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# Modified RBSM for nonlinear analysis

**Shah S.J.**
*Framatome, USA*

## ABSTRACT

This paper reports on the development and application of a rigid body spring model (RBSM) for nonlinear analysis. With this model, the angles of the RBSM bars are defined in the local coordinate system, which subsequently helps in describing the behavior of geometrically nonlinear structures. Effectiveness of the modified RBSM element in nonlinear structural analysis is demonstrated by studying the elastic-plastic behaviors of a semi-circular arch under an incremental load.

## INTRODUCTION

The rigid body spring models (RBSMs) are extremely suitable to the discrete analysis of collapse behaviors of the structures accompanied by the plastic deformation. Material nonlinear problems become much simpler and computationally more efficient with the RBSM than with the other conventional methods.

The basic RBSM model, introduced by Kawai [1], applies to beam two-dimensional problems. This model consists of rigid body elements connected by springs resisting relative movements. The derivation of the RBSM is based on the fact that bending stiffness of a structure is much smaller than the direct stiffness of the same structure and therefore, a bending analysis of these structures can be made under the assumption of large direct stiffness so long as the bending deformation considered is small. In the RBSM, deformations are concentrated at the spring on each end of an element. Thus, the elements are rigid and the joints are flexible.

The stiffness matrix of the original RBSM element is determined relative to the local and the global coordinate system in the same manner as performed for the straight member beam element given by Cook [2]. Usually, the curved members with these types of elements in a structure are replaced by a series of straight members. The disadvantages of this replacement are the great increase in the degrees of freedom of the structure and approximation involved in the analyses. For instance, a static and stability analyses of a semi-circular arch, to an acceptable degree of accuracy, would require a minimum of sixteen straight members. With the use of the modified RBSM element, which is introduced in the following presentation, the same degree of accuracy could be achieved with approximately half of the original RBSM beam element or a two-dimensional beam bending element. Specifically, the modified RBSM element would be advantageous in nonlinear analysis of problems in structural analysis, since it requires considerably less computer time than the conventional finite-element method. The static and buckling analyses of curved members using the original RBSM and the new RBSM beam elements have been well documented by Shah [3]. The combination of the lumped axial force given by Shah and Pilkey [4] with the new and the original RBSM elements were utilized for the buckling analysis.

In the following discussion, the RBSM stiffness matrix is developed for the beam bending element for geometrically curved members. Also discussed is a method for the solution of the elasto-plastic analysis under the assumption of an ideal plastic stress-strain relationship and the von-Mises yield criterion. At first, the stiffness matrix of a beam is determined relative to the local coordinate axes of the member. The rigid members of the RBSM within the element can be defined at some specified angles about the local coordinate system plane as shown in Figure 1. Then, through the usual global coordinate transformation, the members' stiffness matrix is formed for the common coordinate system. The stiffness matrix presented for nonlinear geometric problems is very general, and it can be used for straight members by equating the angles used in defining the rigid members of the new RBSM element in the local coordinate system plane to zero.

## NEW RBSM MODEL FOR GEOMETRICALLY NONLINEAR PROBLEMS

Consider the deformation of two rigid bars connected by rotational springs as shown in Fig. 1. The bars are assumed to be displaced under some applied loading.

Before deformation, $i$ is at $(w_i, u_i, \theta_i)$ and after deformation the centroid $i$ is at $(w_i + \Delta w_i, u_i + \Delta u_i, \theta_i + \Delta\theta_i)$.

Before deformation, the component of vector $\overrightarrow{S_{j1}S_{j2}}$ in the direction of the $Z(w)$ axis is

$$\left(w_i - \frac{\ell_i}{2}(1-\cos\theta_i)\right) - \left(w_{i-1} + \frac{\ell_{i-1}}{2}(1-\cos\theta_{i-1})\right) \tag{1}$$

where $S_{j1}$ represents the right end of the rigid element $i$-1 and $S_{j2}$ represents the left end of the rigid element $i$.

Also, before deformation, the component of vector $\overrightarrow{S_{j1}S_{j2}}$ in the direction of the $X(u)$ axis is

$$\left(u_i + \frac{\ell_i}{2}\sin\theta_i\right) - \left(u_{i-1} - \frac{\ell_{i-1}}{2}\sin\theta_{i-1}\right) \tag{2}$$

After deformation, the component of vector $\overrightarrow{S'_{j1}S'_{j2}}$ in the direction of the $Z(w)$ axis is

$$\left\{w_i + \Delta w_i - \frac{\ell_i}{2}\left(1-\cos(\theta_i + \Delta\theta_i)\right)\right\} - \left\{w_{i-1} + \Delta w_{i-1} + \frac{\ell_{i-1}}{2}\left(1-\cos(\theta_{i-1} + \Delta\theta_{i-1})\right)\right\} \tag{3}$$

where $S'_{j1}$ and $S'_{j2}$ represent the positions of $S_{j1}$ and $S_{j2}$ after deformation.

After deformation, the component of vector $\overrightarrow{S'_{j1}S'_{j2}}$ in the $X(u)$ axis is

$$\left\{u_i + \Delta u_i + \frac{\ell_i}{2}\sin(\theta_i + \Delta\theta_i)\right\} - \left\{u_{i-1} + \Delta u_{i-1} - \frac{\ell_{i-1}}{2}\sin(\theta_{i-1} + \Delta\theta_{i-1})\right\} \tag{4}$$

Let $\delta x$ and $\delta z$ be the relative incremental displacement components in the $x(u)$ and $z(w)$ axes, respectively.

$$\delta z = \Delta w_i - \Delta w_{i-1} - \frac{\ell_i}{2}\{\cos(\theta_i + \Delta\theta_i) - \cos\theta_i\} - \frac{\ell_{i-1}}{2}\{\cos(\theta_{i-1} + \Delta\theta_{i-1}) - \cos\theta_{i-1}\} \tag{5}$$

$$\delta x = \Delta u_i - \Delta u_{i-1} + \frac{\ell_i}{2}\{\sin(\theta_i + \Delta\theta_i) - \sin\theta_{i-1}\} + \frac{\ell_{i+1}}{2}\{\sin(\theta_{i-1} + \Delta\theta_{i-1}) - \sin\theta_{i-1}\} \tag{6}$$

In the Taylor series expansion form

$$\begin{aligned} \cos(\theta_i + \Delta\theta_i) &\simeq \cos\theta_i - \sin\theta_i \Delta\theta_i \\ \sin(\theta_i + \Delta\theta_i) &\simeq \sin\theta_i + \cos\theta_i \Delta\theta_i \end{aligned} \tag{7}$$

Substitute these equations into $\delta z$ and $\delta x$

$$\delta z = \Delta w_i - \Delta w_{i-1} + \frac{\ell_i}{2}\sin\theta_i\,\Delta\theta_i + \frac{\ell_{i-1}}{2}\sin\theta_{i-1}\,\Delta\theta_{i-1} \tag{8}$$

$$\delta x = \Delta u_i - \Delta u_{i-1} + \frac{\ell_i}{2}\cos\theta_i\,\Delta\theta_i + \frac{\ell_{i-1}}{2}\cos\theta_{i-1}\,\Delta\theta_{i-1} \tag{9}$$

$$\delta_m = \Delta\theta_i - \Delta\theta_{i-1} \tag{10}$$

In matrix form,

$$\begin{Bmatrix} \delta_z \\ \delta_x \\ \delta_m \end{Bmatrix} = \begin{bmatrix} -1 & 0 & \frac{\ell_{i-1}}{2}\sin\theta_{i-1} & 1 & 0 & \frac{\ell_i}{2}\sin\theta_i \\ 0 & -1 & \frac{\ell_{i-1}}{2}\cos\theta_{i-1} & 0 & 1 & \frac{\ell_i}{2}\cos\theta_i \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \Delta w_{i-1} \\ \Delta u_{i-1} \\ \Delta\theta_{i-1} \\ \Delta w_i \\ \Delta u_1 \\ \Delta\theta_i \end{Bmatrix} \tag{11}$$

$$\{\delta\}_\ell = [A]\{u\}_\ell \tag{12}$$

In the usual RBSM, relative incremental displacements are given by

$$\begin{Bmatrix} \delta_z \\ \delta_x \\ \delta_m \end{Bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & \frac{\ell_{i-1}}{2} & 0 & 0 & \frac{\ell_i}{2} \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \Delta w_{i-1} \\ \Delta u_{i-1} \\ \Delta\theta_{i-1} \\ \Delta w_i \\ \Delta u_1 \\ \Delta\theta_i \end{Bmatrix} \tag{13}$$

Consider definitions of strains and stresses at the spring $S_j$ as shown in Fig. 2. The strains are defined as

$\Delta\varepsilon$ = incremental axial strain,
$\Delta\gamma$ = incremental shear strain,
$\Delta k$ = incremental bending strain.

Strains at spring $S_j$ as shown in Fig. 2 can be determined as

$$\begin{Bmatrix} \Delta\varepsilon \\ \Delta\gamma \\ \Delta k \end{Bmatrix} = \frac{1}{\ell} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} [A] \begin{Bmatrix} \Delta w_{i-1} \\ \Delta u_{i-1} \\ \Delta\theta_{i-1} \\ \Delta w_i \\ \Delta u_1 \\ \Delta\theta_i \end{Bmatrix} \tag{14}$$

$$\{\varepsilon\} = \frac{1}{\ell}[R][A]\{u\}_\ell$$

$$= [B]\{u\}_\ell$$

where $\ell = \dfrac{\ell_{i-1} + \ell_i}{2}$ , $\phi = \dfrac{\theta_{i-1} + \theta_i}{2}$

If $\theta_{i-1} + \theta_i = 0$, meaning that the two rigid bars form an equal and opposite angle with respect to the local coordinate axis (i.e., $\phi = 0$), then the matrix $[R]$ is not needed.

The incremental stresses can be given by

$$\begin{Bmatrix} \Delta\sigma_n \\ \Delta\tau \\ \Delta\sigma_b \end{Bmatrix} = [D] \cdot \begin{bmatrix} \Delta\varepsilon \\ \Delta\gamma \\ \Delta k \end{bmatrix} = [D][B] \begin{Bmatrix} \Delta w_{i-1} \\ \Delta u_{i-1} \\ \Delta\theta_{i-1} \\ \Delta w_i \\ \Delta u_1 \\ \Delta\theta_i \end{Bmatrix} \tag{15}$$

where $[D]$ is the material law while the beam remains elastic.

$$[D] = \left[D^e\right] = \begin{bmatrix} EA & 0 & 0 \\ 0 & GA & 0 \\ 0 & 0 & EI \end{bmatrix} \tag{16}$$

The stiffness matrix $\left[\bar{k}\right]$ of the element in a local coordinate system can be written in matrix form as given by Yamada and Yoshimara [5].

$$\left[\bar{k}\right]_{local} = \ell\left[B^T\right][D][B] \tag{17}$$

This matrix relates displacement $\{u\}_\ell$ and $\{f\}_\ell$ as

$$\{f\}_\ell = \left[\bar{k}\right]_{local}\{u\}_\ell \tag{18}$$

where $\{f\}_{\ell}^{T} = [\Delta n_{i-1}, \Delta v_{i-1}, \Delta M_{i-1}, \Delta n_i, \Delta v_i, \Delta M_i]$

$\Delta n$ = form in the direction of $Z(w)$ axis,
$\Delta v$ = force in the direction of $X(u)$ axis,
$\Delta M$ = rotational force.

The transformation of the element stiffness matrix from the local to the global system is performed by

$$[k_{global}] = [T^T][\bar{k}]_{local}[T] \tag{19}$$

in which $[T]$ represents the coordinate transformation of the beam element.

Also,

$$\{u\}_{local} = [T]\{u\}_{global} \tag{20}$$

so $\{f\}_{global}^{T} = [T]^T[\bar{k}]_{global}[T]\{u\}_{global}$

where $\{u\}_{global}^{T} = [\Delta w_{i-1}, \Delta u_{i-1}, \Delta\theta_{i-1}, \Delta w_i, \Delta u_i, \Delta\theta_i]_{global}$

$$[k]_{global} = [T]^T[\bar{k}]_{local}[T] = \ell[T]^T[B]^T[D][B][T]$$

where $[k]_{global}$ = the stiffness matrix of the element in a global system.

## PROCEDURE OF CALCULATION FOR ELASTIC-PLASTIC ANALYSIS

Some details of procedures provided by Yamada and Yoshimara [5] and used in the flow-sequence of the program for the elastic-plastic problems are presented.

1. Apply incremental load ΔP to the structure and calculate elastic displacements at the nodes and then elastic strains and stresses.
2. Check for whether the element is elastic by using the following von Mises yielding function

$$f(\sigma_n, \tau, \sigma_b) = \frac{\sigma_n^2}{\sigma_{no}} + \frac{\tau^2}{\tau_o} + \frac{\sigma_b^2}{\sigma_{bo}} - 1 = 0$$

   where $\sigma_n$ = axial stress,
   $\tau$ = shear stress,
   $\sigma_b$ = bending stress.

   $\sigma_{no}$, $\tau_o$, $\sigma_{bo}$ are the yielding stresses.
3. Calculate the plastic-stress-strain matrix $[D^P]$ using the procedure given by Yamada and Yoshimara [5]. Thereafter calculate $[k^P]$ for the post-yield element. Note that $[D^P]$ is to be evaluated using the present stress with each post-yield element obtained at the end of the preceding cycle.
4. Set the net value of the load step by equating

$$P_a = P_b + r_{min}\,\Delta P$$

   where $P_b$ = load before deformation,
   $P_a$ = load after deformation,

$r_{min}$ is calculated by Equation given by Yamada and Yoshimara [5],
$\Delta P$ = incremental load.

The factor $r_{min}$ determines the load increment sufficient to reduce the elastic element of $r_{min}$ to yield. From this step on, this element is treated as a post-yield element, and the remainder of the elements are in the elastic state.

5. Calculate stress increment $\Delta\sigma_{ij}^{T}$ and strain increment $\Delta\varepsilon_{ij}^{T}$ at each element. Modify the stiffness matrix [k] of the whole structure. In doing so, leave the matrices [$D^e$] and [$k^e$] for every element in the elastic state as they were.
6. Steps 1 through 5 are repeated until $P_{set}$ (the load that we want) is achieved or the plastic deformations surpass the derived value.

## NUMERICAL EXAMPLES - NONLINEAR ANALYSIS

Nonlinear analysis was conducted for a semi-circular arch under an incremental load applied near the center at two points as shown in Figure 3. Figure 4 shows comparison of the results of the elastic-plastic analysis (assuming perfectly plastic material) of a semi-circular arch under an incremental load, for fixed-fixed end conditions, using the original RBSM method and the new RBSM method. The results in the elastic range are also compared with the general purpose commercially available finite-element computer program.

The solutions obtained by the new RBSM model agree very well with the exact solution in the elastic range. In the plastic range, the results obtained were compared with the ANSYS computer program [6] which has the capability of performing the nonlinear static stress analysis. The results of the analysis for both deflection and forces were quite satisfactory in the plastic (inelastic) range, also. The stability and convergence of incremental solutions by the RBSM method are found to be superior to the conventional finite-element method. In the RBSM, the stiffness of a given body is lumped on the contact surfaces of neighboring rigid elements, and yielding or failure is assumed to occur on these contact surfaces. Consequently, material nonlinear problems become much simpler than with the finite-element method.

## CONCLUSION

The original RBSM was modified to improve it for elastic and inelastic analysis. The modified RBSM defines the angle of the RBSM rigid bars in the local coordinate system, which helps in defining geometrically nonlinear structures in a more exact form, with fewer elements than the original RBSM or a well-known beam bending element. The mathematical basis of the modified RBSM beam element is explicitly derived.

The greatest use of the modified RBSM element is for solving nonlinear material problems where it helps in defining more precisely the geometry of the nonlinear structure with fewer elements, provides excellent stability convergence, and increases efficiency in computer time for incremental solutions. For the results of the elastic-plastic analysis shown in Figure 4, it was found that the modified RBSM element used much less computer time (approximately 1/10th) than the conventional finite-element method.

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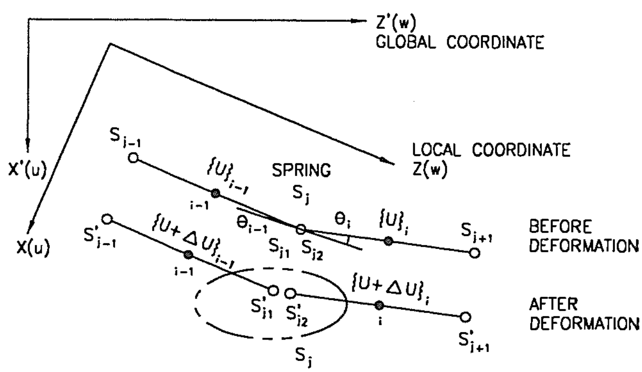
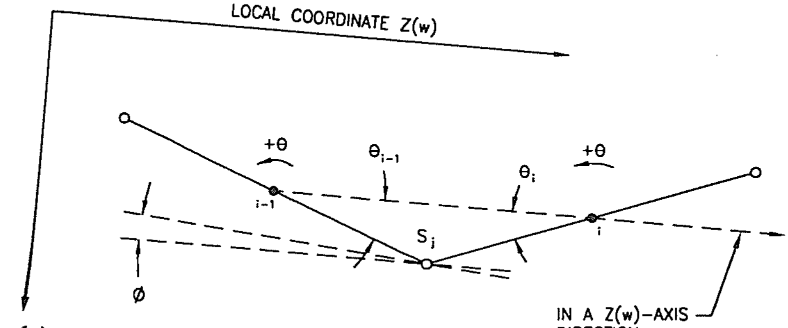



Fig. 1. New RBSM Beam Bending Element




$$\phi = \frac{\theta_{i-1} + \theta_i}{2}$$

Fig. 2. Relative Displacement at Spring $S_j$ in Local Coordinates

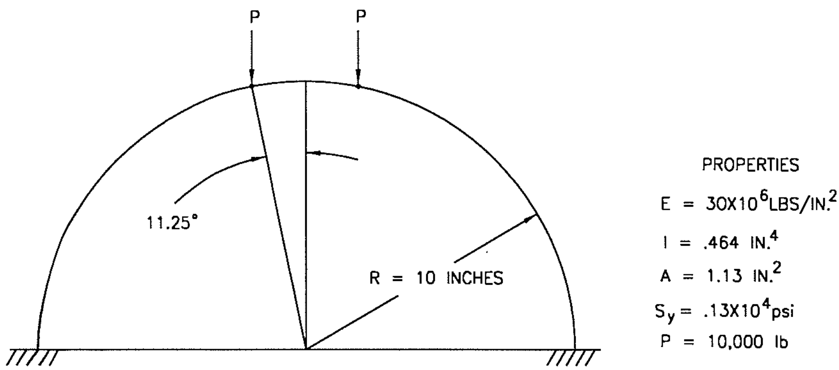



Fig. 3. Semi–Circular Arch Under Concentrated Loadings

Note: 1 in = 25.4 mm, 1 lb = 0.454 kg

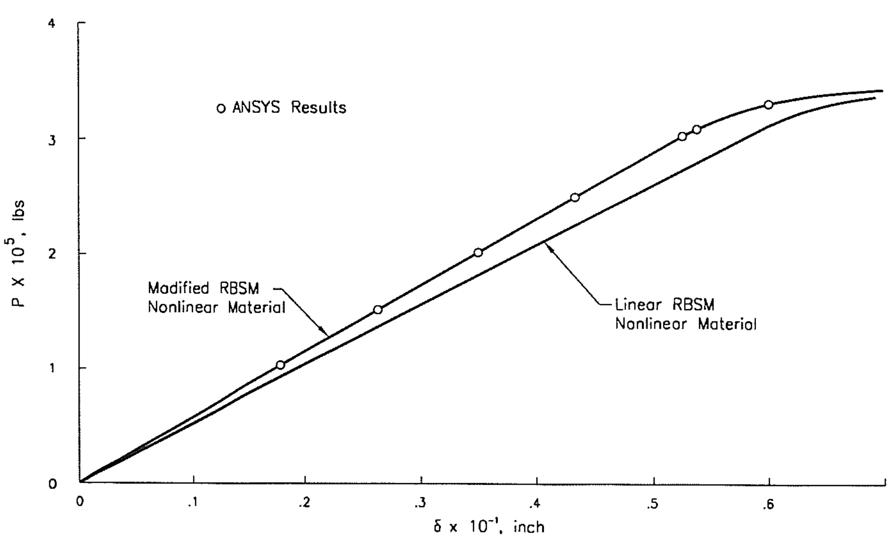



Figure 4

RBSM CURVED BEAM

Load deflection curves of the semi–circular curved beam under loads applied as shown in Figure 3

Note: 1 in = 25.4 mm, 1 lb = 0.454 kg