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MODELING OF A CRACKED BEAM SECTION UNDER BENDING

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ABSTRACT

Numerical simulations are widely used to study the dynamical behaviour of turbines cracked shaft as this event is rare and then doesn't enable to have an useful industrial feedback. A new method, which enables to calculate the constitutive law of a cracked beam subjected to bending was previously proposed. Based on three-dimensional computations taking into account the unilateral contact between both lips of the crack, it consists in defining a (non-linear) behaviour relation between the bending moment applied to the cracked section and the resulting field of displacements, compatible with the beam theory so that it can be used in rotor-dynamics software. The aim of this paper is to complete this first model by adding shear effects. For some crack geometries, a simpler model can be derived, based on the recognition that bending moments and shear forces are uncoupled and the dependence of the behaviour law with respect to the shear forces becomes linear. Developments have been achieved in this case and some results of the validation tests are shown.

Keywords: Turbine, main coolant pump, crack, model, detection

1. INTRODUCTION

The rupture of a rotor shaft through cracking is a rare occurrence, but its economic consequences can be catastrophic. In order to ensure that their equipment is not damaged, rotating machine operators have several tools at their disposal, on-line vibration monitoring being the most commonly used. Based on the measurement of bearing vibrations, it makes it possible to perform a continuous follow-up of the mechanical state of materials as well as an advanced diagnosis of their major faults. Nevertheless, the exceptional nature of rotor cracking phenomena makes it difficult to assess the reliability of algorithms specifically developed for their detection. Numerical modeling methods constitute, therefore, an interesting alternative for exploiting operational feedback since they can be used both for studying the impact of a crack on the dynamic behaviour of a shaft line and for verifying the relevance of the parameters used by monitoring systems.

An original method for calculating the constitutive law of a cracked beam section under bending has been proposed above (Varé *and al.*, 2000) and (Andrieux *and al.*, 2002). Based on three-dimensional computations taking into account the unilateral contact between the lips of the crack, it consists in defining a (non-linear) constitutive relation between the bending moment applied to the cracked section and the resulting field of displacements, compatible with the beam theory so that it can be used in rotor-dynamics software. This approach was validated based on experimental measurements (Audebert *and al.*, 2002). It was then successfully implemented in several industrial cases, more specifically on turbo-alternator units. The contribution of this approach, as compared to existing models in the literature (Wauer, 1991), (Gash, 1993), (Dimarogonas, 1996), is twofolds: it makes it possible to take into consideration (plane) cracks of any shape and any area in the cracked section and it takes precisely into account the conditions of unilateral contact and partial closing of the crack under certain loads. Besides, exploiting the general properties of the three-dimensional solution substantially reduces the amount of 3D computations needed for the identification of the constitutive law.

This analysis carried out on other industrial machines, such as vertical pumping units (Bachschmid *and al.*, 2004), has shown that the stress field applying to sections liable to crack presented a non-negligible shearing component. A new development was thus initiated in order to take into account shearing force in the modeling of a cracked beam section under flexion.

The aim of this paper is to present how shear effects can be implemented in the model of a cracked section. In the first part, we study the properties that can be verified by some crack geometries. In this case, theoretical developments can be done and lead to identification calculations that are easy to process. Finally, we present some results of the comparisons that were made between 3D and 1D computations.

2. SOME PROPERTIES EVIDENCED FROM FINITE ELEMENT COMPUTATIONS

Let us consider a beam which length is 2L, radius R, Young modulus E and quadratic inertia I. This beam has a crack located in its middle. It is clamped at its left edge and free at the other one. A force is applied at the right edge. Unilateral contact conditions are considered on the lips of the crack (Fig. 1).

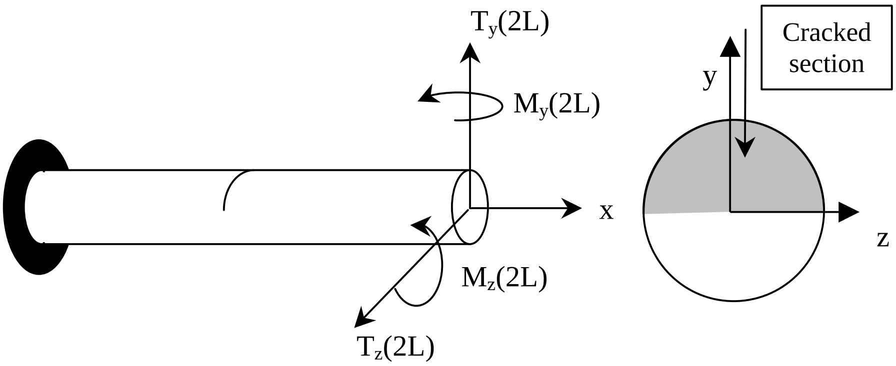

Fig. 1 – 3D Cracked Rotor Model

As for previous studies (Varé *and al.*, 2000) and (Andrieux *and al.*, 2002), the cracked section is modelled by a nodal element which constitutive law has to be identified. The nodal element has 4 degrees of freedom:

$$[q(L)] = \begin{pmatrix} [u_y(L)] \\ [u_z(L)] \\ [\theta_y(L)] \\ [\theta_z(L)] \end{pmatrix} = \begin{pmatrix} u_y(L^+) - u_y(L^-) \\ u_z(L^+) - u_z(L^-) \\ \theta_y(L^+) - \theta_y(L^-) \\ \theta_z(L^+) - \theta_z(L^-) \end{pmatrix} \begin{array}{l} \text{y displacement discontinuity} \\ \text{z displacement discontinuity} \\ \text{y rotation discontinuity} \\ \text{z rotation discontinuity} \end{array} \qquad \text{(Eq. 1)}$$

To identify the constitutive law of the nodal element, it is necessary to establish the relation between [q(L)] and the forces in the beam at x=L.

(El-Arem *and al.*, 2003) have studied the behaviour of a beam submited to in-plane bending. They have observed two properties which enable to dramatically simplify the model. More specifically, they have noted that the relations of the law of behaviour of the cracked section, depending on the bending moment on the one hand and the shearing force on the other, are uncoupled; and that furthermore the behaviour of the cracked section depending on the shearing force is linear, even though the conditions of contact between the lips of the crack have been taken into account. In this part, we present numerical computations in order to explore if these properties remain true when the movement of the beam is not restricted into a plane. For the exploitation of these computations, we use the following (approximate) kinetic relations:

- Rotation discontinuity in y direction: $[\theta_y(L)] = \theta_y(2L) - \theta_y^{safe}(2L)$
- Rotation discontinuity in z direction: $[\theta_z(L)] = \theta_z(2L) - \theta_z^{safe}(2L)$

- Displacement discontinuity in y direction: $\left[u_y(L)\right] = u_y(2L) - L\left[\theta_z(L)\right] - u_y^{safe}(2L)$
- Displacement discontinuity in z direction: $\left[u_z(L)\right] = u_z(2L) + L\left[\theta_y(L)\right] - u_z^{safe}(2L)$

where no exponent states for the results obtained on the cracked beam and the exponent 'safe' is related to the results obtained on the safe beam. These formulas correspond to (Fig. 2):

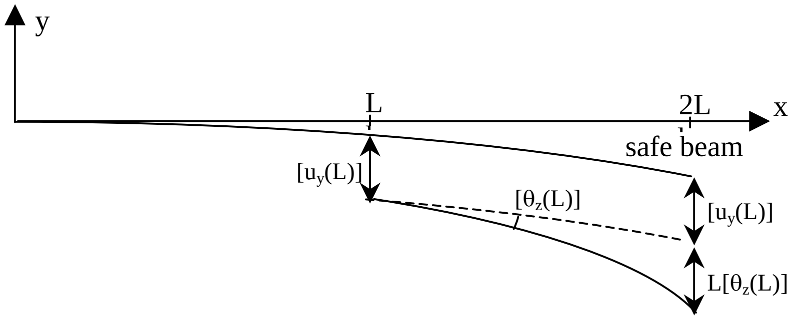

Fig. 2 – Simplified kinetic relations

Below, we present the results we obtained in several cases:

In the first case, the applied load is: $\mathbf{F}(2L) = \begin{pmatrix} Ty \\ Tz \\ My \\ Mz \end{pmatrix} = \begin{pmatrix} \sin\Phi \\ \cos\Phi \\ 2\cos\Phi \\ -2\sin\Phi \end{pmatrix}$

On the cracked section, the load is then: $\mathbf{F}(L) = \begin{pmatrix} Ty \\ Tz \\ My \\ Mz \end{pmatrix} = \begin{pmatrix} \sin\Phi \\ \cos\Phi \\ 0 \\ 0 \end{pmatrix}$

Figure 3 shows the displacement discontinuity with respect with Φ that was obtained:

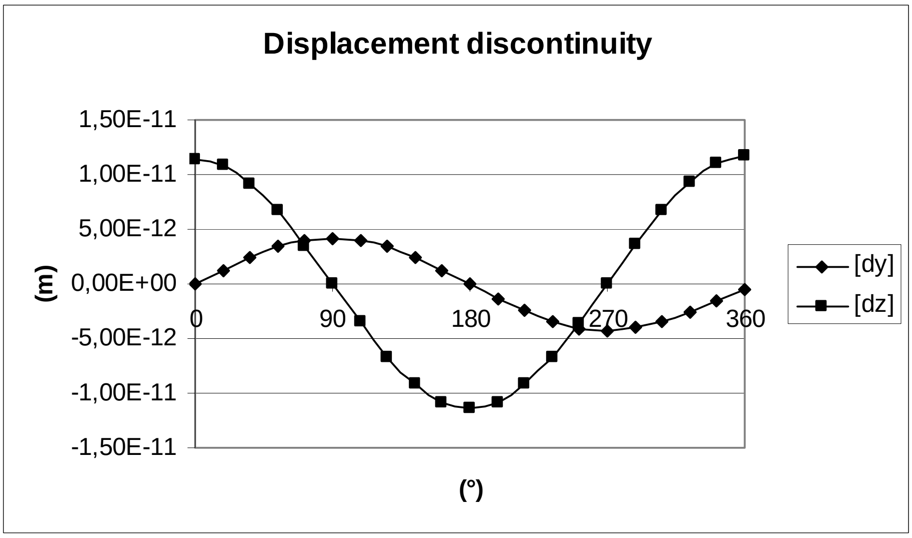

Fig. 3 – Local behaviour of the cracked section – shear forces only [dy]=[u_y(L)], [dz]=[u_z(L)]

In the second case, the applied load is: $\mathbf{F}(2L) = \begin{pmatrix} Ty \\ Tz \\ My \\ Mz \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2\cos\Phi \\ 2\sin\Phi \end{pmatrix}$

On the cracked section, the load is then: $\mathbf{F}(\mathrm{L}) = \begin{pmatrix} \mathrm{Ty} \\ \mathrm{Tz} \\ \mathrm{My} \\ \mathrm{Mz} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2\cos\Phi \\ 2\sin\Phi \end{pmatrix}$

Figure 4 shows the rotation discontinuity with respect with Φ that was obtained:

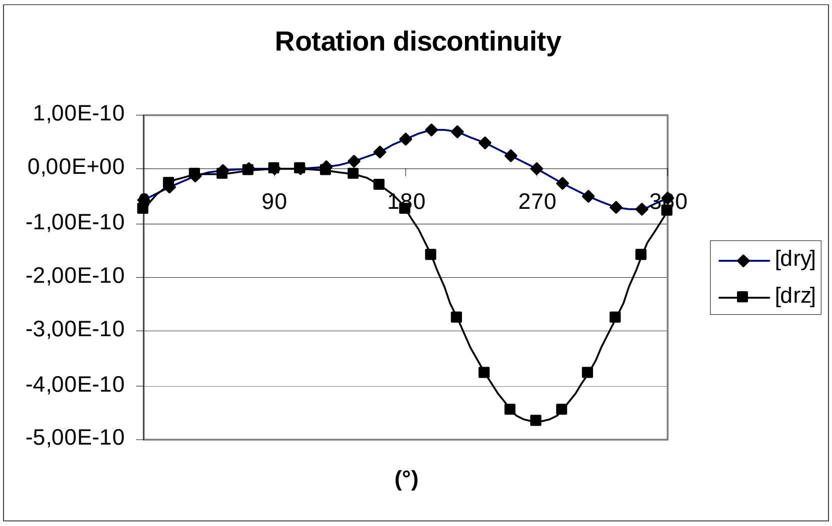

Fig. 4 – Local behaviour of the cracked section – bending moment only
$[\mathrm{dry}]=[\theta_y(\mathrm{L})]$, $[\mathrm{drz}]=[\theta_z(\mathrm{L})]$

In the final case, the applied load is: $\mathbf{F}(2\mathrm{L}) = \begin{pmatrix} \mathrm{Ty} \\ \mathrm{Tz} \\ \mathrm{My} \\ \mathrm{Mz} \end{pmatrix} = \begin{pmatrix} \sin\Phi \\ \cos\Phi \\ 0 \\ 0 \end{pmatrix}$

On the cracked section, the load is then: $\mathbf{F}(\mathrm{L}) = \begin{pmatrix} \mathrm{Ty} \\ \mathrm{Tz} \\ \mathrm{My} \\ \mathrm{Mz} \end{pmatrix} = \begin{pmatrix} \sin\Phi \\ \cos\Phi \\ -2\cos\Phi \\ 2\sin\Phi \end{pmatrix}$

Figure 5a shows the displacement discontinuity with respect with Φ that was obtained:

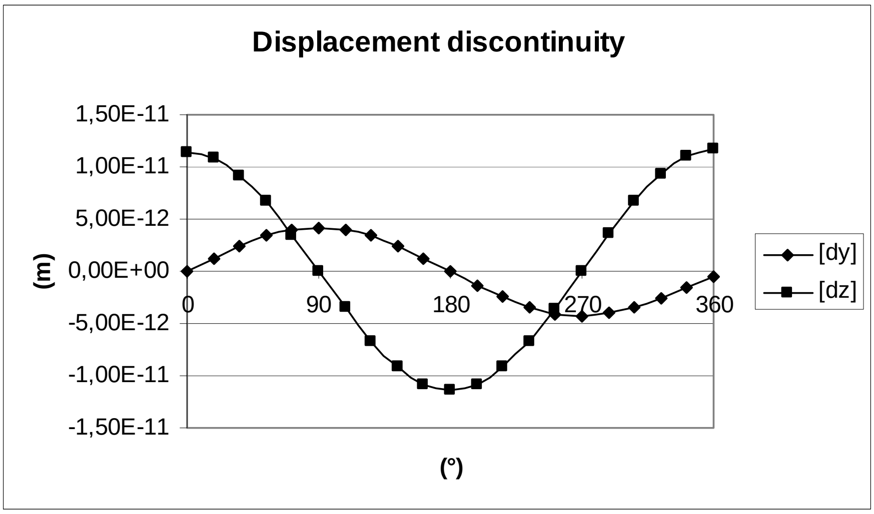

Fig. 5a – Local behaviour of the cracked section – shear forces+bending moment

Figure 5b shows the rotation discontinuity with respect with Φ that was obtained:

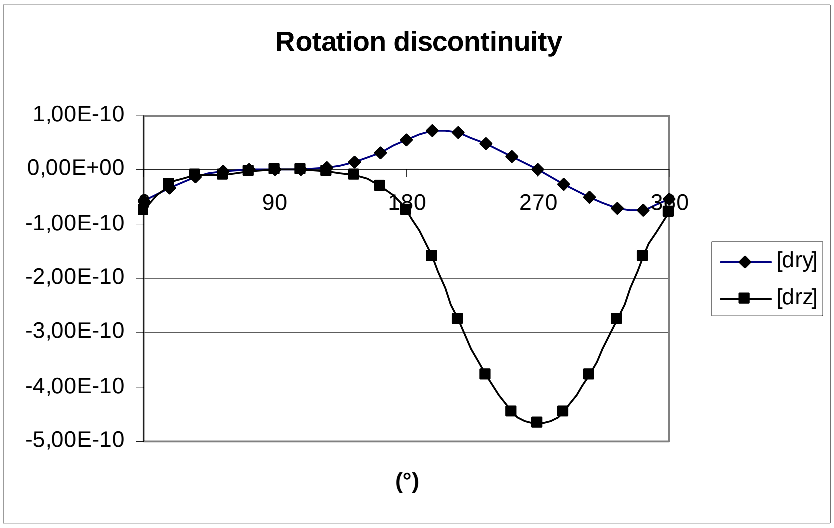

Fig. 5b – Local behaviour of the cracked section – shear forces+bending moment

It can be seen that the displacement discontinuities are sinusoidal just as applied shearing forces. **The linearity of the response with respect to shearing forces is then confirmed.** Concerning rotation discontinuities, in each case, we precisely obtain the results that were obtained in (Varé *and al.*, 2000). **The relations of the law of behaviour of the cracked section, depending on the bending moment on the one hand and the shearing force on the other, are then uncoupled**.

Although, owing to the generality of the possible shapes of cracks, the geometries for which these properties are verified cannot be known *a priori*, it has nevertheless been possible to obtain *an a posteriori* criterion in order to verify whether they are satisfied by a geometry of a cracked section:

The properties hold true if the following solutions of the unilateral problem under only shearing forces, it means: u_y^T corresponding to the load applied at x = 2L: $\mathbf{F}^{u_y^T}(2L) = (0, 1, 0, 0, 0, -L)$
u_z^T corresponding to the load applied at x = 2L: $\mathbf{F}^{u_z^T}(2L) = (0, 0, 1, 0, L, 0)$

verify:

1. $\mathbf{F}^{u_y^T}(2L).\mathbf{q}^{U_1}(2L) = 0$ et $\mathbf{F}^{u_z^T}(2L).\mathbf{q}^{U_1}(2L) = 0$

   where $\mathbf{q}^{U_1}(2L)$ is the solution of the unilateral problem under: $\mathbf{F}^{U_1}(2L) = (1, 0, 0, 0, 0, 0)$

2. $u_x^{u_y^T}(2L) = u_x^{u_z^T}(2L) = 0$ where u_x is the x component of u.

3. STRAIN ENERGY

The strain energy of the 3D cracked beam model is given by the Clapeyron's formula even in the unilateral case):

$$W(\mathbf{F}(2L)) = \frac{1}{2}\left(T_y(2L)u_y(2L) + T_z(2L)u_z(2L) + M_y(2L)\theta_y(2L) + M_z(2L)\theta_z(2L)\right)$$

Then, we consider that the strain energy of the cracked section is the sum of the strain energy of the safe beam and the strain energy due to the crack:

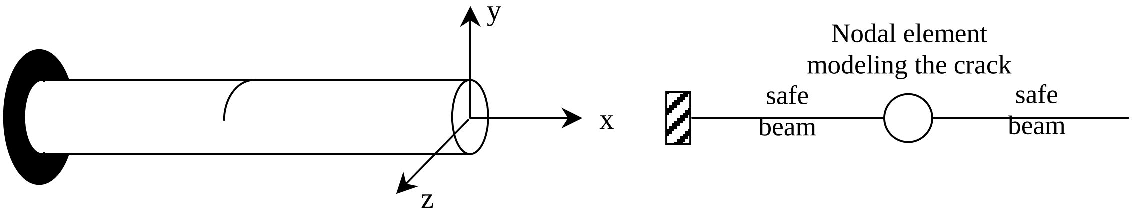

Fig. 6 – Identification of the strain energy of the crack element

$$W(\mathbf{F}(2L)) = W^{\text{safe beams}}(\mathbf{F}(2L)) + W^{\text{cracked section}}(\mathbf{F}(L))$$

where **F**(L) is the internal load at x=L when **F**(2L) is applied to the beam.

If the properties described in the previous part hold true, then the strain energy with respect with the internal loads can be written as follows:

$$W^{\text{cracked section}}(T_y, T_z, M_y, M_z) = W(M_y, M_z) + \frac{1}{2}s_y T_y^2 + \frac{1}{2}s_z T_z^2 + s_{yz}T_yT_z \qquad \text{(Eq. 2)}$$

where W(M_y,M_z) is the strain energy for pure flexion (Varé *and al.*, 2000) and (Andrieux *and al.*, 2002) and s_y, s_z et s_{yz} are the shear flexibilities due to the crack in y, z and yz directions which can be identified by computing the elementary solutions: W(1,0,0,0), W(0,1,0,0) and W(1,1,0,0).

In order to have a relation involving the stiffness and not the flexibility, we calculate the strain energy with respect with the discontinuities w^d. This latter is related with W by the Legendre-Fenchel transform:

Let us consider: $\mathbf{T} = \begin{pmatrix} T_y \\ T_z \end{pmatrix}$, $\mathbf{M} = \begin{pmatrix} M_y \\ M_z \end{pmatrix}$, $[u] = \begin{pmatrix} [u_y] \\ [u_z] \end{pmatrix}$ et $[\theta] = \begin{pmatrix} [\theta_y] \\ [\theta_z] \end{pmatrix}$

The Legendre-Fenchel transform is then: $w^d([u],[\theta]) = \underset{\mathbf{T},\mathbf{M}}{\mathrm{Sup}}\left(\mathbf{T}[u] + \mathbf{M}[\theta] - W(\mathbf{T},\mathbf{M})\right)$

As we have seen before, W exhibits no coupling between component in **T** and **M**.
Then, we have: W(**T**,**M**)=W(**T**,**0**)+W(**0**,**M**).
The Legendre-Fenchel transform reduces to: $w^d([u],[\theta]) = \underset{\mathbf{T},\mathbf{M}}{\mathrm{Sup}}\left(\mathbf{T}[u] + \mathbf{M}[\theta] - W(\mathbf{T},0) - W(0,\mathbf{M})\right)$

Then: $w^d([u],[\theta]) = \underset{\mathbf{T}}{\mathrm{Sup}}\left(\mathbf{T}[u] - W(\mathbf{T},0)\right) + \underset{\mathbf{M}}{\mathrm{Sup}}\left(\mathbf{M}[\theta] - W(0,\mathbf{M})\right)$

Or: $w^d([u],[\theta]) = w^d([u],0) + w^d(0,[\theta])$

The solution for the second term is presented in (Andrieux *and al.*, 2002). It reads:

$$w^d(0,[\theta]) = \frac{EI}{4L}\|[\theta]\|^2 k(\varphi) \quad \text{where} \quad \varphi = \mathrm{Arc}\tan\left(\frac{[\theta_y]}{[\theta_z]}\right)$$

Below, we look for the solution with respect with shear forces **T**. By integrating (Eq. 2) in Legendre-Fenchel transform, we have:

$$w^d([u],0) = \underset{\mathbf{T}}{\mathrm{Sup}}\left(T_y[u_y] + T_z[u_z] - \frac{1}{2}s_yT_y^2 - \frac{1}{2}s_zT_z^2 - s_{yz}T_yT_z\right)$$

Maximum value is obtained when: $\begin{cases} \dfrac{\partial w^d}{\partial T_y} = 0 \\ \dfrac{\partial w^d}{\partial T_z} = 0 \end{cases}$

It leads to : $\begin{cases} [u_y] = s_yT_y + s_{yz}T_z \\ [u_z] = s_{yz}T_y + s_zT_z \end{cases}$ and $\begin{cases} T_y = \dfrac{s_z[u_y] - s_{yz}[u_z]}{s_ys_z - s_{yz}^2} \\ T_z = \dfrac{-s_{yz}[u_y] + s_y[u_z]}{s_ys_z - s_{yz}^2} \end{cases}$ (Eq. 3)

By integrating (Eq. 3) in the strain energy formula, we obtain:

$$w^d([u],0) = \frac{1}{2}\frac{s_z}{s_ys_z - s_{yz}^2}[u_y]^2 + \frac{1}{2}\frac{s_y}{s_ys_z - s_{yz}^2}[u_z]^2 - \frac{1}{2}\frac{s_{yz}}{s_ys_z - s_{yz}^2}[u_y][u_z]$$

We define then k_y, k_z et k_{yz} as following:

$$k_y = \frac{s_z}{s_y s_z - s_{yz}}, \quad k_z = \frac{s_y}{s_y s_z - s_{yz}} \quad \text{and} \quad k_{yz} = -\frac{s_{yz}}{s_y s_z - s_{yz}}$$

It can be shown that $s_y s_z$-s_{yz} can’t be equal to zero as the flexibility matrix is strictly positive.

We then obtain the classical formula: $w^d([u],0) = \frac{1}{2} k_y [u_y]^2 + \frac{1}{2} k_z [u_z]^2 + k_{yz} [u_y][u_z]$

4. CONSTITUTIVE LAW

Finally, the strain energy can be written:

$$w^d([u],[\theta]) = \frac{EI}{4L} \|[\theta]\|^2 k(\varphi) + \frac{1}{2} k_y [u_y]^2 + \frac{1}{2} k_z [u_z]^2 + k_{yz} [u_y][u_z]$$

where $\varphi = \operatorname{Arc}\tan\left(\frac{[\theta_y]}{[\theta_z]}\right)$

The constitutive law is obtained from strain energy function derivatives. Components, with respect with [u] and [θ] being uncoupled, we can write:

$$M_y([\theta_y],[\theta_z]) = \frac{EI}{4L} \frac{\partial\left(\|[\theta]\|^2 k(\varphi)\right)}{\partial[\theta_y]}$$

$$M_z([\theta_y],[\theta_z]) = \frac{EI}{4L} \frac{\partial\left(\|[\theta]\|^2 k(\varphi)\right)}{\partial[\theta_z]}$$

$$T_y([u_y],[u_z]) = \frac{\partial\left(\frac{1}{2} k_y [u_y]^2 + \frac{1}{2} k_z [u_z]^2 + k_{yz} [u_y][u_z]\right)}{\partial[u_y]}$$

$$T_z([u_y],[u_z]) = \frac{\partial\left(\frac{1}{2} k_y [u_y]^2 + \frac{1}{2} k_z [u_z]^2 + k_{yz} [u_y][u_z]\right)}{\partial[u_z]}$$

The derivation with respect with [θ] has been described in (Andrieux *and al.*, 2002). They obtained the following formula:

$$\begin{pmatrix} M_y \\ M_z \end{pmatrix} = \frac{EI}{4L} \begin{pmatrix} k(\varphi) & -\frac{1}{2} k'(\varphi) \\ \frac{1}{2} k'(\varphi) & k(\varphi) \end{pmatrix} \begin{pmatrix} [\theta_y] \\ [\theta_z] \end{pmatrix}$$

which enjoys the symmetry properties: $\frac{\partial M_y}{\partial[\theta_z]} = \frac{\partial M_z}{\partial[\theta_y]}$

The derivation with respect with [u] is obtained easily: $\begin{pmatrix} T_y \\ T_z \end{pmatrix} = \begin{pmatrix} k_y & k_{yz} \\ k_{yz} & k_z \end{pmatrix} \begin{pmatrix} [u_y] \\ [u_z] \end{pmatrix}$

Finally, the constitutive law is written as follows:

$$\begin{pmatrix} M_y \\ M_z \\ T_y \\ T_z \end{pmatrix} = \begin{pmatrix} \frac{EI}{4L} k(\varphi) & -\frac{EI}{8L} k'(\varphi) & 0 & 0 \\ \frac{EI}{8L} k'(\varphi) & \frac{EI}{4L} k(\varphi) & 0 & 0 \\ 0 & 0 & k_y & k_{yz} \\ 0 & 0 & k_{yz} & k_z \end{pmatrix} \begin{pmatrix} [\theta_y] \\ [\theta_z] \\ [u_y] \\ [u_z] \end{pmatrix}$$

4. APPLICATION AND VALIDATION

The previous developments have been implemented in a software developed by Electricité de France (EDF) for rotor-dynamic calculations. We then studied the case of the rotor presented (Fig. 1), rotating slowly and subjected to a force which direction and amplitude are constant (Fig. 7).

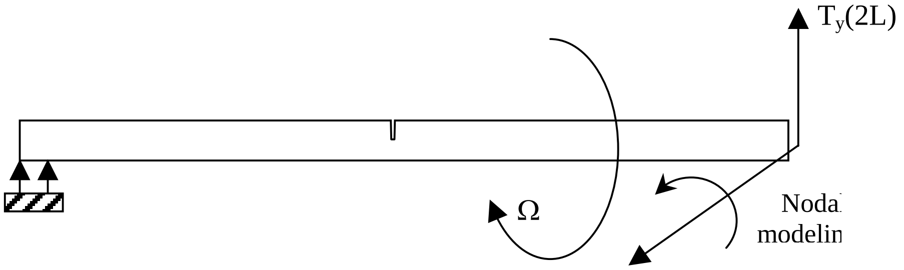

Fig. 7 – Application

Three load cases were successively considered:

- Bending moment on the cracked section
- Bending moment and shear forces on the cracked section
- Shear forces on the cracked section

In the first case, the applied load is: $\mathbf{F}(2L) = \begin{pmatrix} Ty \\ Tz \\ My \\ Mz \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2\cos\Phi \\ 2\sin\Phi \end{pmatrix}$

On the cracked section, the load is then: $\mathbf{F}(L) = \begin{pmatrix} Ty \\ Tz \\ My \\ Mz \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2\cos\Phi \\ 2\sin\Phi \end{pmatrix}$

Figure 8 presents the vertical displacement of the free edge of the beam during one revolution. The evolution we observe is due to the ‘breathing’ behaviour of the crack: it means that the crack opens and closes continuously once per revolution. In particular, when the crack is closed, its flexibility is equal to zero.

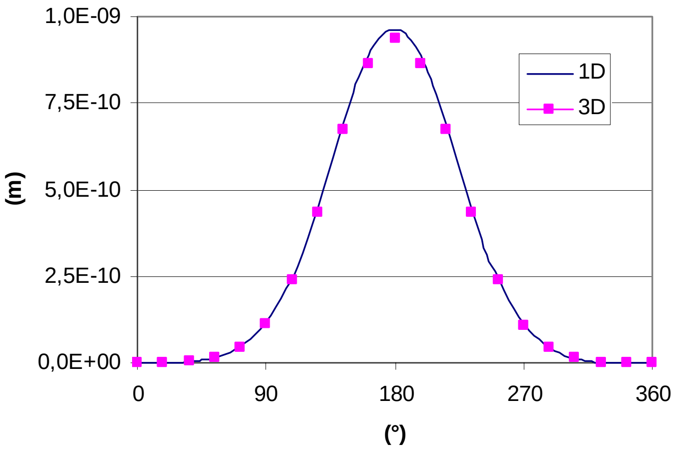

Fig. 8 – Comparison 3D-1D – bending moment only

In the second case, the applied load is: $\mathbf{F}(2L) = \begin{pmatrix} Ty \\ Tz \\ My \\ Mz \end{pmatrix} = \begin{pmatrix} \sin\Phi \\ \cos\Phi \\ 0 \\ 0 \end{pmatrix}$

On the cracked section, the load is then: $\mathbf{F}(L) = \begin{pmatrix} Ty \\ Tz \\ My \\ Mz \end{pmatrix} = \begin{pmatrix} \sin\Phi \\ \cos\Phi \\ -2\cos\Phi \\ 2\sin\Phi \end{pmatrix}$

Figure 9 presents the results we obtained:

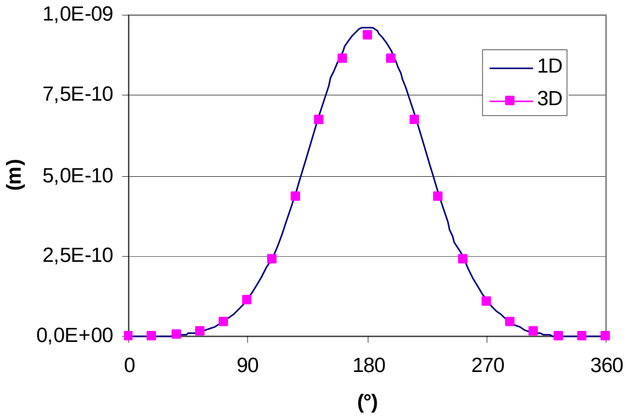

Fig. 9 – Comparison 3D-1D – bending moment + shear forces

In the two cases above, there is a good agreement between 3D and 1D results. We also see that the results are the same. The effects of shear forces seem then to be negligible in these cases.

In the last case, the applied load is: $\mathbf{F}(2L) = \begin{pmatrix} Ty \\ Tz \\ My \\ Mz \end{pmatrix} = \begin{pmatrix} \sin\Phi \\ \cos\Phi \\ 2\cos\Phi \\ -2\sin\Phi \end{pmatrix}$

On the cracked section, the load is then: $\mathbf{F}(L) = \begin{pmatrix} Ty \\ Tz \\ My \\ Mz \end{pmatrix} = \begin{pmatrix} \sin\Phi \\ \cos\Phi \\ 0 \\ 0 \end{pmatrix}$

Figure 10 presents the vertical displacement of the free edge of the beam during one revolution.

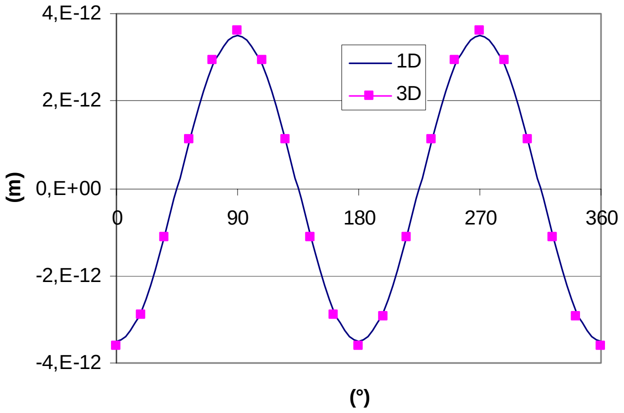

Fig. 9 – Comparison 3D-1D – shear forces only

In this case again, there is a good agreement between 3D and 1D results. So, this last result validates the model when the effects of shear forces are not negligible.

5. CONCLUSION

This paper shows how to implement shear forces in a one-dimensional cracked shaft model which is identified from three-dimensional computations taking into account the unilateral contact between the lips of the crack. For some crack geometries it is possible to simplify the proposed beam model since bending moments and shear forces are uncoupled and the behaviour law associated with the shear forces is linear. Although, owing to the generality of the possible shapes of cracks, the geometries for which these properties are verified cannot be known *a priori*, it has nevertheless been possible to derive a criterion in order to verify whether they are satisfied by a given geometry of a cracked section.

The method which has been developed is applied to a 50% cracked section (straight front and section perpendicular to the beam). In this case, the simplified properties are verified and the model can be used without any approximation. The comparisons between 3D and 1D computations show a very good agreement, which validates the method.

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