A Consistent Linear Theory of Thin Elastic Shells

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Summary:

Shell structures are well-known in structural engineering for their load carrying capacity besides minimum weight. The object of the present paper is to provide a consistent linear shell theory.

Starting from the basic equations of the linear three-dimensional elasticity theory, displacements and strains are expanded in power series in terms of the thickness coordinate. When the kinematic and Hooke's law are statisfied locally for the three-dimensional elasticity theory, the equilibrium equations are satisfied globally by application of an integration procedure. In principle the three-dimensional state of stress and displacement can be approximated accurately even for thick shells.

In order to apply the theory to thin shell structures the question arises, where the series expansion has to be determinated without restricting the validity of the theory considerably. It has been found here, that a quadratic approximation leads to consistent shell equations without requiring any additional assumption.

The circular cylindrical shell is taken as typical example. The system of equations is reduced to only one partial differential equation of eighth order with constant coefficients for the transverse displacement w.

The roots of the "characteristic" equation are compared with solutions of the classical shell theory (Kirchhoff-Love-Hypothesis). The agreement is well and justifies the a-priori assumptions of the classical shell theory.

1. Linear elasticity theory

The basic equations of the linear elasticity theory are in tensor notation [1]:

$$\begin{aligned} &\text{equilibrium} && \tau^{ij}|_j + p^i = 0, \\ &\text{kinematic} && \gamma_{ij} = \tfrac{1}{2}(u_i|_j + u_j|_i), \\ &\text{Hooke's law} && \tau^{ij} = E^{ijkm}\,\gamma_{km}, \\ &\text{deformation energy per unit volume} && U_v = \tfrac{1}{2}\,\tau^{ij}\,\gamma_{ij}. \end{aligned} \tag{1}$$

τ^{ij} is the symmetric stress tensor, γ_{ij} Green's strain tensor, p^i and u_i the vectors of body forces and displacements respectively. Derivation of the deformation energy with respect to a strain component leads to the corresponding stress component and v.v.

$$\frac{\partial U_v}{\partial \gamma_{ij}} = \tau^{ji}, \quad \frac{\partial U_v}{\partial \tau^{ij}} = \gamma_{ji}. \tag{2}$$

Use is made of Einstein's summation convention. Throughout this paper latin indices have the range: 1,2,3, greek indices take values of: 1,2.

2. Stress resultants and equilibrium

Shells are distinguished from the three dimensional continuum by the fact that the theory is applied to bodies bounded by two curved surfaces, the distance between the surfaces being small in comparison with the other dimensions. The shell continuum is referred to the middle surface (curvlinear coordinates ξ^α), which bisects the distance between both shell faces at every point. The middle surface or reference surface is an arbitrary formed surface embeded in the three dimensional Euclidion space. All material points of the shell (position vektor $\vec{R}$) are described by points of the middle surface (position vector $\vec{r}$) and the distance ζ measured normal to the surface (fig.1). All variables of state are expanded in power series in terms of that thickness coordinate ζ

$$\text{e.g.}\quad A_{ij} = {}^0A_{ij} + {}^1A_{ij}\zeta + {}^2A_{ij}\zeta^2 + \ldots = \sum_{\ell=0}^{\infty} {}^\ell A_{ij}\zeta^\ell.$$

Avoiding confusions with tensor indices the expansion coefficients are placed on the upper left-hand side of the symbols. Summarizing the equations of equilibrium multiplication by ζ^ℓ and integration across the volume element leads to

$$\int_V (\tau^{ij}|_j + p^i)\vec{g}_i\zeta^\ell dV = 0.$$

After some straight forward operations the integration with respect to the thickness ζ can be separated. That leads to the definitions of stress resultants:

$${}^\ell m^{ij} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \zeta^\ell\,\mu\,\tau^{ik}\,\mu_k^j\,d\zeta. \tag{3}$$

Hereby μ_k^j is the shifter and μ it's determinant.
The equilibrium equations of the shell element written down in stress resultants read as follows:

$$\begin{aligned} {}^{\ell}m^{\alpha\beta}|_{\alpha} - b_{\alpha}^{\beta}\,{}^{\ell}m^{\alpha 3} - \ell\,{}^{\ell-1}m^{3\beta} + {}^{\ell}p^{\beta} &= 0, \\ {}^{\ell}m^{\alpha 3}|_{\alpha} + b_{\alpha\beta}\,{}^{\ell}m^{\alpha\beta} - \ell\,{}^{\ell-1}m^{33} + {}^{\ell}p^{3} &= 0. \end{aligned} \tag{4}$$

From the symmetry of the stress tensor τ^{ij} we obtain the following equations after application of the same integration scheme:

$$\begin{aligned} {}^{\ell}\bar{m}^{3\beta} &:= \varepsilon_{\alpha\beta}({}^{\ell}m^{\alpha 3} - {}^{\ell}m^{3\alpha} - b_{\gamma}^{\alpha}\,{}^{\ell+1}m^{\gamma 3}) = 0, \\ {}^{\ell}\bar{m}^{33} &:= \varepsilon_{\alpha\beta}({}^{\ell}m^{\alpha\beta} + b_{\gamma}^{\beta}\,{}^{\ell+1}m^{\gamma\alpha}) = 0. \end{aligned} \tag{5}$$

$\varepsilon_{\alpha\beta}$ is the permutation tensor. Equations (4) and (5) correspond with that given by KRÄTZIG [2] and HAMMEL [3] considering some other notations.

3. Strain resultants and compatibility

Let U_A be the deformation energy per unit area of the middle surface, we define the strain resultants ${}^{\ell}\lambda_{ij}$ by analogy with the three-dimensional continuum as

$${}^{\ell}\lambda_{ij} = \frac{\partial U_A}{\partial\,{}^{\ell}m^{ji}}. \tag{6}$$

By virtue of the principle of stationary complementary energy we calculate the geometric field equations

$$\begin{aligned} {}^{\ell}\lambda_{\alpha\beta} &= {}^{\ell}u_{\alpha}|_{\beta} - b_{\alpha\beta}\,{}^{\ell}u_{3}, \\ {}^{\ell}\lambda_{3\alpha} &= {}^{\ell}u_{3}|_{\alpha} + b_{\alpha}^{\beta}\,{}^{\ell}u_{\beta}, \\ {}^{\ell}\lambda_{j3} &= (\ell+1)\,{}^{\ell+1}u_{j}. \end{aligned} \tag{7}$$

The symmetrical part of ${}^{0}\lambda_{\alpha\beta}$ corresponds to the extensions and shear strain of the middle surface, ${}^{1}\lambda_{\alpha\beta}$ with the changes of curvature and the twist. ${}^{\ell}\lambda_{3\alpha}$ are the transverse shear strain resultants and ${}^{\ell}\lambda_{33}$ represents the extensibility in the direction of the normal.
There are nine strain resultants defined for each ℓ computable from three deformation quantities. Therefore the equations of compatibility must be written down:

$$\begin{aligned} \varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}({}^{\ell}\lambda_{\gamma\delta}|_{\beta} + b_{\gamma\delta}\,{}^{\ell}\lambda_{3\beta}) &= 0, \\ \varepsilon^{\alpha\beta}({}^{\ell}\lambda_{3\alpha}|_{\beta} - b_{\alpha}^{\gamma}\,{}^{\ell}\lambda_{\gamma\beta}) &= 0, \\ {}^{\ell}\bar{\Lambda}^{3\alpha} := \varepsilon^{\alpha\beta}((\ell+1)\,{}^{\ell+1}\lambda_{3\beta} - {}^{\ell}\lambda_{33}|_{\beta} - b_{\beta}^{\gamma}\,{}^{\ell}\lambda_{\gamma 3}) &= 0, \\ {}^{\ell}\bar{\Lambda}^{\alpha\beta} := \varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}((\ell+1)\,{}^{\ell+1}\lambda_{\alpha\beta} + {}^{\ell}\lambda_{\beta 3}|_{\alpha}) &= 0. \end{aligned} \tag{8}$$

If we define "permutated strain resultants" ${}^{\ell}\Lambda^{ij}$, for example

$$ {}^{\ell}\Lambda^{\alpha\beta} = \varepsilon^{\alpha\gamma}\, a^{\beta\delta}\, {}^{\ell}\lambda_{\delta\gamma}, \tag{9} $$

an equivalent representation of the compatibility conditions is as follows:

$$ \begin{aligned} {}^{\ell}\Lambda^{\alpha\beta}|_{\alpha} - b_{\alpha}^{\beta}\ {}^{\ell}\Lambda^{\alpha 3} - \ell\ {}^{\ell-1}\Lambda^{3\beta} &= 0, \\ {}^{\ell}\Lambda^{\alpha 3}|_{\alpha} + b_{\alpha\beta}\ {}^{\ell}\Lambda^{\alpha\beta} - \ell\ {}^{\ell-1}\Lambda^{33} &= 0. \\ {}^{\ell}\Lambda^{j\alpha} &= 0. \end{aligned} \tag{10} $$

In this notation the equations of compatibility have the same structure as the homogeneous equations of equilibrium (4) and symmetry (5). So the formal requirements of the static-geometric analogy are met.

4. Elasticity relations and deformation energy

Hooke's law has to be transfered from the three dimensional theory to the two-dimensional shell theory as well. Then the stress resultants are computable from the strain resultants, and the deformation energy per unit area U_A is given as function of the strain resultants. The following constitutive equation is valid

$$ \frac{\partial U_A}{\partial\, {}^{\ell}\lambda_{ij}} = {}^{\ell}m^{ji}. \tag{11} $$

By virtue of the principle of stationary energy the equations of equilibrium are verified and the boundary conditions are determined.

5. Thin shell theory

By means of the equations given above the three-dimensional state of stress and displacement can be approximated accurately even for thick shells. In order to apply the theory to thin shell structures the question arises, where the series expansion has to be determinated without restricting the validity of the theory considerably. The ranges of uncertainty of a linear approximation were shown by KRÄTZIG [4]. It has been found [5], that a consequent quadratic approximation leads to consistent shell equations without requiring any additional assumption.

The number of all variables of state is enlarged in relation to the classical theory. In addition to displacements (${}^{0}u_j$) and rotations (${}^{1}u_{\alpha}$) of the shell middle surface changes of thickness (${}^{1}u_3, {}^{2}u_3$) and warping of the cross section (${}^{2}u_{\alpha}$) are taken into account. The tensor of inplane forces ${}^{0}m^{\alpha\beta}$ and the tensor of moments ${}^{1}m^{\alpha\beta}$ are supplemented by the tensor of warping moments ${}^{2}m^{\alpha\beta}$, apart from transverse shear forces ${}^{0}m^{\alpha 3}$ transverse shear resultents of higher order. ${}^{1}m^{\alpha 3}$ and ${}^{2}m^{\alpha 3}$ are considered. ${}^{\ell}m^{3\alpha}$ and ${}^{\ell}m^{33}$ ($\ell = 0,1,2$) act on cross sections $\zeta = \text{const}$. The constitutive equations (11) and the symmetry conditions (5) are fulfilled identically.

Starting with the equations of equilibrium, inserting Hocke's law and the kinematics, we obtain a system of nine partial differential equations in both

surface coordinates to determine the nine unknown displacements.

6. Exampel: circular cylindrical shell

The circular cylindrical shell is taken as typicalexample(fig.2). Let R be the radius, h the thickness of the shell, the parameter

$$c^2 = \frac{h^2}{12R^2}$$

is a small quantity. The derivatives with respect to the dimensionless coordinates $\xi = \frac{x}{R}$ and φ will be indicated by primes and dots respectively. During the formation of the equations only terms are considered up to the quadratic order, i.e. terms of $O(c^4)$ are neglected. Therefore it's consequent, to neglect terms of $O(c^4)$ during the reduction of the system of partial differential equations. In this way we obtain one partial differential equation with constant coefficients for the transverse displacement w, which is of eighth order in ξ and φ.

For that reason a particular solution can be found immediately, if the load is sufficiently smooth. For closed cylindrical shells the homogeneous equation is solved by expansion of w in circumferential direction in a Fourier series and an exponential function in the longitudinal direction with unknown exponent λ

$$w = \sum_n w_n e^{\lambda\xi} \sin n\varphi. \qquad (12)$$

For each circumferential wave number n we obtain as "characteristic" equation a polynomial of fourth order in λ^2:

$$\lambda^8 - 2(2n^2-\beta_2)\lambda^6 + [\frac{1-\nu^2}{\nu^2} + 6n^2(n^2-\beta_1) + \beta_3]\lambda^4 - 4n^2(n^2-1)^2\lambda^2 + n^4(n^2-1)^2 = 0$$

$$\text{with } \beta_1 = \frac{1}{8}(9+6\nu+5\nu^2), \quad \beta_2 = \frac{3}{8}(1+\nu)(1+5\nu), \quad \beta_3 = \frac{3(1+\nu)}{4(1-\nu)}(5-6\nu+4\nu^2). \qquad (13)$$

The eight solutions of this equation are complex and may be written as follows [6]:

$$\lambda_{1,2,3,4} = \pm\kappa_1 \pm i\mu_1; \quad \lambda_{5,6,7,8} = \pm\kappa_2 \pm i\mu_2. \qquad (14)$$

The roots are compared with solutions of other authors (FLÜGGE [6], DONNELL [7] and MORLEY [8]) for a range of n and the slenderness ratio $\frac{h}{R}$ (table I). Poisson's ratio ν has been taken as 0.3. The agreement is well and justifies the a priori assumptions of the classical theory (Kirchhoff-Love-Hypothesis).

References

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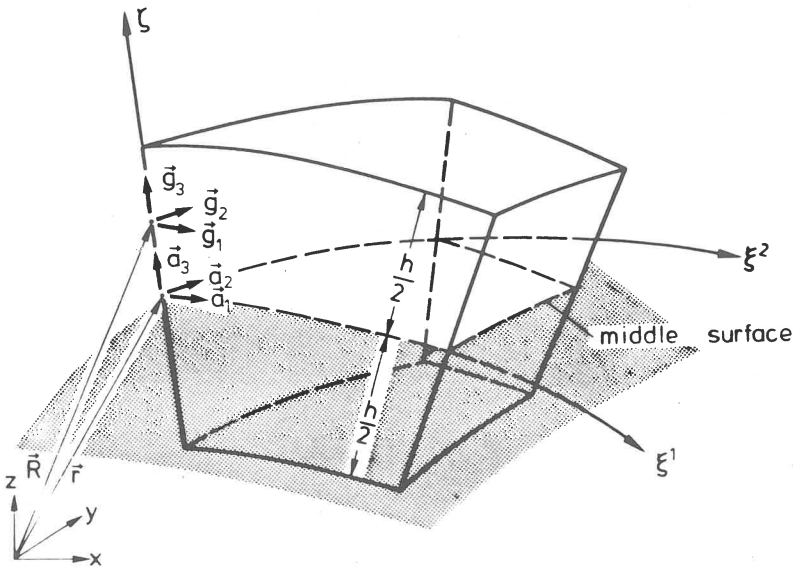

Figure 1: Shell element

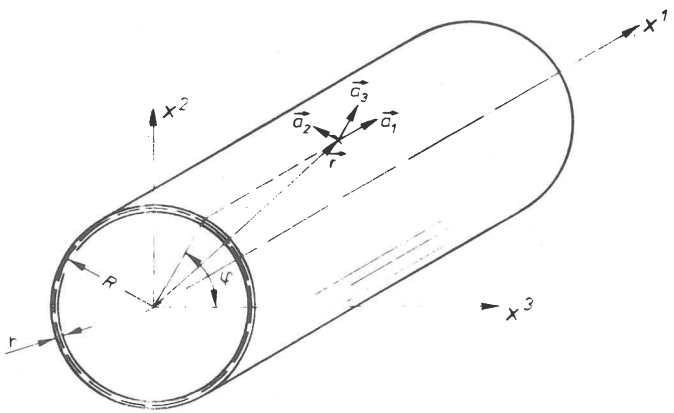

Figure 2: Circular cylindrical shell

TABLE I - Solution of the characteristic equation (13).
Comparison with other authors.

n	$\frac{1-\nu^2}{c^2}$	$\frac{h}{R}$		EQ. (13)	FLÜGGE [6]	DONNELL [7]	MORLEY [8]
0	$2,5 \cdot 10^3$	$\approx \frac{1}{15}$	κ_1	4,94119	4,98498	5,0	4,95025
			μ_1	5,06302	5,01498	5,0	5,05025
0	$4,0 \cdot 10^4$	$\approx \frac{1}{60}$	κ_1	9,96980	9,99250	10,0	9,97503
			μ_1	10,03073	10,00750	10,0	10,02503
0	$2,5 \cdot 10^7$	$\approx \frac{1}{1500}$	κ_1	49,9939	49,9985	50,0	49,9950
			μ_1	50,0061	50,0015	50,0	50,0050
1	$4,0 \cdot 10^4$	$\approx \frac{1}{60}$	κ_1	10,0197	10,0424	10,0502	10,0250
			μ_1	9,9806	9,9574	9,9503	9,9750
			κ_2	0	0	0,0520	0
			μ_2	0	0	0,0497	0
2	$4,0 \cdot 10^4$	$\approx \frac{1}{60}$	κ_1	10,1723	10,1952	10,2038	10,1781
			μ_1	9,8333	9,8104	9,8042	9,8283
			κ_2	0,1757	0,1758	0,2038	0,1761
			μ_2	0,1705	0,1704	0,1958	0,1701
3	$4,0 \cdot 10^4$	$\approx \frac{1}{60}$	κ_1	10,4351	10,4581	10,4683	10,4414
			μ_1	9,5989	9,5760	9,5718	9,5944
			κ_2	0,4397	0,4399	0,4683	0,4406
			μ_2	0,4060	0,4057	0,4282	0,4049
10	$4,0 \cdot 10^4$	$\approx \frac{1}{60}$	κ_1	15,2606	15,2533	15,2909	15,2678
			μ_1	7,4293	7,4851	7,4293	7,4348
			κ_2	5,2658	5,2610	5,2909	5,2678
			μ_2	2,5763	2,5719	2,5707	2,5652