ABSTRACT

JIN, QINGWEI. Quadratically Constrained Quadratic Programming Problems and Extensions. (Under the direction of Dr. Shu-Cherng Fang.)

In this dissertation, quadratically constrained quadratic programming (QCQP) problems and extensions are studied. We first discuss the case of a quadratic programming problem with one quadratic inequality constraint. The zero duality gap property is investigated in a most general setting. Then we handle the inequality constrained QCQP (IQCQP) problems using the concept of "cone of nonnegative quadratic functions." A sufficient condition for a Karush-Kuhn-Tucker (KKT) solution to become optimal is given. Based on this condition, we prove that the optimal Lagrangian multipliers satisfy some maximal property and develop an algorithm for solving the IQCQP problems. A relaxation scheme is proposed to achieve ϵ-optimality in polynomial time. We also extend the proposed approach to study the conic form quadratically constrained quadratically programming (CQCQP) problems. Finally, we focus on the exact formulation of the cones of nonnegative quadratic functions for related conic optimization problems. Suggestions for future research are included at the end.

Quadratically Constrained Quadratic Programming Problems and Extensions

by
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DEDICATION

This dissertation is dedicated to my family for their endless love and support:
Chen Gao, my beloved wife
Xiwen Jin, my dear dad
Yingxin Wang, my dear mom

BIOGRAPHY

I was born in Xi'an on November 28th, 1981. During my childhood, I spent most of my time in Guozhen, a small town surrounded by hills and rivers. My family stayed there for ten years and I could not forget those happy moments I spent on hills, in woods, and even along the railroad! Many years later when I grew up, my mom's friends still remember that I had been such a naughty boy that could hardly be quiet for just one minute. Luckily, when they related my little "shameful" stories to my wife, she could not believe that I was used to be so mischievous. In 1991, we moved back to Xi'an, an ancient city in northwestern part of China, where I was lucky enough to meet my wife and shared the same study desk with her for one year in high school. In 2000, I was admitted to Tsinghua University in Beijing. I spent seven years in Tsinghua, during which I received my B.S. degree in Information and Computing Science and M.S. degree in Operations Research under the supervision of Prof. Wenxun Xing, who inspired me, as a mentor and friend, to pursue the academic path I had followed in Tsinghua. In 2007, I joined North Carolina State University to pursue my Ph.D. degree in the Department of Industrial and Systems Engineering under the supervision of Prof. Shu-Cherng Fang, who set a great example for me of being a rigorous and responsible scholar.

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Chapter 1

Introduction

The aims of this dissertation are to explore the optimality conditions of the quadratically constrained quadratic programming problems, to extend these results to the conic form quadratically constrained quadratic programming problems, and to find the connection between the classes of easy problems and hard problems in quadratic optimization.

1.1 Motivation

The quadratically constrained quadratic programming (QCQP) problems form an important class of optimization problems. The study of QCQP problems originated in 1951 ([66]), if not earlier. They are of great interests to the researchers in theory and practice.

In the theory of computational complexity, if an optimization problem can be solved in the time and space polynomially to its input size, then this problem is in class P, which is the class of problems that can be solved efficiently. A commonly seen problem in P is the linear programming (LP) problem. However, general nonlinear programming (NLP) problems are not expected to be solved in polynomial time unless $P = NP$. Many researchers try to understand the difference between LP and NLP problems in terms of the difficulties involved in solving the problems. QCQP problems, in a sense, are the simplest NLP problems that go beyond LP problems.

Besides the theoretical importance, QCQP problems are of wide applications. Some good examples are:

1. In Statistics, the linear regression model minimizes an unconstrained quadratic function which is a special case of QCQP.

2. In binary integer programming problems, the integer requirements can be formulated as quadratic constraints. Hence, both the binary integer linear programming problems and quadratically constrained quadratic programming problems can be formulated as QCQP prob-

lems.

3. In numerical optimization, at each iteration of the trust region method, a quadratic optimization problem with one elliptic constraint is solved as a subproblem in order to find a moving direction. This subproblem is a special case of QCQP and is known as the trust region subproblem (TRS).

1.2 Statement of The Problems

A quadratically constrained quadratic programming problem can be expressed as

$$\begin{aligned} \min \quad & x^T A_0 x + 2b_0^T x + c_0 \\ s.t. \quad & x^T A_i x + 2b_i^T x + c_i \leq 0, \quad i = 1, \ldots, m_1, \\ & x^T A_i x + 2b_i^T x + c_i = 0, \quad i = m_1 + 1, \ldots, m_1 + m_2, \end{aligned} \qquad \text{(QCQP)} \tag{1.1}$$

where $A_i \in \mathcal{S}^n$, the space of real symmetric square matrices of order n, $b_i \in \mathbb{R}^n$, the n-dimensional real space, and $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, m_1 + m_2$.

When $m_1 = 1$ and $m_2 = 0$, the QCQP problems are known as the generalized trust region (GTRS) subproblems.

When $A_i = 0$ for all $i = 1, \ldots, m_1 + m_2$, the QCQP problems become the quadratic programming (QP) problems. Furthermore, if we also have $A_0 = 0$, then the QCQP problems become the linear programming (LP) problems.

An extension of QCQP is the conic form quadratically constrained quadratic programming (CQCQP) problems, which can be expressed as

$$\begin{aligned} \min \quad & f(x) = x^T A_0 x + 2b_0^T x + c_0 \\ s.t. \quad & g_i(x) \in \mathcal{K}_i \subset \mathbb{R}^{m_i}, \quad i = 1, \ldots, I, \end{aligned} \qquad \text{(CQCQP)} \tag{1.2}$$

where $A_0 \in \mathcal{S}^n$, $b_0 \in \mathbb{R}^n$, $c_0 \in \mathbb{R}$, $\mathcal{K}_i \subset \mathbb{R}^{m_i}$ is a proper cone, and $g_i(x) = [g_{i1}(x), \ldots, g_{im_i}(x)]^T$ satisfying that

$$\begin{aligned} & g_{ij}(x) = x^T A_{ij} x + 2b_{ij}^T x + c_{ij}, \\ & A_{ij} \in \mathcal{S}^n, \ b_{ij} \in \mathbb{R}^n, \ c_{ij} \in \mathbb{R}, j = 1, \ldots, m_i, \end{aligned}$$

for $i = 0, 1, \ldots, I$. When $\mathcal{K}_i$ is $-\mathbb{R}_+$, the set of nonpositive real numbers, for $i = 1, \ldots, I$, CQCQP problems become the quadratic programming problems with inequality quadratic constraints (IQCQP), a subclass of QCQP.

For convenience, in the rest of the dissertation, problems are referred to either by the abbreviations such as QCQP or by the equation numbers such as (1.2).

1.3 Contributions

In this dissertation, several interesting findings related to QCQP are obtained.

First, we explore the structure of the quadratic optimization problem with one quadratic inequality constraint, i.e., GTRS, and find the following main results:

(i) If both GTRS and its Lagrangian dual (or semidefinite dual) are feasible, then there does not exist any duality gap between the primal and dual optimal values.

(ii) Let $\mathcal{F} = \{\lambda \geq 0 \mid A_0 + \lambda A_1 \in \mathcal{S}^n_+\}$, where $\mathcal{S}^n_+$ is the set of positive semidefinite matrices of order n. Under the assumption in (i), if $\mathcal{F}$ is an interval, or if $\mathcal{F}$ is a singleton set with $A_1 \gneqq 0$, i.e., A_1 is positive semidefinite and $A_1 \neq 0$, then there always exists a finite optimal solution to GTRS.

These findings have been published in [57].

Second, we work on finding sufficient conditions for a Karush-Kuhn-Tucker (KKT) solution to be optimal for IQCQP. A linear conic programming approach using the cone of nonnegative quadratic functions over the feasible domain of IQCQP is introduced. The relation between the conic programming problem and IQCQP is studied and the results lead to a sufficient condition called the copositiveness condition which is more general than the well-known positive semidefiniteness condition. Based on this condition, a relaxed conic programming problem is introduced and this relaxation can be solved efficiently. Furthermore, the relaxed linear conic programming problem can handle a larger class of IQCQP problems than the SDP relaxation method can. These results are scheduled to appear in [68].

Third, we extend the results for IQCQP to CQCQP. A conic programming problem using the cone of nonnegative quadratic functions over the feasible domain of CQCQP can be introduced and a similar copositiveness condition can be derived. If the cones used in the constraints and in the relaxation exhibit some special structure, e.g., the second order cone or the positive semidefinite cone, then the relaxed conic programming problem can be solved in polynomial time. These results can also be seen in a working paper [58].

Fourth, we work on the exact formulation of the cones of nonnegative quadratic functions over several sets of our special interests. In particular, we provide an exact representation of the cone of nonnegative quadratic functions over the second order cone constraint. This result is new to the field of mathematical optimization.

1.4 Outline of The Dissertation

The rest of the dissertation are organized as follows. In Chapter 2, theories of nonlinear programming, convex programming, and conic programming are introduced. Results related to

QCQP problems and extensions are reviewed. In Chapter 3, GTRS is studied and the property of zero duality gap is proved. In Chapter 4, IQCQP is studied and the copositiveness condition is introduced and explored. In Chapter 5, the results for IQCQP are extended to CQCQP problems. In Chapter 6, the exact formulations of several cones of nonnegative quadratic functions are studied. In Chapter 7, we provide conclusions with some suggestions for future research.

Chapter 2

Background and Literature Review

From [79], QP problems are NP-hard problems. Therefore, QCQP problems and their extensions are all NP-hard problems. Since we do not expect polynomial time algorithms for QCQP, the studies of QCQP problems can be roughly divided into four categories:

1. Given a feasible solution, find sufficient conditions for it to be optimal. The KKT conditions for NLP problems and the globally optimality properties of the convex programming (CoP) problems are usually applied to exploit these sufficient conditions. (See, e.g. [55], [14].)

2. Characterize the structures of QCQP with few constraints or special constraints. This includes exploring the duality properties of QCQP problems and their dual problems, finding the structures of the optimal solutions or the feasible domain under certain conditions. After fully characterizing their structures, solving these simple structured subclasses of QCQP problems is usually used as subroutines in other methods for more general optimization problems. A typical example is using the TRS problems in trust region method. (See, e.g., [107], [74].)

3. Find a subclass of QCQP problems such that it can be solved in polynomial time. This category has many overlaps with the above two categories. However, because of its great importance in practical computing, it gains special concerns from the researchers. (See, e.g., [110], [65].)

4. Find an approximate solution or a lower bound of the QCQP problems. Various dual problems, relaxation and reformulation techniques are used in this category. The approximate solutions can be used directly as the solutions of the optimization problems in practice. The lower bound and its corresponding relaxed solution can be used as an intermediate, such as the lower bound for branch and bound algorithms or the positive

semidefinite programming solution for the MAX-CUT problems, which allow further exploration in the original problems that are modeled as QCQP problems. (See, e.g. [49], [46].)

In the rest of this chapter, some useful notations, theories and techniques are introduced and major results in the studies of QCQP problems and their extensions are reviewed.

2.1 Notations

The feasible domain of an optimization problem is denoted as $\text{dom}(\cdot)$. For the minimization problems, the feasible domain is defined as the set of feasible solutions with objective values strictly less than $+\infty$. Whereas, for the maximization problems, the feasible domain is defined as the set of feasible solutions with objective values strictly greater than $-\infty$.

Given an optimization problem, suppose its feasible domain is $\mathcal{F}_d \subset \mathbb{R}^n$. For any $x \in \mathcal{F}_d$, if there exists an open set $\mathcal{O}(x) \subset \mathcal{F}_d$ containing x, then $\mathcal{O}(x)$ is an open neighborhood of x and x is an interior point of $\mathcal{F}_d$. The set of feasible interior points is the interior of $\mathcal{F}_d$, which is denoted as $\text{int}(\mathcal{F}_d)$. The smallest closed set containing $\mathcal{F}_d$ is the closure of $\mathcal{F}_d$ which is denoted as $\text{cl}(\mathcal{F}_d)$. Then we have

$$\text{int}(\mathcal{F}_d) \subset \mathcal{F}_d \subset \text{cl}(\mathcal{F}_d), \text{ and cl int}(\mathcal{F}_d) = \text{cl}(\mathcal{F}_d).$$

The boundary of $\mathcal{F}_d$ is defined as

$$\text{bdry}(\mathcal{F}_d) = \text{cl}(\mathcal{F}_d) \backslash \text{int}(\mathcal{F}_d).$$

Suppose that $\mathcal{F}_d$ is a subset of the affine space $\mathcal{A} = \{x \in \mathbb{R}^n \mid Ax = b\}$ and there is no smaller affine space containing $\mathcal{F}_d$. For any $x \in \mathcal{F}_d$, if there exist an open set $\mathcal{O}(x)$ containing x and $\mathcal{O}(x) \cap \mathcal{A} \subset \mathcal{F}_d$, then x is a relative interior point of $\mathcal{F}_d$. The set of feasible relative interior points is the relative interior of $\mathcal{F}_d$ which is denoted as $\text{ri}(\mathcal{F}_d)$.

Given a point x in $\mathcal{F}_d$, for any vector $d \in \mathbb{R}^n$, if there exists a constant $\alpha_0 > 0$ such that for any $0 < \alpha < \alpha_0$, $x + \alpha d \in \mathcal{F}_d$, then d is a feasible direction at x.

Given a sufficiently continuously differentiable function $f(x)$, its first derivative is a $1 \times n$ vector denoted as $\nabla f(x) \in \mathbb{R}^{1\times n}$. Its second derivative is an $n \times n$ matrix denoted as $\nabla^2 f(x) \in \mathbb{R}^{n\times n}$. Write $x = (y, z)$, then $f(x) = f(y, z)$ and its first and second partial derivatives on y can be denoted as $\nabla_y f(y, z)$ and $\nabla^2_{yy} f(y, z)$, respectively.

For any continuously differentiable vector valued function $f(x) \in \mathbb{R}^m$, its first derivative is an $m \times n$ matrix denoted as $\nabla f(x) \in \mathbb{R}^{m\times n}$.

For convenience, we will use $e_i \in \mathbb{R}^n$ to denote a vector with its ith position being one and other elements being zero and use $I_r \in \mathcal{S}^r$ to denote an identity matrix of order r. The vector $e \in \mathbb{R}^n$ is a vector with all elements being one.

2.2 Nonlinear Programming Problems

The QCQP problems are a subclass of NLP problems, therefore, theories in NLP problems can be applied directly to them. The NLP problems are defined as

$$\begin{array}{ll} \min & f(x) \\ s.t. & g_i(x) \geq 0, \quad i = 1, \ldots, m_1, \\ & h_i(x) = 0, \quad i = m_1 + 1, \ldots, m_1 + m_2, \\ & x \in \Omega, \end{array} \qquad \text{(NLP)} \qquad (2.1)$$

where $f(x)$, $g_i(x)$ and $h_i(x)$ are all continuous differentiable functions, and Ω is a subset of $\mathbb{R}^n$ which usually has a simple structure. The optimality conditions below about NLP can be found in many books such as [70] and [78].

The first lemma is a necessary condition for a local minimum.

Lemma 1 (First order necessary condition)**.** *Given x^* being a local minimum of NLP, then for any d being a feasible direction at x^*, we have*

$$\nabla f(x^*) d \geq 0.$$

Given a feasible point x^*, the active set $\mathcal{A}(x^*)$ at this point is define as

$$\mathcal{A}(x^*) = \{1 \leq i \leq m_1 \mid g_i(x^*) = 0\} \cup \{m_1 + 1, \cdots, m_1 + m_2\}.$$

Definition 2 (LICQ)**.** *Given a feasible point x^*, if the vectors $\nabla h_i(x^*)$, $i \in \mathcal{A}(x^*)$ and $\nabla g_i(x^*)$, $i \in \mathcal{A}(x^*)$, are linearly independent, then the NLP problem satisfies linear independence constraint qualification (LICQ) at x^*.*

Under the LICQ condition, several optimality conditions about local minimum solutions can be found.

Theorem 3 (Karush-Kuhn-Tucker condition (KKT))**.** *Let $\Omega = \mathbb{R}^n$ and x^* be a local minimum solution of NLP problem. Suppose the LICQ condition is satisfied at x^*, then there exist $\mu \in \mathbb{R}^{m_2}$ and $\lambda \in \mathbb{R}^{m_1}_+$ such that*

$$\begin{array}{l} \nabla f(x^*) - \sum\limits_{i=1}^{m_1} \lambda_i \nabla g_i(x^*) - \sum\limits_{i=m_1+1}^{m_1+m_2} \mu_i \nabla h_i(x^*) = 0, \\ \lambda_i g_i(x^*) = 0, \quad i = 1, \ldots, m_1. \end{array} \qquad (2.2)$$

Especially, the conditions $\lambda_i g_i(x^*) = 0$, for all $i = 1, \ldots, m_1$, are called complementary slackness conditions.

Define the Lagrangian function as

$$L(x^*, \lambda, \mu) = f(x^*) - \sum_{i=1}^{m_1} \lambda_i g_i(x^*) - \sum_{i=m_1+1}^{m_1+m_2} \mu_i h_i(x^*). \tag{2.3}$$

Then the KKT conditions can be written as

$$\begin{aligned} &\nabla_x L(x^*, \lambda, \mu) = 0, \\ &\lambda_i g_i(x^*) = 0, \quad i = 1, \ldots, m_1, \\ &\lambda \geq 0, \\ &g_i(x^*) \geq 0, \quad i = 1, \ldots, m_1, \\ &h_i(x^*) = 0, \quad i = m_1 + 1, \ldots, m_1 + m_2. \end{aligned} \tag{2.4}$$

Here, λ and μ are called the Lagrangian multipliers of x^* and (x^*, λ, μ) is called a KKT solution. Note that under the LICQ condition, the Lagrangian multipliers at given x^* are unique.

Given a KKT solution (x^*, λ, μ), a cone can be defined as follows

$$\mathcal{C}(x^*, \lambda, \mu) = \left\{ d \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} \nabla h_i(x^*) d = 0, & \text{for all } i = \mathcal{A}(x^*), \\ \nabla g_i(x^*) d = 0, & \text{for all } i \in \mathcal{A}(x^*) \text{ with } \lambda_i > 0, \\ \nabla g_i(x^*) d \geq 0, & \text{for all } i = 1, \ldots, m_1, \text{ with } \lambda_i = 0. \end{array} \right\}, \tag{2.5}$$

which is known as the critical cone.

Theorem 4 (Second order necessary conditions). *Let $\Omega = \mathbb{R}^n$ and $f(x)$, $g_i(x)$ and $h_i(x)$ are all twice continuous differentiable. Given a local minimum x^*, if LICQ is satisfied at x^*, then there exist λ and μ such that (x^*, λ, μ) is a KKT solution and*

$$\nabla^2_{xx} L(x^*, \lambda, \mu) = \nabla^2 f(x^*) - \sum_{i=1}^{m_1} \lambda_i \nabla^2_{xx} g_i(x^*) - \sum_{j=m_1+1}^{m_1+m_2} \mu_j \nabla^2_{xx} h_j(x^*) \tag{2.6}$$

is positive semidefinite on the critical cone $\mathcal{C}(x^*, \lambda, \mu)$.

Theorem 5 (Second order sufficient conditions). *Let $\Omega = \mathbb{R}^n$ and $f(x)$, $g_i(x)$ and $h_j(x)$ are all twice continuous differentiable. Given a point x^*, suppose LICQ is satisfied at x^*. If there exist $\lambda \in \mathbb{R}^{m_1}_+$ and $\mu \in \mathbb{R}^{m_2}$ such that the KKT conditions (2.4) are satisfied and, furthermore,*

$$\nabla^2_{xx} L(x^*, \lambda, \mu) = \nabla^2 f(x^*) - \sum_{i=1}^{m_1} \lambda_i \nabla^2_{xx} g_i(x^*) - \sum_{j=m_1+1}^{m_1+m_2} \mu_j \nabla^2_{xx} h_j(x^*) \tag{2.7}$$

is positive definite on the critical cone $\mathcal{C}(x^*, \lambda, \mu)$, then x^* is a strictly local minimum point.

Besides the local optimality conditions, another useful theory is the duality theory.

The Lagrangian dual of NLP is defined as

$$\begin{array}{ll} \max & \theta(\lambda, \mu) \\ s.t. & \lambda \geq 0, \end{array} \qquad \text{(NLD)} \tag{2.8}$$

where

$$\theta(\lambda, \mu) = \inf\{L(x, \lambda, \mu) \mid x \in \Omega\}. \tag{2.9}$$

From the previous definition, the feasible domain of the NLD is

$$\text{dom(NLD)} = \{(\lambda, \mu) \mid \theta(\lambda, \mu) > -\infty\}. \tag{2.10}$$

One useful result is the weak duality theorem.

Theorem 6 (Weak duality theorem)**.** *Suppose x is a feasible solution of NLP and (λ, μ) is a feasible solution of NLD, then*

$$f(x) \geq \theta(\lambda, \mu). \tag{2.11}$$

From this theorem, the objective value of any feasible dual solution can be used as a lower bound of the primal problem. This fact is very useful in many algorithms such as the branch and bound method or cutting plan methods, etc. Stronger results about duality theory are given in the next section.

2.3 Convex Programming Problems

A set $\mathcal{X} \subset \mathbb{R}^n$ is convex if, for any $x_1, x_2 \in \mathcal{X}$ and $0 \leq \alpha \leq 1$, we have $\alpha x_1 + (1-\alpha)x_2 \in \mathcal{X}$. Given a set $\mathcal{X} \subset \mathbb{R}^n$, define the convex hull $\text{conv}(\mathcal{X})$ of $\mathcal{X}$ as the smallest convex set containing it. According to Theorem 2.3 [97],

$$\text{conv}(\mathcal{X}) = \{x \in \mathbb{R}^n | x = \sum_{i=1}^{m} \lambda_i x^i, \sum_{i=1}^{m} \lambda_i = 1, \lambda_i \geq 0, x^i \in \mathcal{X}, i = 1, \ldots, m, \text{ for some } m.\}$$

A function $f(x)$ is convex if it is defined on a convex set Ω and, for any $x_1, x_2 \in \Omega$ and $0 \leq \alpha \leq 1$, we have $f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2)$. A function $f(x)$ is concave if $-f(x)$ is convex.

In NLP, if $f(x)$ is convex, $g_i(x)$, $i = 1, \ldots, m_1$, are all concave, $h_i(x)$, $i = m_1+1, \ldots, m_1+m_2$, are all affine, and Ω is a convex subset of $\mathbb{R}^n$, then the problems become the convex programming (CoP) problems. For the maximization problems, suppose the objective function is $f(x)$. If a maximization problem becomes CoP after replacing maximization by minimization and $f(x)$ by $-f(x)$, then it is also called CoP problem. CoP problems have much simpler structures than NLP problems and many polynomial time algorithms are based on the convex properties of the problems. Theories of CoP can be found in [97], [29] and [23].

Lemma 7. *If the feasible domain of a CoP problem is nonempty, then it is a convex set.*

Theorem 8. *If the feasible domain of a CoP problem is not empty and the objective function $f(x)$ is a proper convex function, then any local minimum is a global minimum. Here f is proper means there is at least one x in* dom(CoP) *such that $f(x) < +\infty$. In addition, if f is strictly convex, then there exists at most one global minimum.*

Theorem 9. *For an NLP problem, the objective function $\theta(\lambda, \mu)$ of its corresponding NLD problem is concave. Therefore, the NLD problem is a CoP problem.*

In previous section, we know under certain conditions, any local minimum solution of NLP is a KKT point and has its corresponding Lagrangian multipliers. This is only a necessary condition. However, when the optimization problem is a CoP problem, then a point being a KKT point and having Lagrangian multipliers becomes a sufficient condition for global optimality.

Theorem 10. *For any CoP problem with $\Omega = \mathbb{R}^n$, if (x^*, λ, μ) is a KKT solution, then x^* is an optimal solution of the CoP problem and (λ, μ) is an optimal dual solution of the CoP problem. Therefore, the duality gap between CoP and its Lagrangian dual problem is zero.*

Compared to the weak duality theorem in the NLP problems, a great advantage in the CoP problems is the strong duality theorem. It holds under certain constraint qualifications. The Slater condition is one of them.

Definition 11 (Slater condition)**.** *For a CoP problem, if there is a feasible x such that*

$$x \in \mathrm{ri}(\Omega) \text{ and } g_i(x) > 0 \qquad \forall i = 1, \ldots, m_1,$$

then we say CoP satisfies the Slater condition.

From [23], when $\Omega = \mathbb{R}^n$, the Slater condition guarantees that any optimal solution x^* is a KKT point and has its Lagrangian multiplier.

Theorem 12 (Strong duality theorem)**.** *For a CoP problem, if it satisfies the Slater condition, then the optimal values of CoP and its Lagrangian dual problem are equal. Furthermore, there exists $\lambda \geq 0$ and μ such that*

$$\theta(\lambda, \mu) = \inf\{f(x) \mid x \in \mathrm{dom(CoP)}\}.$$

The strong duality theorem establishes a theoretical result of the global optimality conditions for CoP, but it does not provide any algorithm to find the optimal solutions. From [77] and [19], a subclass of CoP can be solved in polynomial time by the ellipsoid method, which was introduced by Yudin and Nemirovskii [117] in 1976 and Shor [100] in 1977 independently, and this subclass of CoP includes a large part of CoP that people could encounter in practice. However, the performance of the ellipsoid method in solving some simple structured but widely used CoP, such as the LP, is not as good as other methods even if they are exponential time algorithms theoretically. This drawback is overcome by the interior point method which was first introduced by Karmarkar [59] in 1984. In [77], Nesterov and Nemirovskii provide a self-concordant based interior point method and show that an ϵ-optimal solution of a given CoP can be obtained in a polynomial number of Newton iterations if the self-concordant barrier function is given. The interior point method has great advantages when solving CoP, not only in theory but in practice, especially in conic programming problems which are introduced in the next section.

2.4 Conic Programming Problems

A cone $\mathcal{C}$ is a subset of $\mathbb{R}^n$ that satisfies

$$\alpha x \in \mathcal{C}, \qquad \text{for all } x \in \mathcal{C} \text{ and } \alpha \geq 0.$$

If it satisfies

$$x \in \mathcal{C} \text{ and } -x \in \mathcal{C} \text{ if and only if } x = 0,$$

then it is called a pointed cone. Cone $\mathcal{C}$ is a solid cone if it has nonempty interior. If it is pointed, solid, closed and convex, then we say the cone $\mathcal{C}$ is proper.

Given a set $\mathcal{X}$, the cone($\mathcal{X}$) is defined as the smallest convex cone containing $\mathcal{X}$. From [97], we know

$$\text{cone}(\mathcal{X}) = \{x \mid x = \sum_{i=1}^{r} \alpha_i x_i, x_i \in \mathcal{X}, \alpha_i \geq 0, i = 1, \ldots, r, \text{ for some } r \in \mathbb{N}\}.$$

The conic form nonlinear programming (CNLP) problems are defined as

$$\begin{array}{ll} \min & f(x) \\ s.t. & g_i(x) \in \mathcal{C}_i \subset \mathbb{R}^{m_i}, \quad i = 1, \ldots, m, \\ & x \in \mathbb{R}^n. \end{array} \qquad \text{(CNLP)} \qquad (2.12)$$

Here $\mathcal{C}_i$ are all closed convex cones.

For the optimality conditions and duality properties about CNLP, one may refer [101], [102], [103], [104], [80], [38], [4], and [3]. Recently, studies on the conic form polynomial programming problems gains more interests. (See, e.g., [63], [64].) For the CNLP on complex numbers, one may refer [1], [20] and [21].

CNLP is a generalization of NLP. The Lagrangian dual problem of the CNLP problem is also a generalization of NLD. To define the Lagrangian function, we first need the concept of dual cone. The dual cone $\mathcal{C}^*$ of the cone $\mathcal{C}$ is defined as

$$\mathcal{C}^* = \{x \mid x^T y \geq 0, \forall y \in \mathcal{C}\}. \tag{2.13}$$

One property of the cone and its dual is

Lemma 13. *A cone $\mathcal{C}$ is pointed if and only if its dual cone $\mathcal{C}^*$ is solid.*

The Lagrangian function here is

$$L(x, \lambda) = f(x) - \sum_{i=1}^{m} \lambda_i^T g_i(x), \tag{2.14}$$

where $\lambda^T = (\lambda_1^T, \ldots, \lambda_m^T)$ and $\lambda_i \in \mathcal{C}_i^*$, $i = 1, \ldots, m$.

The extended KKT conditions for a given local minimum x^* is a necessary one under certain conditions.

Lemma 14 (Extended KKT conditions [3])**.** *Suppose the cones $\mathcal{C}_i$, $i = 1, \ldots, m$, are all solid, closed and convex, and x^* is a local minimum solution of CNLP. If there exists $d \in \mathbb{R}^n$ such that*

$$\nabla g_i(x^*) d \in \text{int}(\mathcal{C}_i), \ \forall i = 1, \ldots, m,$$

then there exists λ such that the extended KKT conditions are satisfied.

$$\begin{aligned} &\nabla_x L(x^*, \lambda) = \nabla f(x^*) - \sum_{i=1}^{m} \lambda_i^T \nabla g_i(x) = 0, \\ &\lambda_i^T g_i(x^*) = 0, \\ &g_i(x^*) \in \mathcal{C}_i, \ \lambda_i \in \mathcal{C}_i^*, \ i = 1, \ldots, m. \end{aligned} \tag{2.15}$$

The Lagrangian dual of CNLP is defined as

$$\begin{array}{lll} \max & \theta(\lambda) & \\ s.t. & \lambda^T = (\lambda_1^T, \ldots, \lambda_m^T), & \text{(CNLD)} \\ & \lambda_i \in \mathcal{C}_i^*, \ i = 1, \ldots, m, & \end{array} \tag{2.16}$$

where

$$\theta(\lambda) = \inf\{L(x, \lambda) \mid x \in \mathbb{R}^n\}. \tag{2.17}$$

The feasible domain of CNLD is

$$\text{dom(CNLD)} = \{\lambda \in \mathcal{C}_1^* \times \cdots \times \mathcal{C}_m^* \mid \theta(\lambda, \mu) > -\infty\}. \tag{2.18}$$

The weak duality theorem always holds in these problems.

Theorem 15 (Weak duality theorem)**.** *Suppose x is a feasible solution of CNLP problem and λ is a feasible solution of CNLD, then*

$$f(x) \geq \theta(\lambda). \tag{2.19}$$

Similar to CoP, there are stronger duality results when the convex concept is used. A function $g : \mathbb{R}^n \to \mathbb{R}^m$ is called $\mathcal{C}$-convex if for any x and y in $\mathbb{R}^n$ and $0 \leq \alpha \leq 1$, we have

$$\alpha g(x) + (1 - \alpha)g(y) - g(\alpha x + (1 - \alpha)y) \in \mathcal{C}.$$

When the cones $\mathcal{C}_i$, $i = 1, \ldots, m$, are all proper, and the functions $-g_i$, $i = 1, \ldots, m$, are $\mathcal{C}_i$-convex, then the CNLP problems become the conic programming (CP) problem. CP problems are highly related to CoP. Theories about CP problems can be found in [77], [29] and [19].

The sufficiency of the KKT conditions for global optimality and the strong duality under the Slater condition still hold in the CP problems.

Theorem 16. *For any CP, if these exist a KKT solution (x^*, λ), then x^* is an optimal solution of CP and λ is an optimal dual solution of CP. Therefore, the duality gap between CP and its Lagrangian dual problem is zero.*

Theorem 17 (Slater condition for CP)**.** *For a CP problem, if there is a feasible x such that*

$$g_i(x) \in \text{int}(\mathcal{C}_i) \qquad \forall i = 1 \ldots, m_1,$$

then we say the CP problem satisfies the Slater condition.

Theorem 18 (Strong duality theorem for CP)**.** *For a CP problem, if it satisfies the Slater condition, then the optimal values of the CP and its Lagrangian dual problem are equal. Furthermore, there exists λ such that*

$$\theta(\lambda) = \inf\{f(x) \mid x \in \text{dom(CP)}\}.$$

Among all CP problems, three subclasses are widely used in theoretical study and practical computing. They are linear programming (LP) problems, second order cone programming (SOCP) problems, and semidefinite programming (SDP) problems. All these three problems can be solved efficiently by the interior point method.

These problems are defined as

$$\begin{array}{ll} \min & c \cdot x \\ s.t. & a_i \cdot x = b_i, \ i = 1, \ldots, m, \\ & x \in \mathcal{C}. \end{array} \tag{2.20}$$

And the dual problems are defined as

$$\begin{array}{ll} \max & \sum_{i=1}^{m} b_i y_i \\ s.t. & c - \sum_{i=1}^{m} y_i a_i \in \mathcal{C}^*. \\ & y \in \mathbb{R}^m. \end{array} \tag{2.21}$$

In LP, x, c and a_i, $i = 1, \ldots, m$, are all vectors in $\mathbb{R}^n$; cones $\mathcal{C}$ and $\mathcal{C}^*$ are $\mathbb{R}^n_+$; the product $c \cdot x$ means $c^T x$.

In SOCP, x, c and a_i, $i = 1, \ldots, m$, are all vectors in $\mathbb{R}^{n+1}$; cones $\mathcal{C}$ and $\mathcal{C}^*$ are second order cones

$$\mathcal{SOC}(n) = \left\{ \begin{bmatrix} t \\ x \end{bmatrix} \middle| \ t^2 \geq x^T x, t \in \mathbb{R}_+, x \in \mathbb{R}^n \right\};$$

the product $c \cdot x$ means $c^T x$.

In SDP, x, c and a_i, $i = 1, \ldots, m$, are all matrices in $\mathcal{S}^n$, the space of the real symmetric square matrices of order n; cones $\mathcal{C}$ and $\mathcal{C}^*$ are positive semidefinite cone

$$\mathcal{S}^n_+ = \{X \in \mathcal{S}^n \mid y^T X y \geq 0, \forall y \in \mathbb{R}^n\};$$

the product $c \cdot x$ means $\mathrm{tr}(c^T x)$. Since people usually use X instead of x to denote a matrix, without ambiguity, we will also use C, X, A_i instead of c, x, a_i to denote the matrices in SDP.

These problems are all linear conic programming problems of which the dual problems have explicit formulation. In particular, LP has a stronger form of duality theorem. (Ref, e.g., [43], [70])

Theorem 19 (LP duality theorem)**.** *For any LP problem, its dual problem is also an LP problem. If the primal problem is feasible and bounded below, then the dual problem is also feasible and the optimal solutions of the two problems have the same objective values. If the primal problem is unbounded below, then the dual problem is infeasible. The proposition is also true for the dual problem.*

2.5 Quadratically Constrained Quadratic Programming Problems

We have already introduced the results of optimality conditions and duality theories for NLP, CoP, and CP problems. These results are widely used in the studies of QCQP.

The studies of the QCQP problems can trace back to 1951 in [66]. When $A_0 \in \mathcal{S}^n_+$, and $A_i \in \mathcal{S}^n_+$, $i = 1, \ldots, m_1 + m_2$, QCQP is a convex programming problem. Around 1970s, several papers worked on it. In [85], [86], [87] and [88], the duality theory using the geometric programming is studied. Based on the geometric programming results, an ϵ-optimal solution can be obtained by solving the geometric dual problem and a linear programming problem in [42]. A cutting plane method is developed in [12]. In [40], the Lagrangian dual is given in an explicit way and a dual method with a penalty function is used to solve the convex QCQP problem. In [116], a Newton method based dual algorithm is developed and the superlinear convergence under certain conditions is proved. After the development of the interior point methods, the convex QCQP problem is shown to be solved in time which is a polynomial of the input size and required precision. (Ref [77] and therein.)

For the nonconvex QCQP problems, since Pardalos and Vavasis prove that QP is NP-hard even if A_0 has only one negative eigenvalue in [79], then QCQP problems are also NP-hard problems.

When there are few constraints, QCQP problem can be used as a subproblem for other NLP methods such as trust region method. When the constraint is a ball constraint, it is the trust region subproblem (TRS). Several algorithms are developed to solve this problem efficiently in practice, e.g., Lagrangian dual based line search algorithm in [74], semidefinite programming based algorithm in [96], and generalized Lanczos method in [51]. (See [45] for a survey.) In particular, Fu et al. [46] prove that with mild assumptions, TRS can be solved with desired precision in polynomial time. If the constraint is a general quadratic inequality, then it is GTRS. Its strong duality property under the Slater condition is studied in [106] and [73] independently. This property can be seen more clearly in the SDP formulation. (Ref [107], [29], [89].) In the next chapter, we will study the zero duality gap property under more relaxed assumption. When the constraints are two quadratic inequalities, if at the minimum solution x^* the Jacobian is not zero, then the Hessian of the Lagrangian function has at most one negative eigenvalue. This result is proved by Yuan [115] for convex constraints and Peng and Yuan [84] for general constraints. In [113], Ye and Zhang prove that under the Slater condition when the two constraints are not binding at the optimal solution, then the SDP relaxation has no gap and can lead to an optimal solution of the QCQP problem. They also show similar results for complementary linear constraints cases and for homogeneous cases without the binding assumption, which is a generalization of the results in [10] and [11]. Burer and Anstreicher [35] provide an exact SDP representation of a ball constraint with two parallel linear constraints. For the similar problem over complex field, the strong duality is proved in [15]. The results in [113] implies that these problem can be solved in polynomial time. In [89], several cases in which the Lagrangian dual is exact are provided and the duality theorem there is called S-Lemma.

For general QCQP problems, polynomial time solvable subclasses are studied in [18] and [60]. In [46], an (1-$\frac{1}{m_1^2}$)-approximation algorithm is presented for inequality QCQP. In [112], a $\frac{4}{7}$-approximation algorithm is developed for cases with unit box constraint and homogeneous diagonal constraints.

There are also many works on relaxation problems and reformulation problems. In [47], the duality property of the SDP relaxation is studied. The reformulation linearization technique (RLT) is developed for QCQP with box constraint in [98] and [99]. In [6], Anstreicher combined the SDP relaxation and the RLT and got better solution. Burer and Letchford [33] studied the geometric properties of the RLT technique and its relation to boolean quadric polytope. Branch and bound methods are proposed in [95] and [2]. In [8], Audet et al. use a branch and cut algorithm to solve QCQP. Duality bound methods are used in [17] and [108]. The successive convex relaxation method is studied in [61] and [62]. In [44], QCQP is reformulated as a bilinear problem, which is a special case of QCQP, and in [30], linear equality constraints together with graph theoretic tools are used to reduce the number of variables for the bilinear reduction.

One important subclass is the binary QCQP problem and its relaxations. In [56], an explicit form of the Lagrangian multiplier is given and, based on the positive semidefiniteness condition, sufficient conditions of the global optimality are developed. In [90] and [91], bounds based on box relaxation, sphere relaxation and eigenvalue relaxation are studied and some equivalence results are proved. In [24] and [25], several reformulation techniques are used to make the objective function been convex and the new problem is solved by standard solver. Helmberg and Rendl [54] developed an SDP based cutting plane method. Sun et al. [105] try to use cell enumeration of the hyperplane arrangement to reduce the gap between the binary QCQP problem and its SDP relaxation. While under special conditions, Malik et al. [71] use the technique developed in [5] to eliminate the duality gap in polynomial time. The copositive representation of the binary constraints is used in [32], which is a special case of the representation in [107].

CQCQP is an extension of QCQP. When $-g_i(x)$ are all $\mathcal{K}_i$-convex, $i = 1, \ldots, m$ and $f(x)$ is convex, then the problem is a conic programming problem and its optimality conditions can be seen in [72], [28], [21] and [9].

When the constraints are all linear functions over polyhedral cones, Parida and Roy give a sufficient condition to verify the existence of the optimal solution in [80]. When the polyhedral cones are relaxed to convex cones, the duality theory is studied in [104]. When problem is a CNLP problem, Fritz-John conditions are studied for polyhedral cones in [38] and [4]. For convex cones, saddle-point conditions are studied in [101]. Based on the Farkas lemma over cones developed in [102], regularity conditions, KKT conditions and complementary slackness are studied in [103] and [3]. For problems on complex number, one may refer [1], [20] and [21]. In [63], a general framework for the convex relaxation of the conic form polynomial programming problems is introduced. The SDP relaxation for the conic form polynomial programming problem is studied in [64].

Chapter 3

Quadratic Programming Problem with One Quadratic Constraint

In this chapter, we focus on QCQP problems that have only one constraint. These problems are well studied when the constraint is an ellipsoid, i.e., the TRS case. The problems with one general quadratic constraint is called the generalized trust region subproblem (GTRS), which is

$$\begin{aligned} \min \quad & f(x) = x^T A_0 x + 2b_0^T x + c_0 \\ s.t. \quad & g(x) = x^T A_1 x + 2b_1^T x + c_1 \leq 0, \end{aligned} \qquad \text{(GTRS)} \qquad (3.1)$$

where A_0 and A_1 are in $\mathcal{S}^n$, b_0 and b_1 are in $\mathbb{R}^n$, and c_0 and c_1 are in $\mathbb{R}$. The global optimality of this problem is studied in this chapter.

3.1 Introduction

As mentioned in Chapter 1, TRS problem serves as a subroutine of the trust region methods [37], which are used for solving NLP problems through quadratic approximation. Since the trust region methods are frequently used, many research results can be found in the literature. In their settings, the global optimality is not a real issue, and most papers aim to solve the problem in a most efficient manner. Among them, Moré and Sorensen [74] provided a dual based algorithm, Rendl and Wolkowicz [96] developed a semidefinite programming based algorithm, and Gould et al. [51] proposed a generalized Lanczos trust region method. (See [45] for a survey.)

GTRS arises when the problem is considered in the indefinite inner product space or Minkowski space (ref [106]). In this general setting, the investigation of global optimality conditions becomes crucial. Stern and Wolkowicz [106] studied GTRS with two-sided boundary under the assumption that the constraint matrix A_1 is nonsingular, i.e.,

$$\begin{aligned} \min \quad & x^T A_0 x + 2b_0^T x + c_0 \\ s.t. \quad & c_1 \leq x^T A_1 x \leq c_2 \end{aligned} \tag{3.2}$$

where $-\infty \leq c_1 \leq c_2 \leq +\infty$. They tackled the problem when A_0 and A_1 can be simultaneously diagonalized and studied global optimality when $A_0 + \lambda A_1 \succ 0$ for some λ, or $A_0 + \lambda A_1 \not\succ 0$ for all λ but $A_0 + \lambda A_1 \succeq 0$ for some λ. Moré [73] studied GTRS in equality form, i.e.,

$$x^T A_1 x + 2b_1^T x + c_1 = 0. \tag{3.3}$$

He derived some global optimality conditions under certain constraint qualifications. Then the results may apply to GTRS with an inequality constraint under the Slater condition as explicitly pointed out in Hiriart-Urruty [53]. The Theorem 3.3 in [73] confirmed the existence of global optimal solutions when $A_0 + \lambda A_1 \succ 0$ for some $\lambda \geq 0$ holds, and the discussion in Section 5 of [73] actually revealed the fact of zero duality gap, though Moré did not point out explicitly. This result will be seen more clearly in our discussion.

Sturm and Zhang [107] formulated the dual problem of GTRS as an SDP problem and obtained similar global optimality results under the Slater condition. They also proposed an SDP based algorithm to find a near optimal solution of GTRS. The key idea is to identify a positive semidefinite matrix which corresponds to an optimal solution of the dual problem, and then use the spectral decomposition to get a near optimal solution to GTRS. A simpler version can be seen in the Appendix B of Boyd and Vandenberghe [29], (or [89]). One can see that the Lagrangian dual and SDP dual of GTRS are essentially the same.

In this chapter, we first discuss the relations between the Lagrangian dual and the SDP dual, and then study when the zero duality gap property holds and when the optimal primal solution can be attained. We define $\mathcal{F} = \{\lambda \geq 0 \mid A_0 + \lambda A_1 \succeq 0\}$. Our main results include:

(i) If both GTRS and its Lagrangian dual problem are feasible, then there does not exist any duality gap between their optimal values.

(ii) Under the assumption in (i), if $\mathcal{F}$ is an interval, or if $\mathcal{F}$ is a singleton set with $A_1 \preceq 0$ and $A_1 \neq 0$, then there always exists a finite optimal solution to GTRS.

The result (i) is a slightly stronger than the result obtained in other literatures, in which the zero duality gap property is usually obtained under the Slater condition. It should be mentioned that our results (i) and (ii) are extensions of the results in [73], and can be implicitly implied from the discussion of section 3, 4, and 5 in [73] (under the assumption $A_0 + \lambda A_1 \succ 0$). But this extension is not a trivial job. Besides, we focus more on the relations between the GTRS problem and its Lagrangian dual problem, and give the conclusion in an explicit form.

3.2 Lagrangian Dual and SDP Dual

3.2.1 Lagrangian dual

The Lagrangian of GTRS here is

$$\begin{aligned} L(x,\lambda) &= f(x) + \lambda g(x) \\ &= x^T(A_0 + \lambda A_1)x + 2(b_0 + \lambda b_1)^T x + c_0 + \lambda c_1 \end{aligned} \tag{3.4}$$

for $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}_+$. For convenience, we use $A(\lambda) = A_0 + \lambda A_1$, $b(\lambda) = b_0 + \lambda b_1$ and $c(\lambda) = c_0 + \lambda c_1$. The Lagrangian dual problem is

$$\begin{array}{ll} \max & \theta(\lambda) \\ s.t. & \lambda \in \mathbb{R}_+, \end{array} \qquad \text{(GTRSD)} \tag{3.5}$$

where

$$\theta(\lambda) = \inf\{L(x,\lambda) \mid x \in \mathbb{R}^n\}. \tag{3.6}$$

It is well known that $\overline{x}$ is a minimum solution of a quadratic function $q(x) = x^T Ax + 2b^T x + c$ if and only if the Hessian of q is positive semidefinite and $\nabla q(\overline{x}) = 2A\overline{x} + 2b = 0$, which means $A \succeq 0$ and $b \in \mathcal{R}(A)$, where $\succeq$ means positive semidefiniteness and $\mathcal{R}(A)$ is the column space of A. Furthermore, the set containing all such minimum solutions becomes $\{x \mid x = -A^+ b + (I - A^+ A)y$ for $y \in \mathbb{R}^n\}$ and the minimum value of q is $q(-A^+ b)$, where A^+ is the Moore-Penrose inverse of A (ref [16]).

Consequently,

$$\begin{aligned} \theta(\lambda) &= \inf_{x \in \mathbb{R}^n} L(x,\lambda) \\ &= \begin{cases} c(\lambda) - b^T(\lambda)A^+(\lambda)b(\lambda), & \text{if } A_0 + \lambda A_1 \succeq 0 \text{ and} \\ & b(\lambda) \in \mathcal{R}(A(\lambda)) \\ & \text{with } \lambda \geq 0, \\ -\infty, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.7}$$

Notice that the effective domains of GTRSD is

$$\text{dom}(GTRSD) = \{\lambda \geq 0 \mid \theta(\lambda) > -\infty\}. \tag{3.8}$$

By equation (3.7), the value of $\theta(\lambda)$ depends on the positive semidefiniteness of $A(\lambda)$. We further recall that the set $\mathcal{F} = \{\lambda \geq 0 \mid A(\lambda) \succeq 0\}$.

Several observations can be made here:

Observation 1 $\mathcal{F}$ is a convex subset of $\mathbb{R}_+$. It can be a closed interval, a singleton set, or the empty set $\varnothing$.

Observation 2 If the interior of $\mathcal{F}$ is an interval, i.e., $\text{int}(\mathcal{F}) = (\lambda_0, \lambda_1)$ for some $0 \le \lambda_0 < \lambda_1$, then, for any $\lambda, \lambda' \in (\lambda_0, \lambda_1)$, we have $\mathcal{R}(A(\lambda)) = \mathcal{R}(A(\lambda'))$, $\mathcal{R}(A(\lambda_0)) \subset \mathcal{R}(A(\lambda))$ and $\mathcal{R}(A(\lambda_1)) \subset \mathcal{R}(A(\lambda))$. As a result, $\mathcal{R}(A_0) \subset \mathcal{R}(A(\lambda))$ and $\mathcal{R}(A_1) \subset \mathcal{R}(A(\lambda))$ for any $\lambda \in (\lambda_0, \lambda_1)$.

Observation 3 If $b(\lambda') \in \mathcal{R}(A(\lambda'))$ and $b(\lambda'') \in \mathcal{R}(A(\lambda''))$ with $\lambda' \neq \lambda''$ and $\lambda', \lambda'' \in \text{int}(\mathcal{F})$, then b_0, $b_1 \in \mathcal{R}(A(\lambda))$ for all $\lambda \in \text{int}(\mathcal{F})$.

Observation 4 Note that dom(GTRSD) is a convex subset of $\mathcal{F}$. It can be an interval, a singleton set or an empty set. In case dom(GTRSD) is an interval, then $\text{int}(\text{dom}(GTRSD)) = \text{int}(\mathcal{F})$. Hence dom(GTRSD) is a convex set.

For Observation 1, if $A(\lambda') \succeq 0$ and $A(\lambda'') \succeq 0$ for any $\lambda', \lambda'' \in \mathcal{F}$, then $A(\alpha\lambda' + (1-\alpha)\lambda'') \succeq 0$ for any $0 \le \alpha \le 1$. Hence $\mathcal{F}$ is a convex set. $\mathcal{F}$ is also closed because the positive semidefiniteness property remains true for the boundary points of $\mathcal{F}$. The cases that $\mathcal{F}$ contains one point or is an empty set are special cases of being a convex set. Thus Observation 1 is true.

To verify Observation 2, we use the linear space orthogonal decomposition technique. For any $\lambda \in (\lambda_0, \lambda_1)$ and $x \perp \mathcal{R}(A(\lambda))$, we have $x^T A(\lambda) x = 0$ and $x^T A(\lambda \pm \epsilon) x \ge 0$ for ϵ being sufficiently small. This implies that $x^T A_1 x = 0$, and, consequently, $x^T A(\lambda') x = x^T (A(\lambda) + (\lambda' - \lambda) A_1) x = 0$ for any $\lambda' \in (\lambda_0, \lambda_1)$. Notice that $A(\lambda') \succeq 0$, hence $A(\lambda') = P^T P$. Moreover, $x^T A(\lambda') x = 0$ implies that $x^T P^T = 0$ and $x^T A(\lambda') = 0$, i.e., $x \perp \mathcal{R}(A(\lambda'))$. Therefore, $\mathcal{R}(A(\lambda')) \subset \mathcal{R}(A(\lambda))$. Due to the arbitrariness of λ and λ', we have

$$\mathcal{R}(A(\lambda)) = \mathcal{R}(A(\lambda')). \tag{3.9}$$

We also see that $x^T A(\lambda_0) x = x^T A(\lambda_1) x = 0$. Hence

$$\mathcal{R}(A(\lambda_0)) \subset \mathcal{R}(A(\lambda)) \text{ and } \mathcal{R}(A(\lambda_1)) \subset \mathcal{R}(A(\lambda)) \tag{3.10}$$

for any $\lambda \in (\lambda_0, \lambda_1)$. Since $\mathcal{R}(A(\lambda')) = \mathcal{R}(A(\lambda))$ for $\lambda' \neq \lambda$, we have

$$\mathcal{R}(A_1) \subset \mathcal{R}(A(\lambda)) \text{ and } \mathcal{R}(A_0) \subset \mathcal{R}(A(\lambda)) \tag{3.11}$$

for any $\lambda \in (\lambda_0, \lambda_1)$.

To verify Observation 3, if $b(\lambda') \in \mathcal{R}(A(\lambda'))$ and $b(\lambda'') \in \mathcal{R}(A(\lambda''))$, then, from Observation 2, $\mathcal{R}(A(\lambda')) = \mathcal{R}(A(\lambda''))$ and consequently, $(\lambda' - \lambda'')b_1 = (b(\lambda') - b(\lambda'')) \in \mathcal{R}(A(\lambda')) = \mathcal{R}(A(\lambda''))$. Therefore,

$$b_1 \in \mathcal{R}(A(\lambda)) \text{ and } b_0 \in \mathcal{R}(A(\lambda)) \tag{3.12}$$

for all $\lambda \in \text{int}(\mathcal{F})$.

For Observation 4, dom(GTRSD)$\subset \mathcal{F}$ is a direct consequence of the definition of dom(GTRSD) and equation (3.7). When dom(GTRSD) is a singleton set or an empty set, it is obviously a convex subset. When dom(GTRSD) contains at least two points, say λ' and λ'', then, from Observation 2 and Observation 3, $\mathcal{F}$ is an interval containing λ' and λ'', and $b(\lambda'), b(\lambda'') \in \mathcal{R}(A(\lambda))$. Therefore, for any $\lambda \in \text{int}(\mathcal{F})$, we have $b(\lambda) \in \mathcal{R}(A(\lambda))$ and $A(\lambda) \succeq 0$. From equation (3.7), we have $\lambda \in \text{dom}(GTRSD)$, which shows int(dom(GTRSD))= int($\mathcal{F}$) and dom$D$ is convex.

3.2.2 SDP dual

Sturm and Zhang [107] formulated the dual problem of GTRS as a semidefinite programming problem (as listed in [29]):

$$\begin{aligned} \max \quad & \gamma \\ s.t. \quad & \lambda \geq 0 \\ & \begin{bmatrix} c_0 + \lambda c_1 - \gamma & (b_0 + \lambda b_1)^T \\ b_0 + \lambda b_1 & A_0 + \lambda A_1 \end{bmatrix} \succeq 0. \end{aligned} \tag{3.13}$$

where $\gamma, \lambda \in \mathbb{R}$. Its corresponding semidefinite dual becomes

$$\begin{aligned} \min \quad & \text{tr}(\begin{bmatrix} c_0 & b_0^T \\ b_0^T & A_0 \end{bmatrix} X) \\ s.t. \quad & \text{tr}(\begin{bmatrix} -c_1 & -b_1^T \\ -b_1^T & -A_1 \end{bmatrix} X) \geq 0 \\ & X_{11} = 1 \end{aligned} \tag{3.14}$$

where $X \in \mathcal{S}_+^{n+1}$, the set of positive semidefinite matrices of order $n+1$, and X_{11} is the element in the first row and first column of X.

Lemma 20. *The problems GTRSD and (3.13) are equivalent.*

Proof. It can be seen that, for any given $\lambda \geq 0$, it is feasible for GTRSD if and only if

$$(\lambda, \gamma_\lambda) \triangleq (\lambda, c_0 + \lambda c_1 - b(\lambda)^T A^+(\lambda) b(\lambda))$$

is a feasible solution of (3.13). Furthermore, γ_λ is also the maximum γ such that (λ, γ) is a feasible solution of (3.13) for the given λ. Therefore, the optimal value of GTRSD is equal to the optimal value of (3.13). □

With this result, when studying the duality properties, we do not need to treat GTRSD and (3.13) separately.

3.3 Duality Gap and Attaining Primal Optimum

In NLP problems, the weak duality theorem between the primal problem and its Lagrangian dual problem is always true, but a zero duality gap may not always hold. In this section, two things are of our concerns. One is the duality gap between the primal problem and its dual; the other is when and how the optimal primal solution can be attained.

According to Observation 4, when GTRSD is feasible, dom(GTRSD) is either an interval or a singleton set. Our discussion will be based on these two cases.

3.3.1 dom(GTRSD) is an interval

When dom(GTRSD) is an interval, Observation 3 and Observation 4 indicate that $\text{int}(\text{dom}(GTRSD)) = \text{int}(\mathcal{F}) = (\lambda_0, \lambda_1)$ for some $0 \le \lambda_0 < \lambda_1$, and $b_0, b_1 \in \mathcal{R}(A(\lambda))$. We then prove the next result.

Lemma 21. *$\theta(\lambda)$ is concave and differentiable over (λ_0, λ_1).*

Proof. Note that $\theta(\lambda) = c(\lambda) - b^T(\lambda)A^+(\lambda)b(\lambda)$, for $\lambda \in (\lambda_0, \lambda_1)$. We first show that $A^+(\lambda)$ is differentiable over (λ_0, λ_1). From Observation 2, we know $\mathcal{R}(A_1) \subset \mathcal{R}(A(\lambda))$ for any $\lambda \in (\lambda_0, \lambda_1)$. For any sufficiently small $\Delta\lambda > 0$, we have $||\Delta\lambda A^+(\lambda)A_1|| < 1$. (Here, the 2−norm of matrices and vectors are used.) From Ben-Israel and Greville [16], we have

$$A^+(\lambda + \Delta\lambda) = (I + \Delta\lambda A^+(\lambda)A_1)^{-1}A^+(\lambda) \tag{3.15}$$

and

$$(I + \Delta\lambda A^+(\lambda)A_1)^{-1} = I - (\Delta\lambda A^+(\lambda)A_1) + o(\Delta\lambda)B(\lambda), \tag{3.16}$$

where $B(\lambda)$ is a matrix in terms of λ. Notice that

$$\lim_{\Delta\lambda \to 0} \frac{||A^+(\lambda + \Delta\lambda) - A^+(\lambda) + \Delta\lambda A^+(\lambda)A_1A^+(\lambda)||}{|\Delta\lambda|} = 0. \tag{3.17}$$

Therefore, $(A^+(\lambda))' = -A^+(\lambda)A_1A^+(\lambda)$.

Let

$$x(\lambda) = -A^+(\lambda)b(\lambda), \tag{3.18}$$

then $x'(\lambda) = -A^{+}(\lambda)b_1 + A^{+}(\lambda)A_1A^{+}(\lambda)b(\lambda)$. Consequently,

$$\theta'(\lambda) = x^T(\lambda)A_1x(\lambda) + 2b_1^Tx(\lambda) + c_1, \tag{3.19}$$

and

$$\theta''(\lambda) = -2x'(\lambda)A(\lambda)x'(\lambda) \leq 0. \tag{3.20}$$

This proves the concavity and differentiability of $\theta(\lambda)$ over (λ_0, λ_1). □

Remark: The concavity of $\theta(\lambda)$ can also be obtained from known results in NLP regarding Lagrangian dual function. But the differentiability of $\theta(\lambda)$ here is not an obvious one. To the best of our knowledge, this result is shown in the literature only when $A(\lambda) \succ 0$, which is a direct corollary of Theorem 6.3.3 in [13] (page 278).

When there is a $\lambda \in (\lambda_0, \lambda_1)$ such that $\theta'(\lambda) = 0$, then λ becomes an optimal solution of the dual problem from Lemma 21. Moreover, $x(\lambda)$ defined by (3.18) is a feasible solution of the primal problem, i.e., $x(\lambda) \in \text{dom(GTRS)}$. In this way, $f(x(\lambda)) = L(x(\lambda), \lambda) = \theta(\lambda)$, which shows $x(\lambda)$ is also an optimal primal solution.

However, the dual problem may achieve its optimum at either λ_0 or λ_1. In this situation, we need to study the boundary properties of the dual problem.

Lemma 22. *For any $\lambda \in (\lambda_0, \lambda_1)$ and $x(\lambda)$ as defined by (3.18), the following statements are true:*

(i) If $b(\lambda_0) \in \mathcal{R}(A(\lambda_0))$, then there is an $x(\lambda_0)$ such that

$$\lim_{\lambda \to \lambda_0 +} x(\lambda) = x(\lambda_0) \tag{3.21}$$

and

$$\lim_{\lambda \to \lambda_0 +} \theta'(\lambda) = x^T(\lambda_0)A_1x(\lambda_0) + 2b_1^Tx(\lambda_0) + c_1. \tag{3.22}$$

Otherwise,

$$\theta'(\lambda) \to +\infty, \text{ when } \lambda \to \lambda_0 + .$$

(ii) If $\lambda_1 < +\infty$ and $b(\lambda_1) \in \mathcal{R}(A(\lambda_1))$, then there is an $x(\lambda_1)$ such that

$$\lim_{\lambda \to \lambda_1 -} x(\lambda) = x(\lambda_1) \tag{3.23}$$

and

$$\lim_{\lambda \to \lambda_1 -} \theta'(\lambda) = x^T(\lambda_1)A_1x(\lambda_1) + 2b_1^Tx(\lambda_1) + c_1. \tag{3.24}$$

Otherwise,

$$\theta'(\lambda) \to -\infty, \text{ when } \lambda \to \lambda_1 - .$$

(iii) If $\lambda_1 = +\infty$ and $b_1 \in \mathcal{R}(A_1)$, then there is an $x(\infty) \in R^n$ such that

$$\lim_{\lambda \to +\infty} x(\lambda) = x(\infty) \tag{3.25}$$

and

$$\lim_{\lambda \to +\infty} \theta'(\lambda) = x^T(\infty) A_1 x(\infty) + 2b_1^T x(\infty) + c_1. \tag{3.26}$$

Otherwise,

$$\theta'(\lambda) \to -\infty, \text{ when } \lambda \to +\infty.$$

Proof. For convenience, we use S_0 and S to denote the spaces of $\mathcal{R}(A(\lambda_0))$ and $\mathcal{R}(A(\lambda))$, respectively.

(i) The key idea to prove this result when $b(\lambda_0) \in S_0$ is to find another set of vectors $\{\hat{x}(\lambda)\}$ such that $(x(\lambda) - \hat{x}(\lambda)) \to 0$ and $\hat{x}(\lambda)$ converges to some vector x_0, as $\lambda \to \lambda_0+$. In this way, $x(\lambda) \to x_0$, as $\lambda \to \lambda_0+$.

To find such $\hat{x}(\lambda)$, we repeatedly use the fact that

$$S_0 + A_1(S_0^\perp \cap S) = S, \tag{3.27}$$

where $A_1(S_0^\perp \cap S)$ is the linear subspace $\{v = A_1 x \mid x \in S_0^\perp \cap S\}$.

On one hand, from Observation 2, we have $S_0 \subset S$ and $\mathcal{R}(A_1) \subset S$. Hence $S_0 + A_1(S_0^\perp \cap S) \subset S$. On the other hand, if $x \perp S_0 + A_1(S_0^\perp \cap S)$, we let $x = z_1 + z_2$, where $z_1 \in S$ and $z_2 \in S^\perp$. Then $z_1 = (x - z_2) \perp S_0 + A_1(S_0^\perp \cap S)$. Hence $z_1 \perp S_0$ and, consequently, $z_1 \in S_0^\perp \cap S$. Moreover, by using the fact of $z_1 \perp S_0 + A_1(S_0^\perp \cap S)$ again, we have $A(\lambda_0) z_1 = 0$ and $z_1^T A_1 z_1 = 0$. Therefore, $z_1^T A(\lambda) z_1 = 0$. Since $A(\lambda) \succeq 0$, we know $A(\lambda) z_1 = 0$. Together with $z_1 \in S$, we have $z_1 \in S \cap S^\perp = \{0\}$. Therefore, $x = z_1 + z_2 = z_2 \perp S$ and equation (3.27) is proved.

Now, when $b(\lambda_0) \in S_0$, we let $\overline{x} \triangleq -A^+(\lambda_0) b(\lambda_0)$, then $A(\lambda_0)\overline{x} = -b(\lambda_0)$. Notice that $\overline{x} \in S_0 \subset S$ because $\mathcal{R}(A(\lambda_0)) = \mathcal{R}(A^+(\lambda_0))$. Since $\mathcal{R}(A_1) \subset S$ and $b_1 \in S$, we have $(A_1\overline{x} + b_1) \in S$. From (3.27), there exist $x' \in S_0$ and $y' \in S_0^\perp \cap S$ such that $A_1\overline{x} + b_1 = x' + A_1 y'$. Denote $x_0 = (\overline{x} - y') \in S$. Then $A(\lambda_0) x_0 = -b(\lambda_0)$ and $A_1 x_0 + b_1 = x' \in S_0$.

Similarly, let $x'' \triangleq A^+(\lambda_0) x'$, then $A(\lambda_0) x'' = x' = A_1 x_0 + b_1$. Since $A_1 x'' \in S$, there exist $x''' \in S_0$ and $y''' \in S_0^\perp \cap S$ such that $A_1 x'' = x''' + A_1 y'''$. Denote $x_1 = (x'' - y''') \in S$, then $A(\lambda_0) x_1 = A_1 x_0 + b_1$ and $A_1 x_1 = x''' \in S_0$.

Continuing this procedure, we can obtain a set of vectors $x_1, \ldots, x_n \in S$ satisfying $A(\lambda_0) x_1 = A_1 x_0 + b_1$ and $A(\lambda_0) x_k = A_1 x_{k-1}$, for $k = 2, \ldots, n$.

Let

$$\hat{x}(\lambda) = \sum_{i=0}^{n} (-1)^i (\lambda - \lambda_0)^i x_i, \tag{3.28}$$

then

$$A(\lambda)\hat{x}(\lambda) = -b(\lambda) + (-1)^n(\lambda - \lambda_0)^{n+1} A_1 x_n. \tag{3.29}$$

From the fact that $\mathbb{R}^n = S \oplus S^\perp$, we can get $\overline{A} = UU^T \succeq 0$ such that $A(\lambda)\overline{A} = 0$ and $A(\lambda) + \overline{A} \succ 0$. Here the columns of U consist of all the base vectors of $S^\perp$. Since $\hat{x}(\lambda) \in S$ and $x(\lambda) \in S$, we have

$$(A(\lambda) + \overline{A})\hat{x}(\lambda) = A(\lambda)\hat{x}(\lambda)$$

and

$$(A(\lambda) + \overline{A})x(\lambda) = A(\lambda)x(\lambda).$$

As a result, we have

$$\lim_{\lambda \to \lambda_0 +} x(\lambda) - \hat{x}(\lambda) = \lim_{\lambda \to \lambda_0 +} (A(\lambda) + \overline{A})^{-1}(-1)^n(\lambda - \lambda_0)^{n+1} A_1 x_n = 0 \tag{3.30}$$

and

$$\lim_{\lambda \to \lambda_0 +} \hat{x}(\lambda) = x_0. \tag{3.31}$$

This implies that

$$\lim_{\lambda \to \lambda_0 +} \theta'(\lambda) = x_0^T A_1 x_0 + 2b_1^T x_0 + c_1. \tag{3.32}$$

Otherwise, suppose that $b(\lambda_0) \notin S_0$, Observation 3 implies that $b(\lambda_0) \in S$ and $b_1 \in S$. Then, from (3.27), there exist $b' \in S_0$ and $b''(\neq 0) \in A_1(S_0^\perp \cap S)$ such that

$$b(\lambda_0) = b' + b''. \tag{3.33}$$

With the same argument used before, there is an x_1 such that

$$x_1(\lambda) = -A^+(\lambda)(b' + (\lambda - \lambda_0)b_1) \to x_1, \tag{3.34}$$

as $\lambda \to \lambda_0+$, and

$$A(\lambda_0)x_1 = -b'. \tag{3.35}$$

Since $b''(\neq 0) \in A_1(S_0^\perp \cap S)$, there is an $x_2(\neq 0) \in S_0^\perp \cap S$ such that $A_1 x_2 = -b''$ and $x_2^T A(\lambda) x_2 > 0$. Moreover, from $x_2^T A(\lambda_0) x_2 = 0$, we know $x_2^T A_1 x_2 > 0$. Let $x_2(\lambda) \triangleq \dfrac{x_2}{\lambda - \lambda_0}$, then

$$A(\lambda)(x_1(\lambda) + x_2(\lambda)) = -(b' + (\lambda - \lambda_0)b_1) - b'' = -b(\lambda). \tag{3.36}$$

Notice that $x_1(\lambda) + x_2(\lambda) \in S$ and $x(\lambda) \in S$, hence $\Delta x(\lambda) = x(\lambda) - (x_1(\lambda) + x_2(\lambda)) \in S$ and

$A(\lambda)\Delta x(\lambda) = 0$. Therefore, $\Delta x(\lambda) = 0$ and

$$x(\lambda) = x_1(\lambda) + x_2(\lambda). \tag{3.37}$$

As a result, we have

$$\theta'(\lambda) = x^T(\lambda)A_1x(\lambda) + 2b_1^Tx(\lambda) + c_1 \to +\infty,$$

as $\lambda \to \lambda_0+$.

The proof of (ii) is similar to that of (i).

(iii) When $\lambda_1 = +\infty$, the equation $(A_0 + \lambda A_1)x(\lambda) = -(b_0 + \lambda b_1)$ can be seen as $(\frac{1}{\lambda}A_0 + A_1)x(\lambda) = -(\frac{1}{\lambda}b_0 + b_1)$. The rest of the proof is again similar to that of (i). $\square$

Lemma 22 is a general but not trivial extension of Theorem 5.4 in [73]. It not only tells what the limits of $x(\lambda)$ and $\theta'(\lambda)$ are when such limits exist, but also gives the necessary and sufficient condition for the limits to exist. Using this lemma, the next theorem follows.

Theorem 23. *If GTRS is feasible and dom(GTRSD) is an interval, then there always exists an optimal primal solution attaining the optimum value of GTRSD, i.e., the duality gap between GTRS and GTRSD is zero.*

Proof. Given int(dom(GTRSD)) $= (\lambda_0, \lambda_1)$, if there is a $\lambda \in (\lambda_0, \lambda_1)$ with $\theta'(\lambda) = 0$, then $x(\lambda)$ is an optimal solution to the primal problem and $f(x(\lambda)) = L(x(\lambda), \lambda) = \theta(\lambda)$. Otherwise, such λ does not exist and the dual optimal is achieved at either λ_0 or λ_1.

(i) If the dual optimal solution is achieved at λ_0, then $b(\lambda_0) \in \mathcal{R}(A(\lambda_0))$ and $\theta'(\lambda_0+) \triangleq x^T(\lambda_0)A_1x(\lambda_0)+2b_1^Tx(\lambda_0)+c_1 \leq 0$. Notice that for any $x' \perp \mathcal{R}(A(\lambda_0))$, we have $(x')^TA(\lambda)x' \geq 0$ and, consequently, $(x')^TA_1x' \geq 0$.

If $\lambda_0 = 0$, then $x(\lambda_0)$ is the optimal primal solution and $f(x(\lambda_0)) = L(x(\lambda_0), \lambda_0) = \theta(\lambda_0)$.

If $\lambda_0 > 0$, then there exists an $x' \in \mathcal{R}(A(\lambda_0))^\perp \cap \mathcal{R}(A(\lambda))$ such that $x'^TA_1x' > 0$. We can choose some proper μ such that $(x(\lambda_0) + \mu x')^TA_1(x(\lambda_0) + \mu x') + 2b_1^T(x(\lambda_0) + \mu x') + c_1 = 0$ and $A(\lambda_0)(x(\lambda_0) + \mu x') = -b(\lambda_0)$. Therefore, $(x(\lambda_0) + \mu x')$ becomes an optimal primal solution and $f(x(\lambda_0) + \mu x') = \lim_{\lambda \to \lambda_0+} \theta(\lambda)$.

(ii) If the dual optimal is achieved at λ_1, then $b(\lambda_1) \in \mathcal{R}(A(\lambda_1))$ and $\theta'(\lambda_1-) = x^T(\lambda_1)A_1x(\lambda_1) + 2b_1^Tx(\lambda_1) + c_1 \geq 0$.

If $\lambda_1 < +\infty$, similar to (i), there is an $x' \in \mathcal{R}(A(\lambda)) \cap \mathcal{R}(A(\lambda_1))^\perp$ satisfying $x'^TA_1x' < 0$. Consequently, an optimal solution can be obtained.

If $\lambda_1 = +\infty$, then from the proof of Lemma 22, we have $A_1x(\infty) + b_1 = 0$. If $\theta'(\lambda_1-) > 0$, then

$$c_1 - b_1A_1^+b_1 = -x^T(\infty)A_1x(\infty) + c_1 = x^T(\infty)A_1x(\infty) + 2b_1^Tx(\infty) + c_1 > 0$$

i.e., GTRS is infeasible. Note that in this case $\theta'(\lambda) > \epsilon$ for some $\epsilon > 0$ and all sufficiently large λ. Therefore $\lim_{\lambda \to +\infty} \theta(\lambda) = +\infty$. If $\theta'(\lambda_1-) = 0$, then $x(\infty)$ is feasible and it becomes an optimum primal solution with

$$\lim_{\lambda \to +\infty} \theta(\lambda) = f(x(\infty)). \tag{3.38}$$

$\square$

Remark: Our analysis and results can be related to Moré's [73]. In his work, he explicitly defined a function $\phi(\lambda)$, which is $\theta'(\lambda)$ here, and proved that it is a decreasing function. If there exists a λ satisfying $\phi(\lambda) = 0$, then $x(\lambda) = -A^{-1}(\lambda)b(\lambda)$ is an optimal primal solution. Otherwise, by the decreasing property of $\phi(\lambda)$, if $\phi(\lambda) > 0$(or < 0) for all $\lambda \in (\lambda_0, \lambda_1)$, then the limit of $\phi(\lambda_1-)$(or $\phi(\lambda_0+)$) exists. Then he found an optimal primal solution corresponding to such boundary point. However, his analysis heavily relied on the assumption of $A(\lambda) \succ 0$ for the result of simultaneously diagonalizable of A_0 and A_1, the differentiability of $\phi(\lambda)$, etc. Our assumption is weaker and the results are more general.

Another thing that worths mentioning is that though we do not use the simultaneously diagonalizable of A_0 and A_1 in our analysis, this property is true when $\mathcal{F}$ is an interval. One can prove that $\mathcal{F}$ being an interval is equivalent to the assumptions in Theorem 8.7.1 in [50], which is a sufficient condition of the simultaneously diagonalizable property.

3.3.2 dom(GTRSD) is a singleton set

When GTRS is feasible but does not satisfy the Slater condition, then it must be the case that $A_1 \succeq 0$, $b_1 \in \mathcal{R}(A_1)$ and $c_1 = b_1^T A_1^+ b_1$. In this case, If dom(GTRSD) contains only a point λ_0, then we know $A(\lambda_0) \succeq 0$ and $b(\lambda_0) \in \mathcal{R}(A(\lambda_0))$. Since $A_1 \succeq 0$, then we have $A(\lambda) = A(\lambda_0) + (\lambda - \lambda_0)A_1 \succeq 0$, for any $\lambda \geq \lambda_0$. Therefore, $\mathcal{F} \supset [\lambda_0, +\infty)$. From Observation 3, we also know $b_1 \in \mathcal{R}(A_1) \subset \mathcal{R}(A(\lambda))$ and $b(\lambda_0) \in \mathcal{R}(A(\lambda_0)) \subset \mathcal{R}(A(\lambda))$. Hence $b(\lambda) \in \mathcal{R}(A(\lambda))$ for any $\lambda \in \mathcal{F}$, and, consequently, for any $\lambda \in \text{int}(\mathcal{F})$, we know λ also belongs to dom(GTRSD). This contradicts to the assumption that dom(GTRSD) is a singleton.

Therefore, if dom(GTRSD) contains only a point, then the Slater condition is satisfied for GTRS and the zero duality gap property can be seen from the literatures (e.g. [73]). For the discussion of attaining optimal primal solution, there are two possible cases: $\mathcal{F}$ is an interval or a singleton set.

Case 1: When $\mathcal{F}$ is an interval with $\text{int}(\mathcal{F}) = (\lambda_0, \lambda_1)$, then dom(GTRSD) can be $\{\lambda_0\}$, $\{\lambda_1\}$, or $\{\lambda\}$ for any $\lambda \in (\lambda_0, \lambda_1)$.

Note that $b_1 \notin \mathcal{R}(A(\lambda))$ for any $\lambda \in (\lambda_0, \lambda_1)$. If not so, the facts of $b(\hat{\lambda}) \in \mathcal{R}(A(\hat{\lambda})) = \mathcal{R}(A(\lambda))$ and $b_1 \in \mathcal{R}(A(\lambda))$ lead to $b(\lambda) \in \mathcal{R}(A(\lambda))$, which contradicts to the assumption of dom(GTRSD) being a singleton set. Observation 2 further implies that $b_1 \notin \mathcal{R}(A(\lambda_0))$ and $b_1 \notin \mathcal{R}(A(\lambda_1))$ (or $b_1 \notin \mathcal{R}(A_1)$, if $\lambda_1 = +\infty$). Hence there is an $x^\perp$ satisfying $A(\lambda)x^\perp = 0$ and $b_1^T x^\perp \neq 0$. Let $x(\hat{\lambda}) = -A^+(\hat{\lambda})(b_0 + \hat{\lambda} b_1)$, then

$$g(x(\hat{\lambda}) + \mu x^\perp) = x^T(\hat{\lambda}) A_1 x(\hat{\lambda}) + 2b_1^T x(\hat{\lambda}) + c_1 + 2\mu b_1^T x^\perp. \tag{3.39}$$

Since $b_1^T x^\perp \neq 0$, we can choose a proper value of μ such that $x^T(\hat{\lambda}) A_1 x(\hat{\lambda}) + 2b_1^T x(\hat{\lambda}) + c_1 + 2\mu b_1^T x^\perp = 0$. Then $x^* = x(\hat{\lambda}) + \mu x^\perp$ becomes an optimal primal solution.

Case 2: When $\mathcal{F}$ is a singleton set, our discussion depends on the constraint matrix A_1.

(i) If $A_1 \preceq 0$ and $A_1 \neq 0$, then dom(GTRSD)$= \{0\}$ and $b_0 \in \mathcal{R}(A_0)$.

In this case, there must exist an x_0 such that $x_0^T A_0 x_0 = 0$ and $x_0^T A_1 x_0 < 0$. Otherwise, we can show that $\mathcal{F}$ is an interval, which becomes a contradiction. To see this, by the semidefiniteness of A_0 and A_1, we have that, for any $x \in \mathbb{R}^n$, if $x^T A_0 x = 0$ then $x^T A_1 x = 0$. This means that if $x \perp \mathcal{R}(A_0)$, then $x \perp \mathcal{R}(A_1)$. Consequently, $\mathcal{R}(A_1) \subset \mathcal{R}(A_0)$. Therefore, there is a $\Pi \in R^{n\times n}$ such that $\Pi^T A_0 \Pi = diag(\Lambda, 0)$ and $\Pi^T A_1 \Pi = diag(\Lambda', 0)$, where $\Lambda \succ 0$, $\Lambda' \preceq 0$ and the dimensions of Λ and Λ' are the same as the dimension of $\mathcal{R}(A_0)$. Since $\Lambda \succ 0$, there exists a $\lambda > 0$ such that $A_0 + \lambda A_1 \succeq 0$. Hence $\mathcal{F}$ is an interval.

Let $x^* = -A_0^+ b_0 + \mu x_0$. Since $x_0 \perp \mathcal{R}(A_0)$ and $x_0^T A_1 x_0 < 0$, when μ is sufficiently large, we have $x^{*T} A_1 x^* + 2b_1^T x^* + c_1 < 0$ and $\theta(0) = c_0 - b_0^T A_0^+ b_0 = f(x^*)$. This shows that x^* is an optimal primal solution.

(ii) If A_1 is indefinite, i.e., $A_1 \sim diag(I_1, -I_2, 0)$ or $A_1 \sim diag(I_1, -I_2)$, then the optimal primal may not be attainable. The following example shows a case.

Example. Let

$$A_0 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix},\ b_0 = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix},\ c_0 = 0,$$

and

$$A_1 = diag(1,1,1,-1),\ b_1 = (0,0,0,0)^T,\ c_1 = 0.$$

Then

$$\theta(\lambda) = \begin{cases} -b_0^T A_0^+ b_0 & \lambda = 0, \\ -\infty & \text{otherwise}, \end{cases}$$

where

$$A_0^+ = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

Note that the optimal value is -1 for both GTRS and its Lagrangian dual GTRSD, but it is not attainable for any $x \in \text{dom(GTRS)}$. However, it can be approximated with an arbitrarily small gap by choosing $x = (1, 0, \alpha, \sqrt{\alpha^2 + 1})^T \in \text{dom(GTRS)}$ with $\alpha < 0$ and $|\alpha|$ being sufficiently large. For the scheme to obtain an approximation solution, one is referred to [89] and [107].

3.4 Summary

From the discussion in Section 3.2.2 and in Section 3.3, we have obtained the following main results:

(i) If both GTRS and its Lagrangian dual (or semidefinite dual) are feasible, then there does not exist any duality gap between their optimal values.

(ii) Under the assumption in (i), if $\mathcal{F}$ is an interval, or if $\mathcal{F}$ is a singleton set with $A_1 \preceq 0$ and $A_1 \neq 0$, then there always exists a finite optimal solution to GTRS.

These results are weaker than the strong duality properties in linear programming. Recall that in linear programming, if either the primal optimum or the dual optimum is finite, so is the other one. Moreover, the duality gap is always zero and the optimum can be achieved on both sides. But for GTRS and its dual, if the optimum of GTRS is finite, the dual problem may still be infeasible.

For example, consider the following problem:

$$\begin{aligned} \min \quad & 2x_1x_2 \\ s.t. \quad & x_2^2 \leq 0. \end{aligned}$$

In this case, GTRS is feasible and its optimal value is a finite number zero. But both problems GTRSD and (3.13) are infeasible.

Chapter 4

Quadratic Programming Problems with Quadratic Inequality Constraints

In this chapter, we focus on the quadratic programming problems with inequality quadratic constraints, i.e. IQCQP. Using the concept of the cone of nonnegative quadratic functions, we develop a sufficient condition, which is more general than the known positive semidefiniteness condition, to certify a KKT point to be a global optimal solution of IQCQP. We also use linear conic programming duality theory to study the properties of the corresponding optimal Lagrangian multipliers. The results lead to a computational scheme that may find an ϵ-optimal solution or a lower bound of IQCQP in polynomial time. Numerical examples are given at the end of the chapter.

4.1 Introduction

IQCQP problems are a subclass of QCQP problems and defined as

$$\begin{aligned} \min \quad & f(x) = x^T A_0 x + 2b_0^T x + c_0 \\ s.t. \quad & g_i(x) = x^T A_i x + 2b_i^T x + c_i \le 0, \quad i = 1, \dots, m, \end{aligned} \qquad \text{(IQCQP)} \quad (4.1)$$

where A_i is in $\mathcal{S}^n$, b_i is in $\mathbb{R}^n$, and c_i is in $\mathbb{R}$, $i = 0, 1, \dots, m$.

When A_i, $i = 0, \dots, m$, are all positive semidefinite matrices, this problem becomes a convex programming problem and can be solved in polynomial time (within a given precision) by SOCP formulation [67]. In general, this problem is NP-hard. Many research papers can be found. S-turm and Zhang [107] reformulated the nonconvex quadratic programming problem with general constraints into a linear conic programming problem and introduced the cone of nonnegative quadratic functions over the feasible domain. They also introduced a matrix decomposition technique. Based on this reformulation and decomposition technique, they identified several polynomial solvable subclasses of QCQP problems with one quadratic constraint. Ye and Zhang [113] studied IQCQP when $m = 2$ using SDP relaxation and decomposition technique in [107] and also gave several polynomial solvable subclasses. Fu et al. [46] proposed an approximation algorithm for IQCQP with elliptic constraints based on SDP relaxation and Tseng [109] also considered approximation algorithms for similar problems with special conditions.

Besides the SDP representation of a quadratic function, many research papers work on the copositive representation. Bomze et al. [26] reformulated the so called standard QCQP into a copositive programming problem. Here the standard means besides the quadratic constraints, an additional standard simplex constraint, i.e.,

$$\{x \in \mathbb{R}^n_+ \mid \sum_{i=1}^{n} x_i = 1\},$$

is included. Burer [32] also used the copositive programming reformulation on quadratic programming problems with linear and binary constraints. So far, no polynomial time algorithm is known for solving the copositive programming problems, but the copositive programming problems can be approximated by a series of linear conic programming problems that could be solved in polynomial time (e.g., see [39], [81]).

There are also many papers working on finding sufficient conditions for global optimality of IQCQP problem and its special cases (e.g., see [14], [111], [71], [55]). These conditions are commonly based on the positive semidefiniteness condition (See, e.g., Theorem 2 in [48]), which is extended to the copositive condition in this chapter.

In the rest of this chapter, we first formulate the linear conic programming problems using the cone of nonnegative quadratic functions over the feasible domain in Section 4.2. Then in Section 4.3 the copositiveness condition is developed, and based on the copositiveness condition a maximal property of the Lagrangian multiplier is proved. A second linear conic programming problem is used to get the Lagrangian multiplier and after that under certain conditions an optimal solution of IQCQP could be obtained by solving a linear equation. In Section 4.4, a relaxed linear conic programming problem is used so that it can be solved in polynomial time (within a given precision). Two examples are given in Section 4.5 to show that the proposed method could solve more problems than the SDP relaxation does. At last, in Section 4.6, summary of the work is discussed.

4.2 KKT Conditions and Conic Programming Relaxation

Given $\lambda \in \mathbb{R}^m_+$, denote

$$\begin{aligned} A(\lambda) &= A_0 + \sum_{i=1}^m \lambda_i A_i, \\ b(\lambda) &= b_0 + \sum_{i=1}^m \lambda_i b_i, \\ c(\lambda) &= c_0 + \sum_{i=1}^m \lambda_i c_i. \end{aligned}$$

The Lagrangian of IQCQP becomes

$$\begin{aligned} L(x,\lambda) \quad &= f(x) + \sum_{i=1}^m \lambda_i g_i(x) \\ &= x^T(A_0 + \sum_{i=1}^m \lambda_i A_i)x + 2(b_0 + \sum_{i=1}^m \lambda_i b_i)^T x + c_0 + \sum_{i=1}^m \lambda_i c_i \\ &= x^T A(\lambda)x + 2b^T(\lambda)x + c(\lambda) \end{aligned} \tag{4.2}$$

Given a feasible solution x^* of IQCQP, if it satisfies LICQ and is a local minimum, then there exists a λ such that the following KKT conditions are satisfied.

$$\begin{aligned} &\nabla_x^T L(x^*,\lambda^*) = 2A(\lambda^*)x^* + 2b(\lambda^*) = 0, \\ &\lambda_i^* g_i(x^*) = \lambda_i^*((x^*)^T A_i x^* + 2b_i^T x^* + c_i) = 0, \\ &g_i(x^*) = (x^*)^T A_i x^* + 2b_i^T x^* + c_i \leq 0,\ \lambda_i^* \geq 0, \quad i = 1,\ldots,m. \end{aligned} \tag{4.3}$$

As we know, the KKT conditions here are merely a necessary condition of a local minimum solution. Given a KKT solution (x^*, λ^*), the point x^* may not even be a local minimum solution. However, if we already know the λ^* in a KKT solution, $A(\lambda^*)$ is invertible, and its corresponding x^* is optimal to IQCQP, then from KKT conditions, we can find x^* by solving

$$x^* = -A^{-1}(\lambda^*)b(\lambda^*).$$

To ensure x^* is an optimal solution, the "positive semidefiniteness condition" can be adopted to obtain the next result (See, e.g., [48]).

Lemma 24 (Positive semidefiniteness condition)**.** *Let (x^*, λ^*) be a KKT solution. If $A(\lambda^*) \in \mathcal{S}^n_+$, then x^* is a global optimal solution of IQCQP.*

This condition can be further extended in the next section. Here, we first introduce a linear conic reformulation of IQCQP.

The feasible domain of IQCQP is

$$\text{dom(IQCQP)} = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, i = 1, \ldots, m\}.$$

Given a quadratic function $f(x) = x^T A_0 x + 2b_0^T x + c_0$, we can write it as

$$f(x) = \begin{bmatrix} 1 \\ x \end{bmatrix}^T \begin{bmatrix} c_0 & b_0^T \\ b_0 & A_0 \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} c_0 & b_0^T \\ b_0 & A_0 \end{bmatrix} \cdot \begin{bmatrix} 1 & x^T \\ x & xx^T \end{bmatrix}.$$

Define two cones as

$$\mathcal{D}_{n+1} = \left\{ U \in \mathcal{S}^{n+1} \;\middle|\; U \cdot \begin{bmatrix} 1 & x^T \\ x & xx^T \end{bmatrix} \geq 0, \forall x \in \text{dom(IQCQP)} \right\}, \tag{4.4}$$

and

$$\mathcal{D}_{n+1}^* = \text{cl cone} \left\{ X = \begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix}^T \;\middle|\; x \in \text{dom(IQCQP)} \right\}. \tag{4.5}$$

The cone $\mathcal{D}_{n+1}$ is the set of coefficients of quadratic functions which are nonnegative over the feasible domain of IQCQP while $\mathcal{D}_{n+1}^*$ is the smallest closed cone that contains the feasible domain of IQCQP in the homogenous formulation. They are closed and convex by definition and Corollary 1 in [107] shows that these two cones are actually dual to each other.

Now consider the following conic programming problem

$$\begin{aligned} \min \quad & \begin{bmatrix} c_0 & b_0^T \\ b_0 & A_0 \end{bmatrix} \cdot Y \\ s.t. \quad & Y_{11} = 1, \\ & Y \in \mathcal{D}_{n+1}^*. \end{aligned} \tag{4.6}$$

Following the discussion made on the problem (MP) in Section 3 in [107], we have the next result.

Theorem 25. *The problem* (4.6) *and IQCQP are equivalent in the sense that they have the same optimal objective values.*

Note that $-\left[\begin{smallmatrix} c_i & b_i^T \\ b_i & A_i \end{smallmatrix}\right]$ belongs to $\mathcal{D}_{n+1}$, $i = 1, \ldots, m$. Therefore, for any $Y \in \mathcal{D}_{n+1}^*$, we have $Y \cdot \left[\begin{smallmatrix} c_i & b_i^T \\ b_i & A_i \end{smallmatrix}\right] \leq 0$, $i = 1, \ldots, m$, which means they are all valid inequalities for the problem (4.6). By adding these redundant constraints to (4.6), we define the following linear conic programming problems.

$$\begin{aligned}
\min \quad & \begin{bmatrix} c_0 & b_0^T \\ b_0 & A_0 \end{bmatrix} \cdot Y \\
s.t. \quad & \begin{bmatrix} c_i & b_i^T \\ b_i & A_i \end{bmatrix} \cdot Y \leq 0, \quad i = 1, \ldots, m, \qquad \text{(CP1)} \qquad (4.7) \\
& Y_{11} = 1, \\
& Y \in \mathcal{D}_{n+1}^*.
\end{aligned}$$

Then this new problem and IQCQP have the same objective value. Moreover, the conic dual of CP1 is

$$\begin{aligned}
\max \quad & \sigma \\
s.t. \quad & \begin{bmatrix} c(\lambda) - \sigma & b^T(\lambda) \\ b(\lambda) & A(\lambda) \end{bmatrix} \in \mathcal{D}_{n+1}, \qquad \text{(CD1)} \qquad (4.8) \\
& \lambda \in \mathbb{R}_+^m.
\end{aligned}$$

The equivalence of IQCQP, CP1, and CD1 can be seen in the next theorem.

Theorem 26. *If IQCQP is bounded below, then the optimal objective values of IQCQP, CP1, and CD1 are equal.*

Proof. We already know IQCQP and CP1 have the same optimal objective value. Since IQCQP is bounded below, let $\lambda = 0$, and

$$\sigma = \inf\{f(x) \mid x \in \mathrm{dom}(IQCQP)\}.$$

Then

$$\begin{bmatrix} c_0 - \sigma & b_0^T \\ b_0 & A_0 \end{bmatrix} \in \mathcal{D}_{n+1}$$

Note that σ is the optimal objective value of IQCQP, and $(\sigma, 0)$ is a feasible solution of CD1. From the weak duality of linear conic programming, σ is also the optimal objective of CD1. □

Remark. We have already shown that the optimal objective value of CP1 equals to that of the IQCQP problem and under the above conditions, the optimal objective value of the CD1 problem also equals to that of the IQCQP problem. Note that the constraints $Y \cdot \left[\begin{smallmatrix} c_i & b_i^T \\ b_i & A_i \end{smallmatrix}\right] \leq 0$, $i = 1, \ldots, m$, are all valid inequalities and are redundant. If the cone $\mathcal{D}_{n+1}^*$ is replaced by a larger cone, say $\mathcal{C}_{n+1}^*$, then these constraints may not be redundant and the resulting primal and dual conic programming problems are relaxations of the IQCQP problem. In particular, when $\mathcal{C}_{n+1}^* = \mathcal{S}_+^{n+1}$, the corresponding relaxation problems are famous SDP relaxations of the IQCQP problem. From this view, CP1 and the CD1 are generalizations of the SDP relaxations for IQCQP.

4.3 Finding Global Optimal Solutions

In this section, we will extend the condition in Lemma 24 which ensures a KKT solution being globally optimal to IQCQP.

Let (x^*, λ^*) be a KKT solution of IQCQP. We also assume x^* satisfies LICQ and its active set is $\mathcal{A}(x^*)$. Correspondingly, define a matrix

$$M(x^*, \lambda^*) = \begin{bmatrix} c(\lambda^*) - f(x^*) & b^T(\lambda^*) \\ b(\lambda^*) & A(\lambda^*) \end{bmatrix}.$$

We can extend the positive semidefiniteness condition to the next condition.

Theorem 27 (Copositiveness condition)**.** *Let (x^*, λ^*) be a KKT solution of IQCQP. Under LICQ, if $M(x^*, \lambda^*) \in \mathcal{D}_{n+1}$, then $(\sigma^*, \lambda^*) = (f(x^*), \lambda^*)$ is an optimal solution of the CD1 and x^* is an optimal solution of IQCQP.*

Proof. From the condition

$$M(x^*, \lambda^*) = \begin{bmatrix} c(\lambda^*) - f(x^*) & b^T(\lambda^*) \\ b(\lambda^*) & A(\lambda^*) \end{bmatrix} \in \mathcal{D}_{n+1},$$

we know $(\sigma^*, \lambda^*) = (f(x^*), \lambda^*)$ is a feasible solution of CD1. Since x^* is feasible for IQCQP and $\sigma^* = f(x^*)$, then x^* is optimal to IQCQP and (σ^*, λ^*) is optimal to CD1. □

Note that the positive semidefiniteness condition is equivalent to

$$M(x^*, \lambda^*) = \begin{bmatrix} c(\lambda^*) - f(x^*) & b^T(\lambda^*) \\ b(\lambda^*) & A(\lambda^*) \end{bmatrix} \in \mathcal{S}_+^{n+1} \subset \mathcal{D}_{n+1},$$

which reveals that the copositiveness condition is indeed an extension of the positive semidefiniteness condition.

Under the copositiveness condition, our objective is to find such λ^* and its corresponding x^* by solving CD1. Since CD1 may have multiple optimal solutions, we need to further study the property of the Lagrangian vector that leads to the exact Lagrangian multiplier we want.

Lemma 28. *If x^* is an optimal solution of IQCQP and $(\bar{\sigma}, \bar{\lambda})$ is an optimal solution of CD1, then x^* is also an optimal solution for the problem*

$$\begin{aligned} \min \quad & L(x, \bar{\lambda}) \\ s.t. \quad & x \in \mathrm{dom(IQCQP)}. \end{aligned} \tag{4.9}$$

Proof. Since x^* and $(\bar{\sigma}, \bar{\lambda})$ are optimal solutions of IQCQP and CD1, then $f(x^*) = \bar{\sigma}$. The feasibility of $(\bar{\sigma}, \bar{\lambda})$ implies

$$\begin{bmatrix} c(\bar{\lambda}) - \bar{\sigma} & b^T(\bar{\lambda}) \\ b(\bar{\lambda}) & A(\bar{\lambda}) \end{bmatrix} \in \mathcal{D}_{n+1},$$

which means, for any $x \in \text{dom(IQCQP)}$,

$$\begin{bmatrix} 1 \\ x \end{bmatrix}^T \begin{bmatrix} c(\bar{\lambda}) - \bar{\sigma} & b^T(\bar{\lambda}) \\ b(\bar{\lambda}) & A(\bar{\lambda}) \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} \geq 0.$$

Therefore,

$$L(x, \bar{\lambda}) \geq \bar{\sigma}, \quad \forall x \in \text{dom(IQCQP)}.$$

Note that, if we let

$$Y = \begin{bmatrix} 1 & (x^*)^T \\ x^* & x^*(x^*)^T \end{bmatrix},$$

then

$$\begin{aligned} 0 \quad &\leq Y \cdot \begin{bmatrix} c(\bar{\lambda}) - \bar{\sigma} & b^T(\bar{\lambda}) \\ b(\bar{\lambda}) & A(\bar{\lambda}) \end{bmatrix} \\ &= Y \cdot \begin{bmatrix} c_0 & b_0^T \\ b_0 & A_0 \end{bmatrix} + \textstyle\sum_{i=1}^m \bar{\lambda}_i Y \cdot \begin{bmatrix} c_i & b_i^T \\ b_i & A_i \end{bmatrix} - \bar{\sigma} \\ &= f(x^*) + \textstyle\sum_{i=1}^m \bar{\lambda}_i Y \cdot \begin{bmatrix} c_i & b_i^T \\ b_i & A_i \end{bmatrix} - \bar{\sigma} \\ &= \textstyle\sum_{i=1}^m \bar{\lambda}_i Y \cdot \begin{bmatrix} c_i & b_i^T \\ b_i & A_i \end{bmatrix} \leq 0; \end{aligned} \tag{4.10}$$

that is,

$$\sum_{i=1}^m \bar{\lambda}_i Y \cdot \begin{bmatrix} c_i & b_i^T \\ b_i & A_i \end{bmatrix} = 0,$$

and, therefore,

$$L(x^*, \bar{\lambda}) = f(x^*) = \bar{\sigma}.$$

This shows that x^* is an optimal solution of the problem

$$\min\{L(x, \bar{\lambda}) \mid x \in \text{dom(IQCQP)}\}.$$

□

Since x^* is an optimal solution of the problem

$$\inf\{L(x, \bar{\lambda}) \mid x \in \text{dom(IQCQP)}\},$$

then by the first order necessary condition, for any $d \in \mathbb{R}^n$ satisfying $\nabla g_i(x^*)d \leq 0$, $\forall i \in \mathcal{A}(x^*)$, we have

$$\nabla_x L(x^*, \bar{\lambda})d \geq 0.$$

From this fact, we claim that the Lagrangian multiplier has the following maximal property.

Lemma 29. *Let x^* be an optimal solution of IQCQP and λ^* be its Lagrangian vector. If (x^*, λ^*) satisfies the copositiveness condition, then $\bar{\lambda} \leq \lambda^*$ for any $(\bar{\sigma}, \bar{\lambda})$ being an optimal solution of CD1.*

Proof. Since (x^*, λ^*) is a KKT solution satisfying the copositiveness condition, from Theorem 27, $(f(x^*), \lambda^*)$ is an optimal solution of CD1 and

$$\begin{aligned} &\nabla_x^T L(x^*, \lambda^*) = 2A(\lambda^*)x^* + 2b(\lambda^*) = 0, \\ &\lambda_i^* g_i(x^*) = \lambda_i^*(x^T A_i x + 2b_i^T x + c_i) = 0, \quad i = 1, \ldots, m. \end{aligned} \tag{4.11}$$

From Lemma 28, we know $\bar{\sigma} = L(x^*, \bar{\lambda}) = f(x^*)$ and $\bar{\lambda}_i g_i(x^*) = 0$, $i = 1, \ldots, m$.

Note that, on one hand, for any $i \notin \mathcal{A}(x^*)$, we have $g_i(x^*) < 0$, and then $\bar{\lambda}_i = \lambda_i^* = 0$. On the other hand, for any $i \in \mathcal{A}(x^*)$, we have $g_i(x^*) = 0$. Since x^* satisfies LICQ, the vectors $\nabla g_j(x^*)$, $j \in \mathcal{A}(x^*)$, are linearly independent. Therefore, there exists $d \in \mathbb{R}^n$ such that

$$\nabla g_i(x^*)d < 0 \text{ and } \nabla g_j(x^*)d = 0, \ \forall j \in \mathcal{A}(x^*)\backslash\{i\}.$$

This implies

$$\nabla_x L(x^*, \bar{\lambda})d = \nabla f(x^*)d + \bar{\lambda}_i \nabla g_i(x^*)d \geq 0.$$

Since $\nabla_x L(x^*, \lambda^*) = 0$, then

$$\nabla_x L(x^*, \lambda^*)d = \nabla f(x^*)d + \lambda_i^* \nabla g_i(x^*)d = 0.$$

Therefore, we have

$$(\bar{\lambda}_i - \lambda_i^*)\nabla g_i(x^*)d \geq 0$$

and this shows $\lambda_i^* \geq \bar{\lambda}_i$. □

According to this property, λ^* can be found by maximizing the sum of its entries.

Theorem 30. *Let x^* be an optimal solution of IQCQP and λ^* be its corresponding Lagrangian vector. If (x^*, λ^*) satisfies the copositiveness condition, then λ^* is the unique optimal solution* *of the maximization problem*

$$\begin{aligned} \max \quad & \sum_{i=1}^m \lambda_i \\ s.t. \quad & \begin{bmatrix} c(\lambda) - \sigma^* & b^T(\lambda) \\ b(\lambda) & A(\lambda) \end{bmatrix} \in \mathcal{D}_{n+1}, \\ & \lambda \in \mathbb{R}_+^m, \end{aligned} \qquad \text{(CD2)}$$

where σ^* is the optimal objective value obtained by solving the CD1.

Proof. Note that λ is a feasible solution of CD2 if and only if (σ^*, λ) is an optimal solution of CD1. Lemma 29 says that $\lambda \le \lambda^*$ for any feasible λ of CD2. Consequently, λ^* is the unique maximizer of CD2. □

After finding the Lagrangian multiplier λ^*, if, further, $A(\lambda^*)$ is invertible, then x^* can be obtained by solving $\nabla_x L(x, \lambda^*) = 0$, i.e.,

$$x^* = -A^{-1}(\lambda^*)b(\lambda^*).$$

4.4 Effective Computational Scheme

The discussions in Sections 4.2 and 4.3 reveal a procedure to find an optimal solution of IQCQP. We could first solve CD1 to obtain σ^*, then solve the corresponding CD2 to find λ^*, and at last solve the linear equation $A(\lambda^*)x^* + b(\lambda^*) = 0$ to get x^*. If (x^*, λ^*) satisfies the copositiveness condition, then x^* is a global optimal solution of IQCQP. Otherwise, the procedure returns a lower bound. However, in general, we do not expect to solve CD1 and CD2 in polynomial time, because there is no known algorithm for solving the general linear conic programming problem with a copositive cone.

Nevertheless, we can consider an inner approximation of $\mathcal{D}_{n+1}$ by a computable $\mathcal{C}_{n+1}$ satisfying $\mathcal{S}_+^{n+1} \subset \mathcal{C}_{n+1} \subset \mathcal{D}_{n+1}$. Then we get the following two approximation problems and they can be solved in polynomial time.

$$\begin{aligned} \max \quad & \sigma \\ s.t. \quad & \begin{bmatrix} c(\lambda) - \sigma & b^T(\lambda) \\ b(\lambda) & A(\lambda) \end{bmatrix} \in \mathcal{C}_{n+1}, \\ & \lambda \in \mathbb{R}_+^m. \end{aligned} \qquad \text{(CD1}'\text{)}$$

and

$$\begin{aligned} \max \quad & \textstyle\sum_{i=1}^m \lambda_i \\ s.t. \quad & \begin{bmatrix} c(\lambda) - \sigma^* & b^T(\lambda) \\ b(\lambda) & A(\lambda) \end{bmatrix} \in \mathcal{C}_{n+1}, \\ & \lambda \in \mathbb{R}^m_+, \end{aligned} \qquad \text{(CD2}')$$

where σ^* is the optimal objective value of CD1′. In this setting, we consider the following computational scheme.

Algorithm 1 (IQCQP algorithm)**.**

STEP 1: Given an IQCQP problem, solve the corresponding CD1′ problem and get the optimal value σ^*. If failed, then stop and note that the IQCQP problem cannot be solved by the current approximation.

STEP 2: Solve the corresponding CD2′ problem to get the optimal λ^*.

STEP 3: Compute $x^* = -A^+(\lambda^*)b(\lambda^*)$, where $A^+(\lambda^*)$ is the Moore-Penrose inverse of $A(\lambda^*)$ [16]. In particular, $A^+(\lambda^*) = A^{-1}(\lambda^*)$ when $A(\lambda^*)$ is nonsingular.

STEP 4: If x^* is a feasible solution and $f(x^*) = \sigma$, then return x^* as a global optimal solution of IQCQP with the objective value $f(x^*) = \sigma^*$. Otherwise, return σ^* as a lower bound of IQCQP.

Remark. Since CD1′ is only an inner approximation of CD1, it may not be feasible and the algorithm could stop at STEP 1. If we choose $\mathcal{C}_{n+1} = \mathcal{S}^{n+1}_+$, then the dual problem of CD1′ is the classic SDP relaxation of IQCQP.

The next theorem validates Algorithm 1.

Theorem 31. *If* Algorithm 1 *returns a solution x^*, then x^* is an optimal solution of the IQCQP problem,. If* Algorithm 1 *returns a value σ^*, then it is a lower bound for IQCQP.*

Proof. If Algorithm 1 returns a feasible x^*, then $f(x^*) = \sigma^*$ and (σ^*, λ^*) is optimal to CD2′. Hence we know $M(x^*, \lambda^*) \in \mathcal{C}_{n+1} \subset \mathcal{D}_{n+1}$ and

$$\begin{aligned} 0 \quad &\le \begin{bmatrix} 1 \\ x^* \end{bmatrix}^T \begin{bmatrix} c(\lambda^*) - f(x^*) & b^T(\lambda^*) \\ b(\lambda^*) & A(\lambda^*) \end{bmatrix} \begin{bmatrix} 1 \\ x^* \end{bmatrix} \\ &= -\sum_{i=1}^m \lambda_i^* g_i(x^*) \le 0. \end{aligned}$$

This implies $\lambda_i^* g_i(x^*) = 0$ for all $i = 1, \ldots, m$. Since $A(\lambda^*)x^* + b(\lambda^*) = 0$, (x^*, λ^*) satisfies the

KKT conditions, i.e.,

$$\begin{aligned}
&\nabla_x^T L(x^*, \lambda^*) = 2A(\lambda^*)x^* + 2b(\lambda^*) = 0, \\
&\lambda_i^* g_i(x^*) = 0, \\
&g_i(x^*) \leq 0,\ \lambda_i^* \geq 0,\ i = 1, \ldots, m.
\end{aligned} \tag{4.12}$$

Consequently, (x^*, λ^*) satisfies the copositiveness condition. By Theorem 27, x^* must be an optimal solution of IQCQP with the objective value $f(x^*) = \sigma^*$.

If Algorithm 1 returns a value σ^*, since $\mathcal{C}_{n+1} \subset \mathcal{D}_{n+1}$, then the feasible domain of problem CD1$'$ is contained in the feasible domain of the problem CD1. Hence σ^* is a lower bound for the problem IQCQP. □

4.5 Numerical Examples

In this section, we use two examples to illustrate the proposed algorithm and to show that it can improve the result obtained by using the positive semidefiniteness condition.

The first example is a box constrained quadratic programming problem.

Example 32.

$$\begin{array}{ll} \min & x^T A x + 2b^T x \\ s.t. & x \in [0,1]^{10}, \end{array}$$

where

$$A = \begin{bmatrix}
123 & 61 & 55 & 230 & -35 & 488 & 691 & -62 & 124 & -92 \\
61 & 114 & 110 & 65 & -61 & 16 & 59 & -15 & -22 & 61 \\
55 & 110 & 334 & 115 & -21 & -44 & -187 & -5 & 189 & 155 \\
230 & 65 & 115 & 246 & 8 & 364 & 626 & -51 & 76 & -96 \\
-35 & -61 & -21 & 8 & 168 & -37 & 22 & -95 & 7 & 123 \\
488 & 16 & -44 & 364 & -37 & 107 & 477 & 81 & -120 & -116 \\
691 & 59 & -187 & 626 & 22 & 477 & 204 & -7 & -216 & -92 \\
-62 & -15 & -5 & -51 & -95 & 81 & -7 & 232 & -44 & -129 \\
124 & -22 & 189 & 76 & 7 & -120 & -216 & -44 & 218 & 4 \\
-92 & 61 & 155 & -96 & 123 & -116 & -92 & -129 & 4 & 451
\end{bmatrix}$$

and

$$b = [-62.05, -157.75, -524.75, -105.9, -66, 176.15, 354.2, 58.05, -359.35, -382.85]^T.$$

The constraint $x \in [0,1]^{10}$ can be written as $2x_i^2 - 2x_i \leq 0$, $i = 1, \ldots, 10$. The Lagrangian function is defined by

$$L(x, \lambda) = x^T(A + 2\Lambda)x + 2(b - \lambda)x$$

where $\Lambda = \text{diag}(\lambda)$ is the diagonal matrix with the entries of λ on its diagonal. The cone $\mathcal{C}_{11} = \mathcal{S}_+^{11} + \mathcal{N}_{11}$ is used in CD1′, where $\mathcal{N}_{11}$ is the cone consisting of symmetric square matrices of order 11 with all nonnegative entries. After running Algorithm 1, we get an optimal solution

$$x^* = (0, 1, 0.5, 0, 0.75, 0, 0, 0.6, 1, 0.5)^T$$

with

$$\lambda^* = (41, 35, 0, 20, 0, 13, 70, 0, 88, 0)^T.$$

Since $A + 2\Lambda^*$ is not positive semidefinite, then the positive semidefiniteness condition is not satisfied and x^* cannot be obtained by solving the SDP relaxation problem.

The second example is an IQCQP problem with two constraints.

Example 33.

$$\begin{aligned} \min \quad & f(x) = x^T A_0 x + 2b_0^T x \\ s.t. \quad & g_1(x) = x^T A_1 x + 2b_1^T x + c_1 \le 0, \\ & g_2(x) = x^T A_2 x + 2b_2^T x + c_2 \le 0, \end{aligned} \tag{4.13}$$

where

$$A_0 = \begin{bmatrix} -2 & 10 & 2 \\ 10 & 4 & 1 \\ 2 & 1 & -7 \end{bmatrix}, \quad b_0 = \begin{bmatrix} -12 \\ -6 \\ 56 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 8 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 0 \\ -2 \\ -64 \end{bmatrix}, \quad c_1 = 512,$$

$$A_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad b_2 = \begin{bmatrix} -2 \\ 0 \\ -16 \end{bmatrix}, \quad c_2 = 128.$$

The optimal objective value of this problem is known to be 448 and the corresponding optimal solution is $x = [0, 0, 8]^T$.

It can be seen that the feasible domain of this example is a subset of $\mathbb{R}_+^3$, i.e.,

$$\left\{ \begin{bmatrix} 1 & x^T \\ x & xx^T \end{bmatrix} \middle| x \in \text{dom}(4.13) \right\} \subset \mathcal{S}_+^4 \cap \mathcal{N}_4.$$

Since $(\mathcal{S}_+^4 \cap \mathcal{N}_4)^* = \mathcal{S}_+^4 + \mathcal{N}_4$, we can choose $\mathcal{C}_4 = \mathcal{S}_+^4 + \mathcal{N}_4$ as the inner approximation of $\mathcal{D}_4$ in CD1′ and CD2′. Solving CD1′ leads to an objective value $\sigma^* = 448$. Using this value and solving CD2$'$, we get the Lagrangian vector $\lambda^* = (1,2)^T$ and the matrices

$$M(x^*,\lambda^*) = \begin{bmatrix} 320 & -16 & -8 & -40 \\ -16 & 4 & 10 & 2 \\ -8 & 10 & 6 & 1 \\ -40 & 2 & 1 & 5 \end{bmatrix} \qquad \text{and} \qquad A(\lambda^*) = \begin{bmatrix} 4 & 10 & 2 \\ 10 & 6 & 1 \\ 2 & 1 & 5 \end{bmatrix}.$$

Since $A(\lambda^*)$ is invertible, we get $x^* = -A^{-1}(\lambda^*)b(\lambda^*) = [0,0,8]^T$ and $f(x^*) = 448$.

Notice that x^* is feasible and $f(x^*) = \sigma^*$, then from Theorem 31, x^* is optimal to IQCQP with the optimal value $f(x^*) = 448$.

One can verify that

$$\begin{aligned} M(x^*,\lambda^*) &= \begin{bmatrix} 320 & -16 & -8 & -40 \\ -16 & 4 & 10 & 2 \\ -8 & 10 & 6 & 1 \\ -40 & 2 & 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 320 & -16 & -8 & -40 \\ -16 & 4 & 0 & 2 \\ -8 & 0 & 6 & 1 \\ -40 & 2 & 1 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathcal{S}_+^4 + \mathcal{N}_4, \end{aligned}$$

and

$$A(\lambda^*) = \begin{bmatrix} 4 & 10 & 2 \\ 10 & 6 & 1 \\ 2 & 1 & 5 \end{bmatrix} \notin \mathcal{S}_+^3.$$

Therefore, this optimal solution cannot be obtained by solving SDP relaxation. Actually, the optimal objective value of the SDP relaxation problem is 445.77, which is strictly less than the optimal value of the IQCQP.

Remark. Ye and Zhang [113] studied several solvable cases of the quadratic programming problems with two quadratic constraints. They studied QCQP with two ball constraints and QCQP with SDP relaxation being non-binding at the optimal solution. However, one can check that in our example, the first constraint is not convex and its SDP relaxation is actually binding at the optimal solution. Therefore, their methods cannot apply to this example.

4.6 Summary

In this chapter, the copositiveness condition for IQCQP, which is a sufficient condition for a KKT solution (x^*, λ^*) being optimal, is studied. This condition is an extension of the positive semidefiniteness condition used frequently in the SDP relaxation of IQCQP.

To apply this condition, we introduce CP1 and CD1 and show that there is not any gap among the optimal values of IQCQP, CP1, and CD1 when IQCQP is bounded below. Based on the copositiveness condition, a relationship between the optimal Lagrangian multipliers and the optimal solution of CD1 is proved. Furthermore, by exploring the maximal property, we can obtain the optimal Lagrangian multipliers λ^* by solving an additional problem CD2. If $A(\lambda^*)$ is invertible, then its corresponding solution x^* can be solved by $x^* = -A^{-1}(\lambda^*)b(\lambda^*)$. Since in general, solving CD1 and CD2 is difficult, their relaxations CD1$'$ and CD2$'$ are introduced and similar optimality condition is also established. Based on this result, we proposed an algorithm that may find an optimal solution for a subclass of IQCQP in polynomial time. Two examples are used to show the advantage of the algorithm.

An extension of this idea to CQCQP is also studied in the next chapter.

Chapter 5

Conic Form Quadratically Constrained Quadratic Programming Problems

In this chapter, we focus on the quadratic programming problem with conic form quadratic constraints, i.e., CQCQP. This problem is a generalization of the IQCQP problem discussed in the last chapter. Using the cone of nonnegative quadratic functions, a linear conic programming relaxation and its dual problem are introduced. We also extend the KKT conditions and establish a sufficient condition to make a KKT solution be optimal. Based on the sufficient condition, we propose an efficient computational scheme which may find an optimal solution or a lower bound of CQCQP in polynomial time.

5.1 Introduction

The CQCQP problem has the formulation:

$$\begin{aligned} \min \quad & f(x) = x^T A_0 x + 2b_0^T x + c_0 \\ s.t. \quad & g_i(x) \in \mathcal{K}_i \subset \mathbb{R}^{m_i}, \quad i = 1, \ldots, I, \end{aligned} \qquad \text{(CQCQP)} \qquad (5.1)$$

where $A_0 \in \mathcal{S}^n$, $b_0 \in \mathbb{R}^n$, $c_0 \in \mathbb{R}$, $\mathcal{K}_i$ is a proper cone in $\mathbb{R}^{m_i}$ and $g_i(x) = [g_{i1}(x), \ldots, g_{im_i}(x)]^T$ in form of

$$g_{ij}(x) = x^T A_{ij} x + 2b_{ij}^T x + c_{ij},$$

with $A_{ij} \in \mathcal{S}^n$, $b_{ij} \in \mathbb{R}^n$, $c_{ij} \in \mathbb{R}$, for $j = 1, \ldots, m_i$ and $i = 1, \ldots, I$.

This problem is a generalization of the IQCQP since the cones can be replaced by $\mathbb{R}_+$.

When $-g_i(x)$ are all $\mathcal{K}_i$-convex, $i = 1, \ldots, I$ and $f(x)$ is convex, then the problem becomes a convex conic programming problem and its optimality conditions have been studied for many years. (See e.g., [72], [28], [21], [9]).

When the constraints are all linear functions over polyhedral cones, Parida and Roy give an sufficient condition to verify the existence of the optimal solution in [80]. When the polyhedral cones are relaxed to convex cones, the duality theory is studied in [104]. When the problem is a CNLP problem, Fritz-John conditions are studied for polyhedral cones in [38] and [4]. For convex cones, saddle-point conditions are studied in [101]. Based on the Farkas' lemma over cones developed in [102], regularity conditions, KKT conditions and complementary slackness are studied in [103] and [3]. For problems on complex numbers, one may refer to [1], [21] and [20]. In [63], a general framework for the convex relaxation of the conic form polynomial programming problems is introduced. The SDP relaxation for the conic form polynomial programming problem is studied in [64]. These results could be used in CQCQP, but we will further utilize the quadratic functions.

In the rest of this chapter, we will study the optimality condition of CQCQP. In Section 5.2, we introduce the KKT condition for CQCQP and a linear conic programming problem using the cones of nonnegative quadratic functions over the feasible domain of CQCQP. In Section 5.3, the global optimality condition which is a generalization of the positive semidefinite condition is developed and a second conic programming problem is used to get the Lagrangian multiplier. After that, the optimal solution of CQCQP could be obtained by solving a linear equation involving the Lagrangian multipliers. In Section 5.4, we discuss using a relaxation to gain polynomial time solvability. At last, the results in this chapter are summarized in Section 5.6.

5.2 Extended KKT conditions and Linear Conic Programming Relaxations

Let $\mathcal{K}_i^*$ be the dual cone of $\mathcal{K}_i$, for $i = 1, \ldots, I$. Given $\lambda_i \in \mathcal{K}_i^*$, we denote

$$\begin{aligned} A(\lambda) &= A_0 - \sum_{i=1}^{I} \sum_{j=1}^{m_i} \lambda_{ij} A_{ij}, \\ b(\lambda) &= b_0 - \sum_{i=1}^{I} \sum_{j=1}^{m_i} \lambda_{ij} b_{ij}, \\ c(\lambda) &= c_0 - \sum_{i=1}^{I} \sum_{j=1}^{m_i} \lambda_{ij} c_{ij}. \end{aligned}$$

Then the Lagrangian of CQCQP becomes

$$
\begin{aligned}
L(x,\lambda) &= f(x) - \sum_{i=1}^{I} \lambda_i^T g_i(x) \\
&= x^T(A_0 - \sum_{i=1}^{I}\sum_{j=1}^{m_i} \lambda_{ij}A_{ij})x + 2(b_0 - \sum_{i=1}^{I}\sum_{j=1}^{m_i}\lambda_{ij}b_{ij})^T x + c_0 - \sum_{i=1}^{I}\sum_{j=1}^{m_i}\lambda_{ij}c_{ij} \\
&= x^T A(\lambda)x + 2b^T(\lambda)x + c(\lambda)
\end{aligned} \tag{5.2}
$$

Note that when $\mathcal{K}_i$ is a matrix cone, any matrix in $\mathcal{K}_i$ is treated as a vector in the above equation.

Given a feasible solution x^* of CQCQP, the extended active set at x^* is defined as

$$\mathcal{A}(x^*) = \{i \mid g_i(x^*) \in \mathrm{bdry}(\mathcal{K}_i)\}.$$

where bdry$(\cdot)$ denotes the boundary of a set. Note that when $\mathcal{K}_i = \mathbb{R}_+$, $\mathcal{A}(x^*)$ coincides with the ordinary definition of the active set for NLP. Parallel to the regularity condition of LICQ, we define an extended LICQ for CQCQP as below.

Definition 34 (Extended LICQ)**.** *Let x^* be a feasible solution of CQCQP. If the vectors $\nabla g_{ij}(x^*) \in \mathbb{R}^{1\times n}$, for $i \in \mathcal{A}(x^*)$ and $j = 1,\ldots,m_i$, are linearly independent, then we say CQCQP satisfies the extended LICQ at x^*.*

We now introduce the extended KKT conditions.

Lemma 35 (Extended KKT conditions)**.** *If x^* is a local minimum solution of* CQCQP *and there exists a vector $d \in \mathbb{R}^n$ such that*

$$\nabla g_i(x^*)d \in \mathrm{int}(\mathcal{K}_i),\ \ \forall i \in \mathcal{A}(x^*),$$

then there exists a λ such that the extended KKT conditions are satisfied:

$$
\begin{aligned}
&\nabla_x L(x^*,\lambda) = \nabla f(x^*) - \sum_{i=1}^{I} \lambda_i^T \nabla g_i(x) = 0, \\
&\lambda_i^T g_i(x^*) = 0, \\
&g_i(x^*) \in \mathcal{K}_i,\ \ \lambda_i = (\lambda_{i1},\ldots,\lambda_{im_i})^T \in \mathcal{K}_i^*,\ \ i = 1,\ldots,I.
\end{aligned} \tag{5.3}
$$

Proof. Notice that x^* is also a local minimum solution of the problem

$$
\begin{aligned}
\min \quad & f(x) = x^T A_0 x + 2b_0^T x + c_0 \\
s.t. \quad & g_i(x) \in \mathcal{K}_i \subset \mathbb{R}^{m_i}, \quad i \in \mathcal{A}(x^*),
\end{aligned} \tag{5.4}
$$

The rest of the proof follows Theorem 2 of [3]. □

If a CQCQP satisfies the extended LICQ at a feasible point x^*, i.e., the vectors $\nabla g_{ij}(x^*)$, for $i \in \mathcal{A}(x^*)$ and $j = 1, \ldots, m_i$, are linearly independent, then for any set of vectors $\{w_i \in \text{int}(\mathcal{K}_i) \mid i \in \mathcal{A}(x^*)\}$, the following system of linear equations

$$\nabla g_i(x^*)d = w_i, \qquad \forall i \in \mathcal{A}(x^*)$$

has at least one feasible solution. Consequently, the condition of the above lemma is met and the extended KKT conditions hold. Therefore, we assume that the extended LICQ is satisfied in the rest of this chapter.

Notice that, the extended KKT conditions are merely a necessary condition of a local minimum solution. Given a KKT solution (x^*, λ^*), the point x^* may not be a local minimum solution. However, similar to those in IQCQP, if we already know the λ^* in a KKT solution, $A(\lambda^*)$ is invertible, and its corresponding x^* is optimal to CQCQP, then we can find x^* by solving

$$x^* = -A^{-1}(\lambda^*)b(\lambda^*).$$

To ensure x^* is indeed globally optimal, the so-called "positive semidefiniteness condition" (E.g., [48]) can be adopted to obtain the next result.

Lemma 36 (Positive semidefiniteness condition)**.** *Let (x^*, λ^*) be a KKT solution. If $A(\lambda^*) \in \mathcal{S}^n_+$, then x^* is a global optimal solution of CQCQP.*

A more general sufficient condition will be discussed in the next section. Here we first introduce a conic reformulation of CQCQP.

Denote the feasible domain of CQCQP by

$$\text{dom(CQCQP)} = \{x \in \mathbb{R}^n \mid g_i(x) \in \mathcal{K}_i, i = 1, \ldots, I.\}.$$

Define the two cones as

$$\mathcal{D}_{n+1} = \left\{ U \in \mathcal{S}^{n+1} \;\middle|\; U \cdot \begin{bmatrix} 1 & x^T \\ x & xx^T \end{bmatrix} \geq 0, \forall x \in \text{dom(CQCQP)} \right\}, \tag{5.5}$$

and

$$\mathcal{D}^*_{n+1} = \text{cl}\,\text{cone} \left\{ X = \begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix}^T \;\middle|\; x \in \text{dom(CQCQP)} \right\}. \tag{5.6}$$

Observe that the cone $\mathcal{D}_{n+1}$ is the set of coefficients of all quadratic functions that are nonnegative over the feasible domain of CQCQP, while $\mathcal{D}^*_{n+1}$ is the smallest closed cone that contains the feasible domain of CQCQP in the homogenous formulation. They are both closed and convex by definition, and Corollary 1 of [107] further shows that they are dual to each other. The next lemma tells when these two cones are proper.

Lemma 37. *If the interior of* dom*(CQCQP) is nonvoid, then the cones* $\mathcal{D}_{n+1}$ *and* $\mathcal{D}^*_{n+1}$ *are proper.*

Proof. Since $\mathcal{S}^{n+1}_+$ is a pointed cone and $\mathcal{D}^*_{n+1} \subset \mathcal{S}^{n+1}_+$, we know $\mathcal{D}^*_{n+1}$ is pointed and its dual cone $\mathcal{D}_{n+1}$ has a nonempty interior. For any

$$U = \begin{bmatrix} c & b^T \\ b & A \end{bmatrix} \in \mathcal{D}_{n+1} \cap -\mathcal{D}_{n+1},$$

we know

$$\begin{bmatrix} 1 \\ x \end{bmatrix}^T U \begin{bmatrix} 1 \\ x \end{bmatrix} = x^T Ax + 2b^T x + c \geq 0,\ \forall x \in \text{dom(CQCQP)},$$

and

$$\begin{bmatrix} 1 \\ x \end{bmatrix}^T (-U) \begin{bmatrix} 1 \\ x \end{bmatrix} = -(x^T Ax + 2b^T x + c) \geq 0,\ \forall x \in \text{dom(CQCQP)}.$$

Therefore,

$$\begin{bmatrix} 1 \\ x \end{bmatrix}^T U \begin{bmatrix} 1 \\ x \end{bmatrix} = x^T Ax + 2b^T x + c = 0,\ \forall x \in \text{dom(CQCQP)}.$$

Since dom(CQCQP) has a nonempty interior and $x^TAx+2b^Tx+c = 0$ for all $x \in$dom(CQCQP), we have in this case $A = 0$, $b = 0$ and $c = 0$, i.e., $U = 0$. This proves that $\mathcal{D}_{n+1}$ is pointed and $\mathcal{D}^*_{n+1}$ has a nonempty interior. Adding the fact that both $\mathcal{D}_{n+1}$ and $\mathcal{D}^*_{n+1}$ are closed and convex, we know they must be proper. □

Remark. Although Lemma 37 will not be used directly in later analysis, the properness of cones is often required in applying the conic duality theorem and interior point methods. Also note that since $\mathcal{S}^{n+1}_+ \subset \mathcal{D}_{n+1}$, any closed convex cone between $\mathcal{S}^{n+1}_+$ and $\mathcal{D}_{n+1}$ is proper, when $\mathcal{D}_{n+1}$ is proper.

Consider the following linear conic programming problem

$$\begin{aligned} \min \quad & \begin{bmatrix} c_0 & b_0^T \\ b_0 & A_0 \end{bmatrix} \cdot Y \\ s.t. \quad & Y_{11} = 1, \\ & Y \in \mathcal{D}^*_{n+1}. \end{aligned} \tag{5.7}$$

Following the discussion made on the problem (MP) in Section 3 of [107], we have the next result.

Theorem 38. *The problems* (5.7) *and* CQCQP *are equivalent in the sense that they have the same optimal objective values.*

Note that in (5.7), the conic form constraints do not appear explicitly. Actually, the information determined by these conic form constraints is implicitly reflected in the cone of $\mathcal{D}^*_{n+1}$. To see this, let G_i denote the matrix form of the ith constraint of CQCQP and Y be a matrix in $\mathcal{D}^*_{n+1}$, we can represent $g_i(x)$ in CQCQP by

$$G_i(Y) = \begin{bmatrix} \begin{bmatrix} c_{i1} & b_{i1}^T \\ b_{i1} & A_{i1} \end{bmatrix} \cdot Y \\ \vdots \\ \begin{bmatrix} c_{im_i} & b_{im_i}^T \\ b_{im_i} & A_{im_i} \end{bmatrix} \cdot Y \end{bmatrix} \in \mathbb{R}^{m_i}.$$

Then we have the next result.

Lemma 39. *If $Y \in \mathcal{D}^*_{n+1}$, then $G_i(Y) \in \mathcal{K}_i$.*

Proof. Given $Y \in \mathcal{D}^*_{n+1}$, then there exists a sequence of $Y_k \in \text{cone}\{X = \left[\begin{smallmatrix}1\\x\end{smallmatrix}\right]\left[\begin{smallmatrix}1\\x\end{smallmatrix}\right]^T \mid x \in \text{dom(CQCQP)}\}$ such that

$$\lim_{k\to\infty} Y_k = Y.$$

Notice that each Y_k has a finite representation of

$$Y_k = \sum_{j=1}^{s_k} \mu_j \begin{bmatrix} 1 \\ x_j \end{bmatrix} \begin{bmatrix} 1 \\ x_j \end{bmatrix}^T, x_j \in \text{dom(CQCQP)}, \mu_j \geq 0.$$

Since $x_j \in \text{dom(CQCQP)}$ and $\mu_j \geq 0$, we have

$$G_i(\begin{bmatrix} 1 \\ x_j \end{bmatrix} \begin{bmatrix} 1 \\ x_j \end{bmatrix}^T) = g_i(x_j) \in \mathcal{K}_i.$$

Consequently,

$$G_i(Y_k) = G_i(\sum_{j=1}^{s_k} \mu_j \begin{bmatrix} 1 \\ x_j \end{bmatrix} \begin{bmatrix} 1 \\ x_j \end{bmatrix}^T) = \sum_{j=1}^{s_k} \mu_j G_i(\begin{bmatrix} 1 \\ x_j \end{bmatrix} \begin{bmatrix} 1 \\ x_j \end{bmatrix}^T) \in \mathcal{K}_i.$$

Since $\mathcal{K}_i$ is closed, we have $G_i(Y) = \lim_{k\to\infty} G_i(Y_k) \in \mathcal{K}_i$.

□

Lemma 39 indicates that

$$G_i(Y) \in \mathcal{K}_i, \quad i = 1, \ldots, I$$

are all valid for problem (5.7). By adding these redundant constraints, we define the following linear conic programming problem:

$$\begin{aligned} \min \quad & \begin{bmatrix} c_0 & b_0^T \\ b_0 & A_0 \end{bmatrix} \cdot Y \\ s.t. \quad & G_i(Y) \in \mathcal{K}_i, \quad i = 1, \ldots, I, \\ & Y_{11} = 1, \\ & Y \in \mathcal{D}_{n+1}^*, \end{aligned} \qquad \text{(CP3)} \qquad (5.8)$$

Then this new problem and CQCQP must have the same objective value. Moreover, the conic dual of problem CP3 becomes

$$\begin{aligned} \max \quad & \sigma \\ s.t. \quad & \begin{bmatrix} c(\lambda) - \sigma & b^T(\lambda) \\ b(\lambda) & A(\lambda) \end{bmatrix} \in \mathcal{D}_{n+1}, \\ & \lambda_i = (\lambda_{i1}, \ldots, \lambda_{im_i})^T \in \mathcal{K}_i^*, \; i = 1, \ldots, I. \end{aligned} \qquad \text{(CD3)} \qquad (5.9)$$

The next theorem ties problems CQCQP, CP3 and CD3 together.

Theorem 40. *If CQCQP is bounded below, then the optimal objective values of the problems CQCQP, CP3 and CD3 are equal.*

Proof. We already know that problems CQCQP and CP3 have the same optimal objective value. Since CQCQP is bounded below, for $\lambda = 0$, we let

$$\sigma = \inf\{f(x) \mid x \in \mathrm{dom}(\mathrm{CQCQP})\},$$

and, therefore,

$$\begin{bmatrix} c_0 - \sigma & b_0^T \\ b_0 & A_0 \end{bmatrix} \in \mathcal{D}_{n+1}.$$

Note that σ is the optimal objective value of both problems CQCQP and CP3, and $(\sigma, 0)$ is a feasible solution of CD3. From the weak duality theorem of linear conic programming, σ is also the optimal objective value of CD3. $\square$

5.3 Finding Global Optimal Solutions

In this section, we derive a sufficient condition to ensure a KKT point to be globally optimal to CQCQP.

Let (x^*, λ^*) be a KKT solution of CQCQP. We assume that x^* satisfies the extended LICQ condition and the extended active set at x^* is $\mathcal{A}(x^*)$. Correspondingly, we define a matrix

$$M(x^*, \lambda^*) = \begin{bmatrix} c(\lambda^*) - f(x^*) & b^T(\lambda^*) \\ b(\lambda^*) & A(\lambda^*) \end{bmatrix}.$$

Then we have the next theorem.

Theorem 41 (Copositiveness condition)**.** *Let (x^*, λ^*) be a KKT solution of CQCQP. Under the extended LICQ condition, if $M(x^*, \lambda^*) \in \mathcal{D}_{n+1}$, then $(\sigma^*, \lambda^*) = (f(x^*), \lambda^*)$ is optimal to CD3 and x^* is optimal to CQCQP.*

Proof. From the copositiveness condition

$$M(x^*, \lambda^*) = \begin{bmatrix} c(\lambda^*) - f(x^*) & b^T(\lambda^*) \\ b(\lambda^*) & A(\lambda^*) \end{bmatrix} \in \mathcal{D}_{n+1},$$

we know $(\sigma^*, \lambda^*) = (f(x^*), \lambda^*)$ is a feasible solution of CD3. Since x^* is feasible for CQCQP and $\sigma^* = f(x^*)$, we know x^* is optimal to CQCQP and (σ^*, λ^*) is optimal to CD3. $\square$

Note that the positive semidefiniteness condition is equivalent to

$$M(x^*, \lambda^*) = \begin{bmatrix} c(\lambda^*) - f(x^*) & b^T(\lambda^*) \\ b(\lambda^*) & A(\lambda^*) \end{bmatrix} \in \mathcal{S}_+^{n+1} \subset \mathcal{D}_{n+1},$$

which shows that the copositiveness condition is indeed an extension of the positive semidefiniteness condition.

Under the copositiveness condition, our objective is to find such λ^* and its corresponding x^* by solving the linear conic problem CD3. Since CD3 may have multiple optimal solutions, we need to further study the property of the Lagrangian vector that leads to the exact Lagrangian multipliers we need.

Lemma 42. *If x^* is an optimal solution of CQCQP and $(\bar{\sigma}, \bar{\lambda})$ is an optimal solution of CD3, then x^* is also an optimal solution to the following problem:*

$$\begin{aligned} \min \quad & L(x, \bar{\lambda}) \\ s.t. \quad & x \in \mathrm{dom(CQCQP)}. \end{aligned} \tag{5.10}$$

Proof. Our assumption directly implies that $f(x^*) = \bar{\sigma}$. The feasibility of $(\bar{\sigma}, \bar{\lambda})$ further implies

$$\begin{bmatrix} c(\bar{\lambda}) - \bar{\sigma} & b^T(\bar{\lambda}) \\ b(\bar{\lambda}) & A(\bar{\lambda}) \end{bmatrix} \in \mathcal{D}_{n+1},$$

which means, for any $x \in \text{dom(CQCQP)}$,

$$\begin{bmatrix} 1 \\ x \end{bmatrix}^T \begin{bmatrix} c(\bar{\lambda}) - \bar{\sigma} & b^T(\bar{\lambda}) \\ b(\bar{\lambda}) & A(\bar{\lambda}) \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} \geq 0.$$

Therefore,

$$L(x, \bar{\lambda}) \geq \bar{\sigma}, \quad \forall x \in \text{dom(CQCQP)}.$$

Note that, if we let

$$Y = \begin{bmatrix} 1 & (x^*)^T \\ x^* & x^*(x^*)^T \end{bmatrix},$$

then

$$\begin{aligned} 0 &\leq Y \cdot \begin{bmatrix} c(\bar{\lambda}) - \bar{\sigma} & b^T(\bar{\lambda}) \\ b(\bar{\lambda}) & A(\bar{\lambda}) \end{bmatrix} \\ &= Y \cdot \begin{bmatrix} c_0 & b_0^T \\ b_0 & A_0 \end{bmatrix} - \sum_{i=1}^{I} \bar{\lambda}_i^T G_i(Y) - \bar{\sigma} \\ &= f(x^*) - \sum_{i=1}^{I} \bar{\lambda}_i^T G_i(Y) - \bar{\sigma} \\ &= - \sum_{i=1}^{I} \bar{\lambda}_i^T G_i(Y) \leq 0. \end{aligned} \tag{5.11}$$

Consequently,

$$\sum_{i=1}^{I} \bar{\lambda}_i^T G_i(Y) = 0,$$

and

$$L(x^*, \bar{\lambda}) = f(x^*) = \bar{\sigma}.$$

This shows that x^* is an optimal solution of the stated problem. □

Knowing that x^* is an optimal solution of the problem (5.10), for any vector $d \in \mathbb{R}^n$ being a feasible direction of CQCQP at x^*, we have

$$\nabla_x L(x^*, \bar{\lambda}) d \geq 0.$$

This fact may help us establish a maximal property of the Lagrangian multipliers.

Lemma 43. *For any vector $d \in \mathbb{R}^n$, if $\nabla g_i(x^*)d \in \operatorname{int}(\mathcal{K}_i)$ for all $i \in \mathcal{A}(x^*)$, then d is a feasible direction of problem CQCQP at x^*.*

Proof. Let d satisfy the assumption of the lemma, then there exists a $\delta_1 > 0$ such that, for any $i \in \mathcal{A}(x^*)$ and $w_i \in \mathbb{R}^{m_i}$,

$$\|\nabla g_i(x^*)d - w_i\| < \delta_1 \quad \Rightarrow \quad w_i \in \operatorname{int}(\mathcal{K}_i).$$

Since

$$\lim_{t \to 0+} \frac{g_i(x^* + td) - g_i(x^*)}{t} = \nabla g_i(x^*)d,$$

we know there exists $\bar{t} > 0$ such that, for any $0 < t < \bar{t}$ and $i \in \mathcal{A}(x^*)$,

$$\left\| \frac{g_i(x^* + td) - g_i(x^*)}{t} - \nabla g_i(x^*)d \right\| < \delta_1.$$

Therefore,

$$\frac{g_i(x^* + td) - g_i(x^*)}{t} \in \operatorname{int}(\mathcal{K}_i).$$

This means

$$g_i(x^* + td) \in \mathcal{K}_i, \quad \forall\ 0 < t < \bar{t}$$

and d is a feasible direction of CQCQP at x^*. □

Moreover, we have the next result.

Lemma 44. *Let x^* be an optimal solution of CQCQP and λ^* be its corresponding Lagrangian vector. If (x^*, λ^*) satisfies the copositiveness condition, then $\lambda^* - \bar{\lambda} \in \mathcal{K}_1^* \times \cdots \times \mathcal{K}_I^*$ for any $(\bar{\sigma}, \bar{\lambda})$ being optimal to CD3.*

Proof. Since (x^*, λ^*) is a KKT solution satisfying the copositiveness condition, from Theorem 41, $(f(x^*), \lambda^*)$ must be an optimal solution of CD3 and

$$\begin{aligned} &\nabla_x^T L(x^*, \lambda^*) = 2A(\lambda^*)x^* + 2b(\lambda^*) = 0, \\ &(\lambda_i^*)^T g_i(x^*) = 0, \quad i = 1, \ldots, I. \end{aligned} \tag{5.12}$$

From Lemma 42, we know $\bar{\sigma} = L(x^*, \bar{\lambda}) = f(x^*)$ and $\bar{\lambda}_i^T g_i(x^*) = 0$, $i = 1, \ldots, I$.

Notice that, on one hand, for any $i \notin \mathcal{A}(x^*)$, we have $g_i(x^*) \in \operatorname{int}(\mathcal{K}_i)$ and then $\bar{\lambda}_i = \lambda_i^* = 0$. On the other hand, since x^* is assumed to satisfy the extended LICQ condition, the vectors $\nabla g_{ij}(x^*)$, $i \in \mathcal{A}(x^*)$ and $j = 1, \ldots, m_i$, are linearly independent. Hence, for any $h_i \in \operatorname{int}(\mathcal{K}_i)$ with $i \in \mathcal{A}(x^*)$, there exists a vector $d \in \mathbb{R}^n$ such that

$$\nabla g_i(x^*)d = h_i, \ \forall i \in \mathcal{A}(x^*).$$

Lemma 43 says that d is a feasible direction at x^*, and, consequently,

$$\nabla_x L(x^*, \bar{\lambda})d = \nabla f(x^*)d - \sum_{i=1}^{I} \bar{\lambda}_i^T \nabla g_i(x^*)d \geq 0.$$

Since $\nabla_x L(x^*, \lambda^*) = 0$, we have

$$\nabla_x L(x^*, \lambda^*)d = \nabla f(x^*)d - \sum_{i=1}^{I} (\lambda_i^*)^T \nabla g_i(x^*)d = 0.$$

Therefore,

$$\sum_{i=1}^{I} (\lambda_i^* - \bar{\lambda}_i)^T \nabla g_i(x^*)d = \sum_{i \in \mathcal{A}(x^*)} (\lambda_i^* - \bar{\lambda}_i)^T h_i \geq 0.$$

By the closeness of the cone $\mathcal{K}_i$, the above expression holds for any $h_i \in \mathcal{K}_i$ with $i \in \mathcal{A}(x^*)$. This means that $\lambda_i^* - \bar{\lambda}_i \in \mathcal{K}_i^*$ for all $i \in \mathcal{A}(x^*)$. Then it is clear that $\lambda^* - \bar{\lambda} \in \mathcal{K}_1^* \times \cdots \times \mathcal{K}_I^*$. □

According to Lemma 44, for any $h = (h_1^T, \ldots, h_I^T)^T$ with $h_i \in \text{int}(\mathcal{K}_i)$, we have $h^T(\lambda^* - \bar{\lambda}) = \sum_{i \in \mathcal{A}(x^*)} h_i^T(\lambda_i^* - \bar{\lambda}_i) > 0$. Therefore, the optimal Lagrangian multipliers can be found by solving a maximization problem stated in the next result.

Theorem 45. *Let x^* be an optimal solution of CQCQP with λ^* being its corresponding Lagrangian vector. If (x^*, λ^*) satisfies the copositiveness condition, then, for any $h^T = (h_1^T, \ldots, h_I^T)$ with $h_i \in \text{int}(\mathcal{K}_i)$, λ^* is the unique optimal solution of the following maximization problem:*

$$\begin{aligned} \max \quad & \textstyle\sum_{i=1}^{I} \lambda_i^T h_i \\ s.t. \quad & \begin{bmatrix} c(\lambda) - \sigma^* & b^T(\lambda) \\ b(\lambda) & A(\lambda) \end{bmatrix} \in \mathcal{D}_{n+1}, \\ & \lambda_i = (\lambda_{i1}, \ldots, \lambda_{im_i})^T \in \mathcal{K}_i^*, \; i = 1, \ldots, I, \end{aligned} \tag{CD4}$$

where σ^* is the optimal objective value obtained by solving CD3.

Proof. Note that λ is a feasible solution of CD4 if and only if (σ^*, λ) is an optimal solution of CD3. Lemma 44 says that $\lambda^* - \lambda \in \mathcal{K}_1^* \times \cdots \times \mathcal{K}_I^*$ holds for any feasible solution λ of CD4. Consequently, λ^* is the unique maximizer of CD4. □

After finding the Lagrangian multiplier λ^*, if we further know that $A(\lambda^*)$ is invertible, then x^* can be obtained by solving $\nabla_x L(x, \lambda^*) = 0$, i.e.,

$$x^* = -A^{-1}(\lambda^*)b(\lambda^*).$$

5.4 Effective Computation Scheme

Similar to IQCQP, the results in the last section suggest a solution method. We could first solve CD3 to obtain σ^*, then solve the corresponding CD4 to find λ^*, and at last solve the linear equation $A(\lambda^*)x^* + b(\lambda^*) = 0$ to get x^*. If (x^*, λ^*) satisfies the copositiveness condition, then x^* is a global optimal solution of the original CQCQP. Otherwise, we have found a lower bound. However, in general, we do not expect that the problems CD3 and CD4 can be solved in polynomial time, because there is no polynomial time algorithm for solving general optimization problems over a copositive cone.

Nevertheless, we can consider an inner approximation of $\mathcal{D}_{n+1}$ by a computable $\mathcal{C}_{n+1}$ satisfying $\mathcal{S}_+^{n+1} \subset \mathcal{C}_{n+1} \subset \mathcal{D}_{n+1}$. If, furthermore, $\mathcal{K}_i,\ i = 1, \ldots, I$, are also computable, then we have the following two corresponding conic approximation problems that can be solved in polynomial time:

$$\begin{aligned} \max \quad & \sigma \\ s.t. \quad & \begin{bmatrix} c(\lambda) - \sigma & b^T(\lambda) \\ b(\lambda) & A(\lambda) \end{bmatrix} \in \mathcal{C}_{n+1}, \\ & \lambda_i = (\lambda_{i1}, \ldots, \lambda_{im_i})^T \in \mathcal{K}_i^*,\ i = 1, \ldots, I. \end{aligned} \tag{CD3$'$}$$

and

$$\begin{aligned} \max \quad & \textstyle\sum_{i=1}^{I} \lambda_i^T h_i \\ s.t. \quad & \begin{bmatrix} c(\lambda) - \sigma^* & b^T(\lambda) \\ b(\lambda) & A(\lambda) \end{bmatrix} \in \mathcal{C}_{n+1}, \\ & \lambda_i = (\lambda_{i1}, \ldots, \lambda_{im_i})^T \in \mathcal{K}_i^*,\ i = 1, \ldots, I. \end{aligned} \tag{CD4$'$}$$

where σ^* is the optimal objective value of CD3$'$ and $h = (h_1^T, \ldots, h_I^T)^T$ with $h_i \in \text{int}(\mathcal{K}_i)$.

In this setting, we consider the following computational scheme:

Algorithm 2 (CQCQP algorithm)**.**

STEP 1: *For a given CQCQP, solve the corresponding CD3$'$ to find its optimal objective value σ^*. If there is no solution, then stop and note that CQCQP cannot be solved by using the current approximation.*

STEP 2: *Choose an $h = (h_1^T, \ldots, h_I^T)^T$ with $h_i \in \text{int}(\mathcal{K}_i)$ and solve problem CD4$'$ to find the optimal λ^*.*

STEP 3: *Compute $x^* = -A^+(\lambda^*)b(\lambda^*)$, where $A^+(\lambda^*)$ is the Moore-Penrose inverse of $A(\lambda^*)$ [16]. Particularly, $A^+(\lambda^*) = A^{-1}(\lambda^*)$ when $A(\lambda^*)$ is nonsingular.*

STEP 4: *If x^* is a feasible solution and $f(x^*) = \sigma^*$, then return x^* as a global optimal solution of CQCQP with the objective value $f(x^*) = \sigma^*$. Otherwise, return σ^* as a lower* *bound for CQCQP.*

Remark. Since problem CD3′ is only an inner approximation of problem CD3, it may not be feasible and the algorithm could stop at STEP 1. If we let $\mathcal{C}_{n+1} = \mathcal{S}_+^{n+1}$, then the dual problem of CD3′ becomes the classic SDP relaxation of the CQCQP problem.

The next theorem validate Algorithm 2.

Theorem 46. *If Algorithm 2 returns a solution x^*, then x^* is an optimal solution of CQCQP. If Algorithm 2 returns a value σ^* then it is a lower bound for CQCQP.*

Proof. If Algorithm 2 returns a feasible x^*, then $f(x^*) = \sigma^*$ and (σ^*, λ^*) is optimal to CD4′. Hence we know $M(x^*, \lambda^*) \in \mathcal{C}_{n+1} \subset \mathcal{D}_{n+1}$ and

$$\begin{aligned} 0 \quad &\leq \begin{bmatrix} 1 \\ x^* \end{bmatrix}^T \begin{bmatrix} c(\lambda^*) - f(x^*) & b^T(\lambda^*) \\ b(\lambda^*) & A(\lambda^*) \end{bmatrix} \begin{bmatrix} 1 \\ x^* \end{bmatrix} \\ &= -\sum_{i=1}^{I} (\lambda_i^*)^T g_i(x^*) \leq 0. \end{aligned}$$

This implies $(\lambda_i^*)^T g_i(x^*) = 0$ for all $i = 1, \ldots, I$. Since $A(\lambda^*)x^* + b(\lambda^*) = 0$, (x^*, λ^*) satisfies the KKT conditions, i.e.,

$$\begin{aligned} &\nabla_x L(x^*, \lambda^*) = 2A(\lambda^*)x^* + 2b(\lambda^*) = 0, \\ &(\lambda_i^*)^T g_i(x^*) = 0, \\ &g_i(x^*) \in \mathcal{K}_i, \ \lambda_i^* \in \mathcal{K}_i^*, \ i = 1, \ldots, I. \end{aligned} \tag{5.13}$$

Consequently, (x^*, λ^*) satisfies the copositiveness condition. By Theorem 41, x^* must be an optimal solution of CQCQP with the objective value $f(x^*) = \sigma^*$.

If Algorithm 2 returns a value σ^*, since $\mathcal{C}_{n+1} \subset \mathcal{D}_{n+1}$, then the feasible domain of problem CD3′ is contained in the feasible domain of CD3. Hence σ^* is a lower bound for CQCQP. □

5.5 Numerical Example

In this section, we use one simple example to illustrate the proposed algorithm and to show that our algorithm indeed improves the result obtained by using the positive semidefiniteness condition. This example is a modification of Example 2 in [68].

Example 47.

$$\begin{aligned} \min \quad & f(x) = x^T A_0 x + 2b_0^T x + c_0 \\ s.t. \quad & \begin{bmatrix} g_{11}(x) & g_{12}(x) \\ g_{12}(x) & g_{22}(x) \end{bmatrix} \in \mathcal{S}_+^2 \end{aligned}$$

where

$$A_0 = \begin{bmatrix} -2 & 10 & 2 \\ 10 & 4 & 1 \\ 2 & 1 & -7 \end{bmatrix}, \quad b_0 = \begin{bmatrix} -12 \\ -6 \\ 56 \end{bmatrix}, c_0 = 0$$

$$A_{11} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad b_{11} = \begin{bmatrix} 1 \\ 0 \\ 8 \end{bmatrix}, c_{11} = -64$$

$$A_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -25 \end{bmatrix}, \quad b_{12} = \begin{bmatrix} 0 \\ 0 \\ 175 \end{bmatrix}, c_{12} = -1200$$

$$A_{22} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}, \quad b_{22} = \begin{bmatrix} 0 \\ 1 \\ 32 \end{bmatrix}, c_{22} = -256$$

The optimal objective value of this problem is known to be 448 and the corresponding optimal solution is $x = [0, 0, 8]^T$.

Since the positive semidefinite cone constraint implies that $g_{11}(x) \geq 0$, the feasible domain is a subset of $\mathbb{R}^3_+$. We can easily check that $\mathcal{S}^4_+ \subset (\mathcal{S}^4_+ + \mathcal{N}^4) \subset \mathcal{D}_4$, where $\mathcal{N}^4$ is the set of 4×4 matrices with nonnegative elements. Therefore, we can choose $\mathcal{C}_4 = \mathcal{S}^4_+ + \mathcal{N}^4$ as the inner approximation of $\mathcal{D}_4$ in the approximation problems CD3′ and CD4′.

Solving CD3′ leads to an objective value of $\sigma^* = 448$. Taking this value into CD4′ for solution, we find the Lagrangian multipliers

$$\lambda^* = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

and the matrices

$$M(x^*, \lambda^*) = \begin{bmatrix} 320 & -16 & -8 & -40 \\ -16 & 4 & 10 & 2 \\ -8 & 10 & 6 & 1 \\ -40 & 2 & 1 & 5 \end{bmatrix} \quad \textit{and} \quad A(\lambda^*) = \begin{bmatrix} 4 & 10 & 2 \\ 10 & 6 & 1 \\ 2 & 1 & 5 \end{bmatrix}.$$

Since $A(\lambda^*)$ is invertible, we get $x^* = -A^{-1}(\lambda^*)b(\lambda^*) = [0, 0, 8]^T$ and $f(x^*) = 448$.

Notice that x^* is feasible and $f(x^*) = \sigma^*$, then from Theorem 46, x^* is an optimal solution of CQCQP with the optimal value $f(x^*) = 448$.

Here, $M(x^*, \lambda^*)$ is not positive semidefinite, but it can be divided into the sum of a positive semidefinite matrix and a nonnegative matrix as following:

$$M(x^*, \lambda^*) = \begin{bmatrix} 320 & -16 & -8 & -40 \\ -16 & 4 & 0 & 2 \\ -8 & 0 & 6 & 1 \\ -40 & 2 & 1 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathcal{C}_4.$$

It is worth mentioning that if we use the classic positive semidefinite relaxation problem, i.e., let $\mathcal{C}_4 = \mathcal{S}^4_+$, then we can only obtain a lower bound of 445.83. This shows the advantage of the copositiveness condition derived in this paper.

5.6 Summary

In this chapter, we have established a new relationship between the optimal Lagrangian multipliers of CQCQP and the dual optimal solutions of its conic reformulations under the copositiveness condition, which is a more general sufficient condition for global optimality than the well known positive semidefiniteness condition in the literature. We also propose an effective computation scheme which may find an optimal solution (within a given precision) or a lower bound for CQCQP by properly choosing a computable approximating cone.

Since the conic programming problem over the cone $\mathcal{D}_{n+1}$ may not be solved efficiently in general, a larger computable $\mathcal{C}_{n+1}$ in the approximation problem will lead to a better result in polynomial time. Also notice that although the problems (5.7) and CP3 are equivalent, their conic dual problems are not. The property of the optimal Lagrangian multipliers can only be obtained from the conic dual of the problem CP3.

Chapter 6

Exact Formulation of The Cone of Nonnegative Quadratic Functions

As we have seen in Chapter 4 and Chapter 5, the cone of nonnegative quadratic functions plays an important role in the proposed linear conic programming problems. In this chapter, we will study the exact formulation of the cone of nonnegative quadratic functions over different domains.

6.1 Introduction

Given a set $\mathcal{F} \subset \mathbb{R}^n$, the cone of nonnegative quadratic functions over $\mathcal{F}$ is denoted as

$$\mathcal{D}_{\mathcal{F}} = \left\{ M \in \mathcal{S}^{n+1} \mid M \cdot \begin{bmatrix} 1 & x^T \\ x & xx^T \end{bmatrix} \geq 0, \forall x \in \mathcal{F} \right\}. \tag{6.1}$$

Recall that given a quadratic function $f(x) = x^T A_0 x + 2b_0^T x + c_0$, we can write it as

$$f(x) = \begin{bmatrix} 1 \\ x \end{bmatrix}^T \begin{bmatrix} c_0 & b_0^T \\ b_0 & A_0 \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} c_0 & b_0^T \\ b_0 & A_0 \end{bmatrix} \cdot \begin{bmatrix} 1 & x^T \\ x & xx^T \end{bmatrix}.$$

Therefore, each element in $\mathcal{D}_{\mathcal{F}}$ is related to a nonnegative quadratic functions over $\mathcal{F}$. In Chapter 4 and Chapter 5, $\mathcal{D}_{\mathcal{F}}$ and its dual cone $\mathcal{D}_{\mathcal{F}}^*$ are used to construct equivalent linear conic programming problems where $\mathcal{F}$ is defined by a set of quadratic inequalities and a set of conic form quadratic constraints, respectively. As we can see, knowing the structures of these cones is quite important in solving the corresponding linear conic programming problems which are related to the quadratic programming problems over $\mathcal{F}$.

In this chapter, we study the structure of $\mathcal{D}_{\mathcal{F}}$ and $\mathcal{D}_{\mathcal{F}}^*$ over a general domain $\mathcal{F}$. When $\mathcal{F}$ has some simple structure, we provide an exact representation of $\mathcal{D}_\mathcal{F}$ and $\mathcal{D}^*_\mathcal{F}$ by using semidefinite constraints and second order cone constraints, which are regarded as computable formulations.

The study of the cone of nonnegative quadratic functions can be found explicitly in Sturm and Zhang [107] and they prove that the problem

$$\begin{aligned} \min \quad & f(x) = x^T A_0 x + 2b_0^T x + c_0 \\ s.t. \quad & x \in \mathcal{F} \end{aligned} \qquad \text{(P0)} \tag{6.2}$$

has the same objective value as

$$\begin{aligned} \inf \quad & \begin{bmatrix} c_0 & b_0^T \\ b_0 & A_0 \end{bmatrix} \cdot Y \\ s.t. \quad & Y_{11} = 1 \\ & Y \in \mathcal{D}^*_\mathcal{F} \end{aligned} \qquad \text{(CP5)} \tag{6.3}$$

and its dual

$$\begin{aligned} \sup \quad & \sigma \\ s.t. \quad & \begin{bmatrix} c_0 & b_0^T \\ b_0 & A_0 \end{bmatrix} - \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{D}_\mathcal{F} \\ & \sigma \in \mathbb{R} \end{aligned} \qquad \text{(CD5)} \tag{6.4}$$

They also study the exact representations of $\mathcal{D}_\mathcal{F}$ and $\mathcal{D}^*_\mathcal{F}$, when

- $\mathcal{F} = \{x \in \mathbb{R}^n | g_1(x) = x^T A_1 x + 2b_1^T x + c_1 \leq 0\}$, or
- $\mathcal{F} = \{x \in \mathbb{R}^n | g_1(x) = x^T A_1 x + 2b_1^T x + c_1 = 0\}$ with $g_1(x)$ being strictly concave/convex, or
- $\mathcal{F} = \{x \in \mathbb{R}^n | g_1(x) = x^T A_1 x + 2b_1^T x + c_1 \leq 0, \text{ and } a^T x \leq a_0\}$ with $g_1(x)$ being convex, or
- $\mathcal{F} = \{x \in \mathbb{R}^n | g_1(x) = x^T A_1 x + 2b_1^T x + c_1 \leq 0, g_2(x) = x^T A_1 x + 2b_2^T x + c_2 \leq 0\}$ with A_1 being positive semidefinite. Ye and Zhang [113] extend the idea of rank-one decomposition in Sturm and Zhang [107]. Based on their results, the exact representations of $\mathcal{D}_\mathcal{F}$ and $\mathcal{D}^*_\mathcal{F}$ can be obtained when
- $\mathcal{F} = \{x \in \mathbb{R}^n | l \leq g_1(x) = x^T A_1 x + 2b_1^T x + c_1 \leq u\}$ with $l < u$, or
- $\mathcal{F} = \{x \in \mathbb{R}^n | \|x\| \leq 1, (a^i)^T x \leq a_0^i, (a_0^i - (a^i)^T x)(a_0^j - (a^j)^T x) = 0, \forall 1 \leq i < j \leq m\}$.

Burer [32] works on formulating the set $\mathcal{D}^*_\mathcal{F} \cap \{Y | Y_{11} = 1\}$ with $\mathcal{F} = \{x \in \mathbb{R}^n | Ax = b, x \in \{0,1\}^n\}$ with a key assumption. Burer [34] and Eichfelder and Povh [41] also extend the results to the case $\mathcal{F} = \{x | Ax = b, x \in \mathcal{K}\}$ with $\mathcal{K}$ being a closed convex cone. These results can be used to construct corresponding $\mathcal{D}^*_\mathcal{F}$. Anstreicher and Burer [7] prove the exact representation of $\mathcal{D}^*_\mathcal{F} \cap \{Y | Y_{11} = 1\}$ when $\mathcal{F}$ is a box with $n \leq 3$ or a simplex with $n \leq 4$. Recently, Burer and Anstreicher [35] also provide the exact representation of $\mathcal{D}_\mathcal{F}$ and $\mathcal{D}^*_\mathcal{F}$ when $\mathcal{F} = \{x | \|x\| \leq 1, l \leq a^T x \leq u\}$ with $l < u$. Burer and Dong [36] consider representing $\mathcal{D}^*_\mathcal{F} \cap \{Y | Y_{11} = 1\}$ for $\mathcal{F}$ with quadratic constraints. In this chapter, we provide exact representations of $\mathcal{D}_\mathcal{F}$ and $\mathcal{D}^*_\mathcal{F}$ when

- $\mathcal{F} = \{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} | \|x\| \le a_1 + a_2^T x + a_3^T y\}$, or
- $\mathcal{F} = \{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} | \|x\| \le a_1 + a_2^T x + a_3^T y, a_1 + a_2^T x + a_3^T y \ge a_4 \ge 0\}$, or
- $\mathcal{F} = \{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} | \|x\| \le a_1 + a_2^T x + a_3^T y, a_5 \ge a_1 + a_2^T x + a_3^T y \ge a_4 \ge 0\}$.

The study of the cone of nonnegative quadratic functions has a closed relation to those of the copositive cone over a domain $\mathcal{F}$ which is denoted as

$$\mathcal{HD}_{\mathcal{F}} = \{M \in \mathcal{S}^n | x^T M x \ge 0, \forall x \in \mathcal{F}\}.$$

Its dual cone has the formulation of

$$\mathcal{HD}^*_{\mathcal{F}} = \mathrm{cl}\,\mathrm{cone}\{xx^T | x \in \mathcal{F}\}.$$

By defining a homogenized set of $\mathcal{F}$

$$\mathcal{H}_{\mathcal{F}} = \mathrm{cl}\left\{ \begin{bmatrix} t \\ x \end{bmatrix} \in \mathbb{R}^{n+1} \middle| x/t \in \mathcal{F}, t > 0 \right\},$$

we have

$$\mathcal{D}^*_{\mathcal{F}} = \mathcal{HD}^*_{\mathcal{H}_{\mathcal{F}}}.$$

The concept of copositivity can be traced back to the work of Motzkin [75] in 1952. For a survey of the study of copositivity and complete positivity in linear algebra literature, one may refer to Berman [22]. In particular, many researches focus on $\mathcal{HD}_{\mathbb{R}^n_+}$ and $\mathcal{HD}^*_{\mathbb{R}^n_+}$. Murty and Kabadi [76] prove that determining if a matrix is in $\mathcal{HD}_{\mathbb{R}^n_+}$ is CoNP-hard. Parrilo [81] uses the concept of sum of squares to construct a sequential inner approximation of $\mathcal{HD}_{\mathbb{R}^n_+}$. He introduces a set of cones $\mathcal{K}^r$ and shows $\mathcal{S}^n + \mathcal{N}_n = \mathcal{K}^0 \subset \mathcal{K}^1 \subset \cdots \subset \mathcal{HD}_{\mathbb{R}^n_+}$ and $\mathrm{int}(\mathcal{HD}_{\mathbb{R}^n_+}) \subset \cup_r \mathcal{K}^r$. Here $\mathcal{N}_n$ is the set of nonnegative symmetric matrices. de Klerk and Pasechnik [39] explore another inner approximation which are all polyhedral cones denoted as $\mathcal{C}^r$ and Peña et al. [83] also propose an inner approximation $\mathcal{Q}^r$. The relations of them can be seen as $\mathcal{C}^r \subset \mathcal{Q}^r \subset \mathcal{K}^r$, for all r. These approximations are all uniform approximations. Bundfuss and Dür [31] develop both inner and outer adaptive approximations of $\mathcal{HD}_{\mathbb{R}^n_+}$ based on continuously partitioning a so called standard simplex. Inspired by their work, Yıldırım [114] proposes an outer approximation of $\mathcal{HD}_{\mathbb{R}^n_+}$ which is denoted as $\mathcal{O}^r$. This approximation is also a uniform approximation. Lu et al. [69] develop an adaptive outer approximation of $\mathcal{D}^*_{\mathcal{F}}$ where $\mathcal{F}$ is a box. The dual of each cone in the approximation leads to an inner approximation of $\mathcal{D}_{\mathcal{F}}$. It worth mentioning that all these uniform approximation can be represented as either LP or SDP, while the sizes of the reformulations grow exponentially when r increases.

Both $\mathcal{D}_{\mathcal{F}}$ and $\mathcal{HD}_{\mathcal{F}}$ play an important role in reformulating NP-hard problems into linear conic programming problems. Quist et al. [94] introduce the copositive cone into the conic relaxation of QCQP problems. Their approach can also be seen as using the cone of nonnegative quadratic function over a polyhedral set. Bomze et al. [26] develop an exact formulation of an NP-hard problem, i.e., the standard quadratic programming, using $\mathcal{HD}^*_{\mathbb{R}^n_+}$. Since then, it has been used in formulating several problems such as calculating stability number [39], chromatic number [52], clique number [27], quadratic assignment [93], 3-partitioning [92], etc.

Recently, more and more researches focus on the cone of nonnegative quadratic functions over a general domain since the work of Sturm and Zhang [107]. In the rest of this chapter, we will first introduce the notations used in this chapter and provide some properties of the cone of nonnegative quadratic functions in Section 6.2; then, we study the exact formulations of several cases in Section 6.3; at last, a brief summary is given in Section 6.4.

6.2 Notation and Properties

Given a set $\mathcal{F} \subset \mathbb{R}^n$, we can define the cone of nonnegative quadratic functions over $\mathcal{F}$ as

$$\mathcal{D}_{\mathcal{F}} = \left\{ M \in \mathcal{S}^{n+1} \;\middle|\; M \cdot \begin{bmatrix} 1 & x^T \\ x & xx^T \end{bmatrix} \geq 0, \forall x \in \mathcal{F} \right\}.$$

From [107], its dual cone can be denoted as

$$\mathcal{D}^*_{\mathcal{F}} = \text{cl cone} \left\{ \begin{bmatrix} 1 & x^T \\ x & xx^T \end{bmatrix} \in \mathcal{S}^{n+1} \middle| x \in \mathcal{F} \right\}$$

In literature, one can define another pair of cones: the copositive cone and the completely positive cone over a set $\mathcal{F}$:

$$\mathcal{HD}_{\mathcal{F}} = \{M \in \mathcal{S}^n | x^T M x \geq 0, \forall x \in \mathcal{F}\},$$

and

$$\mathcal{HD}^*_{\mathcal{F}} = \text{cl cone}\{xx^T \in \mathcal{S}^n | x \in \mathcal{F}\}.$$

One can get the relation of these two pairs of cones by defining a homogenized set of $\mathcal{F}$

$$\mathcal{H}_{\mathcal{F}} = \text{cl} \left\{ \begin{bmatrix} t \\ x \end{bmatrix} \in \mathbb{R}^{n+1} \middle| x/t \in \mathcal{F}, t > 0 \right\},$$

and from Proposition 2 and Corollary 1 in [107], we have the following results.

Lemma 48 ([107]). *Given a nonempty set $\mathcal{F}$, we have*
(1) *$\mathcal{H}_{\mathcal{F}}$ is a closed cone.*

(2) $\mathcal{D}_{\mathcal{F}} = \mathcal{HD}_{\mathcal{H}_{\mathcal{F}}}$.
(3) $\mathcal{D}^*_{\mathcal{F}} = \mathcal{HD}^*_{\mathcal{H}_{\mathcal{F}}}$.
(4) $\mathcal{D}_{\mathcal{F}}$ *and* $\mathcal{D}^*_{\mathcal{F}}$ *are dual to each other.*

The closure operator in the definition of $\mathcal{HD}^*_{\mathcal{F}}$ is less desirable since it is difficult to handle this operator in an optimization problem. In some cases, the closeness requirement is automatically fulfilled without this closure operator. The next lemma provide a necessary and sufficient condition to characterize when the closure operator can be removed from the definition of $\mathcal{HD}^*_{\mathcal{F}}$.

Lemma 49. *Given a nonempty set* $\mathcal{F}$, $\mathcal{HD}^*_{\mathcal{F}} = \text{cone}\{xx^T \in \mathcal{S}^n | x \in \text{cl}\mathcal{F}\}$ *if and only if* $\text{cl}\{tx \in \mathbb{R}^n | x \in \mathcal{F}, t \geq 0\} = \{tx \in \mathbb{R}^n | x \in \text{cl}\mathcal{F}, t \geq 0\}$.

Proof. Clearly, $\mathcal{HD}^*_{\mathcal{F}} \supset \text{cone}\{xx^T \in \mathcal{S}^n | x \in \text{cl}\mathcal{F}\}$ and $\text{cl}\{tx \in \mathbb{R}^n | x \in \mathcal{F}, t \geq 0\} \supset \{tx \in \mathbb{R}^n | x \in \text{cl}\mathcal{F}, t \geq 0\}$ hold.

On one side, suppose $\mathcal{HD}^*_{\mathcal{F}} = \text{cone}\{xx^T \in \mathcal{S}^n | x \in \text{cl}\mathcal{F}\}$, then for any $y \in \text{cl}\{tx \in \mathbb{R}^n | x \in \mathcal{F}, t \geq 0\}$, we have $y = \lim_{i \to +\infty} x^i$, where $x^i \in \{tx \in \mathbb{R}^n | x \in \mathcal{F}, t \geq 0\}$. Define $Y = yy^T$, $X^i = x^i(x^i)^T$, and then $Y = \lim_{i \to +\infty} X^i \in \mathcal{HD}^*_{\mathcal{F}}$. From the assumption, $Y \in \text{cone}\{xx^T \in \mathcal{S}^n | x \in \text{cl}\mathcal{F}\}$ and notice that the rank of Y is only 1. Therefore, $Y = \lambda xx^T$ with $\lambda \geq 0$ and $x \in \text{cl}\mathcal{F}$. This means $y = \lambda^{\frac{1}{2}} x \in \{tx \in \mathbb{R}^n | x \in \text{cl}\mathcal{F}, t \geq 0\}$ and hence $\text{cl}\{tx \in \mathbb{R}^n | x \in \mathcal{F}, t \geq 0\} = \{tx \in \mathbb{R}^n | x \in \text{cl}\mathcal{F}, t \geq 0\}$.

On the other side, if $\text{cl}\{tx \in \mathbb{R}^n | x \in \mathcal{F}, t \geq 0\} = \{tx \in \mathbb{R}^n | x \in \text{cl}\mathcal{F}, t \geq 0\}$, then for any $Y \in \mathcal{HD}^*_{\mathcal{F}}$, we have $Y = \lim_{i \to +\infty} Y^i$, where $Y^i \in \text{cone}\{xx^T \in \mathcal{S}^n | x \in \mathcal{F}\}$ for all i. Each Y^i can be decomposed as $Y^i = \sum_{j=1}^{r_i} (\lambda^i_j x^{ij})(\lambda^i_j x^{ij})^T$, where $r_i \leq \frac{n(n+1)}{2}$, and $\lambda^i_j \geq 0$, $x^{ij} \in \mathcal{F}$, for all i, j. Let $X^i \in \mathbb{R}^{n \times \frac{n(n+1)}{2}}$ be define as follows: the first r_i columns of X^i are $(\lambda^i_j x^{ij})$, $j = 1, \ldots, r_i$, and the rest of columns are all zeros. Since $Y = \lim_{i \to +\infty} Y^i$ and $Y^i = X^i(X^i)^T$, we have $\lim_{i \to +\infty} (X^i \cdot X^i) = \lim_{i \to +\infty} \text{tr}(Y^i) = \text{tr}(Y)$. Therefore, $\{X^i\}$ is a bounded sequence in $\mathbb{R}^{n \times \frac{n(n+1)}{2}}$ and there exists $\bar{X}$ which is the limit of a subsequence of $\{X^i\}$. Hence $Y = \bar{X}\bar{X}^T$. Notice that each column of $\bar{X}$ is an element of $\text{cl}\{tx \in \mathbb{R}^n | x \in \mathcal{F}, t \geq 0\}$ and from $\text{cl}\{tx \in \mathbb{R}^n | x \in \mathcal{F}, t \geq 0\} = \{tx \in \mathbb{R}^n | x \in \text{cl}\mathcal{F}\}$, each nonzero column of $\bar{X}$ can be denoted as $\lambda_j x^j$, where $\lambda_j \geq 0$, $x^j \in \text{cl}\mathcal{F}$. Consequently, $Y = \bar{X}\bar{X}^T \in \text{cone}\{xx^T \in \mathcal{S}^n | x \in \text{cl}\mathcal{F}\}$ and $\mathcal{HD}^*_{\mathcal{F}} = \text{cone}\{xx^T \in \mathcal{S}^n | x \in \text{cl}\mathcal{F}\}$. □

Remark: Given a set $\mathcal{F} \subset \mathbb{R}^n$, notice that $\mathcal{H}_{\mathcal{F}}$ is a closed cone, and hence $\mathcal{D}^*_{\mathcal{F}} = \mathcal{HD}^*_{\mathcal{H}_{\mathcal{F}}} = \text{cone}\{yy^T \in \mathcal{S}^{n+1} | y \in \mathcal{H}_{\mathcal{F}}\} = \text{conv}\{yy^T \in \mathcal{S}^{n+1} | y \in \mathcal{H}_{\mathcal{F}}\} = \{\sum_i y^i(y^i)^T \in \mathcal{S}^{n+1} | y^i \in \mathcal{H}_{\mathcal{F}}\}$.

Similarly, we can provide a necessary and sufficient condition to ensure that the closure in the definition of $\mathcal{D}^*_{\mathcal{F}}$ can be removed.

Lemma 50. *Given a nonempty set $\mathcal{F}$, $\mathcal{D}^*_{\mathcal{F}} = cone \left\{ \begin{bmatrix} 1 & x^T \\ x & xx^T \end{bmatrix} \in \mathcal{S}^n \middle| x \in \mathrm{cl}\mathcal{F} \right\}$ if and only if $\mathcal{F}$ is bounded.*

Proof. We just need to prove $\mathcal{H}_{\mathcal{F}} = \left\{ \begin{bmatrix} t \\ x \end{bmatrix} \in \mathbb{R}^{n+1} \middle| x/t \in \mathrm{cl}\mathcal{F}, t > 0 \right\} \cup \{0\}$ if and only if $\mathcal{F}$ is bounded. Obviously, $\mathcal{H}_{\mathcal{F}} \supset \left\{ \begin{bmatrix} t \\ x \end{bmatrix} \in \mathbb{R}^{n+1} \middle| x/t \in \mathrm{cl}\mathcal{F}, t > 0 \right\} \cup \{0\}$.

Now if $\mathcal{F}$ is bounded, suppose $y = \begin{bmatrix} t \\ x \end{bmatrix} \in \mathcal{H}_{\mathcal{F}}$, then $y = \lim_{i \to +\infty} y^i$ where $y^i = \begin{bmatrix} t^i \\ x^i \end{bmatrix}$ with $t^i > 0$ and $\frac{x^i}{t^i} \in \mathcal{F}$. If $t = 0$, then $\lim_{i \to +\infty} t^i = 0$. Since $\mathcal{F}$ is bounded, the sequence $\{\frac{x^i}{t^i}\}$ is bounded and, therefore, $x = \lim_{i \to +\infty} t^i \frac{x^i}{t^i} = 0$. If $t > 0$, then from the fact that $\{\frac{x^i}{t^i}\}$ is bounded, there exists $z \in \mathrm{cl}\mathcal{F}$ being the limit of a subsequence of $\{\frac{x^i}{t^i}\}$. Hence $x = \lim_{i \to +\infty} t^i \frac{x^i}{t^i} = tz$ and $\mathcal{H}_{\mathcal{F}} = \left\{ \begin{bmatrix} t \\ x \end{bmatrix} \in \mathbb{R}^{n+1} \middle| x/t \in \mathrm{cl}\mathcal{F}, t > 0 \right\} \cup \{0\}$.

If $\mathcal{F}$ is unbounded, then there exists a sequence $\{z^i\}$ in $\mathcal{F}$ such that $\lim_{i \to +\infty} \|z^i\| = +\infty$. Without loss of generality, we can assume none of these vectors is zero. Since the unit ball is bounded, there exists $\bar{z}$ such that a subsequence of $\{\frac{z^i}{\|z^i\|}\}$ converges to $\bar{z}$. We can replace $\{z^i\}$ by such subsequence, i.e., we can assume $\bar{z} = \lim_{i \to +\infty} \frac{z^i}{\|z^i\|} \neq 0$. Now define $y^i = \begin{bmatrix} t^i \\ x^i \end{bmatrix} = \begin{bmatrix} 1/\|z^i\| \\ z^i/\|z^i\| \end{bmatrix}$. Then $\lim_{i \to +\infty} y^i = \begin{bmatrix} 0 \\ \bar{z} \end{bmatrix} \in \mathcal{H}_{\mathcal{F}}$. However, $\begin{bmatrix} 0 \\ \bar{z} \end{bmatrix} \notin \left\{ \begin{bmatrix} t \\ x \end{bmatrix} \in \mathbb{R}^{n+1} \middle| x/t \in \mathrm{cl}\mathcal{F}, t > 0 \right\} \cup \{0\}$. Therefore, $\mathcal{H}_{\mathcal{F}} \neq \left\{ \begin{bmatrix} t \\ x \end{bmatrix} \in \mathbb{R}^{n+1} \middle| x/t \in \mathrm{cl}\mathcal{F}, t > 0 \right\} \cup \{0\}$.

From the discussion above and Lemma 49, we have

$$\mathcal{D}^*_{\mathcal{F}} = \mathrm{cone} \left\{ \begin{bmatrix} 1 & x^T \\ x & xx^T \end{bmatrix} \in \mathcal{S}^n \middle| x \in \mathrm{cl}\mathcal{F} \right\}$$

if and only if

$$\mathcal{H}_{\mathcal{F}} = \left\{ \begin{bmatrix} t \\ x \end{bmatrix} \in \mathbb{R}^{n+1} \middle| x/t \in \mathrm{cl}\mathcal{F}, t > 0 \right\} \cup \{0\}$$

which is equivalent to $\mathcal{F}$ being bounded. □

Below are two examples about $\mathcal{D}_{\mathcal{F}}$ and $\mathcal{D}^*_{\mathcal{F}}$. The cones in these two examples are widely used in literature.

Example 51. *If $\mathcal{F} = \mathbb{R}^n$, then $\mathcal{D}_{\mathcal{F}} = \mathcal{D}^*_{\mathcal{F}} = \mathcal{S}^{n+1}_+$*

Example 52. *When* $\mathcal{F} = \mathbb{R}^n_+$*, then one can verify that* $\mathcal{H}_{\mathbb{R}^n_+} = \mathbb{R}^{n+1}_+$ *and therefore* $\mathcal{D}_{\mathcal{F}} = \mathcal{HD}_{\mathbb{R}^{n+1}_+}$ *and* $\mathcal{D}^*_{\mathcal{F}} = \mathcal{HD}^*_{\mathbb{R}^{n+1}_+}$*. They are the copositive cone and the completely positive cone over* $\mathbb{R}^{n+1}_+$*.*

The cone of nonnegative quadratic functions and its dual cone also have some monotonic properties.

Lemma 53. *If* $\mathcal{F}_1 \subset \mathcal{F}_2$*, then* $\mathcal{D}^*_{\mathcal{F}_1} \subset \mathcal{D}^*_{\mathcal{F}_2}$ *and* $\mathcal{D}_{\mathcal{F}_1} \supset \mathcal{D}_{\mathcal{F}_2}$*. Therefore, for any* $\mathcal{F} \subset \mathbb{R}^n$*,* $\mathcal{D}^*_{\mathcal{F}} \subset \mathcal{S}^{n+1} \subset \mathcal{D}_{\mathcal{F}}$*.*

Proof. The proof is obvious from the definition of these cones and Example 51. □

The next lemma can be used to construct exact formulations or approximations of $\mathcal{D}_{\mathcal{F}}$ and $\mathcal{D}^*_{\mathcal{F}}$ from their components.

Lemma 54. *Suppose* $\mathcal{F} = \cup_{i=1}^k \mathcal{F}_i$ *and each* $\mathcal{F}_i$ *is not empty. Then* $\mathcal{D}_{\mathcal{F}} = \cap_{i=1}^k \mathcal{D}_{\mathcal{F}_i}$ *and* $\mathcal{D}^*_{\mathcal{F}} = \sum_{i=1}^k \mathcal{D}^*_{\mathcal{F}_i}$*.*

Before the proof of the lemma, we will introduce another lemma which is given in Corollary 16.4.2 in [97]. This lemma will be used in the proofs of several results in this chapter.

Lemma 55 (Corollary 16.4.2 in [97])**.** *Let* $K_1, \ldots, K_s$ *be nonempty closed convex cones in* $\mathbb{R}^n$*. Then*

$$(\cap_{i=1}^s K_i)^* = \mathrm{cl}(\sum_{i=1}^s K_i).$$

If the relative interior of $K_i, i = 1, \ldots, s$ *contains a common point, then the closure can be remove from the above statement.*

Proof of Lemma 54. From Lemma 53, we have $\mathcal{D}_{\mathcal{F}} \subset \mathcal{D}_{\mathcal{F}_i}$ for all $i = 1, \ldots, k$ and hence $\mathcal{D}_{\mathcal{F}} \subset \cap_{i=1}^k \mathcal{D}_{\mathcal{F}_i}$. Now if $M \in \cap_{i=1}^k \mathcal{D}_{\mathcal{F}_i}$, then, from $M \in \mathcal{D}_{\mathcal{F}_i}$, we know $M \in \mathcal{D}_{\mathcal{F}}$, then $M \cdot \begin{bmatrix} 1 & x^T \\ x & xx^T \end{bmatrix} \geq 0$, for all $x \in \mathcal{F}_i$. This means $M \cdot \begin{bmatrix} 1 & x^T \\ x & xx^T \end{bmatrix} \geq 0$ for all $x \in \cup_{i=1}^k \mathcal{F}_i = \mathcal{F}$. Therefore, $M \in \mathcal{D}_{\mathcal{F}}$ and $\mathcal{D}_{\mathcal{F}} \supset \cap_{i=1}^k \mathcal{D}_{\mathcal{F}_i}$. Combine the two results, $\mathcal{D}_{\mathcal{F}} = \cap_{i=1}^k \mathcal{D}_{\mathcal{F}_i}$.

By noticing that $\mathcal{D}_{\mathcal{F}_i} \supset \mathcal{S}^{n+1}_+$ for all $i = 1, \ldots, k$, and are all closed convex cones, from Lemma 55, $\sum_{i=1}^k \mathcal{D}^*_{\mathcal{F}_i} = (\cap_{i=1}^k \mathcal{D}_{\mathcal{F}_i})^* = (\mathcal{D}_{\mathcal{F}})^* = \mathcal{D}^*_{\mathcal{F}}$. □

The next lemma reveals a fact that if $\mathcal{F}$ is Cartesian product of $\mathcal{F}_1$ and $\mathbb{R}^m$, then $\mathcal{D}^*_{\mathcal{F}}$ can be expressed by $\mathcal{D}^*_{\mathcal{F}_1}$ and one semidefinite constraint. Therefore, we just need to focus on the representation of $\mathcal{D}^*_{\mathcal{F}_1}$.

Lemma 56. *Given a nonempty set $\mathcal{F}_1 \subset \mathbb{R}^n$ and let $\mathcal{F} = \mathcal{F}_1 \times \mathbb{R}^m$, then*

$$\mathcal{D}^*_{\mathcal{F}} = \left\{ \begin{bmatrix} A_1 & A_2^T \\ A_2 & A_3 \end{bmatrix} \in \mathcal{S}_+^{1+n+m} \middle| A_1 \in \mathcal{D}^*_{\mathcal{F}_1} \right\}.$$

Proof. Let

$$\mathcal{K} = \left\{ \begin{bmatrix} A_1 & A_2^T \\ A_2 & A_3 \end{bmatrix} \in \mathcal{S}_+^{1+n+m} \middle| A_1 \in \mathcal{D}^*_{\mathcal{F}_1} \right\}.$$

Since $\mathcal{H}_{\mathcal{F}} = \mathcal{H}_{\mathcal{F}_1} \times \mathbb{R}^m$, from

$$\mathcal{D}^*_{\mathcal{F}} = \left\{ \sum_i \begin{bmatrix} u^i \\ v^i \end{bmatrix} \begin{bmatrix} u^i \\ v^i \end{bmatrix}^T \in \mathcal{S}_+^{1+n+m} \middle| \begin{bmatrix} u^i \\ v^i \end{bmatrix} \in \mathcal{H}_{\mathcal{F}} = \mathcal{H}_{\mathcal{F}_1} \times \mathbb{R}^m \right\}$$

we have, for any $Y \in \mathcal{D}^*_{\mathcal{F}}$, $Y = \sum_i \begin{bmatrix} u^i \\ v^i \end{bmatrix} \begin{bmatrix} u^i \\ v^i \end{bmatrix}^T = \begin{bmatrix} Y_1 & Y_2^T \\ Y_2 & Y_3 \end{bmatrix} \in \mathcal{S}_+^{1+n+m}$ and $Y_1 = \sum_i u^i (u^i)^T \in \mathcal{D}^*_{\mathcal{F}_1}$. Therefore, $Y \in \mathcal{K}$ and $\mathcal{D}^*_{\mathcal{F}} \subset \mathcal{K}$.

To see the other side, suppose $Y = \begin{bmatrix} Y_1 & Y_2^T \\ Y_2 & Y_3 \end{bmatrix} \in \mathcal{K}$, and then $Y \in \mathcal{S}_+^{1+n+m}$ and $Y_1 \in \mathcal{D}^*_{\mathcal{F}_1}$. There exists k such that $Y_1 = PP^T$ where $P \in \mathbb{R}^{(1+n)\times k}$ and each column of P lies in $\mathcal{H}_{\mathcal{F}_1}$. Suppose $\text{rank}(Y_1) = r$, then there exists $B \in \mathbb{R}^{(1+n)\times r}$ such that $Y_1 = BB^T$. Furthermore, we have $r \leq k$ and $P = BQ$ for some $Q \in \mathbb{R}^{r\times k}$ being full row rank. Since Y is positive semidefinite, then there exists $R \in \mathbb{R}^{r\times m}$ such that $Y_2^T = BR$. We have

$$Y = \begin{bmatrix} BB^T & BR \\ R^TB^T & Y_3 \end{bmatrix} = \begin{bmatrix} BB^T & BR \\ R^TB^T & R^TR \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & Y_3 - R^TR \end{bmatrix}.$$

Notice that $Y \in \mathcal{S}_+^{1+n+m}$ if and only if $Y_3 - R^TR \in \mathcal{S}_+^m$. (Otherwise, suppose $Y_3 - R^TR$ is not positive semidefinite. There exists $\bar{v} \in \mathbb{R}^m$ such that $\bar{v}^T(Y_3 - R^TR)\bar{v} < 0$. One can verify that $\bar{z}^T = [(-B(B^TB)^{-1}R\bar{v})^T \quad \bar{v}^T] \in \mathbb{R}^{1\times(1+n+m)}$ satisfies $\bar{z}^TY\bar{z} = \bar{v}^T(Y_3 - R^TR)\bar{v} < 0$ and, therefore, Y is not positive semidefinite.) Clearly, $\begin{bmatrix} 0 & 0 \\ 0 & Y_3 - R^TR \end{bmatrix} \in \mathcal{D}^*_{\mathcal{F}}$. We need to prove $\begin{bmatrix} BB^T & BR \\ R^TB^T & R^TR \end{bmatrix} \in \mathcal{D}^*_{\mathcal{F}}$. From $BB^T = PP^T = BQQ^TB^T$ we have

$$QQ^T = (B^TB)^{-1}B^T(BQQ^TB^T)B(B^TB)^{-1} = (B^TB)^{-1}B^TBB^TB(B^TB)^{-1} = I_r.$$

Let $U = R^T Q$ and then

$$\begin{bmatrix} P \\ U \end{bmatrix} \begin{bmatrix} P \\ U \end{bmatrix}^T = \begin{bmatrix} PP^T & PU^T \\ UP^T & UU^T \end{bmatrix} = \begin{bmatrix} BB^T & BR \\ R^T B^T & R^T R \end{bmatrix}.$$

Notice that each column of $\begin{bmatrix} P \\ U \end{bmatrix}$ is in $\mathcal{H}_{\mathcal{F}}$, and, therefore, $\begin{bmatrix} BB^T & BR \\ R^T B^T & R^T R \end{bmatrix} \in \mathcal{D}^*_{\mathcal{F}}$. This leads to $Y \in \mathcal{D}^*_{\mathcal{F}}$ and $\mathcal{K} \subset \mathcal{D}^*_{\mathcal{F}}$. Together with $\mathcal{D}^*_{\mathcal{F}} \subset \mathcal{K}$, we get $\mathcal{D}^*_{\mathcal{F}} = \mathcal{K}$. □

When studying the exact formulation of $\mathcal{D}^*_{\mathcal{F}}$, noticing that $\mathcal{D}^*_{\mathcal{F}}$ is a pointed cone and is a subset of $\mathcal{S}^{n+1}_+$, we can always divide $\mathcal{D}^*_{\mathcal{F}}$ into a bounded component and an unbounded component by a cutting plane such that for every point Y^0 in the unbounded component, we can find a point Y^1 in the bounded component and $Y^0 = \lambda Y^1$ for some $\lambda > 0$. One example of such cutting plane is defined by $\mathrm{tr}(Y) = 1$, the bounded component is $\mathcal{D}^*_{\mathcal{F}} \cap \{Y \in \mathcal{S}^{n+1} | \mathrm{tr}\ (Y) \leq 1\}$, and the unbounded component is $\mathcal{D}^*_{\mathcal{F}} \cap \{Y \in \mathcal{S}^{n+1} | \mathrm{tr}\ (Y) \geq 1\}$.. Under this setting, if we can find an exact formulation of the bounded component, then we may easily extend this formulation to the entire cone. From this perspective, it is sufficient to study the properties of the extreme points in the bounded component and the next lemma from Pataki [82] will be useful.

Lemma 57 ([82])**.** *Consider an SDP feasible set*

$$F = \{(X^1, \ldots, X^p) | X^j \in \mathcal{S}^{n_j}, j = 1, \ldots, p, \sum_{j=1}^{p} A^{ij} \cdot X^j = b_i, i = 1, \ldots, m\}.$$

Let $(X^1, \ldots, X^p)$ *be an extreme point of* F*, and* $r_j = \mathrm{rank}(X^j)$*. Then* $\sum_{j=1}^{p} r_j(r_j + 1) \leq 2m$*.*

Another useful technique is the rank-one decomposition and it will be used repeatedly in the proofs.

Lemma 58. *Suppose* $X \in \mathcal{S}^n_+$ *is not zero and* $\mathrm{rank}(X) = r$*. For any vector* $a \in \mathbb{R}^n$*, if* $Xa \neq 0$*, then* $X' = X - \frac{Xaa^T X}{a^T Xa} \in \mathcal{S}^n_+$ *and* $\mathrm{rank}(X') = r - 1$*.*

Proof. Since $Xa \neq 0$ and $X \in \mathcal{S}^n_+$, then $a^T Xa \neq 0$. For any vector $u \in \mathbb{R}^n$, $u^T X' u = u^T Xu - \frac{(u^T Xa)^2}{a^T Xa}$. Suppose $X = Y^T Y$ and let $\bar{a} = Ya$ and $\bar{u} = Yu$. Then from Cauchy-Schwarz inequality, we have $(u^T Xu)(a^T Xa) = \|\bar{u}\|^2 \|\bar{a}\|^2 \geq (\bar{u}^T \bar{a})^2 = (u^T Xa)^2$. Therefore, $u^T X' u \geq 0$ for any $u \in \mathbb{R}^n$ and $X' \in \mathcal{S}^n_+$.

Obviously, $\mathrm{rank}(X') \geq r - 1$. For any u in the null space of X, it is also in the null space of X'. Since $Xa \neq 0$, a is not in the null space of X. Notice that $X'a = 0$, i.e., a is in the null space of X'. Then the dimension of the null space of X' is one more than those of X, and hence $\mathrm{rank}(X') = r - 1$. □

Burer [34] proved that when $\mathcal{F} = \{x \in \mathcal{K} \subset \mathbb{R}^n | Ax = b\}$ with $\mathcal{K}$ being a closed convex cone, then

$$\mathcal{D}^*_{\mathcal{F}} \cap \{Y \in \mathcal{S}^{n+1} | Y_{11} = 1\} = \left\{ Y = \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \in \mathcal{D}^*_{\mathcal{K}} \subset \mathcal{S}^{n+1} \middle| \begin{array}{c} Ax = b \\ \operatorname{diag}(AXA^T) = b \circ b \end{array} \right\}$$

where $\operatorname{diag}(M)$ is a vector with $[\operatorname{diag}(M)]_i = M_{ii}$, $i = 1, \ldots, n$, and $b \circ b$ is a vector with $[b \circ b]_i = b_i^2$, $i = 1, \ldots, n$. Here we give a direct result to $\mathcal{D}^*_{\mathcal{F}}$ and the proof is similar to Burer [34].

Lemma 59. *If $\mathcal{F} = \{x \in \mathcal{K} \subset \mathbb{R}^n | Ax = b\}$ with $\mathcal{K}$ being a closed convex cone is nonempty, then*

$$\mathcal{D}^*_{\mathcal{F}} = \left\{ Y = \begin{bmatrix} \chi & x^T \\ x & X \end{bmatrix} \in \mathcal{D}^*_{\mathcal{K}} \subset \mathcal{S}^{n+1} \middle| \begin{array}{c} Ax = \chi b \\ \operatorname{diag}(AXA^T) = \chi(b \circ b) \end{array} \right\}$$

Proof. Define

$$\mathcal{G} = \left\{ Y = \begin{bmatrix} \chi & x^T \\ x & X \end{bmatrix} \in \mathcal{D}^*_{\mathcal{K}} \subset \mathcal{S}^{n+1} \middle| \begin{array}{c} Ax = \chi b \\ \operatorname{diag}(AXA^T) = \chi(b \circ b) \end{array} \right\}.$$

Since for any $x \in \mathcal{F}$, $\begin{bmatrix} 1 & x^T \\ x & xx^T \end{bmatrix} \in \mathcal{G}$ and $\mathcal{G}$ is a closed convex cone, we have $\mathcal{D}^*_{\mathcal{F}} \subset \mathcal{G}$.

To see the other side, it is sufficient to prove any $Y \in \mathcal{G}$ can be represented as

$$Y = \sum_i y^i (y^i)^T$$

with $y^i \in \mathcal{H}_{\mathcal{F}}$. Since $\mathcal{K}$ is a closed convex cone, we have

$$\mathcal{H}_{\mathcal{K}} = \left\{ \begin{bmatrix} \xi \\ z \end{bmatrix} \in \mathbb{R}^{n+1} \middle| \xi \geq 0, z \in \mathcal{K} \right\}$$

and

$$\mathcal{D}^*_{\mathcal{K}} = \operatorname{cone}\left\{ yy^T \in \mathcal{S}^{n+1} | y \in \mathcal{H}_{\mathcal{K}} \right\} = \left\{ \sum_i y^i (y^i)^T \in \mathcal{S}^{n+1} \middle| y^i \in \mathcal{H}_{\mathcal{K}} \right\}$$

For any $Y \in \mathcal{G}$, we have

$$Y = \sum_i y^i (y^i)^T = \sum_i \begin{bmatrix} \xi^i \\ z^i \end{bmatrix} \begin{bmatrix} \xi^i \\ z^i \end{bmatrix}^T$$

with $\xi^i \geq 0$ and $z^i \in \mathcal{K}$. We claim: (i) if $\xi^i = 0$, then z^i satisfies $Az^i = 0$ and $\begin{bmatrix} 0 \\ z^i \end{bmatrix} \in \mathcal{H}_{\mathcal{F}}$; (ii) if

$\xi^i > 0$, then $x^i = z^i/\xi^i$ satisfies $Ax^i = b$ and $\begin{bmatrix} \xi^i \\ z^i \end{bmatrix} \in \mathcal{H}_{\mathcal{F}}$.

Since $Y \in \mathcal{G}$, then we have

$$(\sum_i (\xi^i)^2)b = \sum_i \xi^i A z^i$$

$$(\sum_i (\xi^i)^2)b \circ b = \sum_i \operatorname{diag}(A(z^i(z^i)^T)A^T) = \sum_i (Az^i) \circ (Az^i)$$

Therefore,

$$(\sum_i \xi^i A z^i) \circ (\sum_i \xi^i A z^i) = (\sum_i (\xi^i)^2) \sum_i (Az^i) \circ (Az^i)$$

By Cauchy-Schwarz inequality, this equal sign holds if and only if there exists $\delta \in \mathbb{R}^m$ such that, for all i, $\xi^i \delta = Az^i$.

When $\xi^i = 0$, we have $Az^i = 0$. Let $\bar{x} \in \mathcal{F}$ and define

$$w^k = \begin{bmatrix} 0 \\ z^i \end{bmatrix} + \frac{1}{k} \begin{bmatrix} 1 \\ \bar{x} \end{bmatrix}.$$

Notice that $A(kz^i + \bar{x}) = b$. Then $w^k \in \mathcal{H}_{\mathcal{F}}$. Since $\lim_{k \to +\infty} w^k = \begin{bmatrix} 0 \\ z^i \end{bmatrix}$, then $\begin{bmatrix} 0 \\ z^i \end{bmatrix} \in \mathcal{H}_{\mathcal{F}}$ and (i) holds.

When $\xi^i > 0$, we just need to prove $\delta = b$.

$$(\sum_j (\xi^j)^2)b = \sum_j \xi^j A z^j = (\sum_j (\xi^j)^2)\delta$$

Since $\xi^i > 0$, then $\sum_j (\xi^j)^2 > 0$, and, therefore, $\delta = b$. This proves claim (ii).

From the claim (i) and (ii), we have $Y \in \mathcal{D}^*_{\mathcal{F}}$ and $\mathcal{D}^*_{\mathcal{F}} \supset \mathcal{G}$. Together with $\mathcal{D}^*_{\mathcal{F}} \subset \mathcal{G}$, we have $\mathcal{D}^*_{\mathcal{F}} = \mathcal{G}$. □

In Lemma 59, $\mathcal{D}^*_{\mathcal{F}}$ is represented by another cone of nonnegative quadratic function $\mathcal{D}^*_{\mathcal{K}}$ and several linear constraints. Notice that the formulation of $\mathcal{D}^*_{\mathcal{K}}$ is not given explicitly. However, according to this lemma, when we study the cone of nonnegative quadratic functions over $\mathcal{F}$, we just need to focus on the inequality parts and conic parts and this simplifies the analysis.

6.3 Exact Formulations

In this section, we will provide both known and new results about exact formulations of $\mathcal{D}_{\mathcal{F}}$ and $\mathcal{D}^*_{\mathcal{F}}$ over different $\mathcal{F}$. We are interested in using linear, second-order-cone, and semidefinite constraints to express $\mathcal{D}_{\mathcal{F}}$ and $\mathcal{D}^*_{\mathcal{F}}$ and this kind of formulations are treated as computable. It worth mentioning that according to the discussion in Section 3 [107], if $\mathcal{F}$ is nonempty, then the corresponding P0, CP5, and CD5 problems will share the same optimal value, and furthermore, either P0 is unbounded below or the optimal value of CD5 is attainable.

6.3.1 One quadratic inequality

Suppose $\mathcal{F} = \{x \in \mathbb{R}^n | g_1(x) = x^T A_1 x + 2b_1^T x + c_1 \leq 0\}$ and $A_1 \neq 0$. From Theorem 1 and Corollary 5 in [107], we have the following results.

Theorem 60 ([107]). *Suppose $\mathcal{F} = \{x \in \mathbb{R}^n | g_1(x) = x^T A_1 x + 2b_1^T x + c_1 \leq 0\}$ is not empty and $A_1 \neq 0$. Then*

$$\mathcal{D}_{\mathcal{F}}^* = \left\{ Y \in \mathcal{S}_+^{n+1} \middle| \begin{bmatrix} c_1 & b_1^T \\ b_1 & A_1 \end{bmatrix} \cdot Y \leq 0 \right\}$$

and

$$\mathcal{D}_{\mathcal{F}} = \mathrm{cl} \left\{ M \in \mathcal{S}^{n+1} \middle| M + \lambda \begin{bmatrix} c_1 & b_1^T \\ b_1 & A_1 \end{bmatrix} \in \mathcal{S}_+^{n+1}, \lambda \geq 0 \right\}.$$

When $\{x \in \mathbb{R}^n | g_1(x) = x^T A_1 x + 2b_1^T x + c_1 < 0\} \neq \emptyset$, we have

$$\mathcal{D}_{\mathcal{F}} = \left\{ M \in \mathcal{S}^{n+1} \middle| M + \lambda \begin{bmatrix} c_1 & b_1^T \\ b_1 & A_1 \end{bmatrix} \in \mathcal{S}_+^{n+1}, \lambda \geq 0 \right\}.$$

Consequently, the two problems

$$\begin{array}{ll} \min & f(x) = x^T A_0 x + 2b_0^T x + c_0 \\ s.t. & g_1(x) = x^T A_1 x + 2b_1^T x + c_1 \leq 0 \\ & x \in \mathbb{R}^n \end{array}$$

and

$$\begin{array}{ll} \min & \begin{bmatrix} c_0 & b_0^T \\ b_0 & A_0 \end{bmatrix} \cdot Y \\ s.t. & Y_{11} = 1 \\ & \begin{bmatrix} c_1 & b_1^T \\ b_1 & A_1 \end{bmatrix} \cdot Y \leq 0 \\ & Y \in \mathcal{S}_+^{n+1} \end{array}$$

have the same optimal value and if there is $\bar{x} \in \mathbb{R}^n$ such that $g_1(\bar{x}) < 0$, then the optimal value *of the problem*

$$\begin{aligned} \max \quad & \sigma \\ s.t. \quad & \begin{bmatrix} c_0 & b_0^T \\ b_0 & A_0 \end{bmatrix} - \sigma \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \lambda \begin{bmatrix} c_1 & b_1^T \\ b_1 & A_1 \end{bmatrix} \in \mathcal{S}^{n+1} \\ & \sigma \in \mathbb{R}, \lambda \in \mathbb{R}_+ \end{aligned}$$

is the same as those of the above two problems and is attainable.

Remark. As we can see, in the formulation of $\mathcal{D}_{\mathcal{F}}$, a closure operator may exist and sometimes cannot be removed. However, the closure operator is difficulty to handle in an optimization problem, so we usually use the formulation without the closure operator. The resulting problem may not contain optimal solutions or is not even feasible. This fact can be seen in the next example.

Example 61. *Consider the following problem.*

$$\begin{aligned} \min \quad & 2x_1x_2 \\ s.t. \quad & x_2^2 \leq 0 \end{aligned}$$

One can check that the optimal value of this problem is 0 and every feasible solution is optimal. From the above theorem, the corresponding CD5 problem is

$$\begin{aligned} \max \quad & \sigma \\ s.t. \quad & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} - \sigma \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix} \in \operatorname{cl} \left\{ M \in \mathcal{S}^3 \middle| M + \lambda \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix} \in \mathcal{S}^3_+, \lambda \geq 0 \right\} \\ & \sigma \in \mathbb{R} \end{aligned}$$

It has an optimal solution $\sigma = 0$ and one can verify that

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \in \operatorname{cl} \left\{ M \in \mathcal{S}^3 \middle| M + \lambda \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix} \in \mathcal{S}^3_+, \lambda \geq 0 \right\}.$$

However, the problem without the closure, i.e.

$$\begin{aligned} \max \quad & \sigma \\ s.t. \quad & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} - \sigma \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix} \in \mathcal{S}^3_+ \\ & \lambda \geq 0, \sigma \in \mathbb{R} \end{aligned}$$

is infeasible. This is due to the fact that $g_1(x) = x_2^2 \geq 0$ *for any* $x \in \mathbb{R}^2$*.*

In the rest of the chapter, after showing the expressions of $\mathcal{D}_{\mathcal{F}}$ and $\mathcal{D}^*_{\mathcal{F}}$, without ambiguity, we will use the term "the corresponding CD5 problem without closure" to denote a formulation relaxing the closure requirement for $\mathcal{D}_{\mathcal{F}}$.

6.3.2 One quadratic equality

When $\mathcal{F} = \{x \in \mathbb{R}^n | g_1(x) = x^T A_1 x + 2b_1^T x + c_1 = 0\}$ with $A_1 \neq 0$. In Sturm and Zhang [107], a strictly convex (concave) quadratic equality is studied. Here we provide a more general result. This form has been studied by Moré [73] and is widely used in the modeling. However, for the completeness, we will provide a detailed proof.

Theorem 62. *Suppose* $\mathcal{F} = \{x \in \mathbb{R}^n | g_1(x) = x^T A_1 x + 2b_1^T x + c_1 = 0\}$ *is not empty and* $A_1 \neq 0$*. Then*

$$\mathcal{D}^*_{\mathcal{F}} = \left\{ Y \in \mathcal{S}^{n+1}_+ \middle| \begin{bmatrix} c_1 & b_1^T \\ b_1 & A_1 \end{bmatrix} \cdot Y = 0 \right\}$$

and

$$\mathcal{D}_{\mathcal{F}} = \mathrm{cl} \left\{ M \in \mathcal{S}^{n+1} \middle| M + \lambda \begin{bmatrix} c_1 & b_1^T \\ b_1 & A_1 \end{bmatrix} \in \mathcal{S}^{n+1}_+, \lambda \in \mathbb{R} \right\}.$$

If there are $x^1 \in \mathbb{R}^n$ *and* $x^2 \in \mathbb{R}^n$ *such that* $g_1(x^1) < 0$ *and* $g_1(x^2) > 0$*, then we have*

$$\mathcal{D}_{\mathcal{F}} = \left\{ M \in \mathcal{S}^{n+1} \middle| M + \lambda \begin{bmatrix} c_1 & b_1^T \\ b_1 & A_1 \end{bmatrix} \in \mathcal{S}^{n+1}_+ \right\}.$$

Consequently, the corresponding P0 and CP5 have the same optimal value and if there are $x^1 \in \mathbb{R}^n$ *and* $x^2 \in \mathbb{R}^n$ *such that* $g_1(x^1) < 0$ *and* $g_1(x^2) > 0$*, then the corresponding CD5 without closure has the same optimal value and is attainable.*

Proof. Let $\mathcal{K} = \left\{ Y \in \mathcal{S}^{n+1}_+ \middle| \begin{bmatrix} c_1 & b_1^T \\ b_1 & A_1 \end{bmatrix} \cdot Y = 0 \right\}$, then K is a closed, convex cone. For every $x \in \mathbb{R}^n$ satisfying $g_1(x) = 0$, we have $Y = \begin{bmatrix} 1 & x^T \\ x & xx^T \end{bmatrix} \in \mathcal{K}$. Therefore, $\mathcal{D}^*_{\mathcal{F}} \subset \mathcal{K}$.

To see the other side, it is sufficient to prove that every extreme point in $\mathcal{K}' = \mathcal{K} \cap \{Y \in \mathcal{S}^{n+1} | \mathrm{tr} Y \leq 1\}$ belongs to $\mathcal{D}^*_{\mathcal{F}}$. Notice that $\mathcal{K}'\}$ can be rewritten as

$$\mathcal{K}' = \left\{ (Y, s_1) \in \mathcal{S}^{n+1}_+ \times \mathcal{S}^1_+ \middle| \begin{bmatrix} c_1 & b_1^T \\ b_1 & A_1 \end{bmatrix} \cdot Y = 0, \sum_{i=1}^{n+1} Y_{ii} + s_1 = 1 \right\}$$

From Lemma 57, let $r_Y = \text{rank}(Y)$, $r_{s_1} = \text{rank}(s_1)$, and then every extreme point satisfies

$$r_Y(r_Y+1) + r_{s_1}(r_{s_1}+1) \leq 4.$$

Therefore, $r_Y \leq 1$, which means, for any Y being an extreme point, $Y = yy^T$ with $y = \begin{bmatrix} t \\ x \end{bmatrix} \in \mathbb{R}^{n+1}$ and $t \geq 0$. If $t > 0$, then $g_1(\frac{x}{t}) = 0$ and hence $Y \in \mathcal{D}^*_{\mathcal{F}}$. Otherwise, if $t = 0$, we need to prove that $Y = \begin{bmatrix} 0 & 0 \\ 0 & xx^T \end{bmatrix}$ is a convergent point in $\mathcal{D}^*_{\mathcal{F}}$.

When $t = 0$, we have $x^T A_1 x = 0$. Without loss of generality, we can always assume A_1 is a diagonal matrix. (Since we can always find an orthogonal matrix such that $P^T A_1 P$ is a diagonal matrix and $P^T P = I_n$.)

If $A_1 x = 0$ and $b_1^T x = 0$, since $\mathcal{F}$ is not empty, there exists $x' \in \mathbb{R}^n$ such that $g_1(x') = 0$. Then $g_1(\alpha x + x') = 0$ for any $\alpha > 0$. Define

$$y^k = \begin{bmatrix} 0 \\ x \end{bmatrix} + \frac{1}{k} \begin{bmatrix} 1 \\ x' \end{bmatrix} \text{ and } Y^k = y^k (y^k)^T.$$

Then $\begin{bmatrix} c_1 & b_1^T \\ b_1 & A_1 \end{bmatrix} \cdot (k^2 Y^k) = g_1(kx + x') = 0$ and $\lim_{k \to +\infty} Y^k = Y$. Hence $Y \in \mathcal{D}^*_{\mathcal{F}}$.

If $A_1 x = 0$ and $b_1^T x \neq 0$, then we can assume $[A_1]_{11} \neq 0$ and define

$$y^k = \begin{bmatrix} 0 \\ x \end{bmatrix} + \frac{1}{\lambda_k} \begin{bmatrix} 1 \\ ke_1 \end{bmatrix} \text{ and } Y^k = y^k (y^k)^T$$

where $e_1 = (1, 0, \ldots, 0)^T$ and $\lambda_k = \frac{-k^2 [A_1]_{11} - 2k b_1^T e_1 - c_1}{2 b_1^T x}$. One can verify that $\begin{bmatrix} c_1 & b_1^T \\ b_1 & A_1 \end{bmatrix} \cdot (\lambda_k^2 Y^k) = g_1(\lambda_k x + k e_1) = 0$ and $\lim_{k \to +\infty} \frac{1}{\lambda_k} = 0$ and $\lim_{k \to +\infty} \frac{k}{\lambda_k} = 0$. Therefore, $\lim_{k \to +\infty} Y^k = Y$ and $Y \in \mathcal{D}^*_{\mathcal{F}}$.

When $A_1 x \neq 0$, without loss of generality, we can assume $[A_1]_{ii} x_i \neq 0$, $i = 1, \ldots, n_1$, $[A_1]_{ii} \neq 0$ but $x_i = 0$, $i = n_1 + 1, \ldots, n_2$, and $[A_1]_{ii} = 0$, $i = n_2 + 1, \ldots, n$. For convenience, define $C_1 = c_1 - \sum_{j=1}^{n_2} \frac{[b_1]_j^2}{[A_1]_{jj}}$ and $C_2 = \sum_{j=n_2+1}^{n} [b_1]_j x_j$. Notice that for any $z \in \mathbb{R}^n$

$$g_1(z) = \sum_{i=1}^{n_2} [A_1](z_i + \frac{[b_1]_i}{[A_1]_{ii}})^2 + \sum_{i=n_2+1}^{n} [b_1]_i z_i + C_1.$$

Define $\bar{x} \in \mathbb{R}^n$ by

$$\bar{x}_i = \begin{cases} -\frac{[b_1]_i}{[A_1]_{ii}} & 1 \leq i \leq n_2 \\ 0 & n_2 + 1 \leq i \leq n \end{cases}$$

If $A_1 x \neq 0$, $C_1 = 0$, and $C_2 = 0$, then $g_1(\alpha x + \bar{x}) = 0$ for all $\alpha > 0$. Define

$$y^k = \begin{bmatrix} 0 \\ x \end{bmatrix} + \frac{1}{k} \begin{bmatrix} 1 \\ \bar{x} \end{bmatrix} \text{ and } Y^k = y^k (y^k)^T.$$

Then $\begin{bmatrix} c_1 & b_1^T \\ b_1 & A_1 \end{bmatrix} \cdot (k^2 Y^k) = g_1(kx + \bar{x}) = 0$ and $\lim_{k \to +\infty} Y^k = Y$. Hence $Y \in \mathcal{D}_{\mathcal{F}}^*$.

If $A_1 x \neq 0$, $C_1 \neq 0$ and $C_2 = 0$, then define

$$y^k = \begin{bmatrix} 0 \\ x \end{bmatrix} + \frac{1}{\lambda_k} \begin{bmatrix} 1 \\ \alpha_k e_1 \end{bmatrix} + \frac{1}{\lambda_k} \begin{bmatrix} 1 \\ \bar{x} \end{bmatrix} \text{ and } Y^k = y^k (y^k)^T$$

where $e_1 = (1, 0, \ldots, 0)^T$, $\lambda_k = \frac{-\alpha_k^2 [A_1]_{11} - C_1}{2\alpha_k [A_1]_{11} x_1}$ and $\lim_{k \to +\infty} \alpha_k = 0$. One can verify that $\begin{bmatrix} c_1 & b_1^T \\ b_1 & A_1 \end{bmatrix} \cdot (\lambda_k^2 Y^k) = g_1(\lambda_k x + \alpha_k e_1 + \bar{x}) = 0$, $\lim_{k \to +\infty} \frac{1}{\lambda_k} = 0$ and $\lim_{k \to +\infty} \frac{\alpha_k}{\lambda_k} = 0$. Therefore, $\lim_{k \to +\infty} Y^k = Y$ and $Y \in \mathcal{D}_{\mathcal{F}}^*$.

If $A_1 x \neq 0$, $C_2 \neq 0$. If there exists i such that $C_1 + \frac{C_2^2}{[A_1]_{ii} x_i^2} \neq 0$, then define

$$y^k = \begin{bmatrix} 0 \\ x \end{bmatrix} + \frac{1}{\lambda_k} \begin{bmatrix} 1 \\ \alpha_k e_i \end{bmatrix} + \frac{1}{\lambda_k} \begin{bmatrix} 1 \\ \bar{x} \end{bmatrix} \text{ and } Y^k = y^k (y^k)^T$$

where $e_i = (0, \ldots, 1, \ldots, 0)^T$ with the ith element being 1, $\lambda_k = \frac{-\alpha_k^2 [A_1]_{ii} - C_1}{2\alpha_k [A_1]_{ii} x_i + 2C_2}$ and $\lim_{k \to +\infty} \alpha_k = -\frac{C_2}{[A_1]_{ii} x_i}$. One can verify that $\begin{bmatrix} c_1 & b_1^T \\ b_1 & A_1 \end{bmatrix} \cdot (\lambda_k^2 Y^k) = g_1(\lambda_k x + \alpha_k e_1 + \bar{x}) = 0$, $\lim_{k \to +\infty} \frac{1}{\lambda_k} = 0$ and $\lim_{k \to +\infty} \frac{\alpha_k}{\lambda_k} = 0$. Therefore, $\lim_{k \to +\infty} Y^k = Y$ and $Y \in \mathcal{D}_{\mathcal{F}}^*$. If such i does not exists, then $[A_1]_{ii} x_i^2 = -\frac{C_2^2}{C_1}$ for all $i = 1, \ldots, n_1$. This leads to a contradiction that $0 = \sum_{i=1}^{n_1} [A_1]_{ii} x_i^2 = -n_1 \frac{C_2^2}{C_1} \neq 0$.

From the discussion above, we get the conclusion that every extreme point of set $\mathcal{K}'$ belongs to $\mathcal{D}_{\mathcal{F}}^*$ and, therefore, $\mathcal{K} \subset \mathcal{D}_{\mathcal{F}}^*$. Together with $\mathcal{D}_{\mathcal{F}}^* \subset \mathcal{K}$, we get $\mathcal{K} = \mathcal{D}_{\mathcal{F}}^*$ and

$$\mathcal{D}_{\mathcal{F}} = (\mathcal{D}_{\mathcal{F}}^*)^* = \text{cl} \left\{ M \in \mathcal{S}^{n+1} \middle| M + \lambda \begin{bmatrix} c_1 & b_1^T \\ b_1 & A_1 \end{bmatrix} \in \mathcal{S}_+^{n+1}, \lambda \in \mathbb{R} \right\}.$$

Notice that $\mathcal{D}_{\mathcal{F}}^* = \mathcal{S}_+^{n+1} \cap \left\{ Y \in \mathcal{S}^{n+1} \middle| \begin{bmatrix} c_1 & b_1^T \\ b_1 & A_1 \end{bmatrix} \cdot Y = 0 \right\}$, and, from Lemma 55, the closure can be removed if the two cones have a common relative interior point, i.e., there exists positive definite matrix Y satisfying $\begin{bmatrix} c_1 & b_1^T \\ b_1 & A_1 \end{bmatrix} \cdot Y = 0$. If there are $x^1 \in \mathbb{R}^n$ and $x^2 \in \mathbb{R}^n$ such that

$g_1(x^1) < 0$ and $g_1(x^2) > 0$, then let $Y^1 = \begin{bmatrix} 1 & (x^1)^T \\ x^1 & x^1(x^1)^T \end{bmatrix}$ and $Y^2 = \begin{bmatrix} 1 & (x^2)^T \\ x^2 & x^2(x^2)^T \end{bmatrix}$, we have

$$\begin{bmatrix} c_1 & b_1^T \\ b_1 & A_1 \end{bmatrix} \cdot Y^1 < 0 \text{ and } \begin{bmatrix} c_1 & b_1^T \\ b_1 & A_1 \end{bmatrix} \cdot Y^2 > 0.$$

Now, given a positive definite matrix Y^0, we can always find $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ such that

$$\begin{bmatrix} c_1 & b_1^T \\ b_1 & A_1 \end{bmatrix} \cdot (Y^0 + \lambda_1 Y^1 + \lambda_2 Y^2) = 0.$$

Since $Y^0 + \lambda_1 Y^1 + \lambda_2 Y^2$ is also positive definite, the condition of Lemma 55 holds and the closure can be removed. Therefore, P0 and CP5 have the same optimal value and the corresponding CD5 without closure has the same optimal value and is attainable. □

6.3.3 One quadratic inequality and one linear inequality

When $\mathcal{F} = \{x \in \mathbb{R}^n | g_1(x) = x^T A_1 x + 2b_1^T x + c_1 \leq 0, [1, x^T]a \geq 0\}$ with $A_1 \in \mathcal{S}_+^n$ and $a \in \mathbb{R}^{n+1}$. From Sturm and Zhang [107], we have the next result.

Theorem 63. *Suppose $\mathcal{F} = \{x \in \mathbb{R}^n | g_1(x) = x^T A_1 x + 2b_1^T x + c_1 \leq 0, [1, x^T]a \geq 0\}$ is not empty and $A_1 \in \mathcal{S}_+^n$ and $a \in \mathbb{R}^{n+1}$. If $\operatorname{rank}(A_1) = r$, then there exists $R \in \mathbb{R}^{r \times n}$ satisfying $R^T R = A_1$. Define*

$$B = \begin{bmatrix} c_1 - 1 & 2b_1^T \\ c_1 + 1 & 2b_1^T \\ 0 & 2R \end{bmatrix} \in \mathbb{R}^{(r+2)\times(n+1)} \text{ and } M_{g_1} = \begin{bmatrix} c_1 & b_1^T \\ b_1 & A_1 \end{bmatrix}.$$

Then

$$\mathcal{D}_{\mathcal{F}}^* = \left\{ Y \in \mathcal{S}_+^{n+1} \middle| M_{g_1} \cdot Y \leq 0, BYa \in \mathcal{SOC}(r+1) \right\}$$

and

$$\mathcal{D}_{\mathcal{F}} = \operatorname{cl}\left\{ M \in \mathcal{S}^{n+1} \middle| M + \lambda M_{g_1} - (a\psi^T B + B^T \psi a^T) \in \mathcal{S}_+^{n+1}, \lambda \geq 0, \psi \in \mathcal{SOC}(r+1) \right\}.$$

If there is $\bar{x} \in \mathbb{R}^n$ such that $g_1(\bar{x}) < 0$ and $[1, \bar{x}^T]a > 0$, then we have

$$\mathcal{D}_{\mathcal{F}} = \left\{ M \in \mathcal{S}^{n+1} \middle| M + \lambda M_{g_1} - (a\psi^T B + B^T \psi a^T) \in \mathcal{S}_+^{n+1}, \lambda \geq 0, \psi \in \mathcal{SOC}(r+1) \right\}.$$

Consequently, the corresponding P0 and CP5 have the same optimal value and if there is $\bar{x} \in \mathbb{R}^n$ such that $g_1(\bar{x}) < 0$ and $[1, \bar{x}^T]a > 0$, then the corresponding CD5 without closure has the same optimal value and is attainable.

Next, we will focus on a nonconvex quadratic constraint, i.e., $\mathcal{F} = \{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} | \|x\| \leq a_1 + a_2^T x + a_3^T y\}$ with $a_1 \in \mathbb{R}$, $a_2 \in \mathbb{R}^{n_1}$, and $a_3 \in \mathbb{R}^{n_2}$.

Theorem 64. *Given a nonempty set $\mathcal{F} = \{(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} | \|x\| \leq a_1 + a_2^T x + a_3^T y\}$ with $a_1 \in \mathbb{R}$, $a_2 \in \mathbb{R}^{n_1}$, and $a_3 \in \mathbb{R}^{n_2}$, let*

$$a^T = [a_1 \ a_2^T \ a_3^T]$$

and we have

$$\mathcal{D}_{\mathcal{F}}^* = \left\{ U = \begin{bmatrix} \chi & x^T & y^T \\ x & X & W^T \\ y & W & Y \end{bmatrix} \in \mathcal{S}_+^{1+n_1+n_2} \middle| \|x\| \leq a_1\chi + a_2^T x + a_3^T y, a^T U a \geq \sum_{i=1}^{n_1} X_{ii} \right\}$$

and

$$\mathcal{D}_{\mathcal{F}} = \mathrm{cl} \left\{ M \in \mathcal{S}^{1+n_1+n_2} \middle| \begin{array}{c} M - \lambda C_1 - \left(e_1 \psi^T C_2^T + C_2 \psi e_1^T\right) \in \mathcal{S}_+^{1+n_1+n_2}, \\ C_2 = \begin{bmatrix} a_1 & 0 \\ a_2 & I_{n_1} \\ a_3 & 0 \end{bmatrix}, C_1 = aa^T - \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{n_1} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \lambda \geq 0, \psi \in \mathcal{SOC}(n_1) \end{array} \right\}.$$

If there is $(\bar{x}, \bar{y}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ such that $\|\bar{x}\| < a_1 + a_2^T \bar{x} + a_3^T \bar{y}$, then the closure in the above equation can be removed.

Consequently, the corresponding P0 and CP5 have the same optimal value and if there is $(\bar{x}, \bar{y}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ such that $\|\bar{x}\| < a_1 + a_2^T \bar{x} + a_3^T \bar{y}$, then the corresponding CD5 without closure has the same optimal value and is attainable.

Before the proof of Theorem 64, we introduce a useful result in [35].

Lemma 65 ([35])**.** *Given $z_0 \in \mathbb{R}$ and $z \in \mathbb{R}^n$, define*

$$\mathrm{Arrow}(z_0, z) = \begin{bmatrix} z_0 I_n & z \\ z^T & z_0 \end{bmatrix}$$

and let $r = \mathrm{rank}(\mathrm{Arrow}(z_0, z))$. Then $\|z\| \leq z_0$ if and only if $\mathrm{Arrow}(z_0, z) \in \mathcal{S}_+^{n+1}$. In addition, if $\|z\| \leq z_0$, then one of the following three cases holds: (i) $(z_0, z) = 0$ and $r = 0$; (ii) $\|z\| = z_0 > 0$ and $r = n$; (iii) $\|z\| < z_0$ and $r = n+1$.

Proof of Theorem 64. Define

$$\mathcal{K} = \left\{ U = \begin{bmatrix} \chi & x^T & y^T \\ x & X & W^T \\ y & W & Y \end{bmatrix} \in \mathcal{S}_+^{1+n_1+n_2} \middle| \|x\| \le a_1\chi + a_2^T x + a_3^T y, a^T U a \ge \sum_{i=1}^{n_1} X_{ii} \right\}$$

The one side $\mathcal{D}_{\mathcal{F}}^* \subset \mathcal{K}$ is obvious.

To see the other side, it is sufficient to prove that all the extreme points of $\mathcal{K}'$ with $\mathcal{K}' = \mathcal{K} \cap \{U \in \mathcal{S}^{1+n_1+n_2} | \text{tr } U \le 1\}$ belong to $\mathcal{D}_{\mathcal{F}}^*$. We need to consider three cases: (i) $\chi = 0$; (ii) $\chi > 0$ and $\|x\| = a_1\chi + a_2^T x + a_3^T y$; (iii) $\chi > 0$ and $\|x\| < a_1\chi + a_2^T x + a_3^T y$.

If (i) holds, then for any U^0 being a nonzero extreme point of $\mathcal{K}'$, we have $(x, y) = 0$. Furthermore, since U^0 is an extreme point of $\mathcal{K}'$, the matrix $Z^0 = \begin{bmatrix} X^0 & (W^0)^T \\ W^0 & Y^0 \end{bmatrix}$ must be the extreme point of

$$\mathcal{L} = \left\{ Z = \begin{bmatrix} X & W^T \\ W & Y \end{bmatrix} \in \mathcal{S}_+^{n_1+n_2} \middle| \text{tr } Z \le 1, \begin{bmatrix} a_2 \\ a_3 \end{bmatrix}^T Z \begin{bmatrix} a_2 \\ a_3 \end{bmatrix} \ge \sum_{i=1}^{n_1} X_{ii} \right\},$$

and hence (Z^0, s_1^0, s_2^0) with

$$s_1^0 = 1 - \text{tr } Z^0, \text{ and } \begin{bmatrix} a_2 \\ a_3 \end{bmatrix}^T Z^0 \begin{bmatrix} a_2 \\ a_3 \end{bmatrix} - s_2^0 = \sum_{i=1}^{n_1} X_{ii}^0,$$

is the extreme point of

$$\mathcal{L}' = \left\{ Z = \begin{bmatrix} X & W^T \\ W & Y \end{bmatrix} \in \mathcal{S}_+^{n_1+n_2}, s_1 \ge 0, s_2 \ge 0 \middle| \text{tr } Z + s_1 = 1, \begin{bmatrix} a_2 \\ a_3 \end{bmatrix}^T Z \begin{bmatrix} a_2 \\ a_3 \end{bmatrix} - s_2 = \sum_{i=1}^{n_1} X_{ii} \right\}.$$

From Lemma 57, let $r_Z = \text{rank}(Z^0)$, $r_1 = \text{rank}(s_1^0)$, $r_2 = \text{rank}(s_2^0)$, and we have

$$r_Z(r_Z + 1) + r_1(r_1 + 1) + r_2(r_2 + 1) \le 4.$$

Therefore, $r_Z = 1$ and $Z^0 = \begin{bmatrix} x' \\ y' \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}^T$ with $a_2^T x' + a_3^T y' \ge \|x'\|$. In this case $U^0 = \begin{bmatrix} 0 \\ x' \\ y' \end{bmatrix} \begin{bmatrix} 0 \\ x' \\ y' \end{bmatrix}^T$.

To see $U^0 \in \mathcal{D}_{\mathcal{F}}^*$, we just need to prove $\begin{bmatrix} 0 \\ x' \\ y' \end{bmatrix} \in \mathcal{H}_{\mathcal{F}}$. Since $\mathcal{F}$ is not empty, there exist $(\bar{x}, \bar{y})$ such that $a_1 + a_2^T x + a_3^T y \ge \|\bar{x}\|$. From $a_2^T x' + a_3^T y' \ge \|x'\|$, we have $\lambda(a_1 + a_2^T x + a_3^T y) + a_2^T x' + a_3^T y' \ge$

$\|\lambda\bar{x} + x'\|$ for all $\lambda \geq 0$. Let $u^k = \frac{1}{k}\begin{bmatrix} 1 \\ \bar{x} \\ \bar{y} \end{bmatrix} + \begin{bmatrix} 0 \\ x' \\ y' \end{bmatrix}$, we have $ku^k \in \mathcal{F}$ for all $k > 0$ and therefore, $\begin{bmatrix} 0 \\ x' \\ y' \end{bmatrix} = \lim_{k\to+\infty} u^k \in \mathcal{H}_{\mathcal{F}}$ and $U^0 \in \mathcal{D}^*_{\mathcal{F}}$.

If (ii) holds, suppose U^0 is an extreme point of $\mathcal{K}'$. Let $u = \frac{1}{\sqrt{\chi^0}}\begin{bmatrix} \chi^0 \\ x^0 \\ y^0 \end{bmatrix}$, and then

$$\begin{aligned} U^0 &= uu^T + \begin{bmatrix} 0 & 0 & 0 \\ 0 & X^0 - \frac{x^0(x^0)^T}{\chi^0} & (W^0)^T - \frac{x^0(y^0)^T}{\chi^0} \\ 0 & W^0 - \frac{y^0(x^0)^T}{\chi^0} & Y^0 - \frac{y^0(y^0)^T}{\chi^0} \end{bmatrix} \\ &= \lambda(\frac{1}{\lambda}uu^T) + (1-\lambda)(\frac{1}{1-\lambda}) \begin{bmatrix} 0 & 0 & 0 \\ 0 & X^0 - \frac{x^0(x^0)^T}{\chi^0} & (W^0)^T - \frac{x^0(y^0)^T}{\chi^0} \\ 0 & W^0 - \frac{y^0(x^0)^T}{\chi^0} & Y^0 - \frac{y^0(y^0)^T}{\chi^0} \end{bmatrix} \end{aligned}$$

with $\lambda = \frac{\|u\|}{\chi^0 + \text{tr } X^0 + \text{tr } Y^0}$. Notice that $\frac{1}{\lambda}uu^T$ and $(\frac{1}{1-\lambda})\begin{bmatrix} 0 & 0 & 0 \\ 0 & X^0 - \frac{x^0(x^0)^T}{\chi^0} & (W^0)^T - \frac{x^0(y^0)^T}{\chi^0} \\ 0 & W^0 - \frac{y^0(x^0)^T}{\chi^0} & Y^0 - \frac{y^0(y^0)^T}{\chi^0} \end{bmatrix}$ are all in $\mathcal{K}'$. Since U^0 is an extreme point in $\mathcal{K}'$, we have $\lambda = 1$, i.e., $U^0 = uu^T$. From $\|x^0\| = a_1\chi^0 + a_2^T x^0 + a_3^T y^0$, we know $u \in \mathcal{H}_{\mathcal{F}}$ and, therefore, $U^0 \in \mathcal{D}^*_{\mathcal{F}}$.

If (iii) holds, suppose U^0 is an extreme point of $\mathcal{K}'$, then (U^0, S^0, s_1^0, s_2^0) with

$$S^0 = \text{Arrow}(a_1\chi^0 + a_2^T x^0 + a_3^T y^0, x^0), s_1^0 = 1 - \text{tr } U^0, a^T U^0 a - s_2^0 = \sum_{i=1}^{n_1} X_{ii}^0$$

is an extreme point of

$$\mathcal{L}'' = \left\{ \begin{array}{l} U = \begin{bmatrix} \chi & x^T & y^T \\ x & X & W^T \\ y & W & Y \end{bmatrix} \in \mathcal{S}_+^{1+n_1+n_2}, \\ S \in \mathcal{S}_+^{1+n_1}, s_1 \geq 0, s_2 \geq 0 \end{array} \middle| \begin{array}{l} \text{tr } U + s_1 = 1, S = \text{Arrow}(a_1\chi + a_2^T x + a_3^T y, x), \\ a^T U a - s_2 = \sum_{i=1}^{n_1} X_{ii} \end{array} \right\}$$

We use r_U, r_S, r_1, r_2 to denote the ranks of U, S, s_1, s_2. From Lemma 57, any extreme point of $\mathcal{L}''$ satisfies

$$r_U(r_U+1) + r_S(r_S+1) + r_1(r_1+1) + r_2(r_2+1) \leq 4 + (1+n_1)(2+n_1).$$

Notice that $r_S = 1 + n_1$, we have $r_U = 1$. Therefore, let $u = \frac{1}{\sqrt{\chi^0}} \begin{bmatrix} \chi^0 \\ x^0 \\ y^0 \end{bmatrix}$ and $U^0 = uu^T$. Since $\|x^0\| < a_1\chi^0 + a_2^T x^0 + a_3^T y^0$, we have $u \in \mathcal{H}_{\mathcal{F}}$, and hence $U^0 \in \mathcal{D}_{\mathcal{F}}^*$.

Combine (i), (ii), and (iii), and we have any extreme point of $\mathcal{K}'$ belongs to $\mathcal{D}_{\mathcal{F}}^*$. Therefore, $\mathcal{K} \subset \mathcal{D}_{\mathcal{F}}^*$ and hence $\mathcal{K} = \mathcal{D}_{\mathcal{F}}^*$.

Notice that

$$\mathcal{D}_{\mathcal{F}}^* = \mathcal{S}_+^{1+n_1+n_2} \cap \{U \in \mathcal{S}^{1+n_1+n_2} | C_2 U e_1 \in \mathcal{SOC}(n_1)\} \cap \{U \in \mathcal{S}^{1+n_1+n_2} | U \cdot C_1 \geq 0\}.$$

From Lemma 55, its dual

$$\begin{aligned} \mathcal{D}_{\mathcal{F}} &= \mathrm{cl}(\mathcal{S}_+^{1+n_1+n_2} + \{U \in \mathcal{S}^{1+n_1+n_2} | C_2 U e_1 \in \mathcal{SOC}(n_1)\}^* + \{U \in \mathcal{S}^{1+n_1+n_2} | U \cdot C_1 \geq 0\}^*) \\ &= \mathrm{cl}\left\{ M \in \mathcal{S}^{1+n_1+n_2} \;\middle|\; \begin{array}{c} M - \lambda C_1 - \left(e_1 \psi^T C_2^T + C_2 \psi e_1^T\right) \in \mathcal{S}_+^{1+n_1+n_2}, \\ C_2 = \begin{bmatrix} a_1 & 0 \\ a_2 & I_{n_1} \\ a_3 & 0 \end{bmatrix}, C_1 = aa^T - \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{n_1} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \lambda \geq 0, \psi \in \mathcal{SOC}(n_1) \end{array} \right\}. \end{aligned}$$

If there is $(\bar{x}, \bar{y}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ such that $\|\bar{x}\| < a_1 + a_2^T \bar{x} + a_3^T \bar{y}$, then let $\bar{u} = \begin{bmatrix} 1 \\ \bar{x} \\ \bar{y} \end{bmatrix}$ and $\bar{U} = \bar{u}\bar{u}^T$. We have $\bar{U} \in \mathcal{S}_+^{1+n_1+n_2}$, $C_2 \bar{U} e_1 \in \mathrm{int}\mathcal{SOC}(n_1)$, and $U \cdot C_1 > 0$. Let $U' = \bar{U} + \tau I_{1+n_1+n_2}$, and when $\tau > 0$ is sufficiently small, we have U' is positive definite, $C_2 U' e_1 \in \mathrm{int}\mathcal{SOC}(n_1)$, and $U' \cdot C_1 > 0$, i.e., $U' \in \mathcal{D}_{\mathcal{F}}^*$. Clearly, U' is an interior point of $\mathcal{D}_{\mathcal{F}}^*$. Therefore, from Lemma 55, the closure can be removed from $\mathcal{D}_{\mathcal{F}}$ and the rest of the lemma holds. □

Remark. Theorem 64 and Lemma 59 can be used to formulate more general $\mathcal{D}_{\mathcal{F}}^*$. However, the dimension of $\mathcal{D}_{\mathcal{F}}^*$ depends on how we construct $\mathcal{F}$.

Example 66 (Second order cone constraint)**.** *In literature, a widely used form of second order cone constraint is* $c^T x + d \geq \|Ax + b\|$*, where* $c \in \mathbb{R}^n$*,* $d \in \mathbb{R}$*,* $A \in \mathbb{R}^{m \times n}$ *and* $b \in \mathbb{R}^m$*.*

One intuitive construction of $\mathcal{F}$ *is* $\mathcal{F}_1 = \{(x, y_0, y) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m | y_0 \geq \|y\|, Ax + b =$

$y, c^T x + d = y_0\}$. *From Lemma 59 and Theorem 64, we have*

$$\mathcal{D}^*_{\mathcal{F}_1} = \left\{ U = \begin{bmatrix} \chi & w^T \\ w & W \end{bmatrix} \in \mathcal{S}^{2+m+n}_+ \;\middle|\; \begin{array}{l} w = \begin{bmatrix} x \\ y_0 \\ y \end{bmatrix}, W = \begin{bmatrix} X & W^T_{xy_0} & W^T_{xy} \\ W_{xy_0} & Y_0 & W^T_{y_0 y} \\ W_{xy} & W_{y_0 y} & Y \end{bmatrix} \\ y_0 \geq \|y\|, Y_0 \geq \operatorname{tr} Y, Bw = \begin{bmatrix} -b\chi \\ -d\chi \end{bmatrix}, \\ \operatorname{diag}\left(BWB^T\right) = \begin{bmatrix} \chi(b \circ b) \\ \chi d^2 \end{bmatrix} \end{array} \right\}$$

where $B = \begin{bmatrix} A & 0 & -I_m \\ c^T & -1 & 0 \end{bmatrix}$, *which is a subset in* $\mathcal{S}^{2+m+n}$.

Now define $C_1 = \begin{bmatrix} A & b \end{bmatrix}$ *and* $r_1 = \operatorname{rank}(C_1)$. *There exists* $C_2 \in \mathbb{R}^{r_1 \times (1+n)}$ *such that* $C_1^T C_1 = C_2^T C_2$. *If* $\operatorname{rank}\left(\begin{bmatrix} c^T & d \\ A & b \end{bmatrix}\right) = r_1$, *then there exists* $a \in \mathbb{R}^{r_1}$ *such that* $a^T C_2 = \begin{bmatrix} c^T & d \end{bmatrix}$. *We can choose* $P = \begin{bmatrix} C_2 \\ C_3 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$ *such that* P *is invertible. Define* $\begin{bmatrix} y \\ z \end{bmatrix} = P \begin{bmatrix} x \\ 1 \end{bmatrix}$, *and then* $\begin{bmatrix} x \\ 1 \end{bmatrix} = P^{-1} \begin{bmatrix} y \\ z \end{bmatrix}$, $\|y\|^2 = \begin{bmatrix} x \\ 1 \end{bmatrix}^T C_2^T C_2 \begin{bmatrix} x \\ 1 \end{bmatrix} = \|Ax - b\|^2$ *and* $c^T x - d = a^T y$. *Let* α^T *be the last row of* P^{-1}, *then we have* $\alpha^T \begin{bmatrix} y \\ z \end{bmatrix} = 1$. *Therefore, we can define*

$$\mathcal{F}_2 = \{(y, z) \in \mathbb{R}^{r_1} \times \mathbb{R}^{n+1-r_1} | \|y\| \leq a^T y, \alpha^T \begin{bmatrix} y \\ z \end{bmatrix} = 1\},$$

and after applying Theorem 64 and Lemma 59, we get $\mathcal{D}^*_{\mathcal{F}_2} \subset \mathcal{S}^{n+2}$. *In this case, when putting the second order cone constraint in an optimization problem, we need to change each variable* x *to* (y, z) *by the above equation and then solving the corresponding optimization problem. Furthermore, if we also have* $\operatorname{rank} A = r_1$, *then* $b = Aw$ *for some* $w \in \mathbb{R}^n$. *We can write* $Ax + b = A(x + w)$. *Let* $C_4 \in \mathbb{R}^{r_1 \times n}$ *and* $a' \in \mathbb{R}^{r_1}$ *satisfying* $C_4^T C_4 = A^T A$ *and* $(a')^T C_4(x + w) = c^T x + d$. *Then we could define* $y = C_4(x + w)$ *and* $z = C_5(x + w)$ *such that* $Q = \begin{bmatrix} C_4 \\ C_5 \end{bmatrix} \in \mathbb{R}^{n \times n}$ *is invertible. Therefore,* $x = Q^{-1} \begin{bmatrix} y \\ z \end{bmatrix} - w$, $\|y\|^2 = \|Ax + b\|^2$, $c^T x + d = (a')^T y$. *If define*

$$\mathcal{F}_3 = \{(y, z) \in \mathbb{R}^{r_1} \times \mathbb{R}^{n-r_1} \mid \|y\|^2 \leq (a')^T y\},$$

then $\mathcal{D}^*_{\mathcal{F}_3}$ *is a subset of* $\mathcal{S}^{n+1}$.

Similarly, if $\operatorname{rank}\begin{bmatrix} c^T & d \\ A & b \end{bmatrix} = r_1 + 1$, *after a linear transformation, we could still represent the cone of nonnegative quadratic function over a set satisfying* $\|Ax + b\| \le c^T x + d$ *in either* $\mathcal{S}^{n+2}$ *or* $\mathcal{S}^{n+1}$ *according to the rank of* $\begin{bmatrix} c^T \\ A \end{bmatrix}$.

Remark. The above example together with Theorem 64 reveals that as long as a second order cone constraint has an interior point solution, we can always find the optimal value of the problem which minimizes a nonconvex quadratic function over the second order cone constraint.

We can add a special linear constraint to the problem.

Theorem 67. *Given a nonempty set* $\mathcal{F} = \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} | \|x\| \le a_1 + a_2^T x + a_3^T y, a_1 + a_2^T x + a_3^T y \ge a_4 \ge 0\}$ *with* $a_1 \in \mathbb{R}$, $a_2 \in \mathbb{R}^{n_1}$, $a_3 \in \mathbb{R}^{n_2}$, *and* $a_4 \ge 0$. *Let*

$$a^T = [a_1 \ a_2^T \ a_3^T] \text{ and } b^T = a - a_4 e_1$$

$$C_1 = [0 \ I_{n_1} \ 0] \text{ and } e_1 = (1, 0, \ldots, 0)^T \in \mathbb{R}^{1+n_1+n_2}$$

and we have

$$\mathcal{D}^*_{\mathcal{F}} = \left\{ U = \begin{bmatrix} \chi & x^T & y^T \\ x & X & W^T \\ y & W & Y \end{bmatrix} \in \mathcal{S}_+^{1+n_1+n_2} \,\middle|\, \begin{array}{l} \|C_1 U e_1\| \le a^T U e_1, U \cdot (aa^T - C_1^T C_1) \ge 0, \\ b^T U e_1 \ge 0, a^T U b \ge \|C_1 U b\| \end{array} \right\}$$

and

$$\mathcal{D}_{\mathcal{F}} = \operatorname{cl} \left\{ M \in \mathcal{S}^{1+n_1+n_2} \,\middle|\, \begin{array}{l} M - \lambda_1 C_3 - \lambda_2 (e_1 b^T + b e_1^T) - \left(e_1 \psi_1^T C_2^T + C_2 \psi_1 e_1^T\right) \\ - \left(b \psi_2^T C_2^T + C_2 \psi_2 b^T\right) \in \mathcal{S}_+^{1+n_1+n_2}, \\ C_2 = [a \ C_1^T], C_3 = aa^T - C_1^T C_1, \lambda_1, \lambda_2 \ge 0, \psi_1, \psi_2 \in \mathcal{SOC}(n_1) \end{array} \right\}.$$

If there is $(\bar{x}, \bar{y}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ *such that* $\|\bar{x}\| < a_1 + a_2^T \bar{x} + a_3^T \bar{y}$ *and* $a_1 + a_2^T \bar{x} + a_3^T \bar{y} > a_4$, *then the closure in the above equation can be removed.*

Consequently, the corresponding P0 and CP5 have the same optimal value and if there is $(\bar{x}, \bar{y}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ *such that* $\|\bar{x}\| < a_1 + a_2^T \bar{x} + a_3^T \bar{y}$ *and* $a_1 + a_2^T \bar{x} + a_3^T \bar{y} > a_4$, *then the corresponding CD5 without closure has the same optimal value and is attainable.*

Proof. Define

$$\mathcal{K} = \left\{ U = \begin{bmatrix} \chi & x^T & y^T \\ x & X & W^T \\ y & W & Y \end{bmatrix} \in \mathcal{S}_+^{1+n_1+n_2} \,\middle|\, \begin{array}{l} a^T U e_1 \ge \|C_1 U e_1\|, U \cdot (aa^T - C_1^T C_1) \ge 0, \\ b^T U e_1 \ge 0, a^T U b \ge \|C_1 U b\| \end{array} \right\}$$

The one side $\mathcal{D}_{\mathcal{F}}^* \subset \mathcal{K}$ is obvious.

To see the other side, it is sufficient to prove that all the extreme points of $\mathcal{K}'$ with $\mathcal{K}' = \mathcal{K} \cap \{U \in \mathcal{S}^{1+n_1+n_2} | \text{tr } U \leq 1\}$ belong to $\mathcal{D}_{\mathcal{F}}^*$. In other word, for each extreme point of $\mathcal{K}'$, we can find a rank one decomposition such that all elements are in $\mathcal{H}_{\mathcal{F}}$.

We will first prove that

$$\mathcal{H}_{\mathcal{F}} \supset \{(t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} | \|x\| \leq a_1 t + a_2^T x + a_3^T y, a_1 t + a_2^T x + a_3^T y \geq a_4, t \geq 0\}.$$

If $t > 0$, then $[1 \ x^T/t \ y^T/t]^T \in \mathcal{F}$ and, therefore, $[t \ x^T \ y^T]^T \in \mathcal{H}_{\mathcal{F}}$. If $t = 0$, since $\mathcal{F}$ is not empty, there exists $\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \in \mathcal{F}$. One can verify that $\begin{bmatrix} 0 \\ x \\ y \end{bmatrix} + \frac{1}{k} \begin{bmatrix} 1 \\ \bar{x} \\ \bar{y} \end{bmatrix} \in \mathcal{H}_{\mathcal{F}}$ and therefore, when k goes to infinity, its limit $\begin{bmatrix} 0 \\ x \\ y \end{bmatrix} \in \mathcal{H}_{\mathcal{F}}$. Hence $\mathcal{H}_{\mathcal{F}} \supset \{(t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} | \|x\| \leq a_1 t + a_2^T x + a_3^T y, a_1 t + a_2^T x + a_3^T y \geq a_4, t \geq 0\}$.

Next, we need to consider five cases: (i) $\chi = 0$; (ii) $\chi > 0$, $a^T U e_1 = \|C_1 U e_1\|$ and $a^T U b = \|CUb\|$; (iii) $\chi > 0$, $a^T U e_1 > \|C_1 U e_1\|$ and $a^T U b = \|CUb\|$; (iv) $\chi > 0$, $a^T U e_1 = \|C_1 U e_1\|$ and $a^T U b > \|CUb\|$; (v) $\chi > 0$, $a^T U e_1 > \|C_1 U e_1\|$ and $a^T U b > \|CUb\|$.

If (i) holds, then for any U^0 being a nonzero extreme point of $\mathcal{K}'$, we have $(x^0, y^0) = 0$. Furthermore, since U^0 is an extreme point of $\mathcal{K}'$, the matrix $Z^0 = \begin{bmatrix} X^0 & (W^0)^T \\ W^0 & Y^0 \end{bmatrix}$ must be the extreme point of

$$\mathcal{L} = \left\{ Z = \begin{bmatrix} X & W^T \\ W & Y \end{bmatrix} \in \mathcal{S}_+^{n_1+n_2} \;\middle|\; \begin{array}{l} \text{tr } Z \leq 1, \begin{bmatrix} a_2 \\ a_3 \end{bmatrix}^T Z \begin{bmatrix} a_2 \\ a_3 \end{bmatrix} \geq \sum_{i=1}^{n_1} X_{ii}, \\ \begin{bmatrix} a_2 \\ a_3 \end{bmatrix}^T Z \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} \geq \|X b_2 + W^T b_3\| \end{array} \right\}$$

If $Z^0 \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} = 0$, then from [107], we can always get a rank one decomposition $Z^0 = \sum_i z^i (z^i)^T$, satisfying

$$\begin{bmatrix} a_2 \\ a_3 \end{bmatrix}^T z^i (z^i)^T \begin{bmatrix} a_2 \\ a_3 \end{bmatrix} \geq \sum_{j=1}^{n_1} (z_j^i)^2.$$

Since Z^0 is positive semidefinite and $\begin{bmatrix} b_2 \\ b_3 \end{bmatrix} Z^0 \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} = 0$ we have $(z^i)^T \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} = 0$ for all i. One can verify that $Z^0 = \sum_i \frac{(z^i)^T z^i}{\text{tr } Z^0} (\frac{\text{tr } Z^0}{(z^i)^T z^i} (z^i (z^i)^T)$ and $\frac{\text{tr } Z^0}{(z^i)^T z^i} (z^i (z^i)^T)$ are all in $\mathcal{L}$. From the fact that Z^0 is an extreme point of $\mathcal{L}$, then $Z^0 = \frac{\text{tr } Z^0}{(z^i)^T z^i}(z^i(z^i)^T)$ for all i, i.e. the rank of Z^0 is 1. Let $Z^0 = z^0(z^0)^T$, then $U^0 = \begin{bmatrix} 0 \\ z^0 \end{bmatrix} \begin{bmatrix} 0 \\ z^0 \end{bmatrix}^T$ and $0 = a^T \begin{bmatrix} 0 \\ z^0 \end{bmatrix} = b^T \begin{bmatrix} 0 \\ z^0 \end{bmatrix}$ and

$$0 = \begin{bmatrix} a_2 \\ a_3 \end{bmatrix}^T z^0(z^0)^T \begin{bmatrix} a_2 \\ a_3 \end{bmatrix} \geq \sum_{j=1}^{n_1} (z_j^0)^2 = 0.$$

Therefore, $\begin{bmatrix} 0 \\ z^0 \end{bmatrix} \in \mathcal{H}_{\mathcal{F}}$, i.e., $U^0 \in \mathcal{D}_{\mathcal{F}}^*$.

If $\begin{bmatrix} a_2 \\ a_3 \end{bmatrix}^T Z^0 \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} = \|X^0 b_2 + (W^0)^T b_3\| > 0$, let $z = Z^0 \begin{bmatrix} b_2 \\ b_3 \end{bmatrix}$. By noticing that

$$Z^0 - \lambda z z^T = (Z^0)^{\frac{1}{2}} \left[I - \lambda (Z^0)^{\frac{1}{2}} \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} \begin{bmatrix} b_2 \\ b_3 \end{bmatrix}^T (Z^0)^{\frac{1}{2}} \right] (Z^0)^{\frac{1}{2}}$$

we have $V = Z^0 - \lambda z z^T$ is positive semidefinite for some $\lambda > 0$. We can rewrite the above equation as

$$Z^0 = \frac{\text{tr } V}{\text{tr } Z^0}\left(\frac{\text{tr } Z^0}{\text{tr } V} V\right) + \frac{\lambda z^T z}{\text{tr } Z^0}\left(\frac{\text{tr } Z^0}{z^T z} z z^T\right).$$

Let $Z^1 = \begin{bmatrix} X^1 & (W^1)^T \\ W^1 & Y^1 \end{bmatrix} = \frac{\text{tr } Z^0}{\text{tr } V} V$ and $Z^2 = \begin{bmatrix} X^2 & (W^2)^T \\ W^2 & Y^2 \end{bmatrix} = \frac{\text{tr } Z^0}{z^T z} z z^T$. Then $\text{tr } Z^1 = \text{tr } Z^2 = \text{tr } Z^0 \leq 1$.

$$\begin{bmatrix} a_2 \\ a_3 \end{bmatrix}^T Z^2 \begin{bmatrix} a_2 \\ a_3 \end{bmatrix} = \frac{\text{tr } Z^0}{z^T z} \left(\begin{bmatrix} a_2 \\ a_3 \end{bmatrix}^T Z^0 \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} \right)^2 = \frac{\text{tr } Z^0}{z^T z} \|X^0 b_2 + (W^0)^T b_3\|^2 = \frac{\text{tr } Z^0}{z^T z} \sum_{i=1}^{n_1} z_i^2 = \sum_{i=1}^{n_1} [Z^2]_{ii}.$$

$$\begin{bmatrix} a_2 \\ a_3 \end{bmatrix}^T Z^1 \begin{bmatrix} a_2 \\ a_3 \end{bmatrix} = \frac{\text{tr } Z^0}{\text{tr } V} \begin{bmatrix} a_2 \\ a_3 \end{bmatrix}^T (Z^0 - \lambda z z^T) \begin{bmatrix} a_2 \\ a_3 \end{bmatrix} \geq \frac{\text{tr } Z^0}{\text{tr } V} \sum_{i=1}^{n_1} (X_{ii}^0 - \lambda z_i^2) = \frac{\text{tr } Z^0}{\text{tr } V} \sum_{i=1}^{n_1} V_{ii} = \sum_{i=1}^{n_1} Z_{ii}^1.$$

$$\begin{bmatrix} a_2 \\ a_3 \end{bmatrix}^T Z^2 \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} = \frac{\text{tr } Z^0}{z^T z} \begin{bmatrix} a_2 \\ a_3 \end{bmatrix}^T Z^0 \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} \begin{bmatrix} b_2 \\ b_3 \end{bmatrix}^T Z^0 \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} = \frac{\text{tr } Z^0}{z^T z} \|X^0 b_2 + (W^0)^T b_3\| (z^T \begin{bmatrix} b_2 \\ b_3 \end{bmatrix})$$
$$= \|X^2 b_2 + (W^2)^T b_3\|.$$

$$\begin{bmatrix} a_2 \\ a_3 \end{bmatrix}^T Z^1 \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} = \frac{\text{tr } Z^0}{\text{tr } V} \begin{bmatrix} a_2 \\ a_3 \end{bmatrix}^T (Z^0 - \lambda z z^T) \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} = \frac{\text{tr } Z^0}{\text{tr } V} \|X^0 b_2 + (W^0)^T b_3\| (1 - \lambda \begin{bmatrix} b_2 \\ b_3 \end{bmatrix}^T Z^0 \begin{bmatrix} b_2 \\ b_3 \end{bmatrix})$$
$$= \|X^1 b_2 + (W^1)^T b_3\|.$$

Therefore, Z^1 and Z^2 are all in $\mathcal{L}$. Since Z^0 is an extreme point of $\mathcal{L}$, then $Z^0 = Z^1 = Z^2$, i.e.,

$Z^0 = \frac{\text{tr } Z^0}{z^T z} z z^T$. Let $u^0 = \begin{bmatrix} 0 \\ u_x^0 \\ u_y^0 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{\frac{\text{tr } Z^0}{z^T z}} z \end{bmatrix}$, and then $U^0 = u^0 (u^0)^T$. Notice that

$$b^T u^0 = a^T u^0 = a_2^T u_x^0 + a_3^T u_y^0 = \sqrt{\frac{\text{tr } Z^0}{z^T z}} \begin{bmatrix} a_2 \\ a_3 \end{bmatrix}^T Z^0 \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} = \sqrt{\frac{\text{tr } Z^0}{z^T z}} \| X^0 b_2 + (W^0)^T b_3 \| = \|u_x^0\|$$

and then we have $u^0 \in \mathcal{H}_{\mathcal{F}}$ and $U^0 \in \mathcal{D}_{\mathcal{F}}^*$.

If $\begin{bmatrix} a_2 \\ a_3 \end{bmatrix}^T Z^0 \begin{bmatrix} b_2 \\ b_3 \end{bmatrix} > \|X^0 b_2 + (W^0)^T b_3\|$, then (Z^0, S^0, s_1^0, s_2^0) with

$$S^0 = \text{Arrow}(\begin{bmatrix} a_2 \\ a_3 \end{bmatrix}^T Z^0 \begin{bmatrix} b_2 \\ b_3 \end{bmatrix}, X^0 b_2 + (W^0)^T b_3), s_1^0 = 1 - \text{tr } Z^0, \text{ and } \begin{bmatrix} a_2 \\ a_3 \end{bmatrix}^T Z^0 \begin{bmatrix} a_2 \\ a_3 \end{bmatrix} - s_2^0 = \sum_{i=1}^{n_1} X_{ii}^0,$$

is an extreme point of

$$\mathcal{L}' = \left\{ \begin{array}{c|c} Z = \begin{bmatrix} X & W^T \\ W & Y \end{bmatrix} \in \mathcal{S}_+^{n_1+n_2} & \text{tr } Z + s_1 = 1, \begin{bmatrix} a_2 \\ a_3 \end{bmatrix}^T Z \begin{bmatrix} a_2 \\ a_3 \end{bmatrix} - s_2 = \sum_{i=1}^{n_1} X_{ii}, \\ S \in \mathcal{S}_+^{1+n_1}, s_1 \geq 0, s_2 \geq 0 & S = \text{Arrow}(\begin{bmatrix} a_2 \\ a_3 \end{bmatrix}^T Z \begin{bmatrix} b_2 \\ b_3 \end{bmatrix}, X b_2 + W^T b_3) \end{array} \right\}$$

From Lemma 57, let $r_Z = \text{rank}(Z^0), r_S = \text{rank}(S^0), r_1 = \text{rank}(s_1^0), r_2 = \text{rank}(s_2^0)$, and then

$$r_Z(r_Z+1) + r_S(r_S+1) + r_1(r_1+1) + r_2(r_2+1) \leq 4 + (n_1+1)(n_1+2).$$

Since $r_S = n_1 + 1$ by Lemma 65, we have $r_Z = 1$ and $Z^0 = \begin{bmatrix} x' \\ y' \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}^T$ with $a_2^T x' + a_3^T y' \geq \|x'\|$. In this case $U^0 = \begin{bmatrix} 0 \\ x' \\ y' \end{bmatrix} \begin{bmatrix} 0 \\ x' \\ y' \end{bmatrix}^T$. Notice that $b^T \begin{bmatrix} 0 \\ x' \\ y' \end{bmatrix} = a^T \begin{bmatrix} 0 \\ x' \\ y' \end{bmatrix} \geq \|x'\|$, then $\begin{bmatrix} 0 \\ x' \\ y' \end{bmatrix} \in \mathcal{H}_{\mathcal{F}}$ and therefore, $U^0 \in \mathcal{D}_{\mathcal{F}}^*$.

After all, in case (i), $U^0 \in \mathcal{D}_{\mathcal{F}}^*$.

If (ii) holds, suppose U^0 is an extreme point of $\mathcal{K}'$. Since $\chi > 0$, we have $U^0 e_1 \neq 0$. Define $U^1 = \frac{U^0 e_1 e_1^T U^0}{e_1^T U^0 e_1}$ and $U^2 = U^0 - U^1$. From Lemma 58 we have U^2 is positive semidefinite.

If $U^2 a = 0$, then $0 = U^2 a = U^0 a - \frac{e_1^T U^0 a}{e_1^T U^0 e_1} U^0 e_1$, i.e. $U^0 a$ and $U^0 e_1$ are linearly dependent.

Therefore, $U^0 b = U^0 a - a_4 U^0 e_1$ is also linearly dependent to $U^0 e_1$. Rewrite

$$U^0 = \frac{\text{tr } U^1}{\text{tr } U^0}\left(\frac{\text{tr } U^0}{\text{tr } U^1}U^1\right) + \frac{\text{tr } U^2}{\text{tr } U^0}\left(\frac{\text{tr } U^0}{\text{tr } U^2}U^2\right)$$

and one can verify $\frac{\text{tr } U^0}{\text{tr } U^1}U^1 \in \mathcal{K}'$. To see $\frac{\text{tr } U^0}{\text{tr } U^2}U^2$ is also in $\mathcal{K}'$, we just need to show U^2 is in $\mathcal{K}$. From $U^2 a = U^2 e_1 = 0$, we get $a^T U^2 e_1 \geq \|C_1 U^2 e_1\|$ and $b^T U^2 e_1 \geq 0$.

$$U^2 \cdot (aa^T - C_1^T C_1) = (U^0 - U^1) \cdot (aa^T - C_1^T C_1) = U^0 \cdot (aa^T - C_1^T C_1) \geq 0$$

From $U^2 a = 0$, we have tr $(C_1 U^2 C_1^T) = 0$ and therefore, $C_1 U^2 = 0$, $a^T U^2 b = 0$, $C_1 U^2 b = 0$. We have $U^2 \in \mathcal{K}$ and $\frac{\text{tr } U^0}{\text{tr } U^2}U^2 \in \mathcal{K}'$. Since U^0 is an extreme point in $\mathcal{K}'$, then either $U^2 = 0$ or $U^0 = \frac{\text{tr } U^0}{\text{tr } U^1}U^1 = \frac{\text{tr } U^0}{\text{tr } U^2}U^2$. Since $U^1 e_1 = U^0 e_1 \neq 0 = U^2 e_1$, we must have $U^2 = 0$ and then $U^0 = U^1 = \frac{U^0 e_1 e_1^T U^0}{e_1^T U^0 e_1}$. From $U^0 \in \mathcal{K}$, one can verify that $\frac{U^0 e_1}{\sqrt{e_1^T U^0 e_1}}$ is in $\mathcal{H}_{\mathcal{F}}$ and therefore, $U^0 \in \mathcal{D}^*_{\mathcal{F}}$.

If $U^2 a \neq 0$, then $a^T U^2 a \neq 0$. Let $U^3 = \frac{U^2 a a^T U^2}{a^T U^2 a}$ and $U^4 = U^2 - U^3 = U^0 - U^1 - U^3$. From Lemma 58, we have U^3 and U^4 are all positive semidefinite.

From tr $(C_1 U^4 C_1^T) \geq 0$, we know

$$\left(a^T U^0 a - \tfrac{(e_1^T U^0 a)^2}{e_1^T U^0 e_1}\right) \text{tr } (C_1 U^0 C_1^T)$$

$$\geq \left(a^T U^0 a - \tfrac{(e_1^T U^0 a)^2}{e_1^T U^0 e_1}\right) \text{tr } (C_1 (U^1 - U^3) C_1^T)$$

$$= \left(a^T U^0 a - \tfrac{(e_1^T U^0 a)^2}{e_1^T U^0 e_1}\right) \tfrac{\|C_1 U^0 e_1\|^2}{e_1 U^0 e_1} + \|C_1 U^0 a - \tfrac{e_1^T U^0 a}{e_1^T U^0 e_1} C_1 U^0 e_1\|^2$$

$$= \tfrac{a^T U^0 a \|C_1 U^0 e_1\|^2}{e_1^T U^0 e_1} + \|C_1 U^0 a\|^2 - \tfrac{2 e_1^T U^0 a}{e_1^T U^0 e_1} e_1^T U^0 C_1^T C_1 U^0 a$$

$$= \left(a^T U^0 a - \tfrac{(e_1^T U^0 a)^2}{e_1^T U^0 e_1}\right) a^T U^0 a$$

$$+ \left[\tfrac{a^T U^0 a \|C_1 U^0 e_1\|^2}{e_1^T U^0 e_1} + \|C_1 U^0 a\|^2 - \tfrac{2 e_1^T U^0 a}{e_1^T U^0 e_1} e_1^T U^0 C_1^T C_1 U^0 a - (a^T U^0 a)^2 + \tfrac{(a^T U^0 a)(e_1^T U^0 a)^2}{e_1^T U^0 e_1}\right]$$

$$= \left(a^T U^0 a - \tfrac{(e_1^T U^0 a)^2}{e_1^T U^0 e_1}\right) a^T U^0 a + 2 \tfrac{(a^T U^0 a)(e_1^T U^0 a)^2}{e_1^T U^0 e_1} - \tfrac{2 e_1^T U^0 a}{e_1^T U^0 e_1} e_1^T U^0 C_1^T C_1 U^0 a$$

$$+ \|C_1 U^0 b + a_4 C_1 U^0 e_1\|^2 - (a^T U^0 b + a_4 a^T U^0 e_1)^2$$

$$= \left(a^T U^0 a - \tfrac{(e_1^T U^0 a)^2}{e_1^T U^0 e_1}\right) a^T U^0 a + 2 \tfrac{(a^T U^0 a)(e_1^T U^0 a)^2}{e_1^T U^0 e_1} - \tfrac{2 e_1^T U^0 a}{e_1^T U^0 e_1} e_1^T U^0 C_1^T C_1 U^0 a$$

$$+ 2 a_4 e_1^T U^0 C_1^T C_1 U^0 (a - a_4 e_1) - 2 a_4 (a^T U^0 e_1) a^T U^0 (a - a_4 e_1)$$

$$= \left(a^T U^0 a - \tfrac{(e_1^T U^0 a)^2}{e_1^T U^0 e_1}\right) a^T U^0 a + 2 \tfrac{(a^T U^0 a)(e_1^T U^0 a) e_1^T U^0 (a - a_4 e_1)}{e_1^T U^0 e_1} - \tfrac{2 e_1^T U^0 (a - a_4 e_1)}{e_1^T U^0 e_1} e_1^T U^0 C_1^T C_1 U^0 a$$

$$= \left(a^T U^0 a - \tfrac{(e_1^T U^0 a)^2}{e_1^T U^0 e_1}\right) a^T U^0 a + 2 \tfrac{e_1^T U^0 b}{e_1^T U^0 e_1} \left((a^T U^0 a)(e_1^T U^0 a) - e_1^T U^0 C_1^T C_1 U^0 a\right)$$

$$\geq \left(a^T U^0 a - \tfrac{(e_1^T U^0 a)^2}{e_1^T U^0 e_1}\right) a^T U^0 a$$

From the above inequality, we have tr $(C_1 U^0 C_1^T) = a^T U^0 a$ if and only if tr $(C_1 U^4 C_1^T) = 0$ and $\begin{bmatrix} a^T U^0 e_1 \\ C_1 U^0 e_1 \end{bmatrix}$ and $\begin{bmatrix} a^T U^0 a \\ C_1 U^0 a \end{bmatrix}$ are linearly dependent. Therefore, $\begin{bmatrix} a^T U^0 e_1 \\ C_1 U^0 e_1 \end{bmatrix}$ and $\begin{bmatrix} a^T U^0 b \\ C_1 U^0 b \end{bmatrix}$ are linearly dependent. Noticing

$$a^T U^2 b = a^T U^2 a - a_4 a^T U^2 e_1 = a^T U^2 a > 0$$

and $\begin{bmatrix} a^T U^2 b \\ C_1 U^2 b \end{bmatrix} = \begin{bmatrix} a^T U^0 b \\ C_1 U^0 b \end{bmatrix} - \frac{e_1^T U^0 b}{e_1^T U^0 e_1} \begin{bmatrix} a^T U^0 e_1 \\ C_1 U^0 e_1 \end{bmatrix}$, we have $a^T U^2 b = \|C_1 U^2 b\|$. Then we can easily verify that U^1 and U^2 are all in $\mathcal{K}$ and from

$$U^0 = \frac{\text{tr } U^1}{\text{tr } U^0}(\frac{\text{tr } U^0}{\text{tr } U^1}U^1) + \frac{\text{tr } U^2}{\text{tr } U^0}(\frac{\text{tr } U^0}{\text{tr } U^2}U^2)$$

$\frac{\text{tr } U^0}{\text{tr } U^1}U^1$ and $\frac{\text{tr } U^0}{\text{tr } U^2}U^2$ are all in $\mathcal{K}'$. Therefore, $U^0 = U^1 = \frac{U^0 e_1 e_1^T U^0}{e_1^T U^0 e_1}$. From $U^0 \in \mathcal{K}$, one can verify that $\frac{U^0 e_1}{\sqrt{e_1^T U^0 e_1}}$ is in $\mathcal{H}_{\mathcal{F}}$ and hence $U^0 \in \mathcal{D}^*_{\mathcal{F}}$.

After all, in case (ii), $U^0 \in \mathcal{D}^*_{\mathcal{F}}$.

If (iii) holds, let $U^1 = \lambda U^0 b b^T U^0$ and $U^2 = U^0 - U^1$ with $\lambda > 0$ being a sufficiently small number. One can easily check that $U^1 \in \mathcal{K}$. Notice that when λ is small enough, U^2 is positive semidefinite. From $a^T U^0 e_1 > \|C_1 U^0 e_1\|$, we know $\begin{bmatrix} a^T U^0 e_1 \\ C_1 U^0 e_1 \end{bmatrix}$ is an interior point of $\mathcal{SOC}(n_1)$. Therefore $\begin{bmatrix} a^T U^2 e_1 \\ C_1 U^2 e_1 \end{bmatrix} = \begin{bmatrix} a^T U^0 e_1 \\ C_1 U^0 e_1 \end{bmatrix} - \lambda(b^T U^0 e_1) \begin{bmatrix} a^T U^0 b \\ C_1 U^0 b \end{bmatrix}$ is also in $\mathcal{SOC}(n_1)$ when λ is small enough, i.e., $a^T U^2 e_1 \geq \|C_1 U^2 e_1\|$. We can also see that

$$\begin{aligned}
&U^2 \cdot (aa^T - C_1^T C_1) = (U^0 - U^1) \cdot (aa^T - C_1^T C_1) = U^0 \cdot (aa^T - C_1^T C_1) \geq 0 \\
&b^T U^2 e_1 = b^T U^0 e_1 - \lambda(b^T U^0 b) b^T U^0 e_1 \geq 0 \\
&a^T U^2 b = (1 - \lambda(b^T U^0 b)) a^T U^0 b \geq \|(1 - \lambda(b^T U^0 b)) C_1 U^0 b\| = \|C_1 U^2 b\|
\end{aligned}$$

Therefore U^2 is also in $\mathcal{K}$. Since U^0 is an extreme point in $\mathcal{K}'$, we have $U^0 = \frac{\text{tr } U^0}{\text{tr } U^1}U^1 = \frac{\text{tr } U^0}{\text{tr } U^2}U^2$. However, $a^T U^0 e_1 = \frac{\text{tr } U^0}{\text{tr } U^1} a^T U^1 e_1 = \frac{\text{tr } U^0}{\text{tr } U^1}\lambda(b^T U^0 e_1)(a^T U^0 b) = \frac{\text{tr } U^0}{\text{tr } U^1}\lambda(b^T U^0 e_1)\|C_1 U^0 b\| = \frac{\text{tr } U^0}{\text{tr } U^1}\|C_1 U^1 e_1\| = \|C_1 U^0 e_1\|$ which contradicts to $a^T U^0 e_1 > \|C_1 U^0 e_1\|$. This means any extreme point of $\mathcal{K}'$ does not satisfy (iii).

The case (iv) is similar to (iii).

If (v) holds and U^0 is an extreme point of $\mathcal{K}'$, then $(U^0, S_1^0, S_2^0, s_1^0, s_2^0, s_3^0)$ with

$$\begin{aligned}
&S_1^0 = \text{Arrow}(a^T U^0 e_1, C_1 U^0 e_1), S_2^0 = \text{Arrow}(a^T U^0 b, C_1 U^0 b) \\
&s_1^0 = U^0 \cdot (aa^T - C_1^T C_1), s_2^0 = b^T U^0 e_1, s_3^0 = 1 - \text{tr } U^0
\end{aligned}$$

is an extreme point of

$$\mathcal{L}'' = \left\{ \begin{array}{l|l} U = \begin{bmatrix} \chi & x^T & y^T \\ x & X & W^T \\ y & W & Y \end{bmatrix} \in \mathcal{S}_+^{1+n_1+n_2} & \begin{array}{l} S_1 = \text{Arrow}(a^T U e_1, C_1 U e_1), \\ S_2 = \text{Arrow}(a^T U b, C_1 U b), \\ s_1 = U \cdot (aa^T - C_1^T C_1), \end{array} \\ S_1, S_2 \in \mathcal{S}_+^{1+n_1}, s_1, s_2, s_3 \geq 0 & s_2 = b^T U e_1, s_3 = 1 - \text{tr } U \end{array} \right\}.$$

From Lemma 57, let $r_U = \text{rank}(U^0), r_{S_1} = \text{rank}(S_1^0), r_{S_2} = \text{rank}(S_2^0), r_1 = \text{rank}(s_1^0), r_2 =$

$\text{rank}(s_2^0), r_3 = \text{rank}(s_3^0)$, and we have

$$r_U(r_U+1)+r_{S_1}(r_{S_1}+1)+r_{S_2}(r_{S_2}+1)+r_1(r_1+1)+r_2(r_2+1)+r_3(r_3+1) \leq 2(n_1+1)(n_1+2)+6.$$

According to the assumption in (v), we have $a^TU^0e_1 > \|C_1U^0e_1\|$ and $a^TU^0b > \|C_1U^0b\|$. From Lemma 65, $r_{S_1} = r_{S_2} = 1 + n_1$. Then the above inequality becomes

$$r_U(r_U + 1) + r_1(r_1 + 1) + r_2(r_2 + 1) + r_3(r_3 + 1) \leq 6.$$

If $r_U = 1$, then one can easily verify that $U^0 \in \mathcal{D}_{\mathcal{F}}^*$. If $r_U = 2$, we will show that U^0 cannot be an extreme point of $\mathcal{K}'$. In this situation, $r_1 = r_2 = r_3 = 0$, i.e., $s_1^0 = s_2^0 = s_3^0 = 0$. From $a^TU^0e_1 > 0$ and $a^TU^0b > 0$, we have $U^0e_1 \neq 0$ and $U^0b \neq 0$. Define $U^1 = \frac{U^0e_1e_1^TU^0}{e_1^TU^0e_1}$ and $U^2 = U^0 - U^1$. From Lemma 58, U^2 is positive semidefinite and $\text{rank}(U^2) = 1$. Since $s_2^0 = b^TU^0e_1 = 0$, then $U^2b = U^0b \neq 0$. Therefore, $U^2 = \frac{U^2bb^TU^2}{b^TU^2b} = \frac{U^0bb^TU^0}{b^TU^0b}$. This means $U^0 = \frac{U^0e_1e_1^TU^0}{e_1^TU^0e_1} + \frac{U^0bb^TU^0}{b^TU^0b}$. Notice that U^0b and U^0e_1 are linearly independent. (Otherwise, $0 \neq b^TU^0b = \tau b^TU^0e_1 = 0$ for some $\tau \neq 0$, which is a contradiction.) One can further verify that $\frac{\text{tr } U^0}{\text{tr } U^1}U^1$ and $\frac{\text{tr } U^0}{\text{tr } U^2}U^2$ are all in $\mathcal{K}'$ and U^0 is the convex combination of these two different points which means U^0 cannot be an extreme point of $\mathcal{K}'$.

From the discussion above, we have $\mathcal{K} \subset \mathcal{D}_{\mathcal{F}}^*$ and hence $\mathcal{K} = \mathcal{D}_{\mathcal{F}}^*$.

Similar to the discussion of Lemma 64, we have the expression of $\mathcal{D}_{\mathcal{F}}$ and when $(\bar{x}, \bar{y})$ is a strict feasible point of $\mathcal{F}$, the closure can be removed and the corresponding CD5 without closure has the same optimal value as P0 and CP5 and is attainable. □

6.3.4 One quadratic inequality and two linear inequalities

In this subsection, we will study the one quadratic inequality and two linear inequalities cases. We first introduce the result for one ball constraint and two parallel linear constraints. Then we study a second order cone constraint with two special linear constraints.

The one ball constraint with two parallel linear constraints case is studied in [113] and [35]. In [113], the authors point out that this problem can be solved efficiently in practice. However, they do not prove the complexity. While in [35], Burer provides an exact formulation of $\mathcal{D}_{\mathcal{F}}^*$ of this problem with linear, second order cone, and semidefinite constraints and therefore proves that, theoretically, this problem can also be solved efficiently.

Theorem 68 ([35]). *When $\mathcal{F} = \{x \in \mathbb{R}^n | \|x\| \leq 1, -1 < l \leq x_1 \leq u < 1\}$ with $l < u$, we have*

$$\mathcal{D}_{\mathcal{F}}^* = \left\{ Y = \begin{bmatrix} \chi & x^T \\ x & X \end{bmatrix} \in \mathcal{S}_+^{n+1} \;\middle|\; \begin{array}{l} X_{11} + lu\chi \leq (l+u)x_1, \|X_{\cdot 1} - lx\| \leq x_1 - l\chi \\ \|ux - X_{\cdot 1}\| \leq u\chi - x_1, \text{tr } X \leq \chi \end{array} \right\}$$

and

$$\mathcal{D}_{\mathcal{F}} = \left\{ M \in \mathcal{S}^{1+n} \;\middle|\; \begin{array}{l} M - \lambda_1 \begin{bmatrix} 1 & \\ & -I_n \end{bmatrix} - \lambda_2(ab^T + ba^T) - (a\psi_1^T + \psi_1 a^T) - (b\psi_2^T + \psi_2^T b) \in \mathcal{S}_+^{1+n} \\ \lambda_1, \lambda_2 \geq 0, \psi_1, \psi_2 \in \mathcal{SOC}(n) \end{array} \right\}$$

with $a = (-l, 1, 0, \ldots, 0)^T \in \mathbb{R}^{n+1}$ *and* $b = (u, -1, 0, \ldots, 0)^T \in \mathbb{R}^{1+n}$. *The corresponding P0, CP5, CD5 have the same optimal value and CD5 is attainable.*

The above results can be generalized to ellipsoid with two parallel linear constraints with general form by proper linear transformation. The next result for second order cone is more restrictive, which requires the two linear constraints parallel to the linear constraint defining the second order cone.

Theorem 69. *Given a nonempty set* $\mathcal{F} = \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} | \|x\| \leq a_1 + a_2^T x + a_3^T y, a_5 \geq a_1 + a_2^T x + a_3^T y \geq a_4 \geq 0\}$ *with* $a_1 \in \mathbb{R}$, $a_2 \in \mathbb{R}^{n_1}$, $a_3 \in \mathbb{R}^{n_2}$, *and* $a_4, a_5 \geq 0$. *Let*

$$a^T = [a_1 \ a_2^T \ a_3^T], \ b = a - a_4 e_1, \ \bar{b} = a_5 e_1 - a,$$

$$C_1 = [0 \ I_{n_1} \ 0] \text{ and } e_1 = (1, 0, \ldots, 0)^T \in \mathbb{R}^{1+n_1+n_2}$$

and we have

$$\mathcal{D}_{\mathcal{F}}^* = \left\{ U = \begin{bmatrix} \chi & x^T & y^T \\ x & X & W^T \\ y & W & Y \end{bmatrix} \in \mathcal{S}_+^{1+n_1+n_2} \;\middle|\; \begin{array}{l} a^T U b \geq \|C_1 U b\|, e_1^T U b \geq 0, \\ a^T U \bar{b} \geq \|C_1 U \bar{b}\|, e_1^T U \bar{b} \geq 0, \\ b^T U \bar{b} \geq 0, U \cdot (aa^T - C_1^T C_1) \geq 0 \end{array} \right\}$$

and

$$\mathcal{D}_{\mathcal{F}} = \text{cl} \left\{ M \in \mathcal{S}^{1+n_1+n_2} \;\middle|\; \begin{array}{l} M - \lambda_1 C_3 - \lambda_2(e_1 b^T + b e_1^T) - \lambda_3(e_1 \bar{b}^T + \bar{b} e_1^T) \\ - \left(b\psi_1^T C_2^T + C_2 \psi_1 b^T\right) - \left(\bar{b}\psi_2^T C_2^T + C_2 \psi_2 \bar{b}^T\right) \in \mathcal{S}_+^{1+n_1+n_2}, \\ C_2 = [a \ C_1^T], C_3 = aa^T - C_1^T C_1, \lambda_1, \lambda_2, \lambda_3 \geq 0, \psi_1, \psi_2 \in \mathcal{SOC}(n_1) \end{array} \right\}.$$

If there is $(\bar{x}, \bar{y}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ *such that* $\|\bar{x}\| < a_1 + a_2^T \bar{x} + a_3^T \bar{y}$ *and* $a_5 > a_1 + a_2^T \bar{x} + a_3^T \bar{y} > a_4$, *then the closure in the above equation can be removed.*

Consequently, the corresponding P0 and CP5 have the same optimal value and if there is $(\bar{x}, \bar{y}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ *such that* $\|\bar{x}\| < a_1 + a_2^T \bar{x} + a_3^T \bar{y}$ *and* $a_5 > a_1 + a_2^T \bar{x} + a_3^T \bar{y} > a_4$, *then the corresponding CD5 without closure has the same optimal value and is attainable.*

Proof. Define

$$\mathcal{K}=\left\{U=\begin{bmatrix}\chi & x^T & y^T\\ x & X & W^T\\ y & W & Y\end{bmatrix}\in\mathcal{S}_+^{1+n_1+n_2}\;\middle|\;\begin{array}{l}a^TUb\ge\|C_1Ub\|, e_1^TUb\ge 0,\\ a^TU\bar{b}\ge\|C_1U\bar{b}\|, e_1^TU\bar{b}\ge 0,\\ b^TU\bar{b}\ge 0, U\cdot(aa^T-C_1^TC_1)\ge 0\end{array}\right\}$$

The one side $\mathcal{D}_{\mathcal{F}}^*\subset\mathcal{K}$ is obvious.

To see the other side, it is sufficient to prove that all the extreme points of $\mathcal{K}'$ with $\mathcal{K}'=\mathcal{K}\cap\{U\in\mathcal{S}^{1+n_1+n_2}|\text{tr }U\le 1\}$ belong to $\mathcal{D}_{\mathcal{F}}^*$. In other word, for each extreme point of $\mathcal{K}'$, we can find a rank one decomposition such that all elements are in $\mathcal{H}_{\mathcal{F}}$.

We will first prove that

$$\mathcal{H}_{\mathcal{F}}\supset\{(t,x,y)\in\mathbb{R}_+\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}|\|x\|\le a_1t+a_2^Tx+a_3^Ty, a_5t\ge a_1t+a_2^Tx+a_3^Ty\ge a_4t\ge 0\}.$$

If $t>0$, then $[1\ x^T/t\ y^T/t]^T\in\mathcal{F}$ and, therefore, $[t\ x^T\ y^T]^T\in\mathcal{H}_{\mathcal{F}}$. If $t=0$, then $x=0$ and $a_3^Ty=0$. Since $\mathcal{F}$ is not empty, there exists $\begin{bmatrix}\bar{x}\\ \bar{y}\end{bmatrix}\in\mathcal{F}$. One can verify that $\begin{bmatrix}0\\0\\y\end{bmatrix}+\frac{1}{k}\begin{bmatrix}1\\ \bar{x}\\ \bar{y}\end{bmatrix}\in\mathcal{H}_{\mathcal{F}}$ and therefore, when k goes to infinity, its limit $\begin{bmatrix}0\\0\\y\end{bmatrix}\in\mathcal{H}_{\mathcal{F}}$. Hence $\mathcal{H}_{\mathcal{F}}\supset\{(t,x,y)\in\mathbb{R}_+\times\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}|\|x\|\le a_1t+a_2^Tx+a_3^Ty, a_5\ge a_1t+a_2^Tx+a_3^Ty\ge a_4t\ge 0\}$.

Next, we need to consider five cases: (i) $\chi=0$; (ii) $\chi>0$, $a^TUb=\|C_1Ub\|$ and $a^TU\bar{b}=\|CU\bar{b}\|$; (iii) $\chi>0$, $a^TUb>\|C_1Ub\|$ and $a^TU\bar{b}=\|CU\bar{b}\|$; (iv) $\chi>0$, $a^TUb=\|C_1Ub\|$ and $a^TU\bar{b}>\|CU\bar{b}\|$; (v) $\chi>0$, $a^TUb>\|C_1Ub\|$ and $a^TU\bar{b}>\|CU\bar{b}\|$.

If (i) holds, then for any U^0 being a nonzero extreme point of $\mathcal{K}'$, we have $(x^0,y^0)=0$. Therefore, $a^TU^0e_1=b^TU^0e_1=\bar{b}^TU^0e_1=0$. From $a^TU^0\bar{b}\ge 0$ and $a^TU^0\bar{b}=a_5a^TU^0e_1-a^TU^0a=-a^TU^0a\le 0$, we have $U^0a=0$ and hence $U^0b=U^0\bar{b}=0$. From $U^0\cdot(aa^T-C_1^TC_1)=-\text{tr }(C_1U^0C_1)=-\text{tr }X^0\le 0$ we have $X^0=0$ and hence $W^0=0$. Furthermore, since U^0 is an extreme point of $\mathcal{K}'$, the matrix Y^0 must be the extreme point of

$$\mathcal{L}=\left\{Y\in\mathcal{S}_+^{n_2}\,\middle|\,\text{tr }Y\le 1, a_3^TYa_3=0\right\}$$

which is a rank-one matrix, i.e., $Y^0=y^0(y^0)^T$ for some $y^0\in\mathbb{R}^{n_2}$ with $a_3^Ty^0=0$. Let $(u^0)^T=[0\ 0\ (y^0)^T]$ and we have $U^0=u^0(u^0)^T$. Notice that $u^0\in\mathcal{H}_{\mathcal{F}}$, and hence $U^0\in\mathcal{D}_{\mathcal{F}}^*$.

If (ii) holds, suppose U^0 is an extreme point of $\mathcal{K}'$.

If $U^0b=0$, then $e_1^TU^0\bar{b}=e_1^TU^0((a_5-a_4)e_1-b)>0$ which means $U^0\bar{b}\ne 0$. Define

$U^1 = \frac{U^0 \bar{b}\bar{b}^T U^0}{\bar{b}^T U^0 \bar{b}}$ and $U^2 = U^0 - U^1$. By noting that

$$U^2 \cdot (aa^T - C_1^T C_1) = U^0 \cdot (aa^T - C_1^T C_1) \geq 0$$

we can check all the conditions in $\mathcal{K}$ and verify that U^1 and U^2 are all in $\mathcal{K}$. Since U^0 is an extreme point of $\mathcal{K}'$, we must have $U^0 = U^1 = \frac{U^0 \bar{b}\bar{b}^T U^0}{\bar{b}^T U^0 \bar{b}}$. Notice that $U^0 \bar{b} \in \mathcal{H}_{\mathcal{F}}$, we have $U^0 \in \mathcal{D}^*_{\mathcal{F}}$.

If $U^0 \bar{b} = 0$, then similar to the situation $U^0 b = 0$, we have $U^0 = \frac{U^0 bb^T U^0}{b^T U^0 b} \in \mathcal{D}^*_{\mathcal{F}}$.

If $U^0 b \neq 0$ and $U^0 \bar{b} \neq 0$, define $U^1 = \frac{U^0 bb^T U^0}{b^T U^0 b}$ and $U^2 = U^0 - U^1$. If $U^2 \bar{b} = 0$, then $U^0 \bar{b} = \frac{b^T U^0 \bar{b}}{b^T U^0 b} U^0 b$. By noting $U^2 b = U^2 \bar{b} = 0$ and

$$U^2 \cdot (aa^T - C_1^T C_1) = U^0 \cdot (aa^T - C_1^T C_1) \geq 0,$$

we have U^1 and U^2 are all in $\mathcal{K}$. Since U^0 is an extreme point of $\mathcal{K}'$, we must have $U^0 = U^1 = \frac{U^0 \bar{b}\bar{b}^T U^0}{\bar{b}^T U^0 \bar{b}}$. Notice that $U^0 \bar{b} \in \mathcal{H}_{\mathcal{F}}$, we have $U^0 \in \mathcal{D}^*_{\mathcal{F}}$. Otherwise, if $U^2 \bar{b} \neq 0$, then $U^2 a = \frac{1}{a_5 - a_4}(a_4 U^2 \bar{b} + a_5 U^2 b) = \frac{1}{a_5 - a_4} a_4 U^2 \bar{b} \neq 0$. let $U^3 = \frac{U^2 aa^T U^2}{a^T U^2 a}$ and $U^4 = U^2 - U^3 = U^0 - U^1 - U^3$. From Lemma 58, U^4 is positive semidefinite. Therefore,

$$U^0 \cdot (C_1^T C_1) = (U^1 + U^3 + U^4) \cdot (C_1^T C_1) \geq (U^1 + U^3) \cdot (C_1^T C_1).$$

Notice that

$$\begin{aligned}
& a^T U^2 a \left[(U^1 + U^3) \cdot (C_1^T C_1) - a^T U^0 a \right] \\
&= a^T U^2 a \left[\left(\frac{U^0 bb^T U^0}{b^T U^0 b} + \frac{(U^0 a - \frac{U^0 b(b^T U^0 a)}{b^T U^0 b})(U^0 a - \frac{U^0 b(b^T U^0 a)}{b^T U^0 b})^T}{a^T U^2 a} \right) \cdot (C_1^T C_1) - a^T U^0 a \right] \\
&= \frac{a^T U^2 a \|C_1 U^0 b\|^2}{b^T U^0 b} + \|C_1 U^0 a - \frac{C_1 U^0 b(b^T U^0 a)}{b^T U^0 b}\|^2 - (a^T U^2 a) a^T U^0 a \\
&= \frac{a^T U^2 a (a U^0 b)^2}{b^T U^0 b} + \|C_1 U^0 a\|^2 - 2\frac{b^T U^0 a}{b^T U^0 b}(b^T U^0 C_1^T C_1 U^0 a) + \frac{(a^T U^0 b)^4}{b^T U^0 b} - (a^T U^2 a) a^T U^0 a \\
&= 2\frac{a^T U^0 a (a U^0 b)^2}{b^T U^0 b} + \|C_1 U^0 a\|^2 - 2\frac{b^T U^0 a}{b^T U^0 b}(b^T U^0 C_1^T C_1 U^0 a) - (a^T U^0 a)^2
\end{aligned}$$

Let $\tau_1 = \frac{a_5}{a_5 - a_4}$ and $\tau_2 = 1 - \tau$ and then $a = \tau_1 b + \tau_2 \bar{b}$. From $a^T U^0 b = \|C_1 U^0 b\|$ and $a^T U^0 \bar{b} = \|C_1 U^0 \bar{b}\|$, we have

$$\|C_1 U^0 a\|^2 = 2\tau_1 (b^T U^0 C_1^T C_1 U^a) + \tau_2^2 (a^T U^0 \bar{b})^2 - \tau_1 (a^T U^0 b) a^T U^0 a + \tau_1 \tau_2 (a^T U^0 b) a^T U^0 \bar{b}$$

and

$$(a^T U^0 a)^2 = \tau_1 (a^T U^0 b) a^T U^0 a + \tau_2^2 (a^T U^0 \bar{b})^2 + \tau_1 \tau_2 (a^T U^0 b) a^T U^0 \bar{b}.$$

Therefore,

$$\begin{aligned}&a^TU^2a\left[(U^1+U^3)\cdot(C_1^TC_1)-a^TU^0a\right]\\&=2\left((a^TU^0a)(a^TU^0b)-b^TU^0C_1^TC_1U^0a\right)\left(\tfrac{a^TU^0b}{b^TU^0b}-\tau_1\right)\\&=2\left((a^TU^0a)(a^TU^0b)-b^TU^0C_1^TC_1U^0a\right)\tfrac{\tau_2\bar{b}^TU^0b}{b^TU^0b}\end{aligned}$$

From

$$a^TU^0a=\tau_1a^TU^0b+\tau_2a^TU^0\bar{b}=\tau_1\|C_1U^0b\|+\tau_2\|C_1U^0\bar{b}\|\geq\|C_1U^0a\|,$$

we have $a^TU^2a\left[(U^1+U^3)\cdot(C_1^TC_1)-a^TU^0a\right]\geq 0$ and, therefore, $U^0\cdot(C_1^TC_1)\geq a^TU^0a$. The equal sign holds if and only if $U^4\cdot(C_1^TC_1)=0$ and the two vectors $\begin{bmatrix}a^TU^0a\\C_1U^0a\end{bmatrix}$ and $\begin{bmatrix}a^TU^0b\\C_1U^0b\end{bmatrix}$ are linearly dependent. This also implies $\begin{bmatrix}a^TU^T\bar{b}\\C_1U^0\bar{b}\end{bmatrix}$ and $\begin{bmatrix}a^TU^0b\\C_1U^0b\end{bmatrix}$ are linearly dependent. From this result, we can verify that U^1 and U^2 are all in $\mathcal{K}$. Since U^0 is an extreme point in $\mathcal{K}'$, we have $U^0=U^1=\frac{U^0bb^TU^0}{b^TU^0b}$. Notice that U^0b is in $\mathcal{H}_{\mathcal{F}}$, and then $U^0\in\mathcal{D}^*_{\mathcal{F}}$.

After all, in case (ii), $U^0\in\mathcal{D}^*_{\mathcal{F}}$.

If (iii) holds, let $U^1=\lambda U^0\bar{b}\bar{b}^TU^0$ and $U^2=U^0-U^1$ with $\lambda>0$ being a sufficient small number. One can easily check that $U^1\in\mathcal{K}$. Notice that when λ is small enough, U^2 is positive semidefinite. From $a^TU^0b>\|C_1U^0b\|$, we know $\begin{bmatrix}a^TU^0b\\C_1U^0b\end{bmatrix}$ is an interior point of $\mathcal{SOC}(n_1)$. Therefore $\begin{bmatrix}a^TU^2b\\C_1U^2b\end{bmatrix}=\begin{bmatrix}a^TU^0b\\C_1U^0b\end{bmatrix}-\lambda(b^TU^0\bar{b})\begin{bmatrix}a^TU^0\bar{b}\\C_1U^0\bar{b}\end{bmatrix}$ is also in $\mathcal{SOC}(n_1)$ when λ is small enough, i.e., $a^TU^2b\geq\|C_1U^2b\|$. We can also see that

$$\begin{aligned}&U^2\cdot(aa^T-C_1^TC_1)=(U^0-U^1)\cdot(aa^T-C_1^TC_1)=U^0\cdot(aa^T-C_1^TC_1)\geq 0\\&b^TU^2e_1=b^TU^0e_1-\lambda(b^TU^0\bar{b})\bar{b}^TU^0e_1\geq 0\\&a^TU^2\bar{b}=(1-\lambda(\bar{b}^TU^0\bar{b}))a^TU^0\bar{b}\geq\|(1-\lambda(\bar{b}^TU^0\bar{b}))C_1U^0\bar{b}\|=\|C_1U^2\bar{b}\|\end{aligned}$$

Therefore U^2 is also in $\mathcal{K}$. Since U^0 is an extreme point in $\mathcal{K}'$, we have $U^0=\frac{\mathrm{tr}\ U^0}{\mathrm{tr}\ U^1}U^1=\frac{\mathrm{tr}\ U^0}{\mathrm{tr}\ U^2}U^2$. However, $a^TU^0b=\frac{\mathrm{tr}\ U^0}{\mathrm{tr}\ U^1}a^TU^1b=\frac{\mathrm{tr}\ U^0}{\mathrm{tr}\ U^1}\lambda(\bar{b}^TU^0b)(a^TU^0\bar{b})=\frac{\mathrm{tr}\ U^0}{\mathrm{tr}\ U^1}\lambda(\bar{b}^TU^0b)\|C_1U^0\bar{b}\|=\frac{\mathrm{tr}\ U^0}{\mathrm{tr}\ U^1}\|C_1U^1b\|=\|C_1U^0b\|$ which contradicts to $a^TU^0b>\|C_1U^0b\|$. This means any extreme point of $\mathcal{K}'$ does not satisfy (iii).

The case (iv) is similar to (iii).

If (v) holds and U^0 is an extreme point of $\mathcal{K}'$, then $(U^0, S_1^0, S_2^0, s_1^0, s_2^0, s_3^0, s_4^0, s_5^0)$ with

$$\begin{aligned} &S_1^0 = \text{Arrow}(a^T U^0 b, C_1 U^0 b), \\ &S_2^0 = \text{Arrow}(a^T U^0 \bar{b}, C_1 U^0 \bar{b}) \\ &s_1^0 = U^0 \cdot (aa^T - C_1^T C_1), \\ &s_2^0 = b^T U^0 e_1, \\ &s_3^0 = \bar{b}^T U^0 e_1, \\ &s_4^0 = \bar{b}^T U^0 b, \\ &s_5^0 = 1 - \text{tr } U^0 \end{aligned}$$

is an extreme point of

$$\mathcal{L}' = \left\{ \begin{array}{l|l} U = \begin{bmatrix} \chi & x^T & y^T \\ x & X & W^T \\ y & W & Y \end{bmatrix} \in \mathcal{S}_+^{1+n_1+n_2} & \begin{array}{l} S_1 = \text{Arrow}(a^T U b, C_1 U b), \\ S_2 = \text{Arrow}(a^T U \bar{b}, C_1 U \bar{b}), \\ s_1 = U \cdot (aa^T - C_1^T C_1), \\ s_2 = b^T U e_1, s_3 = \bar{b}^T U e_1, \\ s_4 = \bar{b}^T U b, s_5 = 1 - \text{tr } U \end{array} \\ S_1, S_2 \in \mathcal{S}_+^{1+n_1}, s_1, s_2, s_3, s_4, s_5 \geq 0 & \end{array} \right\}.$$

From Lemma 57, let $r_U = \text{rank}(U^0), r_{S_1} = \text{rank}(S_1^0), r_{S_2} = \text{rank}(S_2^0), r_i = \text{rank} s_i^0,\ i = 1, \ldots, 5$, and we have

$$r_U(r_U + 1) + r_{S_1}(r_{S_1} + 1) + r_{S_2}(r_{S_2} + 1) + \sum_{i=1}^{5} r_i(r_u + 1) \leq 2(n_1 + 1)(n_1 + 2) + 10.$$

In the assumption in (v), we have $a^T U^0 b > \|C_1 U^0 b\|$ and $a^T U^0 \bar{b} > \|C_1 U^0 \bar{b}\|$. From Lemma 65, $r_{S_1} = r_{S_2} = 1 + n_1$. Furthermore, $s_2^0 = \frac{1}{a_5} b^T U^0 (a + \bar{b}) > 0$ and $s_3^0 = \frac{1}{a_5} \bar{b}^T U^0 (a + \bar{b}) > 0$. Then the above inequality becomes

$$r_U(r_U + 1) + r_1(r_1 + 1) + r_4(r_4 + 1) + r_5(r_5 + 1) \leq 6.$$

If $r_U = 1$, then one can easily verify that $U^0 \in \mathcal{D}_{\mathcal{F}}^*$. If $r_U = 2$, we will show that U^0 cannot be an extreme point of $\mathcal{K}'$. In this situation, $r_1 = r_2 = r_3 = 0$, i.e., $s_1^0 = s_4^0 = s_5^0 = 0$. From $a^T U^0 b > 0$ and $a^T U^0 \bar{b} > 0$, we have $U^0 b \neq 0$ and $U^0 \bar{b} \neq 0$. Define $U^1 = \frac{U^0 b b^T U^0}{b^T U^0 b}$ and $U^2 = U^0 - U^1$. From Lemma 58, U^2 is positive semidefinite and $\text{rank}(U^2) = 1$. Since $s_4^0 = \bar{b}^T U^0 b = 0$, then $U^2 \bar{b} = U^0 \bar{b} \neq 0$. Therefore, $U^2 = \frac{U^2 \bar{b} \bar{b}^T U^2}{\bar{b}^T U^2 \bar{b}} = \frac{U^0 \bar{b} \bar{b}^T U^0}{\bar{b}^T U^0 \bar{b}}$. This means $U^0 = \frac{U^0 b b^T U^0}{b^T U^0 b} + \frac{U^0 \bar{b} \bar{b}^T U^0}{\bar{b}^T U^0 \bar{b}}$. Notice that $U^0 \bar{b}$ and $U^0 b$ are linearly independent. (Otherwise, $0 \neq b^T U^0 b = \tau b^T U^0 \bar{b} = 0$ for some $\tau \neq 0$, which is a contradiction.) One can further verify that $\frac{\text{tr } U^0}{\text{tr } U^1} U^1$ and $\frac{\text{tr } U^0}{\text{tr } U^2} U^2$ are all in $\mathcal{K}'$ and U^0 is the convex combination of these two different points which means U^0 cannot be an extreme point of $\mathcal{K}'$.

From the discussion above, we have $\mathcal{K} \subset \mathcal{D}_{\mathcal{F}}^*$ and hence $\mathcal{K} = \mathcal{D}_{\mathcal{F}}^*$.

Similar to the discussion of Theorem 64, we have the expression of $\mathcal{D}_{\mathcal{F}}$ and when $(\bar{x}, \bar{y})$ is a strict feasible point of $\mathcal{F}$, the closure can be removed and the corresponding CD5 without closure has the same optimal value as P0 and CP5 and is attainable. □

This theorem can be simplified when $a_4 = 0$ in Theorem 69.

Corollary 70. *Given a nonempty set* $\mathcal{F} = \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} | \|x\| \le a_1 + a_2^T x + a_3^T y, a_1 + a_2^T x + a_3^T y \le a_5\}$ *with* $a_1 \in \mathbb{R}$, $a_2 \in \mathbb{R}^{n_1}$, $a_3 \in \mathbb{R}^{n_2}$, *and* $a_5 \ge 0$. *Let*

$$a^T = [a_1 \ a_2^T \ a_3^T], \ \bar{b} = a_5 e_1 - a,$$

$$C_1 = [0 \ I_{n_1} \ 0] \text{ and } e_1 = (1, 0, \ldots, 0)^T \in \mathbb{R}^{1+n_1+n_2}$$

and we have

$$\mathcal{D}_{\mathcal{F}}^* = \left\{ U = \begin{bmatrix} \chi & x^T & y^T \\ x & X & W^T \\ y & W & Y \end{bmatrix} \in \mathcal{S}_+^{1+n_1+n_2} \middle| \begin{array}{l} a^T U \bar{b} \ge \|C_1 U \bar{b}\|, e_1^T U \bar{b} \ge 0, \\ U \cdot (aa^T - C_1^T C_1) \ge 0 \end{array} \right\}$$

and

$$\mathcal{D}_{\mathcal{F}} = \mathrm{cl} \left\{ M \in \mathcal{S}^{1+n_1+n_2} \middle| \begin{array}{l} M - \lambda_1 C_3 - \lambda_2 (e_1 \bar{b}^T + \bar{b} e_1^T) - \left(\bar{b} \psi^T C_2^T + C_2 \psi \bar{b}^T \right) \in \mathcal{S}_+^{1+n_1+n_2}, \\ C_2 = [a \ C_1^T], C_3 = aa^T - C_1^T C_1, \lambda_1, \lambda_2 \ge 0, \psi \in \mathcal{SOC}(n_1) \end{array} \right\}.$$

If there is $(\bar{x}, \bar{y}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ *such that* $\|\bar{x}\| < a_1 + a_2^T \bar{x} + a_3^T \bar{y}$ *and* $a_1 + a_2^T \bar{x} + a_3^T \bar{y} < a_5$, *then the closure in the above equation can be removed.*

Consequently, the corresponding P0 and CP5 have the same optimal value and if there is $(\bar{x}, \bar{y}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ *such that* $\|\bar{x}\| < a_1 + a_2^T \bar{x} + a_3^T \bar{y}$ *and* $a_1 + a_2^T \bar{x} + a_3^T \bar{y} < a_5$, *then the corresponding CD5 without closure has the same optimal value and is attainable.*

Proof. It is sufficient to show that the four constraints in $\mathcal{D}_{\mathcal{F}}^*$ imply the seven constraints in Theorem 69 with $a_4 = 0$. Let $b = a - a_4 e_1 = a$. From $U \cdot (aa^T - C_1^T C_1) \ge 0$ and

$$\begin{bmatrix} a^T \\ C_1 \end{bmatrix} U \begin{bmatrix} a & C_1^T \end{bmatrix} = \begin{bmatrix} a^T U a & a^T U C_1^T \\ C_1 U a & C_1 U C_1^T \end{bmatrix} \in \mathcal{S}_+^{1+n_1},$$

we have $(a^T U a)^2 \ge (a^T U a) \mathrm{tr}\, C_1 U C_1^T \ge \mathrm{tr}\, C_1 U a a^T U C_1^T = \|C_1 U a\|^2$, which shows $a^T U b \ge \|C_1 U b\|$. From $a^T U \bar{b} \ge 0$ and $e_1 = \frac{1}{a_5}(a + \bar{b})$, we have $b^T U e_1 = \frac{1}{a_5} a^T U (a + \bar{b}) \ge 0$. The last constraint $\bar{b}^T U b = \bar{b}^T U a \ge 0$. Therefore, all the seven constraints in Theorem 69 hold and the claim is true. □

6.3.5 Other results

In [113], Ye and Zhang also provided a result about one ball constraint and several complementary linear constraints.

Theorem 71 ([113]). *Suppose* $\mathcal{F} = \{x \in \mathbb{R}^n | \|x\| \le 1, a_1^T x \le a_0, b_1^T x \le b_0, (a_0 - a_1^T x)(b_0 - b_1^T x) = 0\}$. *Let*

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -I_n \end{bmatrix}, \quad a = \begin{bmatrix} a_0 \\ -a \end{bmatrix}, \quad b = \begin{bmatrix} b_0 \\ -b \end{bmatrix}$$

and we have

$$\mathcal{D}_{\mathcal{F}}^* = \left\{ Y \in \mathcal{S}_+^{n+1} \middle| Y \cdot J \ge 0, Ya, Yb \in \mathcal{SOC}(n), a^T Y b = 0 \right\}$$

and

$$\mathcal{D}_{\mathcal{F}} = \mathrm{cl} \left\{ M \in \mathcal{S}^{n+1} \middle| \begin{array}{l} M - \lambda_1 J - \lambda_2 (ab^T + ba^T) - (\psi_1 a^T + a\psi_1^T) - (\psi_2 b^T + b\psi_2^T) \in \mathcal{S}_+^{n+1} \\ \lambda_1 \ge 0, \lambda_2 \in \mathbb{R}, \psi_1, \psi_2 \in \mathcal{SOC}(n) \end{array} \right\}$$

This results can be extended to several linear inequalities with each two of them satisfying complementary constraints.

In [7], Anstreicher and Burer studied the exact representations of $\mathcal{D}_{\mathcal{F}}^*$ of simplex and box constraints in low dimension.

Theorem 72 ([7]). *Suppose* $\mathcal{F}$ *is a simplex in* $\mathbb{R}^n$, *i.e.,* $\mathcal{F} = \{x \in \mathbb{R}^n | x = Ay, y \in \mathbb{R}_+^{n+1}, e^T y = 1\}$, *where* $e = (1, \ldots, 1)^T \in \mathbb{R}^{n+1}$ *and* A *is full row rank. Then we have*

$$\mathcal{D}_{\mathcal{F}} = \left\{ \begin{bmatrix} \chi & e^T X A^T \\ AXe & AXA^T \end{bmatrix} \middle| X \in \mathcal{S}_+^n, X_{ij} \ge 0, \forall i, j, \text{ and } X \cdot (ee^T) = \chi \right\}$$

holds when $n \le 3$.

Theorem 73 ([7]). *Suppose* $\mathcal{F} = \{x \in \mathbb{R}_+^n | e - x \in \mathbb{R}_+^n\}$ *with* $e = (1, \ldots, 1)^T$. *We have*

$$\mathcal{D}_{\mathcal{F}}^* = \left\{ Y = \begin{bmatrix} \chi & x^T \\ x & X \end{bmatrix} \in \mathcal{S}_+^{n+1} \middle| X_{ij} \le x_i, X_{ij} \le x_j, X_{ij} \ge 0, X_{ij} \ge x_i + x_j - 1, \forall i, j \right\}$$

holds when $n = 2$.

6.4 Summary

In this chapter, we explore the exact representations of the cone of nonnegative quadratic functions over different domains. The domains include: one quadratic inequality; one quadratic equality; one convex quadratic inequality with one linear constraint; one second order cone constraint; one second order cone constraint with special linear constraints; one ball constraint with two parallel linear constraints; and some special cases. In particular, this is the first proof of the exact formulation of $\mathcal{D}_{\mathcal{F}}^*$ for the second order cone constraint. In [34], the same formulation is used, but the exactness of this formulation is not known.

These results can be used in the design of algorithms for general NLP problems. For example, the minimization of a quadratic function over a ball constraint can be used to as a subroutine in the well known trust region method. Based on Lemma 53 and Lemma 54, one could cover a region by several pieces with each of them being the cases above, and then get an approximation. Lu et al. [69] study the approximation using one ball constraint to cover each piece and get good results.

Chapter 7

Conclusions

In this dissertation, we have studied quadratically constrained quadratic programming problems and their extensions. Here, we first summarize the results obtained in this dissertation and then suggest some directions for future research.

7.1 Summary of Dissertation

In Chapter 3, we studied QCQP with one quadratic inequality constraint and obtained the following key results:

(i) If both GTRS and its Lagrangian dual (or semidefinite dual) are feasible, then there does not exist any duality gap between the optimal values of GTRS and its dual.

(ii) Under the assumption of (i), if $\mathcal{F}$ is an interval, or if $\mathcal{F}$ is a singleton set with $A_1 \preceq 0$ and $A_1 \neq 0$, then there always exists a finite optimal solution to GTRS.

In Chapter 4, we studied the inequality constrained QCQP problems. We first introduced the cone of nonnegative quadratic functions over the feasible domain, and then proved a new sufficient condition to verify the optimality of a KKT solution. For computations, we introduced a linear conic reformulation based on the cones of nonnegative quadratic functions, established the relation between the optimal KKT solution and the solution of the corresponding linear conic programming problem, and proposed a computational scheme that leads to an efficient algorithm for solving IQCQP problems. We also provided examples to illustrate the advantage of the proposed scheme comparing with the commonly used SDP relaxation.

In Chapter 5, we extended the results of IQCQP to conic form QCQP problems. By introducing a new active set, we derived a copositiveness condition that ensures a KKT solution being optimal. We also introduced linear conic programming problems based on the cone of nonnegative quadratic functions and proposed a computational scheme. The advantage of the proposed scheme was shown by an example.

In Chapter 6, we introduced and studied the exact formulations of different cones of nonnegative quadratic functions. Our results advanced the treatment of elliptic constraints, second order cone constraints, and linear constraints. In particular, we obtained some new results on the second order cone optimization problems.

7.2 Conclusions and Future Research

In this dissertation, we have studied QCQP problems and their extensions based on a linear conic programming framework using the cone of nonnegative quadratic functions. This framework is promising for solving nonconvex QCQP problems. We have also established a relationship between the optimal KKT solution and the optimal solution to the corresponding linear conic programming problem, and proposed a computational scheme that may lead to an efficient algorithm for solving IQCQP problems and CQCQP problems. The main difficulty in designing an efficient algorithm is the representation of the cone of nonnegative quadratic functions. Due to the NP-hardness of QP problems, it is not likely to find a computable representation of the feasible domain of a general quadratic optimization problems. Therefore, we studied the exact representations of the cone of nonnegative quadratic functions for some domains with simple structures. This provided new results for the second order cone optimization problems. Our findings have shown some advantages of the proposed linear conic programming framework and reveal some directions for future research.

First, as we can see, if a computable representation of the cone of nonnegative quadratic functions over $\mathcal{F}$ is known, then the corresponding quadratic programming problem over $\mathcal{F}$ can be solved efficiently. We provided new results for the second order cone constrained problems. However, up to now, when $\mathcal{F}$ is determined by one quadratic inequality constraint and one linear inequality constraint or is determined by two elliptic constraints, no results are known. We intend to investigate these two cases in the future.

Second, it is interesting to see the relation between the Lagrangian multipliers of an optimal KKT solution and the optimal solution of the corresponding linear conic programming problems based on the cone of nonnegative quadratic functions. The resulting copositiveness condition is useful in solving IQCQP and CQCQP problems. We would like to extend our findings to more general settings.

Third, the linear conic programming framework studied in this dissertation has many advantages in solving QCQP and CQCQP problems. Similar discussion has arisen for polynomial optimization problems in recent literature. We would like to extend the linear conic programming framework to study the polynomial optimization problems.

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