adjacent to x by (a_2), and each of these belongs to a unique grand clique through x, the average number of vertices other than x in a grand clique containing x is n-m. Thus the average clique size among the m grand cliques through x is n-m+1. The next two lemmas are needed to show that actually we have $|K| = n-m+1$ for every grand clique K in G.

Let K_0 and K_1 be two intersecting grand cliques in G, and let $x = K_0 \cap K_1$. Denote by $E(K_0,K_1)$ the total number of edges (x_0,x_1) such that $x_0 \in K_0-x$, $x_1 \in K_1-x$.

Lemma 2. For any two intersecting grand cliques K_0, K_1 in G,

$$E(K_0,K_1) \leq 1 + \max\{ |K_0|, |K_1| \}.$$

Proof. Let $x = K_0 \cap K_1$. For each vertex $x_0 \in K_0-x$ denote by $\varphi(x_0)$ the number of vertices in K_1-x which are adjacent to x_0. Also define

$$\varphi = \max_{x_0 \in K_0-x} \{ \varphi(x_0) \}.$$

We shall establish an upper bound for $E(K_0,K_1)$ for three cases corresponding to different values of φ.

Case 1. $\varphi \geq 3$. Let $x_0 \in K_0-x$ be such that $\varphi(x_0) = \varphi$. Denote the φ vertices in K_1-x adjacent to x_0 by $y_1,y_2,\ldots,y_\varphi$. Since $x_0 \notin K_1$ and K_1 is a grand clique and is therefore complete, there exists a vertex $y \in K_1$ not adjacent to x_0. Thus $d(x_0,y) = 2$, and by (a_4) we have $\Delta(x_0,y) = 4$. But $x,y_1,y_2,\ldots,y_\varphi$ are $\varphi+1$ vertices adjacent to both x_0 and y, and so we cannot have $\varphi \geq 4$. Thus $\varphi \geq 3$ implies $\varphi = 3$. Again if $\varphi=3$ and y is adjacent to a vertex $z \in K_0-x$, then x_0,y_1,y_2,y_3, z are five vertices adjacent to both x_0 and y, contradicting (a_4). Hence the only vertices in K_1-x which can be adjacent to vertices in K_0-x are y_1,y_2,y_3. Each of these can be adjacent to at most three vertices in K_0-x (including x_0) by the argument above. Hence if $\varphi \geq 3$ we have

$$E(K_0,K_1) \leq 9. \tag{4}$$

Case 2. $\varphi=2$. Again let $x_0 \in K_0-x$ be such that $\varphi(x_0)=2$, and let y_1,y_2 be the two vertices in K_1-x adjacent to x_0. If y is one of the $|K_1|-3$ vertices in K_1 not adjacent to x_0, then y can be adjacent to at most one vertex in K_0-x, for otherwise we again contradict (a_4). If either y_1 or y_2 is adjacent to three vertices in K_0-x including x_0, then $E(K_0,K_1) \leq 9$ by the argument in Case 1. Otherwise each of y_1 and y_2 is adjacent to at most two vertices in K_0-x including x_0. Hence in any case, if $\varphi=2$ then

$$E(K_0,K_1) \leq \max \{ 9, |K_1|+1 \} \tag{5}$$

Case 3. $\varphi \leq 1$. In this case it is obvious that

$$E(K_0,K_1) \leq |K_0| - 1. \tag{6}$$

Now by Lemma 1 we have $|K_1| \geq n-m(m-1)$. If $|K_1| + 1 < 9$, then $n < m(m-1) + 8$, which contradicts the assumptions $m \geq 3$ and $n > 2m(m-1) + 4$. Hence combining (5) and (6) we have

$$\begin{aligned} E(K_0,K_1) &\leq \max \{ |K_1| + 1, |K_0| - 1 \} \\ &\leq 1 + \max \{ |K_0|, |K_1| \}. \end{aligned}$$

Lemma 3. If K is a grand clique in G, then $|K| = n-m+1$.

Proof. Let x be an arbitrary vertex in G and let $K_0, K_1, \ldots, K_{m-1}$ be the m grand cliques containing x. Let $N_i = |K_i|$ and assume without loss of generality that $N_0 \leq N_1 \leq \ldots \leq N_{m-1}$. Then by (a_2) and Lemma 1 we have

$$\sum_{i=0}^{m-1} (N_i - 1) = m(n-m)$$

or

$$\sum_{i=0}^{m-1} N_i = m(n-m+1). \tag{7}$$

Thus since the minimum clique size can be no greater than the average, we have in particular that

$$N_0 \leq n-m+1. \tag{8}$$

Let us now count the number of edges joining vertices of $K_0 - x$ to vertices of $\bigcup_{i=1}^{m-1} (K_i - x)$. Corresponding to each vertex $x_0 \in K_0 - x$ there are $N_0 - 2$ vertices adjacent to both x and x_0 in K_0. By (a_3) we have $\Delta(x, x_0) = n-2$. Hence there remain $n - N_0$ vertices adjacent to both x and x_0 which lie in $\bigcup_{i=1}^{m-1} (K_i - x)$. Since there are $N_0 - 1$ vertices in $K_0 - x$, the required number of edges is $(n-N_0)(N_0-1)$. But this number is also given by $\sum_{i=1}^{m-1} E(K_0, K_i)$. Hence using Lemma 2 and (7) we have

$$\begin{aligned}(n-N_0)(N_0-1) &= \sum_{i=1}^{m-1} E(K_0, K_i) \\ &\leq \sum_{i=1}^{m-1} (N_i + 1) \\ &= [m(n-m+1) - N_0] + (m-1).\end{aligned}$$

On simplification, this inequality becomes

$$N_0^2 - (n+2) N_0 + [(m+1)(n-m+1) + 2(m-1)] \geq 0.$$

It follows that

$$\left| N_0 - \tfrac{1}{2}(n+2) \right| \geq \tfrac{1}{2} D^{\frac{1}{2}}, \tag{9}$$

where

$$\begin{aligned} D &= (n+2)^2 - 4[(m+1)(n-m+1) + 2(m-1)] \\ &= n^2 - 4mn + 4m^2 - 8m + 8 \\ &= (n-2m-2)^2 + 4(n-4m+1). \end{aligned} \tag{10}$$

If $n - 4m + 1 \leq 0$, then since $n > 2m(m-1) + 4$, we have

$$2m(m-1) + 4 < 4m - 1,$$

i.e.,

$$2(m-1)(m-2) + 1 < 0,$$

which of course is a contradiction. Hence $n-4m+1 > 0$ and thus by (10) we have

(11) $$D^{\frac{1}{2}} > n - 2m - 2.$$

Then from (9) and (11) it follows that

(12) $$| N_0 - \frac{1}{2}(n+2) | > \frac{1}{2}(n-2m-2).$$

Suppose first that $N_0 < \frac{1}{2}(n+2) - \frac{1}{2}(n-2m-2) = m + 2$. By Lemma 1, $N_0 \geq n-m(m-1)$. Thus $N_0 < m+2$ implies $n < m^2 + 2$ and therefore

$$2m(m-1) + 4 < m^2 + 2,$$

i.e. $$(m-1)^2 + 1 < 0,$$

which is again a contradiction. Hence by (12) we must have

(13) $$N_o > \frac{1}{2}(n+2) + \frac{1}{2}(n-2m-2) = n-m.$$

From (8) and (13) and the fact that N_0 must be an integer, it follows that $N_o = n-m+1$. But $N_0 \leq N_1 \leq \cdots \leq N_{m-1}$ and $\sum_{i=0}^{m-1} N_i = mN_0$ implies that $N_0 = N_1 = \cdots = N_m = n-m+1$. Hence each of the m grand cliques containing x contains exactly n-m+1 vertices. Since x is an arbitrary vertex and every grand clique is non-empty, it follows that $|K| = n-m+1$ for every grand clique K in G.

<u>Corollary</u>. For any two intersecting grand cliques K_0, K_1 in G,

$$E(K_0, K_1) \leq n-m+2.$$

<u>Proof</u>. This result follows immediately from Lemmas 2 and 3.

<u>Lemma 4</u>. For any two intersecting grand cliques K_0, K_1 in G,

$$E(K_0,K_1) \geq n-m^2+2.$$

<u>Proof</u>. Let $x = K_0 \cap K_1$ and denote the remaining m-2 grand cliques containing x by $K_2, K_3, \ldots, K_{m-1}$. Let us count the total number of edges connecting vertices of $K_i - x$ to vertices of $K_j - x$ for all $i,j = 0, 1, \ldots, m-1$, $i \neq j$. If y is a vertex in $K_i - x$, then there are n-m-1 vertices in K_i adjacent to both x and y. Since $\Delta(x,y) = n-2$ by (a_3), there are $(n-2) - (n-m-1) = m-1$ vertices adjacent to both x and y and not in K_i. These vertices must therefore lie in $\bigcup_{j=0, j\neq i}^{m-1} (K_j - x)$. By taking each of the m(n-m) possible choices for y, we get a total of m(m-1)(n-m) edges. But each edge is included twice in this count, so that the total number of distinct edges is $\frac{1}{2}m(m-1)(n-m)$. Counting the number of such edges in another obvious way, we obtain the relation

$$\sum_{0\leq i<j\leq m-1} E(K_i, K_j) = \frac{1}{2}m(m-1)(n-m).$$

Thus

(14) $$E(K_0,K_1) = \frac{1}{2}m(m-1)(n-m) - \sum_{i=2}^{m-1} E(K_0,K_i) - \sum_{i=2}^{m-1} E(K_1,K_i) - \sum_{2\le i<j\le m-1} E(K_i,K_j).$$

If we replace each of the $E(\cdot\,,\cdot)$'s on the right-hand side of (14) by its upper bound $n-m+2$ as given by the Corollary, we obtain

$$E(K_0K_1) \ge \frac{1}{2}m(m-1)(n-m) - 2(m-2)(n-m+2) - \frac{1}{2}(m-2)(m-3)(n-m+2) = n-m^2+2.$$

Lemma 5. If K is a grand clique and x is a vertex not in K, then there are at most two vertices in K which are adjacent to x.

Proof. Let t be the number of vertices in K adjacent to x and denote these vertices by $y_1,y_2,\ldots y_t$. The argument used in Case 1 of the proof of Lemma 2 shows that $t \le 4$. Suppose first that $t=4$ and denote by K_1 the grand clique containing y_1 and x. Then since x is a vertex in K_1-y_1 adjacent to three vertices y_2,y_3,y_4 in $K-y_1$, by Case 1 in the proof of Lemma 2 we have $E(K_1,K) \le 9$. Hence by Lemma 4 it follows that $n-m^2+2 \le 9$. Since $n \geq 2m(m-1)+4$, this implies

$$2m(m-1)+4 < m^2 + 7$$

or

$$(m-1)^2 < 4,$$

which contradicts the assumption that $m \ge 3$. Hence we must have $t \le 3$.

Next suppose $t = 3$. Let K_1,K_2,K_3 be the grand cliques containing x which intersect K in y_1,y_2,y_3, respectively. If y is one of the $n-m-2$ vertices in K not adjacent to x, then $d(x,y) = 2$ and hence by (a_4), $\Delta(x,y) = 4$. Thus there is exactly one vertex other than y_1,y_2,y_3 which is adjacent to both x and y. Hence the number of edges joining vertices of $K-y_1-y_2-y_3$ to vertices of $\bigcup_{i=1}^{3}(K_i-x-y_i)$ is at most $n-m-2$. It follows that for at least one K_i, say K_1, the number of edges joining vertices of $K-y_1-y_2-y_3$ to vertices of K_1-x-y_1 is at most $\frac{1}{3}(n-m-2)$. In addition each of y_2,y_3 can be adjacent to at most two vertices in K_1-y_1 including x. Thus we have

(15) $$E(K,K_1) \le \frac{1}{3}(n-m-2) + 4 = \frac{1}{3}(n-m+10).$$

By Lemma 4 (15) implies that $\frac{1}{3}(n-m+10) \ge n-m^2+2$, or $2n \le 3m^2-m+4$. Since $n > 2m(m-1)+4$, $2n > 4m(m-1)+8$ and hence we must have

$$4m(m-1) + 8 < 3m^2 - m + 4,$$

i.e.,

$$(m-1)(m-2) + 2 < 0,$$

which again is a contradiction. This proves the lemma.

Corollary. For any two intersecting grand cliques K_0, K_1 in G,

$$E(K_0,K_1) \leq n-m.$$

Proof. Let $x = K_0 \cap K_1$. By Lemma 5 no vertex in K_0-x can be adjacent to more than one vertex in K_1-x. Thus the maximum number of edges is at most n-m, the number of vertices in K_0-x.

Lemma 6. For any two intersecting grand cliques K_0, K_1 in G,

$$E(K_0,K_1) = n-m,$$

and the vertices of K_0-x (where $x = K_0 \cap K_1$) may be put in one-one correspondence with the vertices of $K_1 - x$ so that corresponding vertices are adjacent.

Proof. To prove the first part we proceed exactly as in the proof of Lemma 4 except that we replace each of the $E(\cdot\ ,\ \cdot)$'s on the right-hand side of (14) by its upper bound n-m as given by the above Corollary. We then obtain

$$\begin{aligned} E(K_0,K_1) &\geq \tfrac{1}{2}\, m(m-1)(n-m) - 2(m-2)(n-m) - \tfrac{1}{2}(m-2)(m-3)(n-m) \\ &= n-m, \end{aligned}$$

which by the Corollary shows that $E(K_0,K_1) = n-m$.

If x_0 is an arbitrary vertex in K_0-x, then by Lemma 5 there is at most one vertex in K_1-x adjacent to x_0. If there exists a vertex in K_0-x which is adjacent to no vertex in K_1-x, then we would have $E(K_0,K_1) < n-m$. Hence every vertex in $K_0 - x$ is adjacent to exactly one vertex in K_1-x. By the same argument, every vertex in K_1-x is adjacent to exactly one vertex in K_0-x, and the lemma is proved.

Lemma 7. The number of grand cliques in G is $\binom{n}{m-1}$.

Proof. Consider pairs (x,K) where x is a vertex in G and K is a grand clique containing x. Since each vertex is contained in m grand cliques, and by (a_1) there are $\binom{n}{m}$ vertices in G, the number of such pairs is $m\binom{n}{m}$. But each grand clique accounts for n-m+1 such pairs, so the total number of grand cliques in G is

$$m\binom{n}{m}/(n-m+1) = \binom{n}{m-1}.$$

Lemma 8. Each grand clique in G is intersected by exactly (m-1)(n-m+1) other grand cliques.

Proof. This follows at once by noting that each of the $n-m+1$ vertices in a grand clique is contained in exactly $m-1$ other grand cliques.

Lemma 9. If K_0 and K_1 are two intersecting grand cliques in G, then there are exactly $n-2$ grand cliques intersecting both K_0 and K_1.
Proof. Let $x = K_0 \cap K_1$. By Lemma 1 there is a one-one correspondence between edges joining vertices of K_0-x to vertices of K_1-x and grand cliques intersecting both K_0-x and K_1-x. Hence by Lemma 6 the number of such grand cliques is $n-m$. In addition there are $m-2$ grand cliques other than K_0 and K_1 which contain x. Thus the total number of grand cliques intersecting both K_0 and K_1 is $(n-m)+(m-2) = n-2$.

Lemma 10. If K_0 and K_1 are two non-intersecting grand cliques which are both intersected by a third grand clique L_1, then there are exactly four grand cliques which intersect both K_0 and K_1, including L_1.
Proof. Let $x_0 = K_0 \cap L_1$, $x_1 = K_1 \cap L_1$. Since K_1 and L_1 are intersecting grand cliques, it follows from Lemma 6 that there is exactly one vertex y_1 in K_1 distinct from x_1 which is adjacent to x_0. Similarly there is exactly one other vertex y_0 in K_0 adjacent to x_1. Let L_2, L_3 be the grand cliques containing the pairs (x_0,y_1), (x_1,y_0), respectively. Suppose y_0 and y_1 are not adjacent. Then $d(y_0,y_1) = 2$. Since L_3 and K_1 are intersecting grand cliques, there exists exactly one vertex z_1 in K_1 distinct from x_1 and y_1 which is adjacent to y_0. Similarly there exists a vertex z_0 in K_0 distinct from x_0 and y_0 which is adjacent to y_1. But for the same reason there must also exist a vertex z_2 in L_2, distinct from x_0 and y_1, which is adjacent to y_1. Thus x_0, x_1, z_0, z_1, z_2 are five distinct vertices adjacent to both y_0 and y_1, which contradicts the fact that $\Delta(y_0,y_1) = 4$ when $d(y_0,y_1) = 2$. Hence it follows that y_0 and y_1 must be adjacent. Let L_4 be the grand clique containing y_0 and y_1.

Suppose there is a fifth grand clique L_5 which intersects both K_0 and K_1. Clearly L_5 cannot contain any of the four vertices x_0,x_1,y_0,y_1 or Lemma 5 would be contradicted. Let $z_0 = K_0 \cap L_5$ and $z_1 = K_1 \cap L_5$. There is a vertex z_2 in K_1 distinct from z_1,x_1,y_1 which is adjacent to z_0.

There is also a vertex z_3 in L_5 distinct from z_0 and z_1 which is adjacent to x_1. But then again we have five vertices x_0, y_0, z_1, z_2, z_3 adjacent to both z_0 and x_1. Thus there cannot exist a fifth clique intersecting both K_0 and K_1 and the lemma is proved.

Theorem. If G is the graph of a T_m association scheme, then G is connected and the following conditions are satisfied:

(a_1) The number of vertices in G is $\binom{n}{m}$;

(a_2) G is regular of valence $m(n-m)$;

(a_3) $\Delta(x,y) = n-2$ if $d(x,y) = 1$;

(a_4) $\Delta(x,y) = 4$ if $d(x,y) = 2$.

Conversely, if G is a connected graph satisfying (a_1)-(a_4) for some integers $m \geq 2$ and $n > 2m(m-1) + 4$, then G is the graph of a T_m association scheme.

Proof. The necessity of the conditions (a_1)-(a_4) follows from (1), (2), and (3) and can easily be verified directly.

The proof of sufficiency of these conditions when $n > 2m(m-1) + 4$ will be established by induction on m. Connor [2] has shown that the theorem holds in the triangular case, i.e., when $m = 2$. Thus let m be an arbitrary integer greater than two and assume that the theorem holds for $2,3,\ldots,m-1$. Then since $m \geq 3$ and $n > 2m(m-1) + 4$, all the results established above hold for G. Let H be a graph whose vertices are the grand cliques of G, two vertices of H being adjacent if and only if the corresponding grand cliques of G intersect. Since G is connected, it is clear that H is also connected. By Lemmas 7,8,9 and 10 it follows that H satisfies the following conditions:

(a_1') The number of vertices in H is $\binom{n}{m-1}$;

(a_2') H is regular of valence $(m-1)[n-(m-1)]$;

(a_3') $\Delta(x,y) = n-2$ if $d(x,y) = 1$.

(a_4') $\Delta(x,y) = 4$ if $d(x,y) = 2$.

But these are just the conditions of the theorem with m-1 replacing m. Also $n > 2m(m-1) + 4$ implies that $n > 2(m-1)[(m-1) - 1] + 4$. Hence by the inductive hypothesis the theorem holds for H, i.e., H is the graph of a T_{m-1} scheme. We can therefore associate the vertices of H with

$(m-1)$-plets from among the symbols $1,2,\ldots,n$ in such a way that two vertices of H are adjacent if and only if their corresponding $(m-1)$-plets contain exactly $m-2$ common symbols. From the correspondence between H and G it follows that we can associate the grand cliques of G with $(m-1)$-plets from among $1,2,\ldots,n$ so that two grand cliques intersect if and only if their corresponding $(m-1)$-plets have exactly $m-2$ symbols in common.

Now let K_0 and K_1 be two intersecting grand cliques in G, and let $(a_1,a_2,\ldots,a_{m-2},\alpha)$ and $(a_1,a_2,\ldots,a_{m-2},\beta)$ be the corresponding $(m-1)$-plets. To the vertex $x = K_0 \cap K_1$ we assign the m-plet $(a_1,a_2\ldots,a_{m-2},\alpha,\beta)$. If K is any one of the other $m-2$ grand cliques containing x, then K intersects both K_0 and K_1, and therefore the $(m-1)$-plet corresponding to K must have $m-2$ symbols in common with both $(a_1,a_2,\ldots,a_{m-2},\alpha)$ and $(a_1,a_2,\ldots,a_{m-2},\beta)$. Hence it must either be of the form $(a_1,a_2,\ldots,a_{i-1},a_{i+1},\ldots,a_{m-2},\alpha,\beta)$ for some i, $1 \leq i \leq m-2$, or else of the form $(a_1,a_2,\ldots,a_{m-2},\gamma)$ where $\gamma \neq \alpha,\beta$. Suppose first that it is of the latter form. The total number of $(m-1)$-plets which contain the $(m-2)$-plet $(a_1,a_2,\ldots,a_{m-2})$ is $n-m+2$, and corresponding to each of these is a grand clique in G. Now there are only m grand cliques containing x and $n-m+2 > m$ since $n > 2m(m-1)+4$, and thus there exists a grand clique L not containing x whose corresponding $(m-1)$-plet has the form $(a_1,a_2,\ldots a_{m-2}, \xi)$, where $\xi \neq \alpha,\beta,\gamma$. Then L is a grand clique which intersects K_0,K_1, and K. Thus x is a vertex not in L which is adjacent to at least three vertices in L, which contradicts Lemma 5. It follows that if K is any grand clique containing x other than K_0 and K_1, then the $(m-1)$-plet corresponding to K must be of the form $(a_1,a_2,\ldots,a_{i-1}, a_{i+1},\ldots,a_{m-2}, \alpha,\beta)$. Hence the m-plet $(a_1,a_2,\ldots a_{m-2},\alpha,\beta)$ associated with x is unambiguously determined by any two of the m grand cliques intersecting in x, and to each of the $\binom{n}{m}$ vertices in G we can associate a unique m-plet.

If two vertices x and y of G are adjacent, then there is a unique grand clique K containing x and y. If $(a_1,a_2,\ldots,a_{m-1})$ is the $(m-1)$-plet corresponding to K, then the m-plets corresponding to x and y must contain the symbols $a_1,a_2,\ldots,a_{m-1}$. Conversely, if the m-plets corresponding to x and y contain the symbols $a_1,a_2,\ldots,a_{m-1}$, then x and y are contained in the grand clique K corresponding to the $(m-1)$-plet $(a_1,a_2,\ldots,a_{m-1})$. Thus two vertices of G are adjacent if and only if the corresponding m-plets have m-1 symbols in common. Hence G must be the graph of a T_m association scheme.

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References

[1] Bose, R. C. and Renu Laskar, A characterization of tetrahedral graphs, Inst. Stat., UNC, Mimeo Ser. No 509. To be published in J. Comb. Theory.

[2] Connor, W. S, The uniqueness of the triangular association scheme, Ann. Math. Stat., 29 (1958), 262-266.

[3] Hoffman, A. J., On the uniqueness of the triangular association scheme, Ann. Math. Stat., 31 (1960), 492-497.

[4] Hoffman, A. J., On the exceptional case in a characterization of the arcs of a complete graph, IBM J., 4 (1960), 487-496.

[5] Li-Chien, Chang, The uniqueness and non-uniqueness of the triangular association schemes, Science Record, Math. New Ser., 3 (1959), 604-613.

[6] Li-Chien, Chang, Association schemes of partially balanced designs with parameters $v = 28$, $n_1 = 12$, $n_2 = 15$, and $p_{11}^2 = 4$, Science Record, Math. New Ser., 4 (1960), 12-18.

[7] Ogasawara, M., A necessary condition for the existence of regular and symmetrical PBIB designs of T_m type, Inst. Stat., UNC, Mimeo Ser. No. 418.

[8] Shrikhande, S. S., On a characterization of the triangular association scheme, Ann. Math. Stat., 30 (1959), 39-47.