ASYMPTOTIC EQUIVALENCE OF REAL PROBABILITY DISTRIBUTIONS

by

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This note is based upon a series of lectures given by the author during Spring Semester 1966; it contains and refines the results of previous works [1,2,3,4], and includes some new results.

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Summary

Several notions of asymptotic equivalence together with corresponding notions of asymptotic independence for real probability distributions are introduced and some of their properties are discussed. These notions seem to have wide applicability in the study of asymptotic approximation problems.

This work is a part of a construction of a general theory of asymptotic equivalence of probability distributions which the present author desires to complete, and many unanswered questions are left open.

Introduction

Let us consider the following questions:

(i) Suppose that a sequence of random variables, $\{X_s\}$ $(s=1,2,\ldots)$, converges in law to some random variable Y as s tends to infinity. Let $\{c_s\}$ $(s = 1,2,\ldots)$ and $\{d_s\}$ $(s = 1,2,\ldots)$ be any given sequences of real numbers, and put $\tilde{X}_s = c_s X_s + d_s$ and $\tilde{Y}_s = c_s Y + d_s$ for every s. $\tilde{Y}_s$ has been called sometimes an asymptotic distribution of $\tilde{X}_s$. Then, in what sense are these two sequences of random variables approximate asymptotically as s tends to infinity?

(ii) Let $\{X_i\}$ $(i = 1,2,\ldots, N)$ be a system of elementary coverages obtained by a random division of the interval $[0,1)$ into $N + 1$ sub-intervals. This is not an independent system of random variables as is evident from the functional form of their joint probability density function. Let us now consider the case when N increases. It is easily noticed that any subsystem of fixed size of the above system forms an independent system in some stronger sense asymptotically as N tends to infinity. This suggests to us the following setting of a question:

Let $n = n(N)$ be a positive integer for each N such that $n \to \infty$ as $N \to \infty$, and choose any subsystem of size n, $\{X_i\}$ $(i = 1,2,\ldots, n)$ say. Then, in order that this subsystem forms an independent system in any given sense asymptotically as $N \to \infty$, how large could be the value of n ?

(iii) The following question concerns a central limit property of a dependent system of random variables. Suppose we are given a system of random variables, $\{X_i^s\}$ $(i = 1,2,\ldots,n_s)$, for each positive integer s, where n_s is assumed to tend to infinity with s. If this is an independent system for each fixed s, then the system would have a central limit property: Under suitable conditions there would exist sets of real numbers, $\{c_i^s\}$ and $\{d_i^s\}$, $i = 1,2,\ldots,n_s$, for each s such that $\sum_{i=1}^{n_s} (c_i^s X_i^s + d_i^s)$ converges in law to the standard normal distribution $N(0,1)$ as $s \to \infty$. We set forth a question in the following way: Is it still true that the system has the same central limit property even if we replace the "independence" of the system by "asymptotic independence" in a certain sense, and if so, what is the weakest such sense of asymptotic independence?

(iv) Let $\left\{X^s_{(n_s)} = (X_1^s,\ldots,X_{n_s}^s)\right\}$ $(s = 1,2,\ldots)$ be a sequence of n_s-dimensional independent normal random variables with mean vector 0 and variance-covariance matrix I_{n_s} (unit matrix). Then, $\sum_{i=1}^{n_s} Y_i^2$ is distributed according to the central chi-square distribution with n_s degrees of freedom. Suppose now that n_s tends to infinity with s, and that for any given positive integer n the sequence of marginal random variables, $\left\{X^s_{(n)} = (X_1^s,\ldots, X_n^s)\right\}$ $(s \geqq s_n;\ n_{s_n} \geqq n)$, converges in law to $Y_{(n)} = (Y_1,\ldots, Y_n)$ as $s \to \infty$. Is it possible in this situation to say that two random variables $\sum_{i=1}^{n_s} (X_i^s)^2$ and $\sum_{i=1}^{n_s} Y_i^2$, are distributed approsimately the same in some meaningful sense asymptotically as $s \to \infty$?

(v) Let $\{(X^s_{(k)}, Z^s_{(m)})\}$ (s = 1, 2, ...) be a sequence of k+m = n dimensional random variables, where k, m and n are assumed to be fixed independently of s. Then for each s the marginal distribution of $Z^s_{(m)}$ has the cumulative distribution function $H_s(z_{(m)}) = \int_{R_{(k)}} F_s(z_{(m)} | x_{(k)}) \, dG_s(X_{(k)})$,

where $G_s(x_{(k)})$ stands for the cumulative distribution function of $X^s_{(k)}$. Let, further, $\{Y^s_{(k)}\}$ (s = 1,2,...) be another sequence of random variables which is "asymptotically equivalent" in some sense to the sequence $\{X^s_{(k)}\}$ (s = 1, 2,...), as $s \to \infty$. If we define $\tilde{H}_s(z_{(m)}) = \int_{R_{(k)}} F_s(z_{(m)} | y_{(k)}) \, d\tilde{G}_s(y_{(k)})$, $\tilde{G}_s(y_{(k)})$ being the cumulative distribution function of $Y^s_{(k)}$, then this gives a culjulative distribution function of a certain random variable, $\tilde{Z}^s_{(m)}$ say. Possibly under some conditions $Z^s_{(m)}$ and $\tilde{Z}^s_{(m)}$ would be asymptotically equivalent in some sense as $s \to \infty$. Then, what types of asymptotic equivalence could be expected between $X^s_{(k)}$ and $Y^s_{(k)}$, and $Z^s_{(m)}$ and $\tilde{Z}^s_{(m)}$?

To give answers for these questions it is necessary to introduce notions of <u>asymptotic equivalence</u> between two sequences of random variables, which are defined independently of the existence of limiting distributions.

For most of the problems in which two sequences of random variables are required to compare, we are given a sequence of events, $\{E_s\}$ (s = 1, 2,...) say, specified by subsets of euclidean spaces, together with two sequences of random variables, $\{X_s\}$ (s = 1, 2,...) and $\{Y_s\}$ (s = 1, 2,...) say, and we are asked to examine whether $P^{X_s}(E_s)$ and $P^{Y_s}(E_s)$ are sufficiently "close" to each other for large values of s. It should be noted that the dimensions of underlying euclidean spaces are not always the same. We shall call the above type of problems the <u>asymptotic approximation problem</u> in general. For a given sequence $\{X_s\}$ (s = 1, 2,...), it is sometimes required to find another sequence $\{Y_s\}$ (s = 1, 2,...) which has the required closeness of $P^{Y_s}(E_s)$ to

$P^{X_s}(E_s)$ for given E_s's or for any sequence of subsets, each of which belongs to a certain class of subsets of the underlying space. This problem will also be included to the asymptotic assprximation problem.

As usual, two kinds of approximation error may be considered for the asymptotic approximation problem stated above: One is the so called "absolute error", $| P^{X_s}(E_s)-P^{Y_s}(E_s) |$, and the other is "relative error", $|P^{X_s}(E_s) - P^{Y_s}(E_s) | / P^{X_s}(E_s)$; the latter will be useful for asymptotic evaluation of an infinite series of probabilities, $\sum_{i=1}^{\infty} a_i P^{X_s} (E_{si})$.

These investigations naturally give us, for constructing a general theory of asymptotic equivalence, concepts of basic spaces (underlying euclidean spaces), basic classes (underlying sequence of classes of subsets of basic spaces), and of two types of asymptotic equivalence, type I and II, corresponding to absolute and relative error consideration.

The author developed in [1] a theory of type I and type II asymptotic equivalence choosing σ-finite measure apaces as basic classes. These notions of asymptotic equivalence appear too strong for some applications: In fact, the weaker, type I, turns out to be a notion of convergence in convergence theory which is equivalent to that of "in the mean" convergence. It is well known that the notion of "in law" convergence plays an important role in convergence theory, and some of the problems the author was required to answer suggested necessity of introducing some weaker notions of asymptotic equivalence, for example, one that is almost equivalent to an asymptotic equivalence theoretic version of "in law" convergence.

In this note we shall confine ourselves to the case where the basic spaces are euclidean, and the parameter involved is integral valued. There is no special reason for this, except that it is more direct for application to problems in usual statistics. One could extend easily the idea developed here to the case of abstract basic spaces, as was done partly in [1].

In Chapter 1 we intend to give definitions of sereral types of asymptotic equivalence and corresponding notions of asymptotic independence for real probability distributions. Some of the implication relations between these notions are exhibited in Chapter 2. Contents of Chapter 3 are essentially the restatement of the results already given in [1] with some new results.

In Chapter 4 a special type of asymptotic equivalence which is important for many applications is discussed. Main substances the author intends to include in a general theory of asymptotic equivalence are as follows:

(a) Unconditional and conditional implication relations of different types of asymptotic equivalence. This problem is of importance not only for theoretical completeness but also for practical applications of the theory.

(b) Some suitable criteria for each type of asymptotic equivalence in terms of characteristic functions, probability density functions, cumulative distribution functions, or of any other easily confirmable quantity. This is indispensable for applications of the theory. Unfortunately, for most of the types of asymptotic equivalence, we have not succeded yet in finding such criteria.

(c) To determine a class of measurable transformations of random variables which transfer any given type of asymptotic equivalence of the original sequences to another given type of asymptotic equivalence of the resulting sequences. This problem seems to have a wide applicability if any solution can be found. In this note we give answers for this problem in a few cases.

There are some problems in the directions the author wants to extend the present work. Throughout the present work, the dimensions of the basic euclidean spaces are assumed to be constants. However, for some cases of statistical procedures it would be desirable to assume the dimensions of basic spaces to be stochastically determined in some manner. Any pre-investigation has not been done yet in this problem, but the author hopes some fruitful results could be obtained and the range of application of our notions of asymptotic equivalence could be widened.

Presentation of a new concept of asymptotic independence of a system of random variables, whose definition will be given in Chapter 1, seems to introduce another aspect of problems different from the usual ones in the field of central limit theorems, as was simply exemplified in the question (iii) stated in the beginning of this section. The author guesses it is an interesting work to investigate these problems in connection with the usual results which have already obtained in the indicated field.

The other problem the author intends to do is to find out a field of applications in the theory of stochastic processes. In this connection Dr. R. M. Meyer has recently found that a notion of "mixing" for strictly stationally stochastic processes is well expressible by our notion of asymptotic independence of a suitable system of random variables, and this makes it easier to check the mixing condition in some practically common situation. To find further such applications is left to future investigations.

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I

Definitions of Asymptotic Equivalence of Probability Distributions and Asymptotic Independence of a System of Random Variables

1. Preliminaries

In this section we shall give some necessary notations and remarks about a basic space, basic class and a family of probability distributions.

For any given positive integer n, let $R_{(n)}$ be the n-dimensional Euclidean space, and $\mathcal{B}_{(n)}$ the usual Borel field of subsets of $R_{(n)}$. Let us denote the family of all probability distributions defined over the measurable space $(R_{(n)}, \mathcal{B}_{(n)})$ by $\mathfrak{J}(R_{(n)}, \mathcal{B}_{(n)})$, the members of which will be designated by random variables, $X_{(n)}, Y_{(n)}, \ldots$, say. A probability measure according which, for example, $X_{(n)}$ is distributed will be designated by $P^{X_{(n)}}$. It may happen that two members of the family $\mathfrak{J}(R_{(n)}, \mathcal{B}_{(n)})$ have the same probability measure.

Further, let $\nu_{(n)}$ be any σ-finite measure defined over the measurable space $(R_{(n)}, \mathcal{B}_{(n)})$, and denote the family of all probability distributions which are absolutely continuous with respect to the basic measure $\nu_{(n)}$ by $\wp(R_{(n)}, \mathcal{B}_{(n)}, \nu_{(n)})$. It is well known that for any member of this family there corresponds a generalized probability density function with respect to $\nu(n)$—— "gpdf($\nu_{(n)}$)" for short——in the usual manner.

Let $C_{(n)}$ be any given (non-empty) subclass of $\mathcal{B}_{(n)}$, and let $X_{(n)}$ and $Y_{(n)}$ be any two members of $\mathfrak{J}(R_{(n)}, \mathcal{B}_{(n)})$. For these two random variables, let us define the following two kinds of quantities

$$(1.1) \qquad d_I(X_{(n)}, Y_{(n)}; C_{(n)}) = \sup_{E_{(n)} \in C_{(n)}} \left| P^{X_{(n)}}(E_{(n)}) - P^{Y_{(n)}}(E_{(n)}) \right|$$

and

$$(1.2) \qquad d_{II}(X_{(n)}, Y_{(n)}; \mathcal{C}_{(n)}) = \sup_{E_{(n)}\in \mathcal{C}_{(n)}} \left| \frac{P^{X_{(n)}}(E_{(n)})}{P^{Y_{(n)}}(E_{(n)})} - 1 \right|$$

using the convention $0/0=1$.

Note that the first quantity defines a distance over the family $\mathfrak{Z}(R_{(n)}, \mathcal{B}_{(n)})$ if we identify those random variables which have the same probability measure over the class $\mathcal{C}_{(n)}$, $X_{(n)}$ and $Y_{(n)}$ being called to have the same probability measure over $\mathcal{C}_{(n)}$ if $P^{X_{(n)}}(E_{(n)}) = P^{Y_{(n)}}(E_{(n)})$ for every member $E_{(n)}$ of the class $\mathcal{C}_{(n)}$. It is also noted that if $X_{(n)}$ and $Y_{(n)}$ belong to the family $\wp(R_{(n)}, \mathcal{B}_{(n)}, \nu_{(n)})$ for a certain $\nu_{(n)}$ and have gpdf $(\nu_{(n)})$'s $f_{(n)}(z_{(n)})$ and $g_{(n)}(z_{(n)})$ respectively, then the equality

$$(1.3) \qquad 2\, d_I(X_{(n)}, Y_{(n)}; \mathcal{B}_{(n)}) = \int_{R_{(n)}} \left| f_{(n)} - g_{(n)} \right| \, d\,\nu_{(n)}$$

holds. This is shown as follows: Putting $A_{(n)} = \left\{ z_{(n)}: f_{(n)}(z_{(n)}) > g_{(n)}(z_{(n)}) \right\}$ and $B_{(n)} = \left\{ z_{(n)}: f_{(n)}(z_{(n)}) \leqq g_{(n)}(z_{(n)}) \right\}$, we have

$$\begin{aligned} \text{RHS} &= P^{X_{(n)}}(A_{(n)}) - P^{Y_{(n)}}(A_{(n)}) + P^{Y_{(n)}}(B_{(n)}) - P^{X_{(n)}}(B_{(n)}) \\ &\leqq \left| P^{X_{(n)}}(A_{(n)}) - P^{Y_{(n)}}(A_{(n)}) \right| + \left| P^{X_{(n)}}(B_{(n)}) - P^{Y_{(n)}}(B_{(n)}) \right| \\ &\leqq \text{LHS}, \end{aligned}$$

because both $A_{(n)}$ and $B_{(n)}$ belong to $\mathcal{B}_{(n)}$. On the other hand, since

$\left| P^{X_{(n)}}(E_{(n)}) - P^{Y_{(n)}}(E_{(n)}) \right|$ is equal to $P^{X_{(n)}}(E_{(n)}) - P^{Y_{(n)}}(E_{(n)})$

or $P^{X(n)}(\bar{E}_{(n)}) - P^{Y(n)}(\bar{E}_{(n)})$ for any $E_{(n)}$ in $\mathcal{B}_{(n)}$ with its complementary set $\bar{E}_{(n)}$, we can easily see that

$$d_I(X_{(n)}, Y_{(n)}; \mathcal{B}_{(n)}) = \sup_{E_{(n)} \in \mathcal{B}_{(n)}} \left\{ P^{X(n)}(E_{(n)}) - P^{Y(n)}(E_{(n)}) \right\}.$$

It is also easily verified that

$$P^{X(n)}(E_{(n)}) - P^{Y(n)}(E_{(n)}) \leqq P^{X(n)}(A_{(n)}) - P^{Y(n)}(A_{(n)})$$

and

$$P^{Y(n)}(E_{(n)}) - P^{X(n)}(E_{(n)}) \leqq P^{Y(n)}(B_{(n)}) - P^{X(n)}(B_{(n)})$$

for any $E_{(n)}$ in $\mathcal{B}_{(n)}$. Since then we can choose, for any given $\varepsilon > 0$, a member $E_{(n)}$ belonging $\mathcal{B}_{(n)}$ such that

$$d_I(X_{(n)}, Y_{(n)}; \mathcal{B}_{(n)}) \leqq \left\{ P^{X(n)}(E_{(n)}) - P^{Y(n)}(E_{(n)}) \right\} + \varepsilon ,$$

we have the following inequalities

$$\begin{aligned} \text{LHS} &\leqq \left\{ P^{X(n)}(E_{(n)}) - P^{Y(n)}(E_{(n)}) \right\} + \left\{ P^{Y(n)}(\bar{E}_{(n)}) - P^{X(n)}(E_{(n)}) \right\} + 2\varepsilon \\ &\leqq \left\{ P^{X(n)}(A_{(n)}) - P^{Y(n)}(A_{(n)}) \right\} + \left\{ P^{Y(n)}(B_{(n)}) - P^{X(n)}(B_{(n)}) \right\} + 2\varepsilon \\ &= \text{RHS} + 2\varepsilon . \end{aligned}$$

Hence (1.3) holds true.

For the quantity defined by (1.2), one can show easily that

$$(1.4)\qquad \frac{d_{II}(Y_{(n)},X_{(n)};\, C_{(n)})}{1 + d_{II}(Y_{(n)},X_{(n)};C_{(n)})} \leqq d_{II}(X_{(n)},Y_{(n)};\, C_{(n)}).$$

Clearly the roles of $X_{(n)}$ and $Y_{(n)}$ are exchangeable in this inequality, and therefore the vanishing of the quantity $d_{II}(X_{(n)},Y_{(n)};\, C_{(n)})$ implies that of $d_{II}(Y_{(n)},X_{(n)};\, C_{(n)})$, and vice versa.

The following inequality will also be checked easily:

$$(1.5)\qquad d_{I}(X_{(n)},Y_{(n)};\, C_{(n)}) \leqq d_{II}(X_{(n)},Y_{(n)};\, C_{(n)}) \quad ,$$

or, more precisely

$$(1.5)'\quad d_{I}(X_{(n)},Y_{(n)};\, C_{(n)}) \leqq \min\left\{d_{II}(X_{(n)},Y_{(n)};\, C_{(n)}),\ d_{II}(Y_{(n)},X_{(n)};C_{(n)})\right\}.$$

Now in the next place we shall give some of the subclasses of the class $\mathcal{B}_{(n)}$.

Let $\mathcal{M}_{(n)}$ be the class of all subsets which are of the form

$$E_{(n)}=\left\{z_{(n)}=(z_1,\ldots,z_n):\ -\infty < z_i < a_i\ ;\ i = 1,\ldots n\right\},$$

where a_i's are real numbers admitting the values $\pm\infty$, and $E_{(n)}=\emptyset$ (empty set) when $a_i = -\infty$, $i = 1,\ldots,n$; $E_{(n)} = R_{(n)}$ when $a_i = +\infty$, $i = 1,\ldots n$. Clearly, the intersection of any finite number of subsets belonging to

$\mathcal{M}_{(n)}$ is again in $\mathcal{M}_{(n)}$.

The class $S_{(n)}$ is defined to be the set-theoretical union of the three classes, $\mathcal{M}_{(n)}$, $\bar{\mathcal{M}}_{(n)}$ and $S^{o}_{(n)}$, where $\bar{\mathcal{M}}_{(n)}$ is the class consisting of all complementary subsets of those which belong to $\mathcal{M}_{(n)}$ and $S^{o}_{(n)}$ represents the class of all subsets of the form

$$E_{(n)} = \left\{ z_{(n)} = (z_1, \ldots, z_n) : b_i \leqq z_i < a_i ;\ i=1,\ldots,n \right\}$$

where a_i and b_i, $i=1,\ldots,n$, are all real numbers such that $b_i \leqq a_i$, $i=1,\ldots,n$.

It is evident then that

(1.6) $$\mathcal{M}_{(n)} \subset S_{(n)} \subset \mathcal{G}_{(n)} \subset \mathcal{B}_{(n)} \quad \text{and}$$

$$\mathcal{M}_{(n)} \subset \mathcal{G}_{(n)} \subset \mathcal{B}_{(n)} .$$

It is also well known that (a) the class $\mathcal{G}_{(n)}$ consists of all finite unions, or equivalently, of all finite disjoint unions of the members $S_{(n)}$, i.e.,

(1.7) $$\mathcal{G}_{(n)} = \left\{ \sum_{i=1}^{N} E_{(n)i} : E_{(n)i} \in S_{(n)},\ E_{(n)i}E_{(n)j} = \emptyset \ (i \neq j);\ i, j=1,\ldots,N : N=1,2,\ldots \right\}$$

and (b) for any member $E_{(n)}$ of $S_{(n)}$, there can be found a positive integer N and a set of members, $F_{(n)1}, \ldots, F_{(n)N}$, of $\mathcal{M}_{(n)}$, both depending only on the set $E_{(n)}$ and n, such that $N \leqq 2^n$ and

$$(1.8) \qquad \nu_{(n)}(E_{(n)}) = \sum_{i=1}^{N} c_i \quad \nu_{(n)}(F_{(n)i}),$$

for any measure over $(R_{(n)}, \mathcal{B}_{(n)})$, where c_i's are constants which take the value 1 or -1 depending only on $E_{(n)}$ and n.

The following statement is also wellknown: For any σ-finite measure $\nu_{(n)}$ over $(R_{(n)}, \mathcal{B}_{(n)})$ and for any member $E_{(n)}$ of $\mathcal{B}_{(n)}$, the $\nu_{(n)}$-measure of $E_{(n)}$ is given by

$$(1.9) \qquad \nu_{(n)}(E_{(n)}) = \inf \left\{ \sum_{i=1}^{\infty} \nu_{(n)} (F_{(n)_i}); \; E_{(n)} \subset \bigcup_{i=1}^{\infty} F_{(n)i}, \; F_{(n)i} \in \mathcal{G}_{(n)}, \; i=1,2,\dots \right\}$$

for which we can take $F_{(n)i}$'s mutually disjoint.

In the final place, a remark will be given on the notion of an independent system of random variables.

Let, as before, $X_{(n)} = (X_1, X_2, \dots X_n)$ be a member of $\mathfrak{F}(R_{(n)}, \mathcal{B}_{(n)})$. Corresponding to a decomposition of the space $R_{(n)}$ in the form

$$R_{(n)} = R_{(n_1)} \times R_{(n_2)} \times \dots \times R_{(n_m)}, \; n_1 + \dots + n_m = n,$$

we have a decompostion of the variable

$$X_{(n)} = (X_{(n_1)}, X_{(n_2)}, \dots, X_{(n_m)}),$$

where $X_{(n_i)}$ belongs to $\mathfrak{F}(R_{(n_i)}, \mathcal{B}_{(n_i)})$ for each i. Consider the set of marginal random variables, $\left\{ X_{(n_1)}, \dots, X_{(n_m)} \right\}$, of the above random variable. We intend to call such a set of random variables <u>a system of random variables</u> and m the size of the system. According to the usual definition, $\left\{ X_{(n_1)}, \dots, X_{(n_m)} \right\}$ is said to be an <u>independent system</u> of random variables if it holds that

$$(1.10) \qquad P^{X_{(n)}}(E_{(n)}) = \prod_{i=1}^{m} P^{X_{(n_i)}}(E_{(n_i)}),$$

for every product $E_{(n_1)} \times \dots \times E_{(n_m)} = E_{(n)}$, $E_{(n_i)} \in \mathcal{B}_{(n_i)}$, $i = 1,2,\dots,m$.

Any subsystem of an independent system of random variables is also independent, but the inverse is not necessarily true.

2. Definitions of stronger notions of asymptotic equivalence in the general case

In the first place we shall state the situation under which our notions of asymptotic equivalence are defined.

Let $\left\{ X^s_{(n_s)} = (X^s_1, \ldots, X^s_{n_s}) \right\}$ $(s=1,2,\ldots)$ and $\left\{ Y^s_{(n_s)} = (Y^s_1, \ldots, Y^s_{n_s}) \right\}$ $(s = 1,2,\ldots)$, be two sequences of random variables, for which $X^s_{(n_s)}$ and $Y^s_{(n_s)}$ belong to $\mathfrak{F}(R_{(n_s)}, \mathcal{B}_{(n_s)})$ for each fixed value of the parameter s. The sequence of Euclidean spaces, $\left\{ R_{(n_s)} \right\}$ $(s = 1,2,\ldots)$, will be called the sequence of basic spaces. We shall call the case where the dimensions n_s are identical to some positive integer n for all values of s the case of equal basic spaces, and the case of unequal basic spaces otherwise. Note that the usual notions of convergence are defined in the former case. An important case of unequal basic spaces is such that n_s tends to infinity as $s \to \infty$.

Corresponding to the above sequence of random variables, let us consider a sequence of classes of subsets of basic spaces, $\left\{ C_{(n_s)} \right\}$ $(s = 1,2,\ldots)$, which will be called the sequence of basic classes, where for each s $C_{(n_s)}$ is assumed to be a subclass of $\mathcal{B}_{(n_s)}$.

Under this situation, we shall give two notions of asymptotic equivalence as follows:

DEFINITION 2.1. Two sequences of random variables, $\left\{ X^s_{(n_s)} \right\}$ $(s = 1,2,\ldots)$ and $\left\{ Y^s_{(n_s)} \right\}$ $(s = 1,2,\ldots)$, are said to be asymptotically equivalent in the sense of type (I, C) —"ASEQ(I,C)" , for short — as $s \to \infty$, and are denoted by

(2.1) $$X^s_{(n_s)} \sim Y^s_{(n_s)} \quad (I,C), \ (s \to \infty),$$

if it holds that

(2.2) $$d_I(X^s_{(n_s)}, Y^s_{(n_s)};\ \mathcal{C}_{(n_s)}) \to 0,\ (s \to \infty).$$

DEFINITION 2.2. Two sequences in the above definition are said to be asymptotically equivalent in the sense of type (II, $\mathcal{C}$) —— "ASEQ(II,$\mathcal{C}$)", for short —— as $s \to \infty$, and are denoted by

(2.3) $$X^s_{(n_s)} \sim Y^s_{(n_s)} \quad (\text{II},\mathcal{C}),\ (s \to \infty),$$

if it holds that

(2.4) $$d_{II}(X^s_{(n_s)}, Y^s_{(n_s)};\ \mathcal{C}_{(n_s)}) \to 0,\ (s \to \infty).$$

In the earlier work [1] these types of asymptotic equivalence were defined by taking $\mathcal{B}_{(n_s)}$ as $\mathcal{C}_{(n_s)}$ in the above definitions for probability distributions which are absolutely continuous with respect to any given δ-finite measures over abstract basic spaces, and were called type I and type II asymptotic equivalence respectively. It is noted that the second definition 2.2 is consistent by the inequality (1.4), though the left hand side of (2.4) is not symmetric with respect to the two random variables involved.

The above definitions work, of course, in the case of equal basic spaces, too. In this case we can also consider the notions of convergence corresponding to the two types of notions defined above, which will be stated in the following.

DEFINITION 2.3. A sequence of random variables, $\left\{X^s_{(n)}\right\}$ $(s = 1,2,\ldots)$ is said to converge in the sense of type (I,$\mathcal{C}$) to $Y_{(n)}$, or converge (I,$\mathcal{C}$) to $Y_{(n)}$, in short, as $s \to \infty$, and is denoted by

(2.5) $$X^s_{(n)} \to Y_{(n)}\ (\text{I},\mathcal{C}),\ (s \to \infty),$$

if it holds that

(2.6) $$d_I(X^s_{(n)}, Y_{(n)};\ \mathcal{C}_{(n)}) \to 0,\ (s \to \infty).$$

DEFINITION 2.4. The sequence in the above definition is said to converge in the sense of type (II,$\mathcal{C}$), or, converge (II,$\mathcal{C}$) to $Y_{(n)}$ as $s \to \infty$, and is denoted by

(2.7) $$X^s_{(n)} \to Y_{(n)} \text{ (II,C)}, \ (s \to \infty),$$

if it holds that

(2.8) $$d_{II}(X^s_{(n)}, Y_{(n)}; \mathcal{C}_{(n)}) \to 0, \ (s \to \infty).$$

Four definitions given above provide a large amount of notions of asymptotic equivalence and of convergence, by specializing the sequence of basic classes $\left\{ C_{(n_s)} \right\}$ (s = 1,2,...) or $C_{(n)}$ in the definitions to any specific sequences of basic classes. Among those only several types of the notions will be investigated in this note, without any special reason except that they seem to be familiar to us, which will be given in the next place.

We take especially the following five sequences of basic classes in the case of unequal basic spaces: $\left\{ \mathcal{M}_{(n_s)} \right\}$ (s=1,2,...), $\left\{ S_{(n_s)} \right\}$ (s=1,2,...), $\left\{ \mathcal{G}_{(n_s)} \right\}$ (s=1,2,...), $\left\{ \mathcal{Q}_{(n_s)} \right\}$ (s=1,2,...), and $\left\{ \mathcal{B}_{(n_s)} \right\}$ (s=1,2,...) whose definitions have been given in the preceding section. In the case of equal basic spaces, it is assumed that each of these turns out to be a single class: $\mathcal{M}_{(n)}$, $S_{(n)}$, $\mathcal{G}_{(n)}$, $\mathcal{Q}_{(n)}$, and $\mathcal{B}_{(n)}$. By these specializations, we have two kinds of stronger notions of asymptotic equivalence, each containing five types of notions, and the corresponding notions of convergence in the case of equal basic spaces: type (I,$\mathcal{M}$), (I,S), (I,$\mathcal{G}$), (I,$\mathcal{Q}$), and type (I,$\mathcal{B}$); type (II,$\mathcal{M}$), (II,S), (II,$\mathcal{G}$), (II, $\mathcal{Q}$), and type (II,$\mathcal{B}$).

Mutual implication realtions of these notions will be investigated in the later section. In the final place of this section, we investigate some of the equivalent conditions for type (I,C) and type (II,C) asymptotic equivalence, which would be helpful to the understanding of these notions.

The following lemma gives equivalent conditions to type $(I,\mathcal{C})$ asymptotic equivalence in the general case.

LEMMA 2.1. Type $(I,\mathcal{C})$ asymptotic equivalence of the sequences, $\left\{ X^s_{(n_s)} \right\}$ (s=1,2,...) and $\left\{ Y^s_{(n_s)} \right\}$ (s=1,2,...), is equivalent to each one of the following conditions.

(a) For every sequence of subsets, $\left\{ E^s_{(n_s)} \right\}$ (s=1,2,...), such that $E^s_{(n_s)}$ belongs to $\mathcal{C}_{(n_s)}$ for each s, it holds that

$$\left| P^{X^s_{(n_s)}}(E^s_{(n_s)}) - P^{Y^s_{(n_s)}}(E^s_{(n_s)}) \right| \to 0, \quad (s \to \infty). \tag{2.9}$$

(b) There exists a sequence of positive numbers, $\left\{ q_s \right\}$ (s=1,2,...), depending only on the sequence of basic classes, $\left\{ \mathcal{C}_{(n_s)} \right\}$ (s=1,2,...), such that $q_s \to 0$ as $s \to \infty$ and

$$\left| P^{X^s_{(n_s)}}(E^s_{(n_s)}) - P^{Y^s_{(n_s)}}(E^s_{(n_s)}) \right| \leq q_s, \quad (s=1,2,\ldots), \tag{2.10}$$

for every sequence of subsets, $\left\{ E^s_{(n_s)} \right\}$ (s=1,2,...), with $E^s_{(n_s)}$ belonging to $\mathcal{C}_{(n_s)}$ for each s.

PROOF: Evidently (a) is not stronger than (b). It is also clear that the condition (b) is necessary and sufficient for $X^s_{(n_s)} \sim Y^s_{(n_s)}$ $(I,\mathcal{C})$, to prove the sufficiency of the condition (a), it suffices to merely point out that, if the sequences are not ASEQ$(I,\mathcal{C})$, there exists a subsequence $\{s'\}$ of $\{s\}$ such that

$$\left| P^{X^{s'}_{(n_{s'})}}(E^{s'}_{(n_{s'})}) - P^{Y^{s'}_{(n_{s'})}}(E^{s'}_{(n_{s'})}) \right| \to q, \quad (S' \to \infty)$$

for some $q > 0$, which contradicts (2.9). This proves the lemma.

In the case of equal basic spaces where $n_s = n$ for all s, the above two conditions reduce to the following:

(a)' For every sequence of subsets, $\{E^s_{(n)}\}$ (s = 1,2,...), belonging to the class $\mathcal{C}_{(n)}$, it holds that

$$(2.11)\qquad |P^{X^s_{(n)}}(E^s_{(n)}) - P^{Y^s_{(n)}}(E^s_{(n)})| \to 0, \ (s \to \infty).$$

(b)' It holds that

$$(2.12)\qquad |P^{X^s_{(n)}}(E_{(n)}) - P^{Y^s_{(n)}}(E_{(n)})| \to 0, \ (s \to \infty)$$

uniformly for all $E_{(n)}$ belonging to $C_{(n)}$.

Parallel results to the above lemma hold also for type (II, C) asymptotic equivalence as will be stated in the following

LEMMA 2.2. Type (II,C) asymptotic equivalence of $\{X^s_{(n_s)}\}$ (s=1,2,...) and $\{Y^s_{(n_s)}\}$ (s=1,2,...) is equivalent to each of the following conditions.

(a) It holds that

$$(2.13)\qquad \left|\frac{P^{X^s_{(n_s)}}(E^s_{(n_s)})}{P^{Y^s_{(n_s)}}(E^s_{(n_s)})} - 1\right| \to 0, \ (s \to \infty),$$

for every sequence of subsets, $\{E^s_{(n_s)}\}$(s=1,2,...), with $E^s_{(n_s)}$ belonging to $C_{(n_s)}$ for each s.

(b) There exists a sequence of positive numbers, $\{q_s\}$ (s=1,2,...), such that $q_s \to 0$ as $s \to \infty$ and

$$(2.14)\qquad | P^{X^s_{(n_s)}}(E^s_{(n_s)}) - P^{Y^s_{(n_s)}}(E^s_{(n_s)})| \leq q_s \cdot P^{Y^s_{(n_s)}}(E^s_{(n_s)}),$$

for every sequence of subsets, $\{E^s_{(n_s)}\}$ (s=1,2,...), with $E^s_{(n_s)}$ belonging to $C_{(n_s)}$ for each s.

In the case of equal basic spaces, these conditions turn out to the following

(a)' It holds that

$$(2.15)\qquad \left|\frac{P^{X^s_{(n)}}(E^s_{(n)})}{P^{Y^s_{(n)}}(E^s_{(n)})} - 1\right| \to 0, \ (s \to \infty),$$

for every sequence of subsets, $\left\{ E^s_{(n)} \right\}$ (s = 1,2,...), belonging to $C_{(n)}$.

(b)' It holds that

$$(2.16) \qquad \left| \frac{P^{X^s_{(n)}}(E_{(n)})}{P^{Y^s_{(n)}}(E_{(n)})} - 1 \right| \to 0, \qquad (s \to \infty)$$

uniformly for all $E_{(n)}$ belonging to $C_{(n)}$.

3. Definitions of weaker notions of asymptotic equivalence in the case of equal basic spaces

As was seen in the last two lemmas of the preceding section, type (I,C) and type (II,C) asymptotic equivalence can be regarded as notions of asymptotic equivalence based upon the uniform (over the basic classes) approximation of two probability distributions to be compared.

In the present section we introduce some weaker types of asymptotic equivalence in the case of equal basic spaces, basing upon the set-wise (over the basic class) approximation.

Let, as before, $\left\{ X^s_{(n)} \right\}$ (s = 1,2,...) and $\left\{ Y^s_{(n)} \right\}$ (s = 1,2,...) be two sequences of the members of $\mathfrak{F}(R_{(n)}, \mathcal{B}_{(n)})$, and let $C_{(n)}$ be any non-empty subclass of $\mathcal{B}_{(n)}$.

First we shall give the following definitions.

DEFINITION 3.1. $\left\{ X^s_{(n)} \right\}$ (s=1,2,...) and $\left\{ Y^s_{(n)} \right\}$ (s=1,2,...) are said to be asymptotically equivalent in the sense of type ((I,C))— "ASEQ((I,C))" in short —, as $s \to \infty$, and are denoted by

$$(3.1) \qquad X^s_{(n)} \sim Y^s_{(n)} \quad ((I,C)), \quad (s \to \infty),$$

if it holds that

$$(3.2) \qquad \left| P^{X^s_{(n)}}(E_{(n)}) - P^{Y^s_{(n)}}(E_{(n)}) \right| \to 0, \quad (s \to \infty)$$

for every subset $E_{(n)}$ belonging to $C_{(n)}$.

DEFINITION 3.2. $\left\{ X^s_{(n)} \right\}$ $(s=1,2,\ldots)$ and $\left\{ Y^s_{(n)} \right\}$ $(s=1,2,\ldots)$ are said to be asymptotically equivalent in the sense of type ((II,C)), — "ASEQ ((II,C))," in short — as $s \to \infty$, and are denoted by

$$(3.3) \qquad X^s_{(n)} \sim Y^s_{(n)} \quad ((II,C)), \ (s \to \infty),$$

if it holds that

$$(3.4) \qquad \left| \frac{P^{X^s_{(n)}}(E_{(n)})}{P^{Y^s_{(n)}}(E_{(n)})} - 1 \right| \to 0, \quad (s \to \infty)$$

for every $E_{(n)}$ belonging to $C_{(n)}$, where we use the convention $0/0 = 1$.

If we consider the case when $Y^s_{(n)}$'s are all identical to some fixed distribution $Y_{(n)}$ belonging to $\mathfrak{F}(R_{(n)}, \mathfrak{B}_{(n)})$, the above definitions give the corresponding notions of convergence.

DEFINITION 3.3. $\left\{ X^s_{(n)} \right\}$ $(s=1,2,\ldots)$ is said to converge in the sense of type ((I,C)), or "converge ((I,C))," in short, to $Y_{(n)}$ as $s \to \infty$, and is designated by

$$(3.5) \qquad X^s_{(n)} \to Y_{(n)} \quad ((I,C)), \ (s \to \infty),$$

if it holds that

$$(3.6) \qquad X^s_{(n)} \sim Y_{(n)} \quad ((I,C)), \quad (s \to \infty).$$

DEFINITION 3.4. $\left\{ X^s_{(n)} \right\}$ $(s=1,2,\ldots)$ is said to converge in the sense of type ((II,C)), or "converge ((II,C))," in short, to $Y_{(n)}$ as $s \to \infty$, and is designated by

$$(3.7) \qquad X^s_{(n)} \to Y_{(n)} \quad ((II,C)), \ (s \to \infty),$$

if it holds that

(3.8) $$X^s_{(n)} \sim Y_{(n)} \qquad ((II,\mathcal{C})),\ (s \to \infty).$$

By specializing the class $\mathcal{C}_{(n)}$ to each of the classes $\mathcal{M}_{(n)}$, $S_{(n)}$, $\mathcal{G}_{(n)}$, $\mathcal{G}_{(n)}$ and $\mathcal{B}_{(n)}$, we get five types of weaker notions of asymptotic equivalence and the corresponding notions of convergence in the case of equal basic spaces. Unfortunately, however, the weakest type of convergence defined above, type $((I,\mathcal{M}))$, is still stronger that that of in law convergence for some cases: In fact, if we take, as the class $\mathcal{C}_{(n)}$ in Definition 3.3 above, the class $\mathcal{M}_{(n)}(Y_{(n)})$, formed by deleting from $\mathcal{M}_{(n)}$ all those members which contain at least one discontinuity point of $Y_{(n)}$, i.e., a point whose $P^{Y_{(n)}}$-measure is positive, then the corresponding convergence gives the same notion as in law convergence. It is clear that if $Y_{(n)}$ is of the continuous type, then our type $((I,\mathcal{M}))$ convergence is exactly the same notion as in law convergence. We shall give an improved result for this later.

4. Notions of asymptotic independence of a system of random variables

Notions of asymptotic equivalence given in the preceding two sections will be used to define those of asymptotic independence of a system of random variables.

Let $\left\{ X^s_{(n_s)} \right\}$ (s=1,2,...) be a sequence of random variables taken from $\mathfrak{F}(R_{(n_s)},\mathcal{B}_{(n_s)})$ for each s. Corresponding to the direct product expression of each basic space in the form

$$R_{(n_s)} = R_{(m^s_1)} \times \dots \times R_{(m^s_{k_s})},\quad m^s_1 + \dots + m^s_{k_s} = n_s,$$

we shall decompose $X^s_{(n_s)}$ in the form

(4.1) $$X^s_{(n_s)} = (X^s_{1(m^s_1)}, \dots, X^s_{k_s(m^s_{k_s})}),$$

where $k_s(\geq 2)$ and $\{m_1^s,\dots,m_k^s\}$ may be dependent on s even in the case of equal basic spaces, and $X^s_{i(m_i^s)}$ belongs to $\mathfrak{F}(R_{(m_i^s)},\mathcal{B}_{(m_i^s)})$ for each i.

Now we shall consider a system of random variables of size k_s formed by all the marginals of (4.1), i.e.,

$$(4.2) \qquad \left\{ X^s_{1(m_1^s)},\dots,X^s_{k_s(m^s_{k_s})} \right\} .$$

It is well known that for each s there exists a member of $\mathfrak{F}(R_{(n_s)},\mathcal{B}_{(n_s)})$

$$(4.3) \qquad Y^s_{(n_s)} = (Y^s_{1(m_1^s)},\dots,Y^s_{k_s(m^s_{k_s})})$$

such that the system of marginal random variables

$$\left\{ Y^s_{1(m_1^s)},\dots,Y^s_{k_s(m^s_{k_s})} \right\}$$

forms an <u>independent system</u>, and $Y^s_{i(m_i^s)}$ is identically distributed with $X^s_{i(m_i^s)}$ for each i.

Let $\{C_{(n_s)}\}$ (s=1,2,...) be any given sequence of basic classes such that $C_{(n_s)}$ is a subclass of $\mathcal{B}_{(n_s)}$ for each s.

Under this situation we give, firstly, stronger notions of asymptotic independence in the general case.

<u>DEFINITION 4.1</u>. A system of random variables (4.2) is said to be <u>asymptotically independent in the sense of type (I,C)</u>, or "<u>ASIN(I,C)</u>" in short, as $s\to\infty$, if it holds that $X^s_{(n_s)} \sim Y^s_{(n_s)}$ (I,C) as $s\to\infty$. Type (II,C) asimptotic independence is defined in a quite similar manner.

The following definition gives weaker types of asymptotic independence, in the case of equal basic spaces.

<u>DEFINITION 4.2</u>. A system of random variables (4.2) is said to be <u>asymptotically independent in the sense of type ((I,C))</u>, or "<u>ASIN((I,C))</u>"

in short, as $s \to \infty$, if it holds that $X^s_{(n)} \sim Y^s_{(n)}$ ((I,C)) as $s \to \infty$. Type ((II,C)) asymptotic independence is also given in a similar manner.

By these definitions we have various types of notions of asymptotic independence. Among those, we consider in the present note only several types of notions, corresponding to the notions of asymptotic equivalence considered in the preceding two sections, i.e., type $(I,\mathcal{M})$, (I,S), $(I,\mathfrak{a})$, $(I,\mathfrak{g})$, $(I,\mathfrak{B})$; $(II,\mathcal{M})$, (II,S), $(II,\mathfrak{a})$, $(II,\mathfrak{g})$, $(II,\mathfrak{B})$, in the general case, and $((I,\mathcal{M}))$, ((I,S)), $((I,\mathfrak{a}))$, $((I,\mathfrak{g}))$, $((I,\mathfrak{B}))$; $((II,\mathcal{M}))$, ((II,S)), $((II,\mathfrak{a}))$, $((II,\mathfrak{g}))$, $((II,\mathfrak{B}))$, in the case of equla basic spaces.

It sometimes happens that the whole system is not asymptotically independent, but its sybsystems are asymptotically independent in a certain given sense. The following definition will be useful for some of such cases.

Suppose that we are given a system of random variables of equal dimensions

$$\left\{ X^s_{1(m)}, \ldots X^s_{N_s(m)} \right\} \tag{4.4}$$

where m may be dependent on s, s=1,2,... .

DEFINITION 4.3. The system (4.4) is said to be asymptotically independent (n_s) in the sense of type (I,C), or "(n_s)-ASIN(I,C)" in short, as $s \to \infty$, if every subsystem of size n_s, $\left\{X^s_{i_1}(m), \ldots, X^s_{i_{n_s}}(m)\right\}$, is ASIN(I,C) as $s \to \infty$.

Quite similarly, we can define (n_s)—ASIN(II,C), in the general case, and (n_s)-ASIN((I,C)) or (n_s)-ASIN((II,C)) in the case of equal basic spaces.

II

Implication Relations of Notions of Asymptotic Equivalence

5. Some properties of a sequence of random variables in the case of equal basic spaces.

It is important to investigate the implication relations among the notions of asymptotic equivalence defined in the previous part and the equivalence conditions, under which two different types of asymptotic equivalence are mutually equivalent.

The purpose of the present section is to give two sorts of properties for a sequence of random variables in the case of equal basic spaces, which are useful to derive equivalence conditions for the notions of asymptotic equivalence.

The first one is a sort of stochastic boundedness of a sequence of random variables.

Let $\left\{X^s_{(n)}\right\}$ (s=1,2,...) be a sequence of n-dimensional random variables. For this sequence let us give the following

DEFINITION 5.1. $\left\{X^s_{(n)}\right\}$ (s=1,2,...) is said to have the property B(S), if for any given ϵ (>0) there exist a member of $S_{(n)}$, $B_{(n)}$ say, whose closure being compact, and a positive integer s_o such that

$$P^{X^s_{(n)}}(B_{(n)}) > \epsilon \tag{5.1}$$

for all $s \geqq s_o$.

In a similar manner as this definition, of course, we can give the property B(C) for any subclass $C_{(n)}$ of $\mathcal{B}_{(n)}$, which contains at least one subset whose closure being compact.

We shall list some of the results on property B(S) in the following

LEMMA 5.1. (a) Type ((I,S))asymptotic equivalence brings over the property B(S), i.e., if $X^s_{(n)} \sim Y^s_{(n)}((I,S))$ as $s \to \infty$ and $\{X^s_{(n)}\}$ (s=1,2,...) has the property B(S), then $\{Y^s_{(n)}\}$ (s=1,2,...) has the same property.

(b) Property B(S) is brought over by any type of asymptotic equivalence which is stronger than or equivalent to type ((I,S)).

(c) Let $C_{(n)}$ be any subclass of $\mathcal{B}_{(n)}$ which contain the class $S_{(n)}$. Then, two properties, B(S) and B(C), are mutually equivalent.

(d) If the sequence $\{X^s_{(n)}\}$ (s=1,2,...) converges ((I,S)) to some limiting distribution, then the sequence has the property B(S).

(e) A sufficient condition for $\{X^s_{(n)}\}$ (s=1,2,...) to have the property B(S) is that, for any given decomposition $X^s_{(n)}=(X^s_{1(m_1)},\dots X^s_{k(m_k)})$, k and $\{m_1,\dots m_k\}$ being independent of s, every sequence of marginals, $\{X^s_{j(m_j)}\}$ (s=1,2,...), has the property B(S), j=1,...,k.

(f) If $\{X^s_{(n)}\}$ (s=1,2,...) has the property B(S), then any sequence of marginals, $\{\tilde{X}^s_{(m)}=(X^s_{i_1},\dots,X^s_{i_m})\}$ (s=1,2,...), has the property B(S), where m is arbitrarily fixed independently of s, while the choice of $\{i_1,\dots,i_m\}$ out of $\{1,\dots,n\}$ may depend on s.

(g) If all the one-dimensional marginals of $X^s_{(n)}$ have finite means and variances uniformly for all s, then $\{X^s_{(n)}\}$ (s=1,2,...) has the property B(S).

The proof of this lemma is straightforward from the definition and will be omitted.

Another property is a sort of absolute continuity of a sequence of random variables. Let $C_{(n)}$ be any subclass of $\mathcal{B}_{(n)}$, for which the following definition is meaningful:

DEFINITION 5.2. A sequence of random variables, $\left\{X^s_{(n)}\right\}$ (s=1,2,...) is said to have the property C(C), if for any given $\epsilon > 0$, there exist a positive number δ and a positive constant s_o such that

(5.2) $P^{X^s(n)}(E_{(n)}) < \epsilon$, for every $E_{(n)}$ in $C_{(n)}$ with $\mu_{(n)}(E_{(n)}) < \delta$,

for all $s \geq s_o$, where $\mu_{(n)}$ designates the usual Lebesgue measure over $(R_{(n)}, \mathcal{B}_{(n)})$.

Identifying the class $C_{(n)}$ in this definition with $S_{(n)}$, $G_{(n)}$, $\mathcal{G}_{(n)}$ or $\mathcal{B}_{(n)}$, we have the properties C(S), C(G), C($\mathcal{G}$) or C($\mathcal{B}$), respectively. Note that C($\mathcal{M}$) is meaningless, because any member of the class $\mathcal{M}_{(n)}$ has infinite Lebesgue measure except for the null set. We shall list below some of the natures of these properties, which are direct consequences of the definition.

LEMMA 5.2. (a) If $\left\{X^s_{(n)}\right\}$ (s=1,2,...) has the property C(C), and $C(n) \supseteq C^*_{(n)}$, then the sequence has the property C(C*).

(b) Property C(C) is brought over by type ((I,C)) asymptotic equivalence, and also by any type of asymptotic equivalence which is stronger than or equivalent to type ((I,C)).

(c) If $\left\{X^s_{(n)}\right\}$ (s=1,2,...) has the property C(C) and $C_{(n)}$ is any one of the four classes mentioned above, then the sequence of any marginals, $\left\{\tilde{X}^s_{(m)}=(X^s_{i_1},\dots,X^s_{i_m})\right\}$ (s=1,2,...), has the same property, where m is fixed independently of s, while the choice of $\left\{i_1,\dots,i_m\right\}$ out of $\left\{1,\dots,n\right\}$ may depend on s.

(d) Property C(C) of $\left\{X^s_{(n)}\right\}$ (s=1,2,...) is equivalent to the following condition: For any given $\epsilon > 0$ and any given positive integer N, there exist a positive number δ and a positive integer s_o, both depending on ϵ and N, and such that the conditions

$$\mu_{(n)}\left(\bigcup_{i=1}^{N} E_{(n)i}\right) \text{ and } E_{(n)i} \in C_{(n)}, \quad i=1,\dots,N$$

imply jointly that

$$P^{X^s_{(n)}}\left(\bigcup_{i=1}^{N} E_{(n)i}\right) < \epsilon$$

for all $s \geqq s_o$, where the sets $\left\{E_{(n)i}\right\}$, $i=1,\ldots,N$, may or may not be mutually disjoint.

(e) Suppose that $\left\{X^s_{(n)}\right\}$ $(s=1,2,\ldots)$ converges $((I,\mathcal{C}))$ to $Y_{(n)}$, an n-dimensional random variable which is of the continuous type, i.e., which is absolutely continuous with respect to the Lebesgue measure. Then, the sequence has the property $C(\mathcal{C})$.

6. Implication relations of certain types of asymptotic equivalence in the general case

In this section we shall consider the implication relations of stronger notions of asymptotic equivalence defined in Section 2, where we have defined two kinds of stronger notions, type $(I,\mathcal{C})$ and type $(II,\mathcal{C})$, assuming that $\left\{\mathcal{C}_{(n_s)}\right\}$ $(s=1,2,\ldots)$ is any given sequence of basic classes such that $\mathcal{C}_{(n_s)} \subseteq \mathcal{B}_{(n_s)}$ for each s.

The followeing lemma gives some of the general implication relations of these notions of asymptotic equivalence.

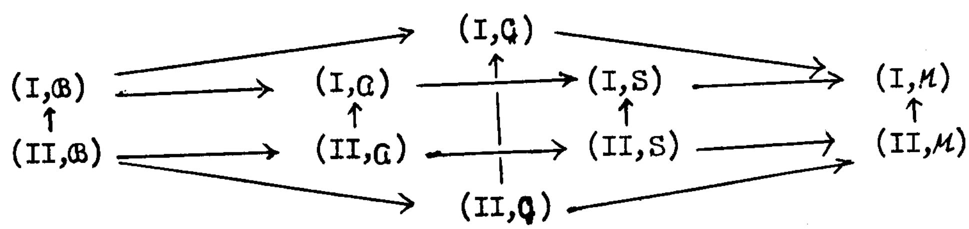
LEMMA 6.1. (a) $(I,\mathcal{C})$ is a weaker notion than $(II,\mathcal{C})$.

(b) Let $\left\{\mathcal{C}_{(n_s)}\right\}$ $(s=1,2,\ldots)$ and $\left\{\mathcal{C}'_{(n_s)}\right\}$ $(s=1,2,\ldots)$ be two sequences of basic spaces such that $\mathcal{C}'_{(n_s)} \subseteq \mathcal{C}_{(n_s)}$ for each s. Then, $(I,\mathcal{C}')$, or $(II,\mathcal{C}')$, is weaker than $(I,\mathcal{C})$, or $(II,\mathcal{C})$, respectively.

Hereafter, we shall use the symbol $\rightarrow$ to denote an implication relation. For example, the assertion(a) of the above lemma will be designated as $(II,\mathcal{C}) \rightarrow (I,\mathcal{C})$, or equivalently as $(I,\mathcal{C}) \leftarrow (II,\mathcal{C})$.

By the lemma stated above, then, we have the following diagram of implication relations among those types of notions of asymptotic equivalence which have been given in Section 2:

(6.1)



Of course, these implication relations are valid for any pair of sequences of random variables without any additional condition.

Now, we shall show the following theorem.

<u>THEOREM 6.1.</u> $(I,\mathfrak{B}) \leftrightarrows (I,\mathfrak{G})$, i.e., these types of notions are mutually equivalent.

<u>PROOF</u> From (6.1), it suffices to show that

$$(6.2) \qquad (I,\mathfrak{B}) \leftarrow (I,\mathfrak{G}).$$

For any given $\epsilon > 0$ and each s, there exists a member of $\mathfrak{B}_{(n_s)}$, $B^s_{(n_s)}$ say, such that

$$(6.3) \qquad 0 \leq d_I\,(X^s_{(n_s)}, Y^s_{(n_s)};\ \mathfrak{B}_{(n_s)}) - \left| P^{X^s_{(n_s)}}(B^s_{(n_s)}) - P^{Y^s_{(n_s)}}(B^s_{(n_s)}) \right| < \epsilon,$$

and there can be found two covering of $B^s_{(n_s)}$, $\mathcal{H}^s = \left\{ H^s_{(n_s)i} \right\}$ (i=1,2,...) and $\mathcal{K}^s = \left\{ K^s_{(n_s)j} \right\}$ (j=1,2,...), consisting of members of the class $\mathfrak{G}_{(n_s)}$, such that

$$\begin{cases} P^{X^s_{(n_s)}}(B^s_{(n_s)}) \leq \sum\limits_{i=1}^{\infty} P^{X^s_{(n_s)}}(H^s_{(n_s)i}) < P^{X^s_{(n_s)}}(B^s_{(n_s)}) + \epsilon, \\ P^{Y^s_{(n_s)}}(B^s_{(n_s)}) \leq \sum\limits_{j=1}^{\infty} P^{Y^s_{(n_s)}}(K^s_{(n_s)j}) < P^{Y^s_{(n_s)}}(B^s_{(n_s)}) + \epsilon, \end{cases}$$

where it can be assumed without any loss of generality that each of the both coverings consists of mutually disjoint members of $\mathcal{G}_{(n_s)}$. Let, for each s, $\mathcal{L}^s = \left\{ L^s_{(n_s)k} \right\}$ (k=1,2,...) be the product covering of the above two coverings, which is obtained by rearranging all subsets of the form $H^s_{(n_s)i} \cap K^s_{(n_s)j}$, i,j,=1,2,..., in any suitable manner. Note that $L^s_{(n_s)k}$ belongs to the class $\mathcal{G}_{(n_s)}$ for each k, and $L^s_{(n_s)k} \cap L^s_{(n_s)k'} = \emptyset$ $(k \neq k')$. Then, putting $L^s_{(n_s)} = \sum_{k=1}^{\infty} L^s_{(n_s)k}$, we have

$$
(6.4) \quad \begin{cases} P^{X^s_{(n_s)}}(B^s_{(n_s)}) \leq P^{X^s_{(n_s)}}(L^s_{(n_s)}) < P^{X^s_{(n_s)}}(B^s_{(n_s)}) + \epsilon, \\ P^{Y^s_{(n_s)}}(B^s_{(n_s)}) \leq P^{Y^s_{(n_s)}}(L^s_{(n_s)}) < P^{Y^s_{(n_s)}}(B^s_{(n_s)}) + \epsilon. \end{cases}
$$

It is noted that $L^s_{(n_s)}$ does not necessarily belong to $\mathcal{G}_{(n_s)}$.

Since, for each s, there exists a positive integer $N=N(\epsilon, s)$ such that

$$
\left| P^{X^s_{(n_s)}}(L^s_{(n_s)}) - P^{X^s_{(n_s)}}(L^s_{N(n_s)}) \right| < \epsilon \text{ and } \left| P^{Y^s_{(n_s)}}(L^s_{(n_s)}) - P^{Y^s_{(n_s)}}(L^s_{N(n_s)}) \right| < \epsilon,
$$

where we have put $L^s_{N(n_s)} = \sum_{k=1}^{N} L^s_{(n_s)k}$, it follows from (6.4) that

$$
\left| P^{X^s_{(n_s)}}(B^s_{(n_s)}) - P^{Y^s_{(n_s)}}(B^s_{(n_s)}) \right| < \left| P^{X^s_{(n_s)}}(L^s_{N(n_s)}) - P^{Y^s_{(n_s)}}(L^s_{N(n_s)}) \right| + 4\epsilon,
$$

from which we obtain

$$
(6.5) \quad d_I(X^s_{(n_s)}, Y^s_{(n_s)}; \mathcal{B}_{(n_s)}) \leq d_I(X^s_{(n_s)}, Y^s_{(n_s)}; \mathcal{G}_{(n_s)}) + 4\epsilon,
$$

because $L^s_{N(n_s)}$ belongs to $\mathcal{G}_{(n_s)}$ for each s.

Since ϵ is arbitrary, (6.5) implies that

$$(6.6)\quad d_I(X^s_{(n_s)}, Y^s_{(n_s)};\ \mathcal{B}_{(n_s)}) = d_I(X^s_{(n_s)},\ Y^s_{(n_s)};\ \mathcal{G}_{(n_s)}),$$

for every s, from which the theorem follows.

Type (I,S) asymptotic equivalence appears to be weaker than that of type (I,$\mathcal{G}$), or equivalently of type (I,$\mathcal{B}$), in an important case: In fact, the in law convergence of the standardized binomial variable with parameters n and p to the standard normal distribution as $n \to \infty$ is a convergence of type (I,S), as will be seen later, but it is not of type (I,$\mathcal{G}$).

Now, in the next place, we shall consider the implication relation between (I,S) and (I,$\mathcal{M}$) in the general case.

The following example shows that these two types of asymptotic equivalence are not necessarily mutually equivalent.

<u>EXAMPLE 6.1</u>. Let $X_1,\dots,X_s$ be a random sample of size s drawn from the uniform distribution on the interval [0,1), while $Y^s_1,\dots,Y^s_s$ be that from a distribution with p.d.f. given by

$$p_s(z)=\begin{cases} 1/s, & \text{if } -1\leq z<0,\\ 1-1/s, & \text{if } 0\leq z<1,\\ 0, & \text{otherwise.}\end{cases}$$

Then, for a set $E_{(s)}$ in $\mathcal{S}_{(s)}$ defined by

$$\mathcal{E}_{(n)}=[0,1)\times\dots\times[0,1)\ ,\ (\text{s times}),$$

one can observe, putting $X_{(s)}=(X_1,\dots,X_s)$ and $Y^s_{(s)}=(Y^s_1,\dots Y^s_s)$, that

$$\left| P^{X_{(s)}}(E_{(s)}) - P^{Y^s_{(s)}}(E_{(s)})\right| = 1-(1-1/s)^s \to 1-1/e,\ (s \to\infty),$$

which shows that $X_{(s)} \sim Y^s_{(s)}$ (I,S), $(s \to \infty)$, does not hold.

On the other hand, it is easy to see that

$$d_I(X_{(s)},Y^s_{(s)};\mu_{(s)}) \leqq 1/s, \text{ for every } s,$$

which implies that these two sequences are ASEQ(I,μ) as $s \to \infty$.

A sufficient condition, under which (I,S) and (I,μ) are mutually equivalent, is given by the following

<u>THEOREM 6.2</u>. If $n_s \leqq K$, $s = 1,2,\ldots$, for some constant K independent of s, then it holds that

$$(6.7) \qquad (I,S) \rightleftarrows (I,\mu)$$

for any pair of sequences $\{X^s_{(n_s)}\}$ $(s = 1,2,\ldots)$ and $\{Y^s_{(n_s)}\}$ $(s = 1,2,\ldots)$ with $X^s_{(n_s)}$, $Y^s_{(n_s)}$ belonging to $\mathfrak{F}(R_{(n_s)},\mathcal{B}_{(n_s)})$ for each s. In particular, the above equivalence is true in the case of equal basic spaces.

The proof of this theorem follows from (6.2), the assumption of the theorem and the following inequality

(6.8) $$d_I(X^s_{(n_s)}, Y^s_{(n_s)}; \mathcal{G}_{(n_s)}) \leq 2^{n_s} d_I(X^s_{(n_s)}, Y^s_{(n_s)}; \mathcal{H}_{(n_s)}).$$

for every s.

As for the implication relation between $(I,\mathcal{G})$ and (I,S), we have the following result, whose proof is easy and will be omitted.

THEOREM 6.3 If the sequences $\{X^s_{(n_s)}\}$ $(s=1,2\ldots)$ and $\{Y^s_{(n_s)}\}$ $(s=1,2,\ldots)$, are such that, for each s, $X^s_{(n_s)}$ and $Y^s_{(n_s)}$ are of the continuous type, i.e., are absolutely continuous with respect to the Lebesgue measure over $(R_{(n_s)}, \mathcal{B}_{(n_s)})$, then it holds for these sequences that

(6.9) $$(I,\mathcal{G}) \rightarrow (I,S).$$

Now, we proceed to the implication relations of type II series of notions of asymptotic equivalence.

The following theorem is due to Dr. R. M. Meyer.

THEOREM 6.4 For any pair of sequences of random variables it holds that

(6.10) $$(II,\mathcal{B}) \rightleftarrows (II,\mathcal{A}) \rightleftarrows (II,S).$$

PROOF. To show (6.10) it suffices to prove that

(6.11) $$(II,S) \rightarrow (II,\mathcal{B}).$$

Since any member of $\mathcal{A}_{(n_s)}$ is expressed as a union of a finite number of the members of $S_{(n_s)}$ for each s, it is easy to see that, for any given member $B^s_{(n_s)}$ of $\mathcal{B}_{(n_s)}$ and any given $\varepsilon > 0$, there exists a covering of $B^s_{(n_s)}$, $\{E^s_{(n_s)i}\}$ $(i = 1,2,\ldots)$ say, consisting of mutually disjoint members of $S_{(n_s)}$, such that

$$\left| P^{X^s_{(n_s)}}(B^s_{(n_s)}) - \sum_{i=1}^{\infty} P^{X^s_{(n_s)}}(E^s_{(n_s)i}) \right| < \varepsilon, \text{ and}$$

$$\left| P^{Y^s_{(n_s)}}(B^s_{(n_s)}) - \sum_{i=1}^{\infty} P^{Y^s_{(n_s)}}(E^s_{(n_s)i}) \right| > \varepsilon \quad ,$$

for each s.

Since the asymptotic equivalence (II,$\wp$) of both sequences of random variables considered above implies the existence of a sequence of positive numbers $\{q_s\}$ (s = 1,2,...) such that $q_s \to 0$ as $s \to \infty$, and

$$\left| P^{X^s_{(n_s)}}(E^s_{(n_s)i}) - P^{Y^s_{(n_s)}}(E^s_{(n_s)i}) \right| \leqq q_s P^{Y^s_{(n_s)}}(E^s_{(n_s)i}),$$

for every i and s, we have

$$\left| \sum_{i=1}^{\infty} P^{X^s_{(n_s)}}(E^s_{(n_s)i}) - \sum_{i=1}^{\infty} P^{Y^s_{(n_s)}}(E^s_{(n_s)i}) \right| \leqq q_s \sum^{\infty} P^{Y^s_{(n_s)}}(E^s_{(n_s)i}) .$$

Hence we get

$$\left| P^{X^s_{(n_s)}}(B^s_{(n_s)}) - P^{Y^s_{(n_s)}}(B^s_{(n_s)}) \right| < q_s P^{Y^s_{(n_s)}}(B^s_{(n_s)}) + (2 = q_s)\varepsilon ,$$

for each s. Since ε is arbitrarily small, and $B^s_{(n_s)}$ is also arbitrary and independent of ε, the above inequality implies that

(6.12) $$\left| P^{X^s_{(n_s)}}(B^s_{(n_s)}) - P^{Y^s_{(n_s)}}(B^s_{(n_s)}) \right| \leqq q_s P^{Y^s_{(n_s)}}(B^s_{n_s}) ,$$

for each s, which shows that the sequences are ASEQ(II,$\mathcal{B}$). This proves (6.11) and hence the theorem.

It is quite easy to see the following

COROLLARY 6.1. If both $X^s_{(n_s)}$ and $Y^s_{(n_s)}$ are of the continuous type, then it holds that

(6.13) $$(\mathrm{II},\mathcal{B}) \rightleftarrows (\mathrm{II},\mathfrak{a}) \rightleftarrows (\mathrm{II},\wp) \rightleftarrows (\mathrm{II},\mathcal{G}).$$

Type (II,μ) is a weaker notion than type (II,$\wp$), as will be seen in the following

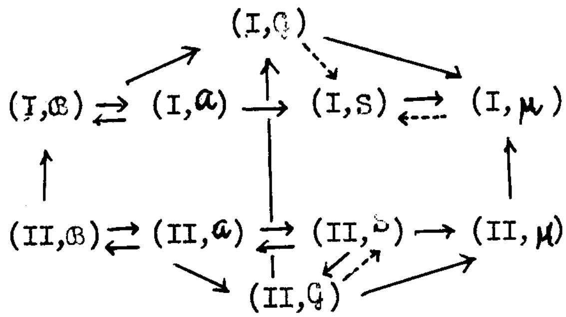
EXAMPLE 6.2. Let X_s be one-dimensional random variable whose p.d f is given by

$$p_s(z) = \begin{cases} s/(s-1) , & \text{if } 0 \leq z < 1-1/s , \\ 0, & \text{otherwise,} \end{cases}$$

for $s = 2,3,\ldots,$ while Y be the uniform distribution on $(0,1)$. Then clearly X_s converges to Y in the sense of type (II,μ), but not in the sense of type (II,S) as $s \nrightarrow \infty$, because it always holds that $d_{II}(X_s, Y; S_{(1)}) = 1$, $s = 1,2,\ldots$

Summarizing the results thus obtained we have the following diagram of implication relations between stronger types of notions of asymptotic equivalence in the general case, where the symbol ' $\longrightarrow$ ' designates an unconditional implication relation, while the symbol ' $\dashrightarrow$ ' does a conditional, for which a sufficient condition has been given in this section.

(6.14)



The following questions are interesting and left open.

(i) To find out useful equivalence conditions under which (I,S) and (I,ℬ) are mutually equivalent.

(ii) To find out equivalence dontitions for (II,S) and (II,μ).

(iii) To find out equivalence dontitions for (I,S) and (II,S), for (I,μ), and (II,μ), and for (I,ℬ) and (II,ℬ).

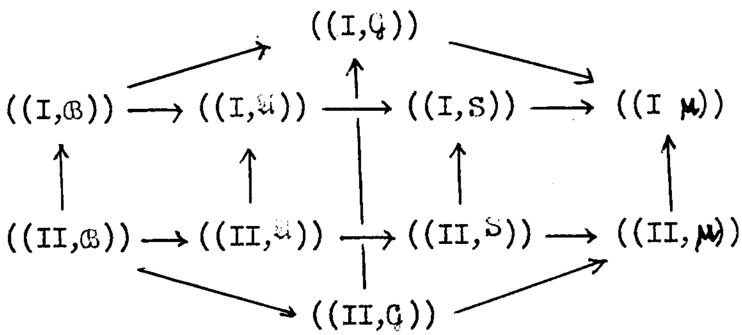
7. Implication relations of certain types of asymptotic equivalence in the case of equal basic spaces

In the case of equal basic spaces, we have defined, in addition to stronger types of notions, some weaker types of notions of asymptotic equivalence. The results obtained in the preceding section can equally be applied in the present case, that is, the implication relations shown by (6.14) hold in the case of equal basic spaces too. Furthermore the mutual equivalence of (I,S) and (I,μ) is unconditional. Equivalence conditions, however, for (II,S) and (II,μ) to be mutually equivalent are still unknown even in the case of equal basic spaces, as is seen from Example 6.2.

It is evident from definitions that $(I,\mathcal{C})$ is a stronger notion than $((I,\mathcal{C}))$ for any $\mathcal{C}_{(n)}$ included by $\mathcal{B}_{(n)}$, and the same is true for $(II,\mathcal{C})$ and $((II,\mathcal{C}))$.

Now we shall consider the implication relations of weaker types of notions. It is clear that the following diagram of implication relations is obtained:

(7.1)



It is shown easily that

<u>THEOREM 7.1</u>. It always holds that

(7.2) $$((I,\mathfrak{A})) \rightleftarrows ((I,S)) \rightleftarrows ((I,\mu))$$

PROOF. To prove (7.2) it is sufficient to show that

(7.3) $$((I,\mathcal{M})) \rightarrow ((I,\mathcal{A})).$$

Since any given member of $\mathcal{A}_{(n)}$ is expressed as the union of a finite number of mutually disjoint members of $S_{(n)}$, and any given member of $S_{(n)}$ is expressed in a form by a finite number of members of $\mathcal{M}_{(n)}$, we have a formal expression such as

$$A_{(n)} = \sum_{i=1}^{N} \sum_{j=1}^{K} c_{ij} E_{(n)ij} ,$$

for any member $A_{(n)}$ of $\mathcal{A}_{(n)}$, where c_{ij} $(=\pm 1)$, N and K are constants depending on $A_{(n)}$, and $E_{(n)ij}$ are the members of $\mathcal{M}_{(n)}$. Hence

$$|P^{X^s(n)}(A_{(n)})-P^{Y^s(n)}(A_{(n)})| \leqq \sum_{i=1}^{N} \sum_{j=1}^{K} |P^{X^s(n)}(E_{(n)ij})-P^{Y^s(n)}(E_{(n)ij})|$$

for each s, from which (7.3) follows.

$((I,\mathcal{B}))$ is stronger than $((I,\mathcal{A}))$ as will be seen in the following

EXAMPLE 7.1 Let X_s and Y_s be one-dimensional random variables whose p.d.f.'s are given respectively by

$$p_s(z) = \begin{cases} 2/s, & \text{if } i \leqq z < i+1/2 , \ i=0,1,\ldots s-1 \\ 0 , & \text{otherwise,} \end{cases}$$

and

$$q_s(z) = \begin{cases} 2/s, & \text{if } i+1/2 \leqq z < i+1, \ i=0,1,\ldots,s-1 \\ 0, & \text{otherwise,} \end{cases}$$

for s=1,2,... Then, it is clear that $\{X_s\}$ (s=1,2,...) and $\{Y_s\}$ (s=1,2,...) are asymptotically equivalent in the sense of type $((I,\mathcal{A}))$, but not in the sense of type $((I,\mathcal{B}))$, as $S \rightarrow \infty$.

In the next place, we shall show the following

THEOREM 7.2 It always holds that

$$((\mathrm{II},\mathcal{A})) \rightleftarrows ((\mathrm{II},\mathrm{S})). \tag{7.3}$$

PROOF. For any given $A_{(n)}$ of $\mathcal{A}_{(n)}$, there exist a positive integer N and N members of $S_{(n)}$, $E_{(n)1},\ldots,E_{(n)N}$ say, such that

$$A_{(n)} = \bigcup_{i=1}^{N} E_{(n)i} ,$$

from which it follows immediately that

$$\left| \frac{P^{X^s_{(n)}}(A_{(n)})}{P^{Y^s_{(n)}}(A_{(n)})} - 1 \right| \leqq \sum_{i=1}^{N} \left| \frac{P^{X^s_{(n)}}(E_{(n)i})}{P^{Y^s_{(n)}}(E_{(n)i})} - 1 \right| ,$$

for each s. This implies that $((\mathrm{II},\mathrm{S})) \rightarrow ((\mathrm{II},\mathcal{A}))$, and the theorem follows from (7.1).

The following two examples shows in turn that ((II,S)) and $((\mathrm{II},\mu))$, and $((\mathrm{II},\mathcal{B}))$ and $((\mathrm{II},\mathcal{A}))$ are not necessarily mutually equivalent.

EXAMPLE 7.2 Let X_s and Y_s be one-dimensional random variables whose p.d.f. being given respectively by

$$p_s(z) = \begin{cases} 1-1/s, & \text{if } -1 \leqq z < 0, \\ 1/s , & \text{if } 0 \leqq z < 1, \\ 0 , & \text{otherwise}, \end{cases}$$

and

$$q_s(z) = \begin{cases} 1-1/s, & \text{if} -1 \leqq z < 0, \\ 1/(2s), & \text{if } 0 \leqq z < 2, \\ 0 , & \text{otherwise}, \end{cases}$$

for s = 1,2,... Then, it is checked easily that these two sequences are $\mathrm{ASEQ}((\mathrm{II},\mu))$, but not ASEQ((II,S)), as $s \rightarrow \infty$.

EXAMPLE 7.3. Let X_s and Y_s be one-dimensional random variables whose p.d.f. being given respectively by

$$p_s(z) = \begin{cases} 1/(2s), & \text{if } 0 \leqq z < 2s, \\ 0, & \text{otherwise}, \end{cases}$$

$$q_s(z) = \begin{cases} (1-\frac{i}{i+s})/(2s), & \text{if } 2i \leqq z < 2i+1;\ i = 0,1,\ldots,s-1, \\ (1+\frac{i}{i+s})/(2s), & \text{if } 2i-1 \leqq z < 2i;\ i = 1,2,\ldots,s, \\ 0, & \text{otherwise}, \end{cases}$$

for $s = 2,3,\ldots$. Then, it is evident that these two sequences are ASEQ((II,S)), or equivalently ASEQ((II,$\mathfrak{A}$)), as $s \to \infty$. If we take, however, the set E defined by

$$E = \sum_{i=0}^{\infty} [2i, 2i+1),$$

which is a member of $\omega_{(1)}$, then it holds that

$$P^{X_s}(E) = 1/2 \quad \text{and} \quad P^{Y_s}(E) = \sum_{i=0}^{s-1} \frac{1}{2s}\left(1-\frac{i}{i+s}\right),$$

for each s. Hence

$$P^{X_s}(E)/P^{Y_s}(E) = \sum_{i=0}^{s-1} \frac{1}{i+s} \to \log 2, \quad (s \to \infty).$$

which shows that the above sequences are not ASEQ((II,$\mathfrak{B}$)).

Now, we return to the implication relation of ((I,$\mathfrak{A}$)) and ((I,$\mathfrak{B}$)), and give an equivalence condition under which they are mutually equivalent.

THEOREM 7.3. If both of the sequences of random variables, $\{X^s_{(n)}\}(s = 1,2,\ldots)$ and $\{Y^s_{(n)}\}(s = 1,2,\ldots)$, have the property C ($\mathfrak{B}$), and at least one of them has the property B(S), then it holds that

(7.4) $\quad$ ((I,$\mathfrak{B}$)) $\rightleftarrows$ ((I,$\mathfrak{A}$)).

PROOF. As we stated in Section 5, it is seen from the assumption that both of the sequences have the property $B(\mathcal{B})$. Hence, in order to prove the theorem, it suffices to show that the asymptotic equivalence of the sequences in the sense of type $((I,\mathcal{A}))$ implies that

$$(7.5) \qquad \left| P^{X^s_{(n)}}(E_{(n)}) - P^{Y^s_{(n)}}(E_{(n)}) \right| \to 0, \ (s \to \infty),$$

for any member $E_{(n)}$ of $\mathcal{B}_{(n)}$ whose closure is compact.

Let $\mu_{(n)}$ be the usual Lebesgue measure over $(R_{(n)}, \mathcal{B}_{(n)})$. Then for any given $\delta > 0$, there exists a covering of $E_{(n)}$, $\{A_{(n)i}\}$ $(i = 1,2,\ldots)$ say, consisting of mutually disjoint members of $\mathcal{A}_{(n)}$, such that

$$(7.6) \qquad \left| \mu_{(n)}(E_{(n)}) - \mu_{(n)}\left(\sum_{i=1}^{\infty} A_{(n)i}\right) \right| < \delta .$$

For any given $\varepsilon > 0$, we can chose, then, by the property $C(\mathcal{B})$ of both sequences, such a value of δ in (7.6) that

$$(7.7) \qquad \begin{aligned} &\left| P^{X^s_{(n)}}(E_{(n)}) - P^{X^s_{(n)}}\left(\sum_{i=1}^{\infty} A_{(n)i}\right) \right| < \varepsilon , \\ &\left| P^{Y^s_{(n)}}(E_{(n)}) - P^{Y^s_{(n)}}\left(\sum_{i=1}^{\infty} A_{(n)i}\right) \right| > \varepsilon , \end{aligned}$$

for all $s \leqq s_o'$ for some positive integer s_o' .

For the same values of ε and δ as above, there exists a positive integer N depending on δ such that

$$\mu_{(n)}\left(\sum_{i=N+1}^{\infty} A_{(n)i}\right) < \delta \qquad ,$$

and hence

$$(7.8) \qquad P^{X^s_{(n)}}\left(\sum_{i=NH}^{\infty} A_{(n)i}\right) < \varepsilon \quad \text{and} \quad P^{Y^s_{(n)}}\left(\sum_{i=NH}^{\infty} A_{(n)i}\right) < \varepsilon \quad ,$$

for all $s \geq s_o''$ for some s_o'' .

From (7.7) and (7.8) it now follows that

$$\left| P^{X^s_{(n)}}(E_{(n)}) - P^{X^s_{(n)}}(\sum_{i=1}^{N} A_{(n)i}) \right| < 2\varepsilon ,$$

and

$$\left| P^{Y^s_{(n)}}(E_{(n)}) - P^{Y^s_{(n)}}(\sum_{i=1}^{N} A_{(n)i}) \right| < 2\varepsilon ,$$

for all $s \leq s_o$, where $s_o = \max(s_o', s_o'')$, from which (7.5) easily follows, and the theorem.

From (7.1), Theorem 7.1 and Theorem 7.3 it immediately follows that

COROLLARY 7.1. Under the same conditions as in Theorem 7.3, it holds that

$$(7.9) \qquad ((I,\mathcal{B})) \rightleftarrows ((I,\mathcal{C})) \rightleftarrows ((I,\mathcal{S})) \rightleftarrows ((I,\mu)) \rightleftarrows ((I,\mathcal{Q})).$$

The following theorem gives a sufficient condition for $((I,\mathcal{Q}))$ and $((I,\mathcal{S}))$ to be mutually equivalent.

THEOREM 7.4. If one of the sequences of $\{X^s_{(n)}\}$ $(s = 1,2,\ldots)$ and $\{Y^s_{(n)}\}$ $(s = 1,2,\ldots)$ has the properties C(S) and B(S), then it holds that

$$(7.10) \qquad ((I,\mathcal{Q})) \rightleftarrows ((I,\mathcal{S})) .$$

PROOF. From Theorem 7.1 and the diagram (7.1), it follows immediately that

$$((I,\mathcal{Q})) \rightarrow ((I,\mathcal{S})),$$

without any additional condition. Hence, in order to show (7.10), it suffices to prove

$$(7.11) \qquad ((I,\mathcal{S})) \rightarrow ((I,\mathcal{Q})).$$

Suppose now that the sequences are ASEQ((I,S)). Then, the properties C(S) and B(S) of one of the sequences are both brought over to the other sequence, and therefore it necessarily follows, under the condition of the theorem, that both of the sequences have the properties C(S) and B(S).

Let $G_{(n)}$ be any given member of $\mathcal{G}_{(n)}$, whose closure being compact Then, for any given $\delta > 0$, there exist positive integers N and N' ($N \leq N'$), and a set of mutually disjoint members of $S_{(n)}$, $\{E_{(n)i}\}$ ($i = 1, \ldots, N, \ldots, N'$) say, such that

$$\sum_{i=1}^{N} E_{(n)i} \subseteq G_{(n)} \subseteq \sum_{i=1}^{N'} E_{(n)i} \quad \text{and} \quad \mu_{(n)}\left(\sum_{i=N+1}^{N'} E_{(n)i}\right) < \delta \quad ,$$

where $\mu_{(n)}$ is the Lebesgue measure over $(R_{(n)}, \mathcal{B}_{(n)})$.

For any given $\varepsilon > 0$, then, we can choose δ above so small that

$$\left| P^{X^s_{(n)}}(G_{(n)}) - P^{X^s_{(n)}}\left(\sum_{i=1}^{N} E_{(n)i}\right) \right| < \varepsilon \quad ,$$

and

$$\left| P^{Y^s_{(n)}}(G_{(n)}) - P^{Y^s_{(n)}}\left(\sum_{i=1}^{N} E_{(n)i}\right) \right| < \varepsilon \quad ,$$

for all $s \geq s_o$ for some s_o depending only on ε, from which it follows that

$$(7.12) \quad \left| P^{X^s_{(n)}}(G_{(n)}) - P^{Y^s_{(n)}}(G_{(n)}) \right| < \sum_{i=1}^{N} \left| P^{X^s_{(n)}}\left(\sum_{i=1}^{N} E_{(n)i}\right) - P^{Y^s_{(n)}}\left(\sum_{i=1}^{N} E_{(n)i}\right) \right| + 2\varepsilon \quad ,$$

for all $s \geq s_o$.

(7.11) now follows from (7.12), and the proof of the theorem is completed.

In the final place, we shall prove the following theorem, which is useful for many cases of application.

THEOREM 7.5. If one of the sequences $\{X^s_{(n)}\}(s = 1,2,\ldots)$ and $\{Y^s_{(n)}\}(s = 1,2\ldots)$ has the properties $C(S)$ and $B(S)$, then it holds that

$$(7.13) \qquad (I,S) \rightleftarrows ((I,S)).$$

PROOF. To prove the theorem it suffices to show that

$$(7.14) \qquad ((I,S)) \rightarrow (I,S).$$

Suppose the sequences are ASEQ((I,S)). Then, both of the sequences have the properties $C(S)$ and $B(S)$. Suppose, on the contrary to (7.14), the sequences are not ASEQ(I,S). Then we can assume, without any harm in the proof below, that there exist a positive constant ρ and a sequence of members of $S_{(n)}$, $\{E^s_{(n)}\}$ $(s = 1,2,\ldots)$ say, such that

$$(7.15) \qquad \left| P^{X^s_{(n)}}(E^s_{(n)}) - P^{Y^s_{(n)}}(E^s_{(n)}) \right| \geqq 2\rho$$

for all s.

By the property $B(S)$ of both sequences, there exist a positive integer s_o and a member $B_{(n)}$ of $S_{(n)}$, whose closure being compact, such that

$$P^{X^s_{(n)}}(B_{(n)}) \geqq 1-\rho/2 \quad \text{and} \quad P^{Y^s_{(n)}}(B_{(n)}) \geqq 1-\rho/2$$

for all $s \geqq s_o$. Thus, putting $A^s_{(n)} = E^s_{(n)} \cap B_{(n)}$ for each s, we have

$$(7.16) \qquad \left| P^{X^s_{(n)}}(A^s_{(n)}) - P^{Y^s_{(n)}}(A^s_{(n)}) \right| \geqq \rho$$

for all $s \geqq s_o$. Here, it is noted that $E^s_{(n)}$'s are members of $S_{(n)}$.

Now, since the closure of $A^s_{(n)}$ is bounded uniformly for all s, there exist a subsequence $\{s'\}(s' \to \infty)$ of $\{s\}$ and a subset $A_{(n)}$ of $R_{(n)}$ such that

$$(7.17) \qquad \lim_{s' \to \infty} A^{s'}_{(n)} \;(= \liminf_{s' \to \infty} A^{s'}_{(n)} = \limsup_{s' \to \infty} A^{s'}_{(n)}\;) = A_{(n)} \,.$$

For this set $A_{(n)}$, there can be found a member $E_{(n)}$ of $S_{(n)}$ such that the symmetric difference $E_{(n)} \triangle A_{(n)}$ is contained in the boundary set of $E_{(n)}$, and hence the closure of $E_{(n)}$ and that of $A_{(n)}$ are identical, and, of course, they are compact.

By the property $C(S)$ of both sequences, then, for any given $\varepsilon > 0$, there exist a set of members of $S_{(n)}$, $\{F_{(n)i}\}(i = 1, \ldots N)$, $N = 2^n$, and a positive integer s_o' such that

$$E_{(n)} \triangle A_{(n)} \subseteqq \bigcup_{i=1}^{N} F_{(n)i}$$

and

$$P^{X^s_{(n)}}\left(\bigcup_{i=1}^{N} F_{(n)i}\right) < \varepsilon \quad \text{and} \quad P^{Y^s_{(n)}}\left(\bigcup_{i=1}^{N} F_{(n)i}\right) < \varepsilon$$

for all $s \geqq s_o'$. Therefore, it holds that

$$(7.18) \quad \begin{cases} \left| P^{X^{s'}_{(n)}}(E_{(n)}) - P^{X^{s'}_{(n)}}(A_{(n)}) \right| < \varepsilon , \\ \\ \left| P^{Y^{s'}_{(n)}}(E_{(n)}) - P^{Y^{s'}_{(n)}}(A_{(n)}) \right| < \varepsilon , \end{cases}$$

for all $s' \geqq s_o'$.

On the other hand, from (7.17) it follows that there exists a positive integer s_o'' such that

$$A_{(n)} \triangle A^{s'}_{(n)} \subseteqq \bigcup_{i=1}^{N} F_{(n)i} ,$$

for all $s \geqq s_o''$. Hence, if $s' \geqq \max(s_o', s_o'')$, it holds that

$$(7.19) \quad \begin{cases} \left| P^{X^{s'}_{(n)}}(A_{(n)}) - P^{X^{s'}_{(n)}}(A^{s'}_{(n)}) \right| < \varepsilon \\ \\ \left| P^{Y^{s'}_{(n)}}(A_{(n)}) - P^{Y^{s'}_{(n)}}(A^{s'}_{(n)}) \right| < \varepsilon \end{cases}$$

(7.18) and (7.19) imply that

$$(7.20) \qquad \left| P^{X^{s'}_{(n)}}(A^{s'}_{(n)}) - P^{Y^{s'}_{(n)}}(A^{s'}_{(n)}) \right| \leq \left| P^{X^{s'}_{(n)}}(E_{(n)}) - P^{Y^{s'}_{(n)}}(E_{(n)}) \right| + 4\varepsilon ,$$

for all $s' \geqq \max(s'_o, s''_o)$, which contradict (7.16), because the first member of the right-hand side of (7.20) tends to zero as $s \to \infty$. This proves the theorem.

Whence we immediately have the following

<u>COROLLARY 7.2</u>. Under the same condition as in the above theorem it holds that

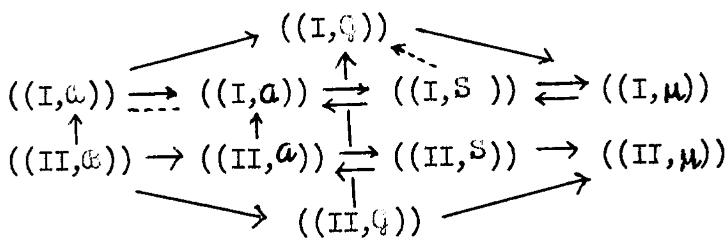
$$(7.21) \qquad (I,S) \rightleftarrows (I,\mu) \rightleftarrows ((I,\mathcal{A})) \rightleftarrows ((I,\mathcal{S})) \rightleftarrows ((I,\mu)) \rightleftarrows ((I,\mathcal{G})).$$

From Theorem 7.5 it is straightforward that

<u>COROLLARY 7.3</u>. If a sequence of random variables, $\{X^s_{(n)}\}(s = 1,2,\ldots)$, converges in law to a certain distribution, $Y_{(n)}$, which is of the continuous type, then this convergence is of type (I,S).

We summarize the results thus obtained to get the following diagram of implication relations of weaker notions of asymptotic equivalence in the case of equal basic spaces.

(7.22)



Some questions are left open for completing the above diagram of implication relations, among which it is interesting to find out sufficient conditions (as weak as possible) for $((I,S))$ and $((II,\mathcal{S}))$, for $((I,\mu))$ and $((II,\mu))$, for $((II,\mathcal{G}))$ and $((II,\mathcal{A}))$, or for $((II,S))$ and $((II,\mu))$, to be mutually equivalent.

8 Implication relation of notions of asymptotic independence.

For the notions of asymptotic independence defined in Section 4, the second sequence $\{Y^s_{(n_s)}\}$ (s = 1,2,...) is determined completely from the first sequence $\{X^s_{(n_s)}\}$ (s = 1,2, ...), whose marginals are defined to be asymptotically independent, in a manner described in that section.

Hence, among the conditional implication relations of notions of asymptotic equivalence obtained in the preceding two sections, those results which have conditions imposed on the dimensions of basic spaces and/or on either one of the sequences to be compared are transferred to the same implication relations of notions of asymptotic independence with slight modification on the statement of the conditions.

It is clear that all results on the absolute implication relations obtained before are transferred to the corresponding results on those of notions of asymptotic independence.

In the first place, we shall prepare the following lemmas:
Let, as before, $\{X^s_{(n)} = (X^s_1(m^s_1),\ldots,X^s_{k_s}(m^s_{k_s}))\}$ (s = 1,2,...) be a sequence of random variables, whose marginals

$$\{X^s_1(m^s_1), \ldots, X^s_{k_s}(m^s_{k_s})\} \tag{8.1}$$

for s = 1,2, ..., are under investigation, while let $Y^s_{(n)}=(Y^s_1(m^s_1), \ldots, Y^s_{k_s}(m^s_{k_s}))$ (s = 1,2,...) be a sequence of random variables such that the i-th marginal is identically distributed with that of $X^s_{(n)}$ for each s and i; i = 1, ..., k_s, and the set of marginals of $Y^s_{(n)}$ corresponding to (8.1) is an independent system for each s.

Under this situation, it is clear that

LEMMA 8.1 If $\{X^s_{(n)}\}$ (s = 1,2, ...) has property B(S), then $\{Y^s_{(n)}\}$

(s = 1,2,...) does the same property.

From Lemma 5 2 (b), it follows easily that

<u>LEMMA 8 2</u>. Suppose that the system of random variables (8 1) is ASIN((I,$\mathcal{C}$)) as $s \to \infty$, and the sequence $\{X^s_{(n)}\}$(s = 1,2,) has property C($\mathcal{C}$), where $\mathcal{C}_{(n)}$ is any subclass of $\mathcal{B}_{(n)}$ for which the stated property is well defined. Then, the sequence $\{Y^s_{(n)}\}$(s = 1,2,) has the same property C($\mathcal{C}$)

In the general case, the following lemma is easily verified:

<u>LEMMA 8.3</u> If, for each s, $X^s_{(n_s)}$ in (4 1) is of the continuous type, the corresponding variable $Y^s_{(n_s)}$ in (4 3) is also of the same type

Now, Theorems 6 1,6 2 and 6.4 hold for notions of asymptotic independence without any change of the statement Theorem 6 3 can be restated, by Lemma 8.3, as follows:

<u>COROLLARY 8 1</u> If $X^s_{(n_s)}$ is of the continuous type for each s, then, for two notions of asymptotic independence, (I,$\mathcal{G}$) and (I,S), it holds that

(8 2) $\qquad$ (I,$\mathcal{G}$) $\to$ (I,S)

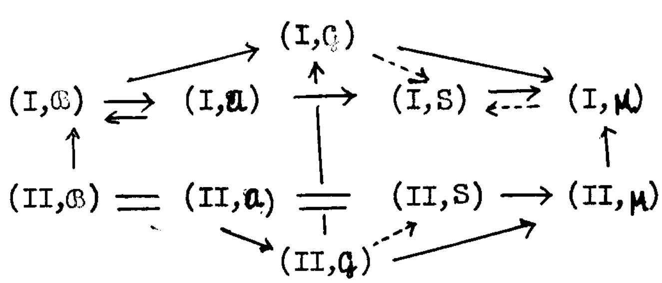
It is also easy to see, corresponding to Corollary 6 1, that

<u>COROLLARY 8 2</u> Under the same condition of Corollary 8 1, it holds that

(8 3) $\qquad$ (II,$\mathcal{G}$) $\rightleftarrows$ ((II,S).

Thus we have the same diagram as that of (6 14) on the implication relations of notions of asymptotic independence in the general case:

(8.4)



In the case of equal basic spaces, we have the same diagram as above except for one relation: In this case, (I,S) and (I,μ) are mutually equivalent without any additional condition

As for the implication relations of weaker types of notions of asymptotic independence in the case of equal basic spaces, we have the following results by using Lemmas 8.1 and 8 2

First, corresponding to Theorem 7 4, we get

<u>COROLLARY 8 3</u>. If the sequence $\{X^s_{(n)}\}(s = 1,2, \ldots)$ has both of the properties C(S) and B(S), then it holds that

(8 5) $$((I,\mathfrak{G})) \rightleftarrows ((I,S))$$

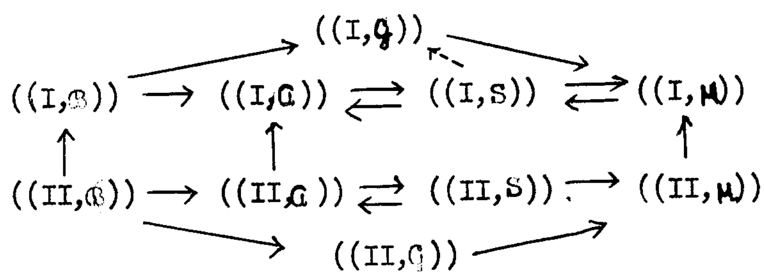
From Theorem 7.5 we have

<u>COROLLARY 8 4</u> If $\{X^s_{(n)}\}(s = 1,2, \ldots)$ has the properties C(S) and B(S), then it holds that

(8.6) $$(I, S) \rightleftarrows ((I,S)).$$

Hence, we obtain the following diagram of implication relations of weaker types of notions of asymptotic independence in the case of equal basic spaces.

(8.7)



In concluding this section, it is noted that we have not any result on equivalence conditions, under which type ((I,$\mathfrak{B}$)) and type ((I,$\mathfrak{A}$)) asymptotic independence are mutually equivalent. If we can prove the result of Lemma 8 2, without the first condition that (8.1) is ASIN((I,C)), we can obtain a corresponding result to Theorem 7 3

III

Type $(I,\mathfrak{B})$ Asymptotic Equivalence and some of their Applications

9. Some properties of type $(I,\mathfrak{B})$ asymptotic equivalence and type $(I,\mathfrak{B})$ asymptotic independence

Notion of type $(I,\mathfrak{B})$ asymptotic equivalence defined in Section 2 is seen, from (1 3), to be equivalent to that of type I defined in the earlier paper [1], if we confine ourselves to Euclidean basic spaces and to the family of all probability distributions which are absolutely continuous with respect to any given σ-finite measure over each basic space

In this section we shall sketch the fundamental properties of type $(I,\mathfrak{B})$ asymptotic equivalence, most of which have been presented in [1]

Let, as before, $\{X^s_{(n_s)}\}(s = 1,2, \quad)$ and $\{Y^s_{(n_s)}\}(s = 1,2 \quad)$ be two sequences of random variables, where $X^s_{(n_s)}$ and $Y^s_{(n_s)}$ belong $\mathfrak{P}(R_{(n_s)}, \mathfrak{B}_{(n_s)})$ for each s. Type $(I,\mathfrak{B})$ asymptotic equivalence of these two sequences was defined by the condition

$$(9.1) \quad d_I(X^s_{(n_s)},Y^s_{(n_s)}) = \sup_{E_{(n_s)}\in\mathfrak{B}_{(n_s)}} \left| P^{X^s_{(n_s)}}(E_{(n_s)}) - P^{Y^s_{(n_s)}}(E_{(n_s)})\right| \to 0 \;,\; (s\to\infty)$$

In the case of equal basic spaces where $n_s = n$ for all s, this gives a notion of type $(I,\mathfrak{B})$ convergence, if we take a certain fixed distribution $Y_{(n)}$ instead of $Y^s_{(n)s}$.

The following lemma gives equivalent conditions to type $(I,\mathfrak{B})$ asymptotic equivalence in the general case, which is a restatement of Lemma 2.1.

LEMMA 9.1. The condition (9.1) for type $(I,\mathfrak{B})$ asymptotic equivalence is equivalent to each one of the following conditions.

(a) For every sequence of subsets, $\{E^s_{(n_s)}\}$ (s = 1,2...), with $E^s_{(n_s)} \in \mathcal{B}_{(n_s)}$ for each s, it holds that

$$(9.2) \qquad \left| P^{X^s_{(n_s)}}(E^s_{n_s}) - P^{Y^s_{(n_s)}}(E^s_{(n_s)}) \right| \to 0, \quad (s \to \infty).$$

(b) There exists a sequence of positive numbers, $\{q_s\}$(s = 1,2,...), depending only on $\{\mathcal{B}_{(n_s)}\}$(s = 1,2,...), such that $q_s \to \infty$ $(s \to \infty)$ and

$$(9.3) \qquad \left| P^{X^s_{(n_s)}}(E^s_{(n_s)}) - P^{Y^s_{(n_s)}}(E^s_{(n_s)}) \right| \leqq q_s, \quad (s = 1,2,\ldots).$$

for any sequence $\{E^s_{(n_s)}\}$(s = 1,2,...) with $E^s_{(n_s)} \in \mathcal{B}_{(n_s)}$ for each s.

Now, let $(R_{(m_s)}, \mathcal{B}_{(m_s)})$ (s = 1,2,...) be another sequence of Euclidean spaces with Borel fields, where we assume $n_s \geqq m_s$ for each s, Let further $f_{(n_s,m_s)}(z_{(n_s)})$ be a measurable transformation with defining space $R_{(n_s)}$ and range space $R_{(m_s)}$ for each s. This transformation gives us sequences of transformed variables,

$$(9.4) \qquad \bar{X}^s_{(m_s)} = f^s_{(n_s,m_s)}(X^s_{(n_s)}) \quad \text{and} \quad \bar{Y}^s_{(m_s)} = f^s_{(n_s,m_s)}(Y^s_{(n_s)}),$$

s = 1,2,..., where $\bar{X}^s_{(m_s)}$ and $Y^s_{(m_s)}$ belong to $\mathfrak{F}(R_{(m_s)}, \mathcal{B}_{(m_s)})$ for each s.

Then it is immediate to see the following

<u>THEOREM 9.1.</u> $X^s_{(n_s)} \sim Y^s_{(n_s)}$ $(I,\mathcal{B})$, $(s \to \infty)$, implies that $\bar{S}^s_{(m_s)} \sim \bar{Y}^s_{(m_s)}(I,\mathcal{B})$, $(s \to \infty)$. The coverse is true if the transformation $f^s_{(n_s,m_s)}$ is non-singular for each s.

The following is immediate from this theorem or the definition of type $(I,\mathcal{B})$ asymptotic equivalence itself.

COROLLARY 9.1. Let $X^s_{(k_s)} = X^s_{i_{k_s}}$ and $Y^s_{(k_s)} = (Y^s_{i_1}, \ldots, Y^s_{i_{k_s}})$ be any marginals of $X^s_{(n_s)}$ and $Y^s_{(n_s)}$ respectively for each s, where k_s and the choice of $\{i_1, \ldots, i_{k_s}\}$ out of $\{1, \ldots, n_s\}$ may depend on s. Then, $X^s_{(n_s)} \sim Y^s_{(n_s)}$ $(I, \mathcal{B})$, $(s \to \infty)$, implies that $X^s_{(k_s)} \sim Y^s_{(k_s)} (I, \mathcal{B})$, $(s \to \infty)$.

Next, we shall proceed to the asymptotic equality of moments of asymptotically equivalent $(I, \mathcal{B})$ sequences of random variables, $\{X^s_{(n_s)}\}(s = 1,2,\ldots)$ and $\{Y^s_{(n_s)}\}(s = 1,2,\ldots)$ in the general case. Let $f^s_{(n_s,1)}(z_{(n_s)})$ be a measurable transformation from $R_{(n_s)}$ to $R_{(1)}$ for each s. Put

$$\bar{X}_s = f^s_{(n_s,1)}(X^s_{(n_s)}) \quad \text{and} \quad \bar{Y}_s = f^s_{(n_s,1)}(Y^s_{(n_s)}),$$

for each s. Then, Theorem 9.1 assures us that if the original sequences are ASEQ$(I, \mathcal{B})$ as $s \to \infty$, then $\{\bar{X}_s\}(s = 1,2,\ldots)$ and $\{\bar{Y}_s\}(s = 1,2,\ldots)$ are also ASEQ $(I, \mathcal{B})$

Now let us ask what conditions guarantee the symptotic equality of expected values of these two sequences of induced random variables.

This problem will be turn out to the following: Let $\{X_s\}(s = 1,2, \ldots)$ and $\{Y_s\}(s = 1,2,\ldots)$ be sequences of one-dimensional random variables, which are ASEQ $(I, \mathcal{B})$ as $s \to \infty$, and let us denote the expected values of X_s and Y_s by $M(X_s)$ and $M(Y_s)$, respectively, for each s. Then the problem is to find out the conditions under which it holds that

$$\left| M(X_s) - M(Y_s) \right| \to 0, \quad (s \to \infty). \tag{9.5}$$

This question is not a trivial one: For example, the existence of the expected values $M(X_s)$ and $M(Y_s)$ is not necessarily sufficient for (9.5), as is seen in the following example.

EXAMPLE 9.1. Suppose X_s and Y_s be distributed according to the normal distributions $N(0,\delta_s^2)$ and $N(1,\delta_s^2)$, respectively, for each s, where $\delta_s \to \infty$ as $s \to \infty$. Then, by using Theorem 1.4.2 of [1] or Theorem 11.2 in later section, it is seen that $X_s \sim Y_s$ $(I,\mathcal{B})$ as $s \to \infty$. However, $M(X_s) = 0$ and $M(Y_s) = 1$ for all s, which shows that (9.5) does not hold.

To answer the above question, we prepare the following lemma which is an extension of the relation (1.3).

LEMMA 9.2 Let $X_{(n)}$ and $Y_{(n)}$ be two members of $\mathfrak{F}(R_{(n)},\mathcal{B}_{(n)})$, and let $P_{(n)}$ and $Q_{(n)}$ be the corresponding probability measures. Then, it holds that

$$2\, d_I(X,Y;\mathcal{B}) = \int_R |d(P - Q)|, \tag{9.6}$$

where we have dropped the suffices for dimension, and the right hand member is given by

$$\int_R |d(P-Q)| = \lim_{\delta \to 0} \sum_{E \in \Gamma_\delta(R,\mathcal{B})} |P(E)-Q(E)| \tag{9.7}$$

$\Gamma_\delta(R,\mathcal{B})$ being a partition of R consisting of (denumerable number of) members of $\mathcal{B}_{(n)}$ whose radii are all less than δ.

PROOF. For any given δ and a partition $\Gamma_\delta(R,\mathcal{B})$, let Γ_δ^* and Γ_δ^{**} be subclasses of $\Gamma_\delta(R,\omega)$ consisting of all such subsets E in $\Gamma_\delta(R,\mathcal{B})$ that $P(E)-Q(E) > 0$ and < 0, respectively. Then, we have

$$\begin{aligned}\sum_{E \in \Gamma_\delta(R,\mathcal{B})} |P(E)-Q(E)| &= \sum_{E \in \Gamma_\delta^*} |P(E)-Q(E)| + \sum_{E \in \Gamma_\delta^{**}} |P(E)-Q(E)| \\ &= (P(F^*) - Q(F^*)) + (Q(F^{**}) - P(F^{**})),\end{aligned}$$

where F^* and F^{**} denote the unions of all members of Γ_δ^* and Γ_δ^{**} respectively. Since F^* and F^{**} belong to $\mathcal{B}_{(n)}$, it follows from the above equality that

$$2\, d_I(X,Y;\mathcal{B}) \geqq \int_R |d(P-Q)| \tag{9.8}$$

For any given $\mathcal{E} > 0$, there exists a member of $\mathcal{B}_{(n)}$, B say, such that

$$(9.9) \qquad 2\, d_I(X,Y;\mathcal{B}) < 2\,(P(B)-Q(B)) + \mathcal{E}\ .$$

The first member of the right-hand side becomes

$$(9.10) \qquad 2\,(P(B)-Q(B)) = (P(B)-Q(B)) + (Q(\overline{B})-P(\overline{B}))\ ,$$ where $\overline{B}$ denotes the complimentary set of B.

For any given partition $\Gamma_\delta(R,\mathcal{B})$, let $\Gamma^*_\delta(R,\mathcal{B})$ be the product partition of $\Gamma_\delta(R,\mathcal{B})$ and $\{B,\overline{B}\}$. Then, $\Gamma^*_\delta(R,\mathcal{B})$ can be divided into two subclasses, $\nabla_\delta(B,\mathcal{B})$ and $\nabla_\delta(\overline{B},\mathcal{B})$, which are partitions of B and of $\overline{B}$, respectively. We then have

$$P(B)-Q(B) = \sum_{E\in\nabla_\delta(B,\mathcal{B})} (P(E)-Q(E)) \text{ and } Q(\overline{B})-P(\overline{B}) = \sum_{E\in\nabla_\delta(\overline{B},\mathcal{B})} (Q(E)-P(E)).$$

Hence, from (9.9) and (9.10) it follows that

$$(9.11) \qquad 2\, d_I(X,Y;\mathcal{B}) < \sum_{E\in\Gamma^*_\delta(R,\mathcal{B})} |\,P(E)-Q(E)| + \mathcal{E}\ ,$$

which yields

$$(9.12) \qquad 2\, d_I(X,Y;\mathcal{B}) \leqq \int_R |d(P-Q)| + \mathcal{E}\ .$$

(9.6) now follows from (9.8) and (9.12), which proves the lemma

It is evident that, when $P_{(n)}$ and $Q_{(n)}$ are dominated by some σ-finite measure $\nu_{(n)}$ over $(R_{(n)},\mathcal{B}_{(n)})$, (9.6) becomes (1.3).

Now it is easy to show that

<u>THEOREM 9.2</u>. Let $\{X_s\}(s = 1,2,\ldots)$ and $\{Y_s\}(s = 1,2,\ldots)$ be two sequences of one-dimensional real random variables, which are ASEQ$(I,\mathcal{B})$ as $s \to \infty$. Suppose that the second order moments of X_s and of Y_s are finite uniformly for all s. Then, the expected values of X_s and Y_s are asymptotically equal in the sense of (9.5).

<u>PROOF</u>. This follows from the inequalities

$$|\,M(X_s)-M(Y_s)|^2 \leqq \int_{R_{(1)}} x^2 |d(P_s-Q_s)| \cdot \int_{R_{(1)}} |d(P_s-Q_s)|$$

$$\leq 2\,(M(X_s^2) + M(Y_s^2)) \cdot d_I(X_s, Y_s; \mathfrak{W}_{(1)}) ,$$

where $M(X_s^2)$ and $M(Y_s^2)$ stand for the second order moments of X_s and Y_s respectively,

It is interesting to ask a similar question as above for the asymptotic equality of expected values in the sense that

(9.13) $\quad M(X_s)/M(Y_s) \rightarrow 0 , \; (s \rightarrow \infty)$, which is left open.

In the last half of this section, we give some properties of type $(I, \mathfrak{W})$ asymptotic independence.

Let $\{X^s_{(n_s)} = (X^s_{1(m^s_1)}, \ldots, X^s_{k_s(m^s_{k_s})})\}$ $(s = 1,2,\ldots)$ be a sequence of random variables, whose marginals

(9.14) $$\{ X^s_{1(m^s_1)}, \ldots, X^s_{k_s(m^s_{k_s})} \}$$

are subjected for investigating the asymptotic independence $(I, \mathfrak{B})$. The corresponding sequence $\{Y^s_{(n_s)}\}$ $(s = 1,2,\ldots)$ is given by the same as in the beginning of Section 4.

Suppose now that there is a sequence of random variables, $\{Z^s_{(n_s)} = (Z^s_{1(m^s_1)}, \ldots, Z^s_{k_s(m^s_{k_s})})\}$ $(s = 1,2,\ldots)$, whose marginals

(9.15) $$\{Z^s_{1(m^s_1)}, \ldots, Z^s_{K_s(m^s_{k_s})}\}$$

form an independent system of random variables for each s .

Under this situation, we can show the following

<u>THEOREM 9.3</u>. (a) If the sequences $\{X^s_{(n_s)}\}$ $(s = 1,2,\ldots)$ and $\{Z^s_{(n_s)}\}$ $(s = 1,2, \ldots)$ are ASEQ$(II, \mathfrak{B})$ as $s \rightarrow \infty$ and the condition $(1 + d_{II}(X^s_{(n_s)}, Z^s_{(n_s)}; \mathfrak{B}_{(n_s)})^{k_s} \rightarrow 1, (s \rightarrow \infty)$, is satisfied,

then the system of random variables (9.14) is ASIN(I,$\mathcal{B}$) as $\to\infty$, (b) If, in particular, n_s are finite uniformly for all s, then it is sufficient for (9.14) to be ASIN(I,$\mathcal{B}$) as $s\to\infty$, that

$$\{X^s_{(n_s)}\}\ (s = 1,2,\dots) \text{ and } \{Z^s_{(n_s)}\}\ (s = 1,2,\dots) \text{ are ASEQ(I},\mathcal{B}) \text{ as } s\to\infty$$

PROOF. (a) Since (II,$\mathcal{B}$) is not weaker than (I,$\mathcal{B}$), it follows from Lemma 2.1 (a) that

$$(9.16)\qquad |P^{X^s_{(n_s)}}(E^s_{(n_s)}) - P^{Z^s_{(n_s)}}(E^s_{(n_s)})| \to 0,\ (s\to\infty),$$

for every sequence $\{E^s_{(n_s)}\}$ (s = 1,2,...), each of which being of the form

$$E^s_{(n_s)} = E^s_{1(m^s_1)}\times\dots\times E^s_{k_s(m^s_{k_s})} \qquad \text{with } E^s_{I(m^s_i)}\in\mathcal{B}_{(m^s_i)},\ i = 1,\dots,k_s.$$

The assumption of the theorem implies that

$$\left|\prod_{i=1}^{k_s} P^{Z^s_{i(m^s_i)}}(E^s_{i(m^s_i)}) - \prod_{i=1}^{k_s} P^{Y^s_{i(m^s_i)}}(E^s_{i(m^s_i)})\right| \to 0,\ (s\to\infty),$$ or equivalently,

$$(9.17)\qquad \left|P^{Z^s_{(n_s)}}(E^s_{(n_s)}) - P^{Y^s_{(n_s)}}(E^s_{(n_s)})\right| \to 0,\ (s\to\infty).$$

Hence, it follows from (9.16) that

$$(9.18)\qquad \left|P^{X^s_{(n_s)}}(E^s_{(n_s)}) - P^{Y^s_{(n_s)}}(E^s_{(n_s)})\right| \to 0,\ (s\to\infty),$$

for any sequence $\{E^s_{(n_s)}\}$ (s = 1,2,...), each of which belonging to the direct product, $\mathcal{B}_{(n_s)} = \mathcal{B}_{(m^s_1)}\times\dots\times\mathcal{B}_{(m^s_{k_s})}$.

Let, for each s, $\mathcal{B}^{**}_{(n_s)}$ be the class of all finite disjoint unions of subsets belonging to the class $\mathcal{B}^*(n_s)$. Then it is clear that the class $\mathcal{B}_{(n_s)}$ is the smallest σ-field over the field $\mathcal{B}^{**}_{(n_s)}$.

Furthermore, it is observed that (9.18) holds true for every sequence $\{E^s_{(n_s)}\}$ (s = 1,2,...) with $E^s_{(n_s)}$ belonging to $\mathcal{B}^{**}_{(n_s)}$.

Hence the result (a) is shown by quite a similar proof to that of Theorem 6.1.

(b) If n_s is finite uniformly for all s, it is easy to see that the asymptotic equivalence $(I,\mathcal{B})$ of $\{X^s_{(n_s)}\}$ (s = 1,2,...) and $\{Z^s_{(n_s)}\}$ (s = 1,2,...) guarantees (9.17).

Thus the theorem is proved.

The result (b) in the theorem can immediately be applied to get the following

<u>COROLLARY 9.2</u>. Suppose that $\{X^s_{(n)} = (X^s_1,\dots,X^s_n)\}$ (s = 1,2,...), converges $(I,\mathcal{B})$ to $Y_{(n)} = (Y_1,\dots,Y_n)$ as $s\to\infty$. Then, the system of marginal random variables, $\{X^s_1,\dots,X^s_n\}$, is ASIN$(I,\mathcal{B})$ as $s\to\infty$, if and only if $\{Y_1,\dots,Y_n\}$ is an independent system.

Now, as before, we shall consider a measurable transformation from $R_{(n_s)}$ to $R_{(m_s)}$, $f^s_{(n_s,m_s)}(z_{(n_s)})$, for each s. Let $\overline{X}^s_{(m_s)}$ and $\overline{Y}^s_{(m_s)}$ be induced random variables defined by (9.4).

Then, the following theorem is an immediate consequence of Theorem 9.1, which proof is omitted.

<u>THEOREM 9.4</u>. If, for every s, the transformation is such that a set of marginals of $\overline{Y}^s(m_s)$, $\{\overline{Y}^s_{1(\ell^s_1)},\dots,\overline{Y}^s_{h_s(\ell^s_{h_s})}\}$, say, forms an independent system, then the corresponding set of marginals of $\overline{X}^s_{(m_s)}$, $\{\overline{X}^s_{1(\ell^s_1)},\dots,\overline{X}^s_{h_s(\ell^s_{h_s})}\}$ is ASIN$(I,\mathcal{B})$ as $s\to\infty$, under the condition that the marginals of original random variable $X^s_{(n_s)}$, $\{X^s_{1(m^s_1)},\dots,X^s_{k_s(m^s_{k_s})}\}$,

is ASIN(I,$\mathcal{B}$) as $s \to \infty$.

If, in particular, we take a projection as the transformation, the independence property of the system $\left\{ Y^s_{1(m^s_1)}, \ldots, Y^s_{k_s}(m^s_{k_s}) \right\}$ is preserved, which gives us the following

COROLLARY 9.3. Any subsystem of asymptotically independent (I,$\mathcal{B}$) system of random variables is asymptotically independent (I,$\mathcal{B}$) under the same limiting process.

10. Some criteria for type (I,$\mathcal{B}$) asymptotic equivalence.

In this section, we shall give two kinds of criteria for type (I,$\mathcal{B}$) asymptotic equivalence, which have been presented in [1].

Let $\left\{ X^s_{(n_s)} \right\}$ (s = 1,2,...) and $\left\{ Y^s_{(n_s)} \right\}$ (s = 1,2,...) be two sequences of random variables with $X^s_{(n_s)}$ and $Y^s_{(n_s)}$ in $\mathcal{P}\,(R_{(n_s)}, \mathcal{B}_{(n_s)}, \nu_{(n_s)})$ for some σ-finite measure $\nu_{(n_s)}$ over $(R_{(n_s)}, \mathcal{B}_{(n_s)})$, for each s. Let further $f^s_{(n_s)}$ and $g^s_{(n_s)}$ be gpdf $(\nu_{(n_s)})$'s of $X^s_{(n_s)}$ and $Y^s_{(n_s)}$ respectively.

Let us introduce the following quantities:

$$\rho(X^s_{(n_s)}) = \int_{R_{(n_s)}} \sqrt{f^s_{(n_s)} g^s_{(n_s)}}\; d\,\nu_{(n_s)} \;, \tag{10.1}$$

$$I(X^s_{(n_s)} : Y^s_{(n_s)}) = \int_{R_{(n_s)}} f^s_{(n_s)} \log(f^s_{(n_s)}/g^s_{(n_s)})\; d\,\nu_{(n_s)} \;, \tag{10.2}$$

for s = 1,2,..., which are called the "affinity" and the "Kullback-Leibler mean information" respectively. Note that the latter is a directed quantity, and of course one can define an analogous quantity by interchanginging the roles of $X^s_{(n_s)}$ and $Y^s_{(n_s)}$. In both of these definitions, it whould be understood that the difference of carriers of gpdf's are taken into account:

For example, in the definition (10.2), if $\nu_{(n_s)}(D(f^s_{(n_s)})-D(g^s_{(n_s)})) > 0$, then the quantity is defined to be infinity, while if $\nu_{(n_s)}(D(g^s_{(n_s)})-D(f^s_{(n_s)})) > 0$, this contributes nothing to the quantity where $D(f^s_{(n_s)})$ and $D(g^s_{(n_s)})$ designate the carriers of $f^s_{(n_s)}$ and $g^s_{(n_s)}$ respectively.

Now, for these quantities the following is easy to verify ([1]).

LEMMA 10.1. It holds always that

(10.3) $$0 \leqq 1-\rho(X^s_{(n_s)},Y^s_{(n_s)}) \leqq d_I(X^s_{(n_s)},Y^s_{(n_s)};\mathcal{B}_{(n_s)})$$

$$\leqq \sqrt{1-\rho(X^s_{(n_s)},Y^s_{(n_s)})^2} \leqq \min.(\sqrt{I(X^s_{(n_s)}):Y^s_{(n_s)})}, \sqrt{I(Y^s_{(n_s)}:X^s_{(n_s)})}),$$

for each s.

The proof of this lemma is omitted.

From this lemma we immediately have the following two criteria for type $(I,\mathcal{B})$ asymptotic equivalence.

THEOREM 10.1. In order that $\{X^s_{(n_s)}\}$ $(s = 1,2,\ldots)$ and $\{Y^s_{(n_s)}\}$ $(s = 1,2,\ldots)$ are ASEQ$(I,\mathcal{B})$ as $s\to\infty$, it is necessary and sufficient that

(10.4) $$\rho(X^s_{(n_s)},Y^s_{(n_s)}) \to 1, \quad (s\to\infty),$$

and an error estimation in this case is given by

(10.5) $$d_I(X^s_{(n_s)},Y^s_{(n_s)};\mathcal{B}_{(n_s)}) \leqq \sqrt{1-\rho(X^s_{(n_s)},Y^s_{(n_s)})^2}.$$

THEOREM 10.2. In order that two sequences in the above theorem are ASEQ$(I,\mathcal{B})$ as $s\to\infty$, it is sufficient that

(10.6) $$I(X^s_{(n_s)}:Y^s_{(n_s)}) \to 0, \quad (s\to\infty),$$

or

(10.7) $$I(Y^s_{(n_s)}:X^s_{(n_s)}) \to 0, \quad (s\to\infty),$$

where an error estimation is given by

(10.8) $$d_I(X^s_{(n_s)}, Y^s_{(n_s)}; \mathcal{B}_{(n_s)}) \leqq \min.\ \left(\sqrt{I(X^s_{(n_s)}:Y^s_{(n_s)})},\ \sqrt{I(X^s_{(n_s)}, Y^s_{(n_s)})}\right)$$

In the earlier paper [1] we have also given the following criterion:

THEOREM 10.3. In order that $\{X^s_{(n_s)}\}$ (s = 1,2,) and $\{Y^s_{(n_s)}\}$ (s = 1,2,...) are ASEQ($I,\mathcal{B}$) as $s \to \infty$, it is sufficient that

(10.9) $$P^{X^s_{(n_s)}}(D(f^s_{(n_s)}) - D(g^s_{(n_s)})) \to 0,\ P^{Y^s_{(n_s)}}(D(g^s_{(n_s)}) - D(f^s_{(n_s)})) \to 0,$$

(10.10) $$\text{ess.sup} \left| f^s_{(n_s)}/g^s_{(n_s)} - 1 \right| \to 0,$$

as $s \to \infty$, where 'ess.sup' is taken over $D(f^s_{(n_s)}) \cap D(g^s_{(n_s)})$ with respect to $\nu_{(n_s)}$.

Any other criteria in terms, for example, of characteristic functions, moment-generating functions, or of distribution functions have not been available yet, and these are left open.

Now, in the final place, we give a remark for type ($I,\mathcal{B}$) convergence criteria.

Let us consider the sequence $\{X^s_{(n)}\}$ (s = 1,2,...) and a random variable $Y_{(n)}$ in the case of equal basic spaces, where all of $X^s_{(n)}$'s and $Y_{(n)}$ belong to $\rho(R_{(n)}, \mathcal{B}_{(n)}\nu_{(n)})$, let the corresponding gpdf($\nu_{(n)}$)'s of $X^s_{(n)}$ and $Y_{(n)}$ be $f^s_{(n)}$ and $g_{(n)}$ respectively.

Type ($I,\mathcal{B}$) convergence of $\{X^s_{(n)}\}$ (s = 1,2,...) to $Y_{(n)}$ is equivalent to the following

(10.11) $$P^{X^s_{(n)}}(E_{(n)}) \to P^{Y_{(n)}}(E_{(n)}),\ (s \to \infty),$$

uniformly for all $E_{(n)}$ in $\mathcal{B}_{(n)}$. For this convergence, we have had a familiar criterion

(10.12) $$f^s_{(n)} \to g_{(n)},\ \text{a.e.}(\nu_{(n)}) \text{ over } R_{(n)},$$

as $s \to \infty$.

It has been shown that our criterion (10.6), or (10.7) given in Theorem 10.2 above is incomparable with the above criterion (10.12), i.e., one cannot say that (10.12) is not weaker than, or not stronger than (10.6), or than (10.7).

11. Some applications of type $(I,\mathcal{B})$ asymptotic independence.

In this section we shall show some applications of notion of asymptotic independence $(I,\mathcal{B})$, the last two of which have been seen in [1]. We omit here the detailed calculations or proofs.

(a) Suppose, first, that we are concerned with the Liapounoff central limit theorem, which states that if a system of one-dimensional random variables

(11.1) $$\left\{ X_1^s, \ldots, X_{n_s}^s \right\}, \text{ (with } n_s \to \infty \text{ as } s \to \infty),$$

is an independent system for each s, s = 1,2,..., and the Liapounoff condition

(11.2) $$\left[\sum_{i=1}^{n_s} (\beta_i^s)^3 \right]^{1/3} \left[\sum_{i=1}^{n_s} (\sigma_i^s)^2 \right]^{1/2} \to 0, \quad (s \to \infty)$$

with $M(X_i^s) = m_i^s$, $\mathrm{Var}(X_i^s)^2 = (\sigma_i^s)^2$ and the 3-rd order absolute moment of $X_i^s - m_i^s = (\beta_i^s)^3$ (provided that these are all exist), is satisfied, then the standardized sum

(11.3) $$\sum_{i=1}^{n_s} (X_i^s - m_i^s) \Big/ \sqrt{\sum_{i-1}^{n_s} (\sigma_i^s)^2}$$

converges in law to the standard normal distribution N(0,1) as $s \to \infty$, and hence in the sense of type (I,S) by Corollary 7.3.

It is easily seen from Theorem 9.1 that this theorem still holds true, even if the independence property of the system (11.1) is replaced by type $(I,\mathcal{B})$ asymptotic independence property, because the condition (11.2) depends on (separated) properties of marginals, not on the inner structure of the whole system (11.1).

It is interesting to ask the following questions: (1°) What is the weakest notion of asymptotic independence of (11.1) which guarantees the Liapounoff central limit theorem? (2°) How small could be the values of ℓ_s for which (ℓ_s) - asymptotic independence (I,$\mathcal{B}$) of (11.1) assures us the theorem ?

These questions are left open.

(b) Suppose that we are given a system of elementary coverages, $\{C_i\}$ (i = 1,2,...,N), which is obtained by random division of the interval $[0,1)$ into N+1 subintervals. Any sybsystem of size n of the whole system, $\{C_{i_1},\ldots C_{i_n}\}$, has the pdf. given by

$$(11.4) \qquad f_{(n)}(z_{(n)}) = \frac{\Gamma(N+1)}{\Gamma(N-n+1)}\left(1-\sum_{i=1}^{n} z_i\right)^{N-n}, \left(0 \leqq z_i, \sum_{i=1}^{n} z_i < 1\right),$$

which is independent of the choices of $\{i_1,\ldots,i_n\}$.

From (11.4) it follows, putting n = 1, that each marginal has pdf. given by

$$(11.5) \qquad f(z) = N\,(1-z)^{N-1}, \quad (0 \leqq z < 1).$$

Then, Theorem 10.2 can be applied to get the following result:

The system $\{C_i\}$ (i = 1,...,N) is (n)-ASIN(I,$\mathcal{B}$) as $N \to \infty$, provided that $n = o(\sqrt{N})$.

(c) Let $X_1 \leqq \cdots \leqq X_N$ be an order statistics drawn from a one-dimensional probability distribution of the continuous type, and let $\underline{X}_{(n)} = (X_1,\ldots,X_n)$ and $\overline{X}_{(m)} = (X_{N-m+1},\ldots,X_N)$ be joint distributions of n lower extremes and that of m upper extremes, respectively.

Then by making use of Theorems 9.1 and 10.2 we can show that a system of random variable, $\{\underline{X}_{(n)}, \overline{X}_{(m)}\}$, is ASIN(I,$\mathcal{B}$) as $N \to \infty$,provided that $n + m = o(\sqrt{N})$; we shall say this as ' $\underline{X}_{(n)}$ and $\overline{X}_{(m)}$ are mutually asymptotically independent (I,$\mathcal{B}$) as $N \to \infty$. '

IV

Type (I,S) Asymptotic Equivalence

12. Some properties of type (I,S) asymptotic equivalence.

In this section we shall show some properties of type (I,S) asymptotic equivalence specially in relation to the usual in probability convergence.

Firstly, it is clear from the definitions of in probability convergence and type (I,S) asymptotic equivalence that

LEMMA 12.1. If $\{X^s_{(n)}\}$ $(s = 1,2,\ldots)$ and $\{Y^s_{(n)}\}$ $(s = 1,a,\ldots)$ are ASEQ(I,S) as $s \to \infty$, and if one of these sequences converges in probability to a point $c_{(n)} = (c_1,\ldots,c_n)$ as $s \to \infty$, then the other converges in probability to the same point as $s \to \infty$.

Note that this lemma still holds even if we replace 'ASEQ(I,S)' in the lemma by 'ASEQ((I,S))'.

Sometimes we meet with the following question: For a given sequence of random variables, $\{X^s_{(n)} = (X^s_1,\ldots,X^s_n)\}(s = 1,2,\ldots)$, let us put

$$(12.1) \qquad \overline{X}^s_{(n)} = (c^s_1 X^s_1,\ldots,c^s_n X^s_n), \quad s = 1,2\ldots,$$

where c^s_i are constants such that $c^s_i \to 1$ as $s \to \infty$, for each i. Under this situation, it is asked that $\{X^s_{(n)}\}$ $(s = 1,2,\ldots)$ and $\{\overline{X}^s_{(n)}\}$ $(s = 1,2,\ldots)$ are ASEQ(I,S) as $s \to \infty$.

An answer for this question is given by the following

THEOREM 12.1 If $\{X^s_{(n)}\}(s = 1,2,\ldots)$ has both of properties C(S) and B(S), then $\{X^s_{(n)}\}(s = 1,2,\ldots)$ and $\{\overline{X}^s_{(n)}\}(s = 1,a,\ldots)$ are ASEQ(I,S) as $s \to \infty$.

PROOF. First, it is noted that there is no harm in assuming that $c^s_i > 0$ for all i and s.

From property B(S) of $\{X^s_{(n)}\}(s = 1,2,\ldots)$, there exist, for any given $\varepsilon > 0$, a positive integer s_o and a member $M_{(n)}$ of $S_{(n)}$, whose closure being compact, such that

$$(12.2) \qquad P^{X^s_{(n)}}(M_{(n)}) > 1 - \varepsilon ,$$

for all $s \geqq s_o$.

Now, put

$$E_{(n)}(a_1,\ldots,a_n) = \{z_{(n)} = (z_1,\ldots,z_n) \mid z_i > a_i,\ i = 1,\ldots,n\} ,$$

for any real numbers a_i , $i = 1,\ldots,n$. Then, clearly

$$(12.3) \quad \left| P^{X^s_{(n)}}(E_{(n)}(a_1,\ldots,a_n)) - P^{\overline{X}^s_{(n)}}(E_{(n)}(a_1,\ldots,a_n)) \right| \leqq P^{X^s_{(n)}}(A^s_{(n)}),$$

for all s, where $A^s_{(n)}$ is the symmetric difference defined by

$$A^s_{(n)} = E_{(n)}(a_1,\ldots,a_n) \Delta E_{(n)}(a_1/c^s_1,\ldots,a_n/c^s_n).$$

Put, for each s,

$$F^s_{(n)} = M_{(n)} \cap A^s_{(n)} \quad .$$

Then, this is a disjoint union of at most n members of $S_{(n)}$, whose closures are all compact. Furthermore, since $M_{(n)}$ in (12.2) is independent of s, we can assume that there exist a positive number K, and a positive integer s'_o such that if $|a_i| > K$ for some i, then the set $F^s_{(n)}$ is empty for all $s \geqq s'_o$.

Hence, by property C(S) of $\{X^s_{(n)}\}(s = 1,2,\ldots)$, there exists a positive integer s''_o such that

$$(12.4) \qquad P^{X^s_{(n)}}(F^s_{(n)}) > \varepsilon ,$$

uniformly for all $(a_1,\ldots,a_n)$ and for all $s \geqq s''_o$.

From (12.2), (12.3) and (12.4), it follows that

$$d_I(X^s_{(n)}, \overline{X}^s_{(n)}; \mathcal{M}_{(n)}) > 2\,\varepsilon ,$$

for all $s \geqq \max(s_o, s_o', s_o'')$, which proves the theorem, because type (I,μ) and type (I,S) asymptotic equivalence are mutually equivalent in the case of equal basic spaces.

The above result could be extended to the case of unequal basic spaces, though it is still an open question.

The following examples show that the conditions of the theorem are not necessarily removable.

EXAMPLE 12.1. Let X_s be a normal random variable with mean m_s and variance 1, for $s = 1,2,\ldots$, with $m_s \to \infty$ as $s \to \infty$. Put $c_s = 1-1/m_s$, and $\bar{X}_s = c_s X_s$, for each s. Then $c_s \to 1$ as $s \to \infty$, and the sequence $\{X_s\}$ $(s = 1,2,\ldots)$ has property C(S), but not B(S). It is easily seen that both sequences are not ASEQ(I,S) as $s \to \infty$.

EXAMPLE 12.2. Let X_s be a discrete random variable such that $P(X_s = 1/s) = 1$, $s = 1,2,\ldots$, and let $\bar{X}_s = c_s X_s$, where $c_s > 1$ for all s, and $c_s \to 1$ as $s \to \infty$. Then, $\{X_s\}(s = 1,2,\ldots)$ has property B(S), but not C(S). Again we can see that they are not ASEQ(I,S) as $s \to \infty$.

Hereafter in this section we shall restrict ourselves to the case $n = 2$.

Firstly, we show the following

THEOREM 12.2. Let $\{(X_s,Y_s)\}(s = 1,2,\ldots)$ be a sequence of two-dimensional random variables satisfying the following conditions:

(i) The first marginals, $\{X_s\}(s = 1,2,\ldots)$, has property C(S).

(ii) The second marginals, $\{Y_s\}(s = 1,2,\ldots)$, converges in probability to a certain constant c.

Then it holds that

$$X_s + Y_s \sim X_s + c \ (I,S), \ (s \to \infty). \quad (12.5)$$

PROOF. Since, for any given $\delta > 0$ and any given real number a,

$$P(X_s+Y_s < a) = P(X_s < a-Y_s, |Y_s-c| > \delta) + P(X_s < a-Y_s, |Y_s-c| \leqq \delta)$$

and

$$P(X_s < a-c-\delta) \leqq P(X_s < a-Y_s, |Y_s-c| \leqq \delta) \leqq P(X_s < a-c+\delta)$$
$$P(X_s < a-Y_s, |Y_s-c| > \delta) \leqq P(|Y_s-c| > \delta),$$

we have

(12.6) $$|P(X_s+Y_s < a)-P(X_s+c < a)| \leqq P(a-c-\delta \leqq X_s < a-c+\delta)+P(|Y_s-c|> \delta),$$

for each s.

By the condition (i) of the theorem, there exist, for any given $\varepsilon > 0$, a value of δ, δ_0 say, and a positive integer s_0 such that

(12.7) $$\sup_{-\infty < a < \infty} P(a-c-\delta_0 \leqq X_s < a-c+\delta_0) < \varepsilon$$

for all $s \geqq s_0$.

On the other hand, by the condition (ii) of the theorem, there exists a positive integer s_0' such that

(12.8) $$P(|Y_s-c| > \delta_0) < \varepsilon$$

for all $s \geqq s_0'$.

It follows from (12.6), (12.7) and (12.8) that

$$\sup_{-\infty < a < \infty} |P(X_s+Y_s < a)-P(X_s+c < a)| < 2\varepsilon$$

for all $s \geqq \max(s_0, s_0')$, which shows that

$$X_s + Y_s \sim X_s + c \ (I,\mu), \ (s \to \infty),$$

or equivalently (12.5).

This completes the proof of the theorem.

The following corollary is an immediate consequence of this theorem.

COROLLARY 12.1. (a) Suppose that the two conditions, (i) and (ii), of the above theorem are satisfied, and $\{X_s\}(s = 1,2,\ldots)$ is ASEQ(I,S) to another sequence $\{\overline{X}_s\}(s = 1,2,\ldots)$ as $s \to \infty$.

Then, it holds that

(12.9) $\quad X_s + Y_s \sim \bar{X}_s + c \quad (I,S)$, $(s \to \infty)$.

(b) In the above theorem, if the condition (i) is replaced by

(i') $\{X_s\}(s = 1,2,\ldots)$ converges in law to a certain distribution, Z, of the continuous type as $s \to \infty$,

then it holds that

(12.10) $\quad X_s + Y_s \to Z + c$ (I,S), $(s \to \infty)$.

PROOF. (b) This follows from the above theorem and Corollary 7.3.

Note that the classical convergence theorem due to Cramér states that if X_s converges in law to Z (not necessarily of the continuous type) and Y_s converges in probability to a constant c as $s \to \infty$, then $X_s + Y_s$ converges in law to Z + c as $s \to \infty$. If Z is of the continuous type, then this reduces to (b) of the above corollary.

Now, we shall prove the following

THEOREM 12.3. Let $\{(X_s,Y_s)\}(s = 1,2,\ldots)$ be a sequence of two-dimensional real random variables satisfying the following two conditions:

(i) $\{X_s\}$ $(s = 1,2,\ldots)$ has both of properties C(S) and B(S).

(ii) $\{Y_s\}(s = 1,2,\ldots)$ converges in probability to a certain non-zero constant c.

Then, it holds that

(12.11) $\quad X_s / Y_s \sim X_s / c$ (I,S), $(s \to \infty)$.

and

(12.12) $\quad X_s \quad Y_s \sim cX_s \quad$ (I,S), $(s \to \infty)$.

PROOF. It is evident that $Y_s \to c$ (in prob.) implies that $1/Y_s \to c^{-1}$ (in prob.), and vice versa. Hence (12.12) follows from (12.11). We therefore prove (12.11), and in doing this, there is no harm in assuming that c = 1 and Y_s are all positive.

From the inequality

$$| P(X_s < aY_s)-P(X_s < aY_s, | Y_s-1| < \delta)| \leq P(| Y_s-1| > \delta),$$

δ being any given positive number, it follows that

$$(12.13) \quad | P(X_s/Y_s < a)-P(X_s < a)| \leq \sup_{| x-x'| < 2| a| \delta} \{P(x \leq X_s < x')+P(| X_s-1| > \delta)$$

for all s.

Since $\{X_s\}$ (s = 1,2,...) has property $B(S)$ by assumption, there exist, for any given $\varepsilon > 0$, a positive number M and a positive integer s_o such that

$$P(|X_s|\geq M) < \varepsilon/3 ,$$

for all $s \geq s_o$. Furthermore, property C(S) of $\{X_s\}(s = 1,2,...)$ assures us that there exist a constant $\delta > 0$ and a positive integer s_o' such that

$$\sup_{| x-x| < \delta_o} P(x \leq X_s < x') < \varepsilon/3$$

for all $s \geq s_o'$.

Because of the inprobability convergence of $\{Y_s\}(s = 1,2,...)$ to 1, there exists a positive integer s_o'' such that

$$P(| Y_s-1| > \delta_o/(2M)) < \varepsilon/3,$$

for all $s \geq s_o''$.

Hence, putting $\delta = \delta_o/(2M)$ for δ in (12.13), we have

$$(12.14) \quad \sup_{-\infty < a < \infty} | P(X_s/Y_s < a) - P(X_s < a) | < \varepsilon$$

for all $s \geq \max(s_o, s_o', s_o'')$, which proves (12.11) and hence the theorem.

As an immediate consequence of this theorem, we have the following

<u>COROLLARY 12.2</u>. (a) Suppose that the conditions, (i) and (ii), of the above theorem hold, and that $\{X_s\}(s = 1,2,...)$ be ASEQ(I,S) to another $\{\overline{X}_s\}(s = 1,2,...)$ as $s \to \infty$. Then, it holds that

$$(12.15) \quad X_s/Y_s \sim \overline{X}_s/c \ (I,S) , \ (s\to\infty),$$

and

(12.16) $X_s Y_s \sim c\bar{X}_s$ (I,S) , $(s \to \infty)$.

(b) If we replace the condition (i) of the above theorem by the condition

(i') $\{X_s\}$(s = 1,2,...) converges in law to a certain distribution, Z, of the continuous type as $s \to \infty$,

then it holds that

(12.17) $X_s/Y_s \to Z/c$ (I,S), $(s \to \infty)$,

and

(12.18) $X_s Y_x \to cZ$ (I,S), $(s \to \infty)$

The proofs of these results are omitted.

Given a sequence of two-dimensional random variables $\{(X_s,Y_s)\}$ (s = 1,2,...) with $Y_s \to c$ (in prob.) as $s \to \infty$, one may ask the following question: What conditions on the sequence $\{X_s\}$(s = 1,2,...) guarantee that

(12.19) $Y_s/X_s \sim c/X_s$ (I,S), $(s \to \infty)$,

holds ?

This is still an unsolved question.

13. Properties of type (I,S) asymptotic independence.

The purpose of this section is to show some properties of type (I,S) asymptotic independence, which are similar to that of type (I,$\mathcal{B}$) asymptotic independence given in the last of Section 9.

Let, as in Section 4, $\{X^s_{(n_s)} = (X^s_{1(m^s_1)},\dots,X^s_{k_s(m^s_{k_s})})\}$(s = 1,2,...) be a sequence of random variables, the system of whose marginals

(13.1) $\left\{ X^s_{1(m^s_1)},\dots,X^s_{k_s(m^s_{k_s})}\right\}$

are subjected for investigation, and let $\left\{ Y^s_{(n_s)} = (Y^s_{1(m^s_1)},\dots,Y^s_{k_s(m^s_{k_s})})\right\}$ (s = 1,2...) be the corresponding sequence such that $Y^s_{i(m^s_i)}$ is identically distributed with $X^s_{i(m^s_i)}$ for each i and s and the system

(13.2) $$\left\{ Y^s_{1(m^s_1)}, \dots, Y^s_{k_s(m^s_{k_s})} \right\}$$

is independent for each s.

Now, let $\left\{ Z^s_{(n_s)} = (Z^s_{1(m^s_1)}, \dots, Z^s_{k_s(m^s_{k_s})}) \right\}$ (s = 1,2,...) be another sequence of random variable, whose marginals

(13.3) $$\left\{ Z^s_{1(m^s_1)}, \dots, Z^s_{k_s(m^s_{k_s})} \right\}$$

is an independent system for each s.

Under this situation one can prove the following

THEOREM 13.1. (a) If $\left\{ X^s_{(n_s)} \right\}$ (s = 1,2,...) and $\left\{ Z^s_{(n_s)} \right\}$ (s = 1,2,...) are ASEQ(II,S) as $s \to \infty$ and the condition $(1 + d_{II}(X^s_{(n_s)}, Z^s_{(n_s)}; S_{(n_s)}))^{k_s} \to 1$, $(s \to \infty)$ is satisfied, then the system (13.1) is ASIN(I,S) as $s \to \infty$.

(b) If n_s are bounded uniformly for all s, and $\{X^s_{(n_s)}\}$ (s = 1,2,...) and $\{Z^s_{(n_s)}\}$ (s = 1,2,...) are ASEQ(I,S) as $\to \infty$, then the system (13.1) is ASIN(I,S) as $s \to \infty$.

PROOF. The proof is quite similar to that of Theorem 9.3, as will be seen below.

(a) We have to show that $X^s_{(n_s)} \sim Y^s_{(n_s)}$ (I,S) as $s \to \infty$, or equivalently

(13.4) $$\left| P^{X^s_{(n_s)}}(E^s_{(n_s)}) - P^{Y^s_{(n_s)}}(E^s_{(n_s)}) \right| \to 0, \quad (s \to \infty),$$

for any sequence $\{ E^s_{(n_s)} \}$ (s = 1,2,...) with $E^s_{(n_s)}$ in $S_{(n_s)}$ for each s.

Since every $E^s_{(n_s)}$ is expressible in the form

$$E^s_{(n_s)} = E^s_{1(m^s_1)} \mathrm{X} \dots \mathrm{X}\, E^s_{k_s(m^s_{k_s})}, \quad E^s_{i(m^s_i)} \in S_{(m^s_i)},$$

and

(13.5) $$\left| P^{Z^s_{i(m^s_i)}}(E^s_{i(m^s_i)}) - P^{Y^s_{i(m^s_i)}}(E^s_{i(m^s_i)}) \right| \leq d_{II}(X^s_{(n_s)}, Z^s_{(n_s)}; S_{(n_s)})\, P^{Z^s_{i(m^s_1)}}(E^s_{i(m^s_i)})$$

for each i and s, it is easy to see that

$$(13.6)\qquad \left| P^{Z^s_{(n_s)}}(E^s_{(n_s)}) - P^{Y^s_{(n_s)}}(E^s_{(n_s)}) \right| \to 0, \ (s \to \infty),$$

from which (13.4) follows.

(b) In this case, we need solely

$$(13.7)\qquad \left| P^{Z^s_{i(m^s_i)}}(E^s_{i(m^s_i)}) - P^{Y^s_{i(m^s_i)}}(E^s_{i(m^s_i)}) \right| \leqq d_I(X^s_{(n_s)}, Z^s_{(n_s)}; S_{(n_s)})$$

instead of (13.5), the right-hand side of which tends to zero as $s \to \infty$, by the assumption of the theorem.

Hence the proof of the theorem is completed.

It is not known yet whether the condition in (a) of the above theorem can be weakened or not.

As an immediate consequence of the above theorem we have the following

COROLLARY 13.1. In the case of equal basic spaces, if a sequence $\{X^s_{(n)} = (X^s_{1(m_1)}, \ldots, S^s_{k(m_k)})\}$ $(s = 1,2,\ldots)$ converges (I,S) to some distribution $Z_{(n)} = (Z_{1(m_1)}, \ldots, Z_{k(m_k)})$, whose marginals $\{Z_{1(m_1)}, \ldots, Z_{k(m_k)}\}$ being an independent system, then the system $\{X^s_{1(m_1)}, \ldots, X^s_{k(m_k)}\}$ is ASIN(I,S) as $s \to \infty$, where k and m_i 's are all independent of s.

The following result is also seen at once :

For any given $\{X^s_{(n_s)} = (X^s_{1(m^s_1)}, \ldots, X^s_{k_s(m^s_{k_s})})\}(s = 1,2,\ldots)$, and the corresponding sequence $\{Y^s_{(n_s)}\}(s = 1,2,\ldots)$ as was given in the beginning of this section, let us put

$$(13.8)\qquad \overline{X}^s_{(m_s)} = f^s_{(n_s,m_s)}(X^s_{(n_s)}) \text{ and } \overline{Y}^s_{(m_s)} = f^s_{(n_s,m_s)}(Y^s_{(n_s)})$$

for each s, where $f^s_{(n_s,m_s)}(z_{(n_s)})$ is a measurable transformation from $R_{(n_s)}$ to $R_{(m_s)}$. Then we have

THEOREM 13.2. Suppose that the transformation preserves type (I,S) asymptotic equivalence and is such that a system of marginals of $\overline{Y}^s_{(m_s)}$, $\{\overline{Y}^s_{1(\ell^s_1)},\ldots,\overline{Y}^s_{h_s(\ell^s_{h_s})}\}$ say, is independent for each s.

Then, the corresponding system of marginals of $\overline{X}^s_{(m_s)}$, $\{\overline{X}^s_{1(\ell^s_1)},\ldots,\overline{X}^s_{h_s(\ell^s_{h_s})}\}$, is ASIN(I,S) as $s\rightarrow\infty$.

The proof of this theorem is omitted.

As an immediate consequence of this theorem we have the following

COROLLARY 12.2. Any subsystem of asymptotically independent (I,S) system is asymptotically independent (I,S).

14. Measurable transformations preserving type (I,S) asymptotic equivalence in the general case.

Let $\{X^s_{(n_s)} = (X^s_1,\ldots,X^s_{n_s})\}$ (s = 1,2,...) and $\{Y^s_{(n_s)} = (Y^s_1,\ldots,Y^s_{n_s})\}$ (s = 1,2,...) be two sequences of random variables with $X^s_{(n_s)}$, $Y^s_{(n_s)}$ belonging to $\mathfrak{F}(R_{(n_s)},\mathcal{B}_{(n_s)})$ for each s. Let, further,

(14.1) $$f^s_{(n_s,m_s)}(z_{(n_s)}) = (f^s_{(n_s,1)1}(z_{(n_s)})),\ldots,f^s_{(n_s,1)m_s}(z_{(n_s)}))$$

be a measurable transformation from $R_{(n_s)}$ to $R_{(m_s)}$ for each s, where $n_s \geqq m_s$. These transformations define new sequences of induced random variables, $\{U^s_{(m_s)} = (U^s_1,\ldots,U^s_{m_s})\}$ (s = 1,2,...) and $\{V^s_{(m_s)} = (V^s_1,\ldots,V^s_{m_s})\}$ (s = 1,2,...):

(14.2) $$U^s_{(m_s)} = f^s_{(n_s,m_s)}(X^s_{(n_s)}) \text{ and } V^s_{(m_s)} = f^s_{(n_s,m_s)}(Y^s_{(n_s)}) .$$

Clearly, these random variables belong to $\mathfrak{F}(R_{(m_s)},\mathcal{B}_{(m_s)})$.

In this section we shall decide some of the classes of measurable transformations which preserves type (I,S) asymptotic equivalence in the general (equal and unequal) basic spaces.

In the first place, we shall consider a class of measurable transformations of the form

$$(14.3) \qquad f^s_{(n_s,n_s)}(z_{(n_s)}) = (f^s_1(z_1),\ldots,f^s_{n_s}(z_{n_s})) ,$$

for each s.

For this type of transformations, we can prove the following

THEOREM 14.1. If, in (14.3), $f^s_i(x)$ is a continuous and monotone non-decreasing function of x on the real line for each i and s, then it holds that

(a) $X^s_{(n_s)} \sim Y^s_{(n_s)}(I,\mu),(s \to \infty)$ implies that $U^s_{(n_s)} \sim V^s_{(n_s)}$ (I,μ), $(s \to \infty)$, and

(b) $X^s_{(n_s)} \sim Y^s_{(n_s)}(I,S),(s \to \infty)$ implies that $U^s_{(n_s)} \sim V^s_{(n_s)}$ (I,S), $(s \to \infty)$.

PROOF. If the function (14.3) satisfies the condition of the theorem, then it holds that

$$f^{s^{-1}}_{(n_s,n_s)}(\mu_{(n_s)}) \subseteqq \mu_{(n_s)} \text{ and } f^{s^{-1}}_{(n_s,n_s)}(S_{(n_s)}) \subseteqq S_{(n_s)}$$

for each s, which yield

$$d_I(X^s_{(n_s)},Y^s_{(n_s)};\mu_{(n_s)}) \geqq d_I(U^s_{(n_s)},V^s_{(n_s)};\mu_{(n_s)})$$

and

$$d_I(X^s_{(n_s)},Y^s_{(n_s)};S_{(n_s)}) \geqq d_I(U^s_{(n_s)},V^s_{(n_s)};S_{(n_s)})$$

for each s. These imply in turn (a) and (b) of the theorem.

As a special case of this theorem we have the following

COROLLARY 14.1. Suppose that the function (14.3) is of the form

$$(14.4) \qquad f^s_{(n_s,n_s)}(z_{(n_s)}) = (c^s_1 z_1+d^s_1,\ldots\ldots\ldots,c^s_{n_s} z_{n_s}+d^s_{n_s})$$

for each s, where $c^s_i > 0$ for all i and s and d^s_i are constants. Then, the result (a) and (b) of the above theorem hold true.

It is not clear in general whether we can weaken the contitions in the above theorem. For a special case, however, where the distributions involved are of the continuous type, we can see the following

THEOREM 14.2. If $X^s_{(n_s)}$ and $Y^s_{(n_s)}$ are of the continuous type for all s, and if $f^s_i(x)$ in (14.3) is continuous and monotone non-decreasing or non-increasing in x on the real line for each i and s, then $X^s_{(n_s)} \sim Y^s_{(n_s)}$ (I,S) implies that $U^s_{(n_s)} \sim V^s_{(n_s)}$ (I,S) as $s \to \infty$.

The proof of this theorem is easy and omitted.

It should be noted that, in both of the above two theorems, if, furthermore, the transformation (14.3) is from $R_{(n_s)}$ onto itself and non-singular (one-to-one and inverse transformation is also measurable), then the inverse conclusions of the results in both of the theorems are also true.

EXAMPLE 14.1. The transformations

$$f_{(n_s,n_s)}(z_{(n_s)}) = (e^{z_1},\dots,e^{z_{n_s}}) \tag{14.5}$$

and

$$f_{(n_s,n_s)}(z_{(n_s)}) = (\log z_1,\dots,\log z_{n_s}), \text{ for } z_i > 0, \tag{14.5}$$

preserve type (I,S) asymptotic equivalence.

EXAMPLE 14.2. So far as we are concerned with sequences of random variables of the continuous type, the transformations

$$f_{(n_s,n_s)}(z_{(n_s)}) = (-z_1,\dots,-z_{n_s}) \ , \tag{14.6}$$

preserve type (I,S) asymptotic equivalence.

EXAMPLE 14.3. Consider the transformations

$$f_{(n_s,n_s)}(z_{(n_s)}) = (\ 1/z_1,\dots,1/z_{n_s}\ ) \ , \text{ for } Z_i > 0. \tag{14.7}$$

Then, since $1/x = e^{-\log x}$, it is seen that if $X^s_{(n_s)}$ and $Y^s_{(n_s)}$ are of the continuous type and their components are all positive for all s, then the transformations defined by (14.7) preserve type (I,S) asymptotic equivalence of $\{X^s_{(n_s)}\}(s = 1,2,\dots)$ and $\{Y^s_{(n_s)}\}(s = 1,2,\dots)$.

These examples come out from the preceding two theorems.

It is difficult to decide the classes of measurable transformations which preserve type (I,S) asymptotic equivalence, except for the cases we treated above. However, we shall give a slight investigation to the case when the transformations are of the form

$$(14.8) \qquad f^s_{(n_s,m)}(z_{(n_s)}) = (f^s_1(z_{(n_s)}),\dots,f^s_m(z_{(n_s)})) ,$$

where m is fixed independently of s and $m \leqq n_s$ for at least sufficiently large values of s.

In this case, corresponding to the original sequences, $\{X^s_{(n_s)}\}$ $(s = 1,2,\dots)$ and $\{Y^s_{(n_s)}\}(s = 1,2,\dots)$, we have those of new random variables of a fixed dimension, i.e., $\{U^s_{(m)}\}(s = 1,2,\dots)$ and $\{V^s_{(m)}\}(s = 1,2,\dots)$. Then, type (I,S) asymptotic equivalence is equivalent to type $(I,\mathfrak{M})$ for sequences $\{U^s_{(m)}\}(s = 1,2,\dots)$ and $\{V^s_{(m)}\}(s = 1,2,\dots)$. Since any subset belonging to $\mathfrak{M}_{(m)}$ is open and the inverse image of any open set with respect to a continuous transformation is open, we have the following

THEOREM 14.3. If the transformation (14.8) is continuous for each s, then type $(I,\mathfrak{G})$ asymptotic equivalence of $\{X^s_{(n_s)}\}(s = 1,2,\dots)$ and $\{Y^s_{(n_s)}\}(s = 1,2,\dots)$ implies type (I,S) asymptotic equivalence of $\{U^s_{(m)}\}$ $(s = 1,2,\dots)$ and $\{V^s_{(m)}\}$ $(s = 1,2,\dots)$.

Thus, if one can find conditions under which $(I,\mathfrak{G})$ and (I,S) are mutually equivalent, then under such conditions continuous transformations of the form (14.8) preserve type (I,S) asymptotic equivalence. It is thus an interesting question to find out conditions under which two notions of asymptotic equivalence, $(I,\mathfrak{G})$ and (I,S), are mutually equivalent.

There is another question in connection with type (I,S) asymptotic equivalence preserving transformations: Let us consider a situation, in which n_s increases monotonically to infinity with increasing s, and $\{X^s_{(n_s)}\}$ $(s = 1,2,\dots)$ and $\{Y^s_{(n_s)}\}$ $(s = 1,2,\dots)$ themselves are not ASEQ(I,S) as $s \to \infty$, but any marginals of finite dimension, $\{\overline{X}_{(n)} = (X^s_{i_1},\dots,X^s_{i_n})\}$ $(s = 1,2,\dots)$ and $\{\overline{Y}^s_{(n)}\} = (Y^s_{i_1},\dots,\overline{Y}^s_{i_n})$ $(s = 1,2,\dots)$ are ASEQ(I,S) as

$s \to \infty$ for any given positive integer n and any given choice of $i_1, \ldots, i_n$ both independently of s. (This is equivalent to the following statement: For any given positive integer n, $\{\overline{X}^s_{(n)} = (X^s_1, \ldots, X^s_n)\}(s = 1,2,\ldots)$ and $\{\overline{Y}^s_{(n)} = (Y^s_1, \ldots, Y^s_n)\}(s = 1,2,\ldots)$ are ASEQ(I,S) as $s \to \infty$.) One may call this type of asymptotic equivalence of $\{X^s_{(n_s)}\}(s = 1,2,\ldots)$ and $\{Y^s_{(n_s)}\}(s = 1,2,\ldots)$ the finite asymptotic equivalence (I,S) and say that these sequences are finitely asymptotically equivalent (I,S) as $s \to \infty$.

Now, we take measurable transformations which are of the form (14.8), and suppose that there exists a sequence of sets of real numbers $\{c^s_i\}(i = 1,\ldots,m;\ s = 1,2,\ldots)$ such that the sequence $\{\overline{V}^s_{(m)} = (c^s_1 V^s_1, \ldots, c^s_m V^s_m)\}(s = 1,2,\ldots)$ has properties C(S) and B(S). The question is then to find out conditions on functions (14.8) under which the finite asymptotic equivalence of $\{X^s_{(n_s)}\}(s = 1,2,\ldots)$ and $\{Y^s_{(n_s)}\}(s = 1,2,\ldots)$ implies the type (I,S) asymptotic equivalence of $\{U^s_{(m)}\}(s = 1,2,\ldots)$ and $\{V^s_{(m)}\}(s = 1,2,\ldots)$.

The following example relates to this question.

EXAMPLE 14.4. The present author once met with the following problem: Let $\{X^s_{(n_s)}\}(s = 1,2,\ldots)$ be a sequence of real random variables with $n_s \uparrow \infty$ as $s \to \infty$, such that $\{\overline{X}^s_{(n)} = (X^s_1, \ldots, X^s_n)\}(s = 1,2,\ldots)$ converges in law to n-dimensional independent normal distribution with mean vector $\underline{0}$ and variance-covariance matrix I_n (unit matrix) as $s \to \infty$, where n is any positive integer fixed independently of s. Then, for any fixed n, the variable $\sum_{i=1}^{n} (X^s_i)^2$ converges in law to the chi-square distribution of degrees of freedom n. Now, the question is to ask whether the variable $\sum_{i=1}^{n_s} (X^s_i)^2$ is asymptotically approximated by the chi-square distribution of degrees of freedom n_s as $s \to \infty$, in the sense of type (I,S), which is still an open question.

15. Measurable transformations preserving type (I,S) asymptotic equivalence in the case of equal basic spaces.

In this section we shall confine ourselves to the case of equal basic spaces where $n_s = n$ for all s, and consider the same type of problems as in the preceding section.

Suppose we are given two sequences of random variables, $\{X^s_{(n)}\}$ (s = 1,2,...) and $\{Y^s_{(n)}\}$(s = 1,2...), and let us consider measurable transformations from $R_{(n)}$ to $R_{(m)}$ ($m \leqq n$), which are of the form

$$(15.1) \qquad f^s_{(n,n)}(z_{(n)}) = (f^s_{(n,1)}(z_{(m)}),\ldots,f^s_{(n,1)m}(z_{(m)}),$$

for s = 1,2,... .

Firstly, we shall prepare some necessary lemmas.

Let $\mathfrak{J}_{(n)}$ be the class of all subsets of $R_{(n)}$ which are of the form

$$(15.2) \left\{ z_{(n)} \;\middle|\; \begin{array}{l} z_{i_k} = b_{i_k} \quad k = 1,\ldots,\ell \text{, and } b_{j_m} \leqq z_{j_m} < a_{j_m} \text{, or} \\ b_{j_m} \leqq z_{j_m} < \infty \text{, or } -\infty < z_{j_m} < z_{j_m}^{\ell} \text{, or } -\infty < z_{j_m} < \infty \text{;} \\ m = 1,\ldots, n-\ell \end{array} \right.$$

where ℓ is any integer such that $1 \leqq \ell \leqq n$, $\{i_1,\ldots,i_\ell\}$ is any subsequence of $\{1,\ldots,n\}$ with complementary subsequence $\{1,\ldots,n\}$ with complementary subsequence $\{j_1,\ldots,j_{n-\ell}\}$, and a_i, b_i, $i = 1,\ldots,n$, are real numbers. Clearly, any member of this class has $\mu_{(n)}$-measure zero.

Furthermore, let $S^*_{(n)}$ be the class of all n-dimensional intervals, open, semi-open and closed. Then, it is evident that this class contains $S_{(n)}$ as a subclass, and that for any member $E_{(n)}$ of $S^*_{(n)}$ there can be found a member $F_{(n)}$ of $S_{(n)}$ such that the symmetric difference $E_{(n)} \Delta F_{(n)}$ is expressed as a union of at most 2n mutually disjoint members of $\mathfrak{J}_{(n)}$.

We now prove the following

<u>LEMMA 15.1</u>. If $\{X^s_{(n)}\}$(s = 1,2,...) and $\{Y^s_{(n)}\}$ (s = 1,2,...) are ASEQ(I,S) as $s \to \infty$, then they are ASEQ(I,$\mathfrak{J}$) as $s \to \infty$.

PROOF. One has to show that

$$(15.3) \quad d_I(X^s_{(n)},Y^s_{(n)};\mathfrak{J}_{(n)}) = \sup_{E_{(n)}\in \mathfrak{J}_{(n)}} |P^{X^s_{(n)}}(E_{(n)})-P^{Y^s_{(n)}}(E_{(n)})| \to 0$$

as $s \to \infty$.

Let $E_{(n)}$ be any given set belonging to $\mathfrak{J}_{(n)}$ with the form (15.2), and $E^p_{(n)}$ be the subset obtained from $E_{(n)}$ by changing the first ℓ conditions in (15.2) into ℓ conditions, $b_{i_k} \leqq z_{i_k} < 1/p$; $k = 1,\ldots,\ell$, where $p = 1,2\ldots.$ Then, evidently, $E^p_{(n)}$ are members of $S_{(n)}$ and $E^p_{(n)} \downarrow E_{(n)}$ as $p \to \infty$.

Hence, by the condition of the theorem, there exists a sequence of positive numbers, $\{q_s\}$ $(s = 1,2,\ldots)$, with $q_s \to 0$ as $s \to \infty$, such that

$$\left| P^{X^s_{(n)}}(E^p_{(n)}) - P^{Y^s_{(n)}}(E^p_{(n)}) \right| < q_s \text{ , for all p and s,}$$

where q_s are independent of p and $E_{(n)}$.

Letting p tend to infinity, we then get

$$(15.4) \quad \left|P^{X^s_{(n)}}(E_{(n)}) - P^{Y^s_{(n)}}(E_{(n)}) \right| \leqq q_s,$$

for all s, which is equivalent to (15.3), and the proof of the lemma is completed.

By this lemma, it is immediate that

LEMMA 15.2. Two types of asymptotic equivalence, $(I,\overset{*}{S})$, and (I,S), are mutually equivalent.

PROOF. This follows from the inequality

$$d_I(X^s_{(n)},Y^s_{(n)};\overset{*}{S}_{(n)}) \leqq d_I(X^s_{(n)},Y^s_{(n)};S_{(n)}) + 2nd_I(X^s_{(n)},Y^s_{(n)};\mathfrak{J}_{(n)}), \text{ for each s.}$$

Now we shall consider the case where the transformations (15.1) are specially of the form

$$(15.5) \quad f^s_{(n,n)}(z_{(n)}) = (f^s_1(z_1),\ldots,f^s_n(z_n)),$$

and put

$$(15.6) \quad U^s_{(n)} = f^s_{(n,n)}(X^s_{(n)}) \text{ and } V^s_{(n)} = f^s_{(n,n)}(Y^s_{(n)}), \text{ for each s.}$$

Then one can see easily that

THEOREM 15.1. Suppose that the i-th component of (15.5), $f_i^s(x)$, is a monotone (non-decreasing or non-increasing) function of x on the real line for each i and s. Then $X_{(n)}^s \sim Y_{(n)}^s$ (I,S) implies that $U_{(n)}^s \sim V_{(n)}^s$ (I,S) as $s \to \infty$.

The proof of this theorem is easy and is omitted.

As immediate consequences of this theorem we have the following

COROLLARY 15.1. If $\{X_{(n)}^s\}(s = 1,2,\dots)$ converges (I,S) to some $Y_{(n)}$, or converges in law to some $Y_{(n)}$ of the continuous type, then, under the condition of the theorem, it holds that $U_{(n)}^s \sim V_{(n)}^s$ (I,S) as $s \to \infty$, where $V_{(n)}^s$ is given by changing $Y_{(n)}^s$ into $Y_{(n)}$ in the second definition of (15.6). Furthermore, if the transformation (15.5) is independent of s, then under the above conditions it holds that $U_{(n)}^s \to V_{(n)}$ (I,S) as $s \to \infty$.

COROLLARY 15.2. Suppose that the function (15.5) takes a special form:

$$(15.7) \qquad f_{(n,n)}^s(z_{(n)}) = (c_1^s z_1 + d_1^s, \dots, c_n^s z_n + d_n^s),$$

where c_i^s and d_i^s are any real numbers. Then $X_{(n)}^s \sim Y_{(n)}^s$ (I,S) implies that $U_{(n)}^s \sim V_{(n)}^s$ (I,S) as $s \to \infty$.

The proofs of these corollaries are omitted.

We shall state another special case in the following

EXAMPLE 15.1. Let the function (15.5) be given by

$$(15.8) \qquad f_{(n,n)}(z_{(n)}) = (1/z_1, 1/z_n) \text{ for all } z_i \neq 0, \text{ and arbitrary when } z_i = 0$$

for some i. Then, it is easy to see that, if $P(X_i^s = 0) = 0$ for each i and s, $X_{(n)}^s \sim Y_{(n)}^s$ (I,S) implies that $U_{(n)}^s \sim V_{(n)}^s$ (I,S) as $s \to \infty$.

Evidently, the above result follows from the fact that the function $1/x$ consists of two monotonic parts, though the function itself is not monotone as a whole. Similar argument can be applied to get the following

THEOREM 15.2. Suppose that the i-th component of the function (15.8), $f_i^s(x$, consists of k_i^s monotonic parts for each i and s, and $k_i^s \leqq K$ uniformly for all i and s. Then the transformations preserve type (I,S) asymptotic equivalence.

The proof of this theorem is easy by using Lemma 15.2, and is omitted.

EXAMPLE 15.2. If $f_i^s(x)$ in (15.5) is a polinomial in x of degree k_i^s for each s and i, and k_i^s are bounded uniformly for all i and s, then (15.5) preserve type (I,S) asymptotic equivalence.

Now, in the next place, we shall consider continuous transformations.

Let us consider a measurable transformation

$$f_{(n,m)}(z_{(n)}) = (f_1(z_{(n)}),\dots,f_m(z_{(n)}) \quad , (n \geqq m), \tag{15.9}$$

and let

$$U^s_{(m)} = f_{(n,m)}(X^s_{(n)}) \text{ and } V^s_{(m)} = f_{(n,m)}(Y^s_{(n)}),$$

and, in a convergence case where $Y^s_{(n)}$ are identical with some $Y_{(n)}$, $V_{(m)} = f_{(n,m)}(Y_{(n)})$.

A classical result states that

LEMMA 15.3. Suppose that the function (15.9) is continuous, i.e., every component of the function is continuous in $z_{(n)}$ over $R_{(n)}$. Then, in law convergence of $\{X^s_{(n)}\}$ (s = 1,2,...) to $Y_{(n)}$ implies the same type of convergence of $\{U^s_{(n)}\}$ (s = 1,2,...) to $Y_{(n)}$ as $s \to \infty$.

A version of this result in our case is given by the following

THEOREM 15.3. Suppose that the function (15.9) is continuous and one of the sequences, $\{X^s_{(n)}\}$ (s = 1,2,...) and $\{Y^s_{(n)}\}$ (s = 1,2,...), has properties C(S) and B(S). Then, $X^s_{(n)} \sim Y^s_{(n)}$ ((I,M)) implies $U^s_{(m)} \sim V^s_{(m)}$ ((I,S)) as $s \to \infty$. If, moreover, the function transfers the properties C(S) and B(S) of the original sequences, then $X^s_{(n)} \sim Y^s_{(n)}$ (I,S) implies $U^s_{(m)} \sim V^s_{(m)}$ (I,S) as $s \to \infty$.

PROOF. First, it is noted that, under the conditions in the first part of the theorem, type $((I,\mu))$ asymptotic equivalence of $\{X^s_{(n)}\}$ $(s = 1,2,\ldots)$ and $\{Y^s_{(n)}\}$ $(s = 1,2,\ldots)$ is equivalent to type $(I,\mathfrak{S})$ asymptotic equivalence of the same sequences, and under the condition in the second part of the theorem, $((I,\mathfrak{S}))$ and (I,S) are mutually equivalent for $\{U^s_{(m)}\}$ $(s = 1,2,\ldots)$ and $\{V^s_{(m)}\}$ $(s = 1,2,\ldots)$. Hence the second part follows immediately from the first part.

We shall show the first part. To prove this, it suffices to show that

$$U^s_{(m)} \sim V^s_{(m)} \ ((I,\mu)), \ (s \to \infty). \tag{15.10}$$

Since the inverse image of any member of $\mu_{(m)}$ with respect to the function (15.9) is open and hence belongs to $\mathcal{G}_{(n)}$, (15.10) follows easily from Theorem 7.4, or Corollary 7.2, and the conditions of this theorem.

This proves the theorem.

From this theorem, it is immediate that

COROLLARY 15.3. If the function (15.9) is continuous, then in law convergence of $\{X^s_{(n)}\}$ $(s = 1,2,\ldots)$ to some distribution $Y_{(n)}$ of the continuous type implies type (I,S) convergence of $\{U^s_{(m)}\}$ $(s = 1,2,\ldots)$ to $V_{(m)}$.

Some of those transformations which transfer the properties $C(S)$ and $B(S)$ simulteneously are shown in the following

EXAMPLE 15.3. The following transformations are easily seen to transfer the described properties: $f(x) = e^x, (-\infty < x < \infty)$; $f(x) = e^{-x}$, $(-\infty < x < \infty)$; $f(x) = x^k$, $(-\infty < x < \infty)$, for any given positive integer k; $f(x) = \log(1 + x)$, $(0 < x < \infty)$, $= 0$, (otherwise).

It should be noted that if we take transformations depending on s instead of (15.9), the problem become quite difficult in general. If, however, we could find suitable conditions under which type (I,S) and type

(I,G) asymptotic equivalence are mutually equivalent, it will be possible to clearify the type (I,S) asymptotic equivalence preserving property of the transformations, which is left to future investigations.

Now, in the final place, we shall consider the following type of transformations.

$$(15.11) \qquad f^s_{(n,2)}(z_{(n)}) = \left(\sum_{i=1}^{n} c_i^s z_i \,,\ \sum_{i=1}^{n} d_i^s z_i^2\right),$$

where c_i^s and d_i^s (> 0) are constants for all s and i, i=1,...,n; s = 1,2,...

Let, further, $U^s_{(2)} = f^s_{(n,2)}(X^s_{(n)})$ and $V^s_{(2)} = f^s_{(n,2)}(Y^s_{(n)})$ for all s.

Then, we can show the following

THEOREM 15.4. Under the situation stated above, if one of the sequences, $\{X^s_{(n)}\}$ (s = 1,2,...) and $\{Y^s_{(n)}\}$ (s = 1,2,...), has properties C(S) and B(S), then $X^s_{(n)} \sim Y^s_{(n)}$ (I,S) implies $U^s_{(2)} \sim V^s_{(2)}$ (I,S) as $s\to\infty$.

PROOF. Firstly, it is noted that there is no harm in assuming that c_i^s and d_i^s are bounded uniformly for all i and s, because type (I,S) asymptotic equivalence of $U^s_{(n)}$ and $V^s_{(n)}$ is equivalent, by Corollary 15 2, to that of $\left(\sum_{i=1}^{n} \bar{c}_i^s X_i^s, \sum_{i=1}^{n} \bar{d}_i^s (X_i^s)^2\right)$ and $\left(\sum_{i=1}^{n} \bar{c}_i^s Y_i^s, \sum_{i=1}^{n} \bar{d}_i^s (Y_i^s)^2\right)$, where

$\bar{c}_i^s = c_i^s/p_s$ and $\bar{d}_i^s = d_i^s/q_s$ with $p_s = \left(\sum_{i=1}^{n} (c_i^s)^2\right)^{1/2}$ and $q_s = \left(\sum_{i=1}^{n} (d_i^s)^2\right)^{1/2}$.

Let us take any convergent subsequences of $\{(c_1^s,\ldots,c_n^s)\}$ (s = 1,2, . .) and $\{(d_1^s,\ldots,d_n^s)\}$ (s = 1,2,...) , and without any loss of generality we suppose the original sequences are convergent, i.e.,

$$c_i^s \to c_i \quad \text{and} \quad d_i^s \to d_i \quad , \text{ as } s \to \infty ,$$

for each i and for some c_i and $d_i (\geqq 0)$.

Let a and b(> 0) be any given real numbers, and let $A^s_{(n)}$ and $B^s_{(n)}$ be subsets of $R_{(n)}$ defined by

$$A^s_{(n)} = \left\{ z_{(n)} \,\middle|\, \sum_{i=1}^{n} c_i^s z_i < a \right\} \quad \text{and} \quad B^s_{(n)} = \left\{ z_{(n)} \,\middle|\, \sum_{i=1}^{n} d_i^s z_i^2 < b \right\},$$

for each s . Then, it is evident that the set $A^s_{(n)} \cap B^s_{(n)}$ is the inverse image of the set $E_{(2)} = \{z_{(2)} \mid z_1 < a\ ,\ z_2 < b\}$ with respect to the function (15.11).

Let us put

$$A_{(n)} = \{z_{(n)} \mid \sum_{i=1}^{n} c_i z_i < a \quad \text{and} \quad B_{(n)} = \{ z_{(n)} \mid \sum_{i=1}^{n} d_i z_i^2 > b\} .$$

Then, the sets $A^s_{(n)}$ and $B^s_{(n)}$ coincide in the limit as $s \to \infty$ with $A_{(n)}$ and $B_{(n)}$ up to their boundry sets, respectively. Hence $A^s_{(n)} \cap B^s_{(n)}$ coincides with $A_{(n)} \cap B_{(n)}$ up to their boundary set in the limit as $s \to \infty$

By the assumption of the theorem for any given $\varepsilon > 0$, there exist a positive integer s_o and a member $M_{(n)}$ of $S_{(n)}$, whose closure being compact, such that

$$(15.12) \qquad P^{X^s_{(n)}}(M_{(n)}) < \varepsilon \quad \text{and} \quad P^{Y^s_{(n)}}(M_{(n)}) < \varepsilon$$

for all $s \geqq s_o$.

Now, put

$$K^s_{(n)} = A^s_{(n)} \cap B^s_{(n)} \cap M_{(n)} \quad \text{and} \quad K_{(n)} = A_{(n)} \cap B_{(n)} \cap M_{(n)} ,$$

for each s. Then, $K^s_{(n)}$ coincides with $K_{(n)}$ up to their boundary sets in the limit as $s \to \infty$, and the closure of $K_{(n)}$ is compact. Note that $K_{(n)}$ and $K^s_{(n)}$ are all dependent on the values of a and b we have given.

Now, it is seen that, for any given positive δ , we can find a partition of $M_{(n)}$ consisting of the members of $S_{(n)}, \{E_{(n)i}\}$ $(i = 1, \ldots, N'')$ say, such that

$$(15.13) \qquad \sup_{-\infty \leqq a, b \leqq \infty} \{\mu_{(n)}(\sum_j E_{(n)j}) \mid \overline{E}_{(n)j} \cap \dot{K}_{(n)} \neq \emptyset \} < \delta \qquad ,$$

where $\overline{E}$ and $\dot{E}$ denote in general the closure and the boundary set of the set E

For any given such finite partition, the supremum in (15.13) is attained for some values of a and b, a_o and b_o say, because there exists positive number h such that $h < a$ and $h < b$ imply that $K_{(n)} = M_{(n)}$, and $a < -h$ implies that $K_{(n)} = \emptyset$. For all values of a and b, any point of $\dot{K}_{(n)}$ is an inner point of the set $\sum_j E_{(n)j}$, unless it lies on the boundary of $M_{(n)}$.

Giving a finite partition of $M_{(n)}$ satisfying the condition (15.13), let, for each s, $\{E_{(n)i_k}\}$ $(k = 1,\ldots,N'_s)$ be the set of all such members of the partition that $\bar{E}_{(n)i_k} \cap \bar{K}^s_{(n)} \neq \emptyset$, and among these, let $\{E_{(n)i_k}\}$ $(k = 1,\ldots,N_s)$ be the set of all such that $\bar{E}_{(n)i_k} \cap \dot{K}_{(n)} = \emptyset$. Clearly the choice of $\{i_1,\ldots,i_{N'_s}\}$ is dependent on a, b and s. Then, we have

$$(15.14) \quad \left| P^{X^s_{(n)}}(K^s_{(n)}) - P^{Y^s_{(n)}}(K^s_{(n)}) \right| \leqq \left| P^{X^s_{(N)}}\left(\sum_{k=1}^{N_s} E_{(n)i_k}\right) - P^{Y^s_{(n)}}\left(\sum_{k=1}^{N_s} E_{(n)i_k}\right) \right| + P^{X^s_{(n)}}\left(\sum_{k=N_s+1}^{N'_s} E_{(n)i_k}\right) + P^{Y^s_{(n)}}\left(\sum_{k=N_s+1}^{N'_s} E_{(n)i_k}\right)$$

for each s.

Choosing δ in (15.13) sufficiently small and a corresponding partition of $M_{(n)}$, it follows from the property C(S) of the original sequences that

$$\overline{\lim_{s\to\infty}} P^{X^s_{(n)}}\left(\sum_{k=N_s+1}^{N'_s} E_{(n)i_k}\right) < \varepsilon \quad \text{and} \quad \overline{\lim_{s\to\infty}} P^{Y^s_{(n)}}\left(\sum_{k=N_s+1}^{N'_s} E_{(n)i_k}\right) < \varepsilon ,$$

uniformly for all values of a and b.

On the other hand, since $N_s \leqq N''$, the condition $X^s_{(n)} \sim Y^s_{(n)}$ (I,S) as $s \to \infty$ implies that the first term on the right-hand side of (15.14) tends to zero as $s \to \infty$.

Hence it follows from (15.14) that

$$(15.15) \quad \overline{\lim_{s\to\infty}} \left| P^{X^s_{(n)}}(K^s_{(n)}) - P^{Y^s_{(n)}}(K^s_{(n)}) \right| \leqq 2\varepsilon ,$$

from which, by (15.12), we have

$$(15.16) \quad \lim_{s\to\infty} \left| P^{X^s_{(n)}}(A^s_{(n)} \cap B^s_{(n)}) - P^{Y^s_{(n)}}(A^s_{(n)} \cap B^s_{(n)}) \right| \leqq 4\varepsilon ,$$

uniformly for all a and b.

This means that $\{U^s_{(2)}\}$ (s = 1,2,...) and $\{V^s_{(2)}\}$ (s = 1,2...) are asymptotically equivalent in the sense of type (I,μ), and hence in the sense of type (I,S) as $s \to \infty$, which proves the theorem.

The following result is immediate from this theorem.

COROLLARY 15.4. Let $\{X^s_{(n)}\}$ (s = 1,2,...) be a sequence of random variables which converges in law to a certain distribution $Y_{(n)}$ of the continuous type as $s \to \infty$, and let $U^s_{(2)} = f^s_{(n,2)}(X^s_{(n)})$ and $V^s_{(2)} = f^s_{(n,2)}(Y_{(n)})$. Then, $U^s_{(2)} \sim V^s_{(2)}$ (I,S) as $s \to \infty$.

Taking marginals of $U^s_{(2)}$ and $V^s_{(2)}$ we have

COROLLARY 15.5. Under the same condition as that of the above theorem, $X^s_{(n)} \sim Y^s_{(n)}$ (I,S) implies that

$$(15.15) \qquad \sum_{i=1}^{n} c_i^s X_i^s \sim \sum_{i=1}^{n} c_i^s Y_i^s \quad (I,S)$$

as $s \to \infty$.

COROLLARY 15.6. If $\{X^s_{(n)}\}$ (s = 1,2,...) converges in law to $Y_{(n)}$ which is of the continuous type as $s \to \infty$, then it holds that

$$(15.16) \qquad \sum_{i=1}^{n} c_i^s X_i^s \sim \sum_{i=1}^{n} c_i^s Y_i \quad (I,S), \ (s \to \infty).$$

In the last place, we shall consider the following example.

EXAMPLE 15.5. Let $\{X^s_{(n)} = (X^s_1,\ldots,X^s_n)\}$ (s = 1,2,...) be a sequence of random variables which converges in law to the n-dimensional independent normal distribution with mean vector $\underline{0}$ and variance-covariance matrix I_n, and put

$$(15.17) \qquad U_s = \sum_{i=1}^{n} (X_i^s + c_i^s)^2 \quad \text{and} \quad V_s = \sum_{i=1}^{n} (Y_i + c_i^s)^2$$

for all s, c_i^s are any given constants. Clearly, V_s is distributed according to a non-central chi-square distribution of degrees of freedom n with non-centrality parameter $p_s^2 = \sum_{i=1}^{n} (c_i^s)^2$, for each s.

For (15.17) we can see that

$$(15.18) \qquad U_s \sim V_s \quad (I,S),\ (s\to\infty).$$

In fact, these variables can be rewritten as

$$(15.19) \qquad \begin{aligned} U_s &= \sum_{i=1}^{n} (X_i^s)^2 + 2\sum_{i=1}^{n} c_i^s X_i^s + p_s^2 \\ V_s &= \sum_{i=1}^{n} Y_i^2 + 2\sum_{i=1}^{n} c_i^s Y_i + p_s^2 \end{aligned}$$

for each s. Then, by Corollary 14.1 or Corollary 15.2, one can see that the condition (15.18) is equivalent to the following

$$(15.20) \qquad \sum_{i=1}^{n} (X_i^s)^2 + 2\sum_{i=1}^{n} c_i^s X_i^s \sim \sum_{i=1}^{n} Y_i^2 + 2\sum_{i=1}^{n} c_i^s Y_i \quad (I,S),\ (s\to\infty).$$

Now, Corollary 15.4 assures us that

$$\Big(\sum_{i=1}^{n} (X_i^s)^2, \sum_{i=1}^{n} c_i^s X_i^s\Big) \sim \Big(\sum_{i=1}^{n} Y_i^2, \sum_{i=1}^{n} c_i^s Y_i\Big) \quad (I,S),\ (s\to\infty).$$

But, since $\{(\sum_{i=1}^{n} Y_i^2, \sum_{i=1}^{n} c_i^s Y_i)\}$ (s = 1,2,...)(s = 1,2,...) has properties C(S) and B(S)(unless c_i^s are all zero for i=1,...,n and for an infinite number of values of s), (15.20) follows from Corollary 15.5.

16. <u>Type (I,S) Asymptotic equivalence of marginal random variables.</u>

General formulation of the problem we want to treat in this section is as follows: Let $\{(X_{(k)}^s, Y_{(\ell)}^s, Z_{(m)}^s)\}$ (s = 1,2,...) be a sequence of n (= k + ℓ + m) dimensional random variables with fixed k, ℓ, and m. For this sequence, suppose that

(i) for the first marginals, $\{X_{(k)}^s\}$ (s = 1,2,...), there is another sequence of k-dimensional random variables, $\{\bar{X}_{(k)}^s\}$ (s = 1,2,...), which is ASEQ (I,S) to the original sequence as $s\to\infty$, and

(ii) for the second marginals, $\{Y_{(\ell)}^s\}$ (s = 1,2,...), there can be found a sequence of real vectors $\{c_\ell^s=(c_1^s,\dots,c_\ell^s)\}$ (s = 1,2,...), such that the sequence $\{(Y_1^s/c_1^s,\dots,Y_\ell^s/c_\ell^s)\}$ (s = 1,2,...) converges in probability to the point (1,...,1).

Under this situation, suppose that we replace the first two marginals of $(X_{(k)}^s, Y_{(\ell)}^s, Z_{(m)}^s)$ by $\bar{X}_{(k)}^s$ and $c_{(\ell)}^s$ respectively for each s, keeping the dependence structure between three marginals of the original variable undestroyed in a sense, to get the new variable $(\overline{X}^s_{(K)}, c^s_{(\ell)}, \overline{Z}^s_{(m)})$.

The problem is now to ask whether the third marginals of them, $\{Z^s_{(m)}\}(s = 1,2,\dots)$ and $\{\overline{Z}^s_{(m)}\}$ $(s = 1,2,\dots)$ are ASEQ(I,S) as $s \to \infty$, probably under some additional conditions.

In the first place, some necessary lemmas are stated.

The following two lemmas are easy to verify and the proofs will be omitted.

LEMMA 16.1. Let $\{u_s(x_{(n)})\}(s = 1,2,\dots)$ be a sequence of real valued functions defined over $R_{(n)}$, and suppose that the following conditions are satisfied:

(i) For each s, $0 \leqq u_s(x_{(n)}) \leqq$ for all $x_{(n)}$ in $R_{(n)}$.

(ii) For each s, $u_s(x_{(n)})$ is continuous over $R_{(n)}$.

Then, there exist a convergent subsequence $\{u_{s'}(x_{(n)}\}$ $(s' \to \infty)$ and a continuous function $u_o(x_{(n)})$ such that $0 \leqq u_o(x_{(n)}) \leqq 1$ over $R_{(n)}$ and $u_{s'}(x_{(n)})$ as $s' \to \infty$ for all $x_{(n)}$ in $R_{(n)}$.

LEMMA 16.2. The convergence $u_{s'}(x_{(n)}) \to u_o(x_{(n)})$ $(s \to \infty)$ in the above lemma is uniform on any given compact subset of $R_{(n)}$, that is, for any compact subset $E_{(n)}$ of $R_{(n)}$ and any given $\varepsilon > 0$, there exists a positive integer s_o, depending only on ε and $E_{(n)}$, such that $|u_{s'}(x_{(n)}) - u_o(x_{(n)})| < \varepsilon$ for all $x_{(n)}$ in $E_{(n)}$ and for all $s' \geqq s_o$.

Now, let $f(x_{(n)}, y_{(m)})$ be a real valued function defined over $R_{(n)} \times R_{(m)}$, the $(n + m)$-dimensional euclidean space. Then, it is easy to see the following

LEMMA 16.3. Suppose that the function $f(x_{(n)}, y_{(m)})$ is continuous with respect to $(x_{(n)}, y_{(m)})$ over $R_{(n+m)}$. Let $E_{(n)}$ be any given subset of $R_{(n)}$, whose closure is compact. Then, for any given point $y_{(m)}$ in $R_{(m)}$

and any given $\rho > 0$, there exists a positive number δ, depending on ρ, $E_{(n)}$ and $Y_{(m)}$, such that $|y_{(m)} - y'_{(m)}| < \delta$ implies that $|f(x_{(n)}, y_{(m)}) - f(x_{(n)}, y'_{(m)})| < \rho$ uniformly for all $x_{(n)}$ in $R_{(n)}$.

PROOF. Put

$$(16.1) \qquad v(y_{(m)}, y'_{(m)}) = \sup_{x_{(n)} \in E_{(n)}} |f(x_{(n)}, y_{(m)}) - f(x_{(n)}, y'_{(m)})|$$

Suppose now that the assertion of the lemma is false. Then, there exist a positive number ρ, a point $y^o_{(m)}$ in $R_{(m)}$ and a sequence of points $\{y^i_{(m)}\}$ $(i = 1,2,\ldots)$ in $R_{(m)}$ such that $y^i_{(m)} \to y^o_{(m)}$ $(i \to \infty)$ and

$$(16.2) \qquad v(y^o_{(m)}, y^i_{(m)}) \geqq \rho \ , \ i = 1,2,\ldots$$

Since $\bar{E}_{(n)}$, the closure of $E_{(n)}$, is compact by assumption, there exists, for any given $y_{(m)}$ and $y'_{(m)}$, a point $x^o_{(n)}$ in $\bar{E}_{(n)}$, depending probably on $y_{(m)}$ and $y'_{(m)}$, such that

$$v(y_{(m)}, y'_{(m)}) = |f(x^o_{(n)}, y_{(m)}) - f(x^o_{(n)}, y'_{(m)})| .$$

Then, corresponding to the sequence $\{x^i_{(m)}\}$ $(i = 1,2,\ldots)$ in $\bar{E}_{(n)}$ such that

$$(16.3) \qquad v(y^o_{(m)}, y^i_{(m)}) = |f(x^i_{(n)}, y^o_{(m)}) - f(x^i_{(n)}, y^i_{(m)})|$$

for each i. This sequence has a convergent subsequence $\{x^{i'}_{(n)}\}$ $(i' \to \infty)$ whose limit point $x^*_{(n)}$ being, of course, in $\bar{E}_{(n)}$.

Since $f(x_{(n)}, y_{(m)})$ is continuous, $(x^i_{(n)}, y^{i'}_{(m)}) \to (x^*_{(n)}, y^o_{(m)})$ and $(x^{i'}_{(n)}, y^o_{(m)}) \to (x^*_{(n)}, y^o_{(m)})$ as $i' \to \infty$, we have

$$|f(x^{i'}_{(n)}, y^{i'}_{(m)}) - f(x^*_{(n)}, y^o_{(m)})| \to 0, \ (i' \to \infty),$$

and

$$|f(x^{i'}_{(n)}, y^o_{(m)}) - f(x^*_{(n)}, y^o_{(m)})| \to 0, \ (i' \to \infty).$$

Hence

$$v(y^o_{(m)}, y^{i'}_{(m)}) = |f(x^{i'}_{(n)}, y^o_{(m)}) - f(x^{i'}_{(n)}, y^{i'}_{(m)})| \to 0, \ (i' \to \infty),$$

which contradicts (16.2).

This proves the lemma.

Using this lemma one can show the following

LEMMA 16.4. Suppose that a function $f(x_{(n)},y_{(m)})$ defined over $R_{(n)} \times R_{(m)}$ satisfies the following conditions.

(i) $f(x_{(n)},y_{(m)}) \geqq 0$ over $R_{(n+m)}$, and $= 0$ over $(R_{(n)}-D_{(n)}) \times R_{(m)}$ with $D_{(n)} = \{x_{(n)} \mid K_i \leqq x_i, i = 1, \ldots, k\}$ for some constants, $K_1, \ldots, K_n$.

(ii) $f(x_{(n)},y_{(m)})$ is continuous in $(X_{(n)},y_{(m)})$ over the set $D_{(n)} \times R_{(m)}$.

(iii) For any given $y_{(m)}$ in $R_{(m)}$, $f(x_{(n)},y_{(m)})$ is integrable over $R_{(n)}$ with respect to the ordinary Lebesgue measure $\mu_{(n)}$,

and

$$\int_{R_{(n)}} f(x_{(n)},y_{(m)})\, d\mu_{(n)} = 1 ,$$

for every $y_{(m)}$ in $R_{(m)}$.

Then, for any given $z_{(n)} = (z_1,\ldots,z_n)$ in $R_{(n)}$, the function of $y_{(m)}$

$$u(z_{(n)},y_{(m)}) = \int_{E_{(n)}(z_{(n)})} f(x_{(n)},y_{(m)})\, d\mu_{(n)} \qquad (16.4)$$

is continuous with respect to $y_{(m)}$ over $R_{(m)}$, where the set $E_{(n)}(z_{(n)})$ is given by

$$E_{(n)}(z_{(n)}) = \{x_{(n)} = (x_1,\ldots,x_n) \mid x_i < z_i , i = 1,\ldots,n\},$$

and z_i's are allowable to be $+\infty$ for some or all i.

PROOF. Firstly, let us consider the case when $z_i < \infty$ for all i. In this case, we have, for any given $z_{(n)}$, $y_{(m)}$ and $y'_{(m)}$,

$$| u(z_{(n)},y_{(m)})-u(z_{(n)},y'_{(m)})|$$
$$\leqq \int_{D_{(n)}\cap E_{(n)}(z_{(n)})} | f(x_{(n)},y_{(m)})-f(x_{(n)},y'_{(m)})|\, d\mu_{(n)}$$
$$\leqq \sup|f(x_{(n)},y_{(m)})-f(x_{(n)},y'_{(m)})|\mu_{(n)}(A_{(n)}\cap E_{(n)}(z_{(n)})),$$

where $D_{(n)}$ designates the same set as in the condition (i) and the 'sup' is taken with respect to $x_{(n)}$ over the set $D_{(n)}\cap E_{(n)}(z_{(n)})$. Since the closure of this set is compact, Lemma 16.3 can be easily applied to get the result.

Suppose next that some z_i's are $+\infty$, $z_{k+1} = \cdot\ z_n = +\infty$ say In this case, putting $z^i_{(n)} = (z_1, \cdot\ \cdot, z_k, z^i_{k+1}, \ldots, z^i_n)$ with any sequence $s^i_j \to \infty$ $(i \to \infty)$, $j=k+1,\ldots,n$, we have by the condition (iii)

$$u(z^i_{(n)}, y_{(m)}) \to u(z_{(n)}, y_{(m)}), \quad (i \to \infty)$$

But, since $u(z^i_{(n)}, y_{(m)})$ is a continuous function of $y_{(m)}$ for each i as was proved above, and since $u(z_{(n)}, y_{(m)})$ exists by (iii), we can conclude that the limit function $u(z_{(n)}, y_{(m)})$ is a continuous function of $y_{(m)}$ for any fixed $z_{(n)}$ in our case.

Finally, in the case when $z_i = +\infty$ for all i, the result follows from (iii) This completes the proof of the lemma.

Now, we shall return to the problem stated in the beginning of this section.

In the first place, let us consider the case when $\ell = 0$.

Let $\{(X^s_{(n)}, Z^s_{(m)})\}(s = 1,2,\ldots)$ be a sequence of $(n + m)$-dimensional random variables, for which n and m are assumed to be fixed independently of s. Cumulative distribution functions of $Z^s_{(m)}$ and $X^s_{(n)}$ are denoted respectively by $H_s(z_{(m)})$ and $F_s(x_{(n)})$, and the conditional cumulative distribution function of $Z^s_{(m)}$ given $X^s_{(n)} = x_{(n)}$ is denoted by $P_s(z_{(m)}|x_{(n)})$, for each s. Clearly

$$H_s(z_{(m)}) = \int_{R_{(n)}} P_s(z_{(m)}|x_{(n)})\, dF_s(x_{(n)}). \tag{16.5}$$

Let us consider another sequence of n-dimensional random variables $\{\overline{X}^s_{(n)}(s = 1,2,\ldots)$ with cdf.'s $\overline{F}_s(x_{(n)})$, and let us put

$$\overline{H}_s(z_{(m)}) = \int_{R_{(n)}} P_s(z_{(m)}|x_{(n)}\qquad)\, d\overline{F}_s(x_{(n)}), \tag{16.6}$$

for each s. Then it is evident that this gives a cdf. of some n-dimensional random variable, $\overline{Z}^s_{(m)}$ say, for each s.

Under this situation, one can prove the following

<u>THEOREM 16.1</u>. Suppose that the following conditions are satisfied:

(i) For any given $z_{(m)}$ in $R_{(m)}$, $P_s(z_{(m)} | x_{(n)})$ is a continuous function of $x_{(n)}$ over $R_{(n)}$

(ii) $\{X^s_{(n)}\}$ $(s = 1,2,\dots)$ and $\{\overline{X}_{(n)}\}$ $(s = 1,2,\dots)$ are ASEQ(I,S) as $s \to \infty$.

(iii) There exist two sequences of real vectors, $\{c^s_{(n)} = (c^s_1,\dots, c^s_n)\}$ $(s = 1,2,\dots)$ and $\{d^s_{(n)} = (d^s_1,\dots,d^s_n)\}$ $(s = 1,2\dots)$, such that for one of the sequences in (ii) above, $\{\overline{X}^s_{(n)}\}$ $(s = 1,2,\dots)$ say, the random variables.

$$\{\overline{U}^s_{(n)} = ((\overline{X}^s_1-d^s_1)/c^s_1,\dots,(\overline{X}^s_n-d^s_n)/c^s_n)\} (s = 1,2\dots).$$

converges (I,S) to some fixed distribution $U_{(n)}$ as $s \to \infty$.

Then it holds that

(16.7) $$Z^s_{(m)} \sim \overline{Z}^s_{(m)} \text{ (I,S)}$$

as $s \to \infty$.

PROOF. If we put

$$U^s_{(n)} = ((X^s_1-d^s_1)/c^s_1,\dots,(X^s_n-d^s_n)/c^s_n) ,$$

for each s, the condition (ii) of this theorem and Corollary 15.2 assure us that $U^s_{(n)} \sim \overline{U}^s_{(n)}$(I,S) as $s \to \infty$, and hence

(16.8) $$U^s_{(n)} \to U_{(n)} \text{ (I,S)}, \ (s \to \infty).$$

Hence, of course, both of the sequences $\{U^s_{(n)}\}$ $(s = 1,2,\ \ )$ and $\{\overline{U}^s_{(n)}\}$ $(s = 1,2,\dots)$ have property B(S). Let us denote the cdf.'s of $U^s_{(n)}$, $\overline{U}^s_{(n)}$ and $U_{(n)}$ by $M_s(x_{(n)})$, $\overline{M}_s(x_{(n)})$ and $M(x_{(n)})$, respectively.

In order to show (16.7), it suffices to prove that

(16.9) $$\sup_{z_{(m)} \in R_{(m)}} | H_s(z_{(m)}) - \overline{H}_s(z_{(m)})| \to 0, \ (s \to \infty),$$

because two types of asymptotic equivalence, (I,S) and (I,μ), are mutually equivalent in our present case.

Let ε be any given positive number. Then, one can find a point $a^s_{(m)} = (a^s_1,\dots,a^s_m)$ in $R_{(m)}$, some of the components of which might be $+ \infty$, such that

$$(16.10) \qquad \sup_{z_{(m)} \in R_{(m)}} | H_s(z_{(m)}) - \overline{H}_s(z_{(m)}) | < | H_s(a^s_{(m)}) - \overline{H}_s(a^s_{(m)}) | + \varepsilon .$$

It is seen, putting

$$(16.11) \qquad u^s_{(n)} = (c^s_1 x_1 + d^s_1, \ldots, c^s_n x_n + d^s_n)$$

for $x_{(n)} = (x_1, \ldots, x_n)$, that

$$(16.12) \qquad M_s(x_{(n)}) = F_s(u^s_{(n)}) \text{ and } \overline{M}_s(x_{(n)}) = \overline{F}_s(u^s_{(n)}) ,$$

for all s. It is also noted that the condition (iii) and (16.8) give us

$$(16.13) \qquad \sup_{x_{(n)} \in R_{(n)}} | M_s(x_{(n)}) - M(x_{(n)}) | \to 0, \ (s \to \infty),$$

and

$$(16.14) \qquad \sup_{x_{(n)} \in R_{(n)}} | \overline{M}_s(x_{(n)}) - M(x_{(n)}) | \to 0, \ (s \to \infty).$$

Now, it follows from (16.5), (16.6) and (16.12) that

$$| H_s(a^s_{(m)}) - \overline{H}_s(a^s_{(m)}) |$$

$$(16.15) = \left| \int_{R_{(n)}} P_s(a^s_{(m)} | u^s_{(n)}) \, dM_s(x_{(n)}) - \int_{R_{(n)}} P_s(a^s_{(m)} | u^s_{(n)}) \, d\overline{M}_s(x_{(n)}) \right| .$$

Since, for each s, $P_s(a^s_{(m)} | u^s_{(n)})$ is a function of $x_{(n)}$, let us put this function as

$$(16.16) \qquad v_s(x_{(n)}) = P_s(a^s_{(m)} | u^s_{(n)}) ,$$

for each s. Then, by the condition (i) of the theorem, $v_s(x_{(n)})$ is a continuous function of $x_{(n)}$ over $R_{(n)}$ for each s, and

$$(16.17) \qquad 0 \leq v_s(x_{(n)}) \leq 1 \quad \text{over } R_{(n)}$$

uniformly for all s. Hence, by Lemmas 16.1 and 16.2, there exist a subsequence $\{v_{s'}(x_{(n)})\}$ $(s' \to \infty)$ and a limit function $v(x_{(n)})$ such that

$$(16.18) \qquad 0 \leq v(x_{(n)}) \leq 1, \text{ and } v_{s'}(x_{(n)}) \to v(x_{(n)}), \ (s \to \infty).$$

This last convergence is uniform on any compact subset of $R_{(n)}$.

By property B(S) of both sequences, $\{ \overline{U}^s_{(n)} \}$ $(s = 1,2, \ldots)$ and $\{U^s_{(n)}\}$ $(s = 1,2,\ldots)$, there can be found a member $B_{(n)}$ of $S_{(n)}$, whose closure being compact, such that

(16.19) $P^{U^s_{(n)}}(B_{(n)}) > 1-\varepsilon$, $P^{\overline{U}^s_{(n)}}(B_{(n)}) > 1-\varepsilon$ and $P^{U_{(n)}}(B_{(n)}) > 1-\varepsilon$,

for all $s \geq s_o$ for some positive integer s_o, where ε is the same as that of (16.10).

From (16.17) and (16.19), it then follows that

$$\left| \int_{R_{(n)}} v_s(x_{(n)})\, dM_s(x_{(n)}) - \int_{B_{(n)}} v_s(x_{(n)})\, dM_s(x_{(n)}) \right| < \varepsilon \ ,$$

$$(16.20) \quad \left| \int_{R_{(n)}} v_s(x_{(n)})\, d\overline{M}_s(x_{(n)}) - \int_{B_{(n)}} v_s(x_{(n)})\, d\overline{M}_s(x_{(n)}) \right| < \varepsilon \ ,$$

$$\left| \int_{R_{(n)}} v_s(x_{(n)})\, dM(x_{(n)}) - \int_{B_{(n)}} v_s(x_{(n)})\, dM(x_{(n)}) \right| < \varepsilon \ ,$$

for all $s \geq s_o$.

Since, as was stated above, $v_{s'}(x_{(n)})$ converges to $v(x_{(n)})$ uniformly over $B_{(n)}$, there exists a positive integer s_o' such that

$$(16.21) \quad \begin{aligned} &\left| \int_{B_{(n)}} v_{s'}(x_{(n)})\, dM_{s'}(x_{(n)}) - \int_{B_{(n)}} v(x_{(n)})\, dM_{s'}(x_{(n)}) \right| < \varepsilon \ , \\ &\left| \int_{B_{(n)}} v_s(x_{(n)})\, d\overline{M}_{s'}(x_{(n)}) - \int_{B_{(n)}} v(x_{(n)})\, d\overline{M}_{s'}(x_{(n)}) \right| < \varepsilon \ , \end{aligned}$$

for all $s' \geq s_o'$.

On the other hand, by (16.13) and (16.14), the Helly-Bray theorem can be used to show that there exists a positive integer s_o'' such that

$$(16.22) \quad \begin{aligned} &\left| \int_{B_{(n)}} v(x_{(n)})\, dM_s(x_{(n)}) - \int_{B_{(n)}} v(x_{(n)})\, dM(x_{(n)}) \right| < \varepsilon \ , \\ &\left| \int_{B_{(n)}} v(x_{(n)})\, d\overline{M}_s(x_{(n)}) - \int_{B_{(n)}} v(x_{(n)})\, dM(x_{(n)}) \right| < \varepsilon \ , \end{aligned}$$

for all $s \geq s_o''$.

It now follows from (16.20), (16.21) and (16.22) that

$$(16.23) \quad \left| H_{s'}(a^{s'}_{(m)}) - \overline{H}_{s'}(a^{s'}_{(m)}) \right| < 6\,\varepsilon \ ,$$

or equivalently, by (16.10),

(16.24) $$\sup_{z_{(m)} \in R_{(m)}} | H_{s'}(z_{(m)}) - \bar{H}_{s'}(z_{(m)}) | < 7\varepsilon$$

for all $s' \geqq \max(s_0, s_0', s_0'')$.

The above argument tells us that, if (16.9) is not true, then we always have a contradiction, which completes the proof of the theorem.

It should be noted that the last condition, (iii), of this theorem can be replaced by the following

(iii)* $\bar{U}^s_{(n)}$ converges in law to some distribution $U_{(n)}$ of the continuous type.

Under the same situation as above, suppose that $Z^s_{(m)}$ has a conditional probability density function given $X^s_{(n)} = x_{(n)}$, $p_s(z_{(m)}|x_{(n)})$, and hence the pdf. of the marginal $Z^s_{(m)}$ is given by

(16.25) $$h_s(z_{(m)}) = \int_{R_{(n)}} p_s(z_{(m)}|x_{(n)})\, dF_s(x_{(n)}) ,$$

for each s.

For the sequence $\{\bar{X}^s_{(n)}\}$ (s = 1,2,...), put

(16.26) $$\bar{h}_s(z_{(m)}) = \int_{R_{(n)}} p_s(z_{(m)}|x_{(n)})\, d\bar{F}_s(x_{(n)}) ,$$

for each s, and let $\{\bar{Z}^s_{(m)}\}$ (s = 1,2,...) be the corresponding sequence.

Then, by using Lemma 16.4, it is immediate from the above theorem that

<u>COROLLARY 16.1</u>. Suppose that the conditions (ii) and (iii) of the preceding theorem and the following condition are satisfied.

(i') For each s, $p_s(z_{(m)}|x_{(n)})$ is continuous with respect to $(z_{(m)}, x_{(n)})$ over $R_{(m)} \times D_{(n)}$, with $D_{(n)} = \{x_{(n)} | K_i \leqq x_i, i=1,.,n\}$ for some set of constants K_i 's, and $p_s(z_{(m)}|x_{(n)}) = 0$ over $R_{(m)} \times (R_{(n)} - D_{(n)})$, for all s

Then, it holds that $Z^s_{(m)} \sim \bar{Z}^s_{(m)}$ (I,S), as $s \to \infty$.

The proof of this result is omitted.

In the second place, we shall consider the case when n = 0 in the general formulation of the problem stated in the beginning of this section.

Let $\{(Y^s_{(\ell)}, Z^s_{(m)})\}$ $(s = 1,2,\ldots)$ be a sequence of random variables, for which ℓ and m are fixed independently of s. Further, let $\{c^s_{(\ell)} = (c^s_1,\ldots,c^s_\ell)\}$ $(s = 1,2,\ldots)$ be a sequence of vectors whose components are all positive, for which it is assumed that, for each s, the point $c^s_{(\ell)}$ lies in the defining subset $D^s_{(\ell)}$ of $Y^s_{(\ell)}$, i.e., the subset of $R_{(\ell)}$ on which $Y^s_{(\ell)}$ is defined.

Let $H_s(z_{(m)})$, $P_s(z_{(m)}|y_{(\ell)})$ and $G_s(y_{(\ell)})$ be cdf. of $Z^s_{(m)}$, conditional cdf. of $Z^s_{(m)}$ given $Y^s_{(\ell)} = y_{(\ell)}$ and cdf. of $Y^s_{(\ell)}$, respectively. Then clearly

(16.27) $$H_s(z_{(m)}) = \int_{R_{(\ell)}} P_s(z_{(m)}|y_{(\ell)})\, dG_s(y_{(\ell)})$$

Let us now put

(16.28) $$\bar{H}_s(z_{(m)}) = P_s(z_{(m)}|c^s_{(\ell)}) .$$

Then this is a cdf. of some m-dimensional distribution, $\bar{Z}^s_{(m)}$ say.

Under this situation, one can show the following

THEOREM 16.2. Suppose that the following conditions are satisfied:

(i) For each s, and for any given $z_{(m)}$ in $R_{(m)}$, the conditional cdf. $P_s(z_{(m)}|y_{(\ell)})$ is a continuous function of $y_{(\ell)}$ over $R_{(\ell)}$.

(ii) The sequence of ℓ-dimensional random variable given by

(16.29) $$\bar{Y}^s_{(\ell)} = (Y^s_1/c^s_1,\ldots,Y^s_\ell/c^s_\ell) , \quad s = 1,2,\ldots ,$$

converges in probability to the point $1_{(\ell)} = (1,\ldots,1)$ as $s \to \infty$.

Then, it holds that

(16.30) $$Z^s_{(m)} \sim \bar{Z}^s_{(m)} \quad (I,S)$$

as $s \to \infty$.

PROOF. Put, for $y_{(\ell)} = (y_1,\ldots,y_\ell)$,

$$v^s_{(\ell)} = (c^s_1 y_1,\ldots,c^s_\ell y_\ell) .$$

Then the cdf. of $\bar{Y}^s_{(\ell)}$ of (16.29) is given by

$$\bar{G}_s(y_{(\ell)}) = G_s(v^s_{(\ell)}) ,$$

and hence the cdf. of $Z^s_{(m)}$ given by (16.27) is expressed as

$$H_s(z_{(m)}) = \int_{R_{(\ell)}} P_s(z_{(m)} | v^s_{(\ell)})\, d\bar{G}_s(y_{(\ell)}) \tag{16.31}$$

For any given $\varepsilon > 0$, there exists a point $a^s_{(m)}$ in $R_{(m)}$ such that

$$\sup_{z_{(m)} \in R_{(m)}} | H_s(z_{(m)}) - \bar{H}_s(z_{(m)}) | < | H_s(a^s_{(m)}) - \bar{H}_s(a^s_{(m)}) | + \varepsilon , \tag{16.32}$$

for each s.

Now, let δ be any given positive number. Then, it holds that

$$\begin{aligned} &| H_s(a^s_{(m)}) - \bar{H}_s(a^s_{(m)}) | \\ &= \left| \int_{R_{(\ell)}} (P_s(a^s_{(m)} | v^s_{(\ell)}) - P_s(a^s_{(m)} | c^s_{(\ell)}))\, d\bar{G}_s(y_{(\ell)}) \right| \\ &\leqq \int_{|y_{(\ell)} - 1_{(\ell)}| \leqq \delta} | u_s(y_{(\ell)}) |\, d\bar{G}_s(y_{(\ell)}) \\ &+ \int_{| y_{(\ell)} - 1_{(\ell)} | > \delta} | u_s(y_{(\ell)}) |\, d\bar{G}_s(y_{(\ell)}) , \end{aligned} \tag{16.33}$$

where we have put

$$u_s(y_{(\ell)}) = P_s(a^s_{(m)} | y^s_{(\ell)}) - P_s(a^s_{(m)} | c^s_{(\ell)}) . \tag{16.34}$$

Evidently, $v_s(y_{(\ell)})$ is a continuous and bounded function of $y_{(\ell)}$ over $R_{(\ell)}$, and $u_s(1_{(\ell)}) = 0$.

By Lemmas 16.1 and 16.2, there exist a subsequence $\{ u_{s'}(y_{(\ell)}) \}$ ($s' - 1,2,\ldots$) and a continuous function $u_o(y_{(\ell)})$ such that $u_{s'}(y_{(\ell)}) \to u_o(y_{(\ell)})$ as $s \to \infty$; this convergence is uniform for all $y_{(\ell)}$ such that $| y_{(\ell)} - 1_{(\ell)} | \leqq \delta$ and $u_o(1_{(\ell)}) = 1$. It follows then that there exists a positive integer s_o such that

$$\sup_{| y_{(\ell)} - 1_{(\ell)} | \leqq \delta} | u_{s'}(y_{(\ell)}) | < \varepsilon , \tag{16.35}$$

for all $s' \geqq s_o$.

Therefore, for the first member of the last expression of (16.33) we have

(16.36) $$\int_{|y_{(\ell)}-1_{(\ell)}| \leqq \delta} |u_{s'}(y_{(\ell)})| \, d\bar{G}_{s'}(y_{(\ell)}) < \varepsilon$$

for all $s' \geqq s_o$.

From the in probability convergence of $\{\bar{Y}^s_{(\ell)}\}$ $(s = 1,2,\ldots)$ to the point $1_{(\ell)}$, it is evident that there exists a positive integer s'_o such that

(16.37) $$\int_{|y_{(\ell)}-1_{(\ell)}| > \delta} |u_{s'}(y_{(\ell)})| \, d\bar{G}_{s'}(y_{(\ell)}) < \varepsilon$$

for all $s' \geqq s'_o$.

It follows from (16.32), (16.33), (16.36) and (16 37) that

(16.38) $$\sup_{z_{(m)} \in R_{(m)}} | H_{s'}(z_{(m)}) - \bar{H}_{s'}(z_{(m)})| < 3\varepsilon ,$$

for all $s \geqq \max(s_o, s'_o)$, which shows that

(16.39) $$\sup_{z_{(m)} \in R_{(m)}} | H_s(z_{(m)}) - \bar{H}_s(z_{(m)})| \to 0, \ (s \to \infty),$$

for, otherwise, one has always a contradiction.

Since the condition (16.39) is equivalent to that of (16.30), the proof of the theorem is completed.

In the case where $Z^s_{(m)}$ has the conditional pdf. $p_s(z_{(m)}| y_{(\ell)})$, and hence the pdf.

(16.40) $$h_s(z_{(m)}) = \int_{R_{(\ell)}} p_s(z_{(m)}| y_{(\ell)}) \, dG_s(y_{(\ell)}) ,$$

we define

(16.41) $$\bar{h}_s(z_{(m)}) = p_s(z_{(m)}| c^s_{(\ell)}),$$

and denote the corresponding variable by $\bar{Z}^s_{(m)}$.

Then, the following is an immediate consequence of the above theorem.

<u>COROLLARY 16.2</u>. Suppose that the condition (ii) of the above theorem and the following condition are satisfied.

(i') For each s, $p_s(z_{(m)}| y_{(\ell)})$ is continuous with respect to $(z_{(m)}, y_{(\ell)})$ over $R_{(m)} \times D_{(\ell)}$, with $D_{(\ell)}=\{Y_{(\ell)} | K_j \leqq y_j, \ j=1,\ldots,\ell\}$ for some constants K_j's, and $p_s(z_{(m)}| y_{(\ell)}) = 0$ over $R_{(m)} \times (R_{(\ell)}-D_{(\ell)})$.

Then it holds that

(16.42) $$Z^s_{(m)} \sim \overline{Z}^s_{(m)} \quad (I,S)$$

as $s \to \infty$.

Now, in the last place, we shall consider the general case when n, $\ell \neq 0$. For this, Theorems 16.1 and 16.2 can be combined to get the following theorem, whose proof is easy and is omitted.

<u>THEOREM 16.3</u>. Let $\{X^s_{(n)}, Y^s_{(\ell)}, Z^s_{(m)}\}$ $(s = 1,2,\ldots)$ be a sequence of $(n + \ell + m)$-dimensional random variables, where n, ℓ and m are fixed independently of s. Suppose that the following conditions are satisfied.

(i) For the second marginals, $\{Y^s_{(\ell)} = (Y^s_1, \ldots, Y^s_\ell)\}$ $(s = 1,2,\ldots)$ there can be found a sequence of real vectors, $\{c^s_{(\ell)} = (c^s_1, \ldots, c^s_\ell)\}$ $(s = 1,2,\ldots)$, with $c^s_i > 0$ for all i and s, such that the sequence

$$\overline{Y}^s_{(\ell)} = (Y^s_1/c^s_1, \ldots, Y^s_\ell/c^s_\ell), \quad s = 1,2,\ldots$$

converges in probability to the point $1_{(\ell)} = (1,\ldots,1)$ as $s \to \infty$, where $c^s_{(\ell)}$ is assumed to lie in the subset of $R_{(\ell)}$ on which $Y^s_{(\ell)}$ is defined.

(ii) For the conditional distribution of $X^s_{(n)}$ given $Y^s_{(\ell)} = c^s_{(\ell)}$, $c^s_{(\ell)}$ being the same as in (i) above, there exist two sequences of real vectors, $\{d^s_{(n)} = (d^s_1, \ldots, d^s_n)\}$ $(s = 1,2,\ldots)$ with $d^s_i > 0$ and $\{e^s_{(n)} = (e^s_1, \ldots, e^s_n)\}$ $(s = 1,2,\ldots)$, and a sequence of n-dimensional random variables, $\{U^s_{(n)} = (U^s_1, \ldots, U^s_n)\}$ $(s = 1,2,\ldots)$, which converges (I,S) to some distribution $U_{(n)}$ as $s \to \infty$, such that

(16.43) $$\overline{X}^s_{(n)} \sim \tilde{X}^s_{(n)} \quad (I,S),\ (s \to \infty),$$

where $\tilde{X}^s_{(n)}$ stands for the conditional distribution of $X^s_{(n)}$ given $Y^s_{(\ell)} = c^s_{(\ell)}$; while $\overline{X}^s_{(n)} = (\overline{X}^s_1, \ldots, \overline{X}^s_n)$ denotes the distribution given by

(16.44) $$\overline{X}^s_i = d^s_i \, U^s_i + e^s_i, \quad i=1,\ldots,n.$$

(iii) Conditional cdf. of $Z^s_{(m)}$ given $X^s_{(n)} = x_{(n)}$ and $Y^s_{(\ell)} = y_{(\ell)}$,

$P_s(z_{(m)} \mid x_{(n)}, y_{(\ell)})$, is continuous with respect to $(x_{(n)}, y_{(\ell)})$ for any fixed $z_{(m)}$.

Then, for two sequences of random variables, $\{Z^s_{(m)}\}$ $(s = 1,2,\ldots)$ and $\{\overline{Z}^s_{(m)}\}(s = 1,2,\ldots)$, whose cdf.'s being given respectively by

(16.45) $$H_s(z_{(m)}) = \int_{R_{(n)} \times R_{(\ell)}} P_s(z_{(m)} \mid x_{(n)}, y_{(\ell)}) \, dL_s(x_{(n)}, y_{(\ell)})$$

with cdf. of $(X^s_{(n)}, Y^s_{(\ell)})$, $L_s(x_{(n)}, y_{(\ell)})$, and

(16.46) $$\overline{H}_s(z_{(m)}) = \int_{R_{(n)}} P_s(z_{(m)} \mid x_{(n)}, c^s_{(\ell)}) \, dF_s(x_{(n)}),$$

$s = 1,2,\ldots$, with $F_s(x_{(n)})$, the cdf. of $X^s_{(n)}$, it holds that

(16.47) $$Z^s_{(m)} \sim \overline{Z}^s_{(m)} \ (I, S), \ (s \to \infty).$$

It should be remarked that the condition (i) of Theorem 16.2 pre-assumes that $P_s(z_{(m)} \mid y_{(\ell)})$ is defined all over the space $R_{(\ell)}$, and therefore, if the set $D^s_{(\ell)}$ on which $Y^s_{(\ell)}$ is defined is not identical with $R_{(\ell)}$, one has to extend the original conditional cdf. in any way but continuously from $D^s_{(\ell)}$ to the whole space $R_{(\ell)}$. A similar remark should be made for Theorem 16.3.

It is also noted that, in both of Theorems 16.2 and 16.3, $c^s_{(\ell)}$ may not necessarily be included in the set $D^s_{(\ell)}$, but solely in the set on which $P_s(z_{(m)} \mid y_{(\ell)})$ is defined to be a cdf. of some probability distribution.

Some applications of the results obtained here will be seen in [5, 6].

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