ANALYSIS OF DYNAMIC INSTABILITY OF PLATES BY THE FINITE ELEMENT METHOD

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SUMMARY

The non-conservative stability of elastic systems has been of considerable interest to the power generating industry. The basic method of determining the non-conservative characteristics of elastic structures is the dynamical method, which is based on the investigation of the oscillations of systems close to their position of equilibrium. One needs to determine the eigenvalues of a non-self-adjoint boundary-value problem. The finite dimensional analog of the system would be a set of algebraic equations in which the matrices of coefficients in generalized coordinates contain non-symmetric components.

The study of non-conservative problems of elastic stability is closely related to the theory of linear non-self-adjoint differential operators. For elastic plates, exact solutions are infeasible and, hence, numerical methods are required. One such numerical method, the finite element method, has been shown to be an effective analysis technique for both linear and nonlinear statical problems and for conventional conservative stability problems. It is expected to be also an effective technique for non-conservative stability problem.

In this work the non-conservative stability of arbitrarily shaped flat plates is investigated using the finite element method with isoparametric triangular elements. The basis of derivation of the finite element equations is Hamilton's Principle extended to include non-conservative forces. Eigenvalues of the resulting non-symmetric matrices are calculated by first reducing the matrices to upper Hessenburg form and then applying the QR algorithm. While this method is not efficient computationally for symmetric problems, non-conservative stability problems leading to non-self-adjoint eigenvalue problems require such an approach. After calculation of the system eigenvalues, the natural frequencies of the total plate and the critical buckling loads are determined.

Several sample problems are solved to verify the technique developed, and to illustrate the analysis procedure. Various loading conditions, plate geometries and support conditions are investigated. The sensitivity of the numerical results to such factors as modelling techniques, boundary approximations, assumed mass distributions, etc. is explored.

1. Introduction

One of the most important non-conservative problems of elastic stability is the problem of the stability of flat plates situated in a flow of gas or liquid, e.g. bending torsional flutter of a cantilevered plate in a flow of gas. Since most problems of practical importance involve complicated geometries and/or boundary conditions, methods extant for developing closed-form or analytic approximate solutions are not applicable and numerical solution methods must be employed instead. Toward that end, a finite element model is presented herein for use in predicting stability regimes of flat plates subjected to non-conservative loads.

Let us first consider the classical problems of non-conservative stability of elastic systems. E. L. Nikolai (1) first introduced the concept of follower force excitation in 1928. Numerous work has been reported since that time, e.g. works by Reut (2) and B.L. Nikolai (3). See the texts by Bolotin (4,5) for details. Bolotin (4) has studied various non-conservative stability problems involving different structural systems, forces and solution techniques.

The general equations of the stability of elastic systems is discussed by Bolotin (5) wherein is also given an extensive bibliography. For the particular case of plate stability, exact methods of solution were used for some special cases. He suggested that for more complex problems one use the Galerkin variational method. More recently, Freodosev (6) used a series solution expansion to establish bending flutter stability conditions for the one-dimensional problem of a projectile being accelerated through space by a tangential thrust. Nemat-Nasser and Hermann (7) considered non-conservative torsion flutter of beams including warping.

The works discussed above were limited to simple geometric shapes for which exact or semi-closed form solutions could be generated. In order to consider complex geometries, numerical methods need be used. One numerical method of structural analysis developed in recent years is the finite element method. Since 1960 numerous applications of the finite element approach to a wide variety of linear structural problems have been made. An extensive bibliography is given in Przemieniecki (8). Turner, Dill, Martin and Melosh (9) were among the first to apply the finite element method to the analysis of geometrically nonlinear problems. Argyris (10), Gallagher and Padlog (11), and Martin (12) have applied the successive correction technique to large deflection and stability problems. A general formula for geometric stiffness matrices was presented by Oden (13).

The stability of plates under static conservative forces by the finite element method has been studied by Kapur and Hartz (14) using the Rayleigh-Ritz approach. Carson and Newton (15) have investigated plate stability with fully compartible finite elements. Anderson, Irons and Zienkiewicz (16) have studied conservative dynamic stability using the finite element method. Hutt (17) considered plates subjected to in-plane pulsating loads. Lagrangian equations were used to develop the governing finite element equations for rectangular, isotropic plates. Vance and Sitchin (18) have solved dynamical system by the direct application of Hamilton's principle.

Finite element analogs of one-dimensional bodies under non-conservative loads have been considered recently. Barsoum (19), and Mote (20) applied the finite element method to uncoupled non-conservative stability problems by using energy methods and the extended Hamilton's principle. Both Barsoum and Mote considered the classical Beck (21) problem, i.e. the bending stability of a clamped-free column subjected to a follower force. Barsoum (19) further considered the stability of a cantilever beam. Mote (20) also solved the problem of dynamic stability of tubes containing high speed fluids. Both studies were restricted to members with double cross-sectional symmetry.

Mote and Matsumato (22) studied coupled, non-conservative problems of stability by the finite element method. They investigated the stability of axisymmetrical, coupled columns discretized by the finite element method and subjected to non-conservative lateral and end loadings. The significance of the coupling contributions to the stability analysis were emphasized.

2. Mathematical Model

In this section, the non-conservative stability of plates is investigated using the finite element method with isoparametric triangular elements. The basis for derivation is the extended Hamilton's principle as described by Mote and Matsumoto (22). Hamilton's principle, (23) states that the time integrated variation in total energy of a system during some time interval is zero, i.e.

$$\int_{t_0}^{t_1} (\delta T - \delta\Omega - \delta U + \delta W)dt = 0 \qquad (1)$$

where δT = variation in kinetic energy

$\delta\Omega$ = variation in potential energy of external conservative forces.

δU = variation in strain energy

δW = variation in external work of non-conservative forces.

In a conservative system, $W = 0$.

For an elastic plate, neglecting in-plane inertial forces,

$$T = \frac{1}{2}\iint \rho \left(\frac{\partial w}{\partial t}\right)^2 dt\ dy \qquad (2)$$

where w = transverse displacement, ρ = mass density per unit area of middle surface, and x, y = Cartesian coordinates on middle surface of plate. The strain energy of bending in the plate is

$$U = \frac{1}{2}\int\int [X]^T [D] [X]\ dx\ dy \qquad (3)$$

where

$$[X] = \begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ \frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \qquad (4)$$

$$[D] = \frac{Eh3}{12(1\upsilon^2)} \begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & 2(1-\upsilon) \end{bmatrix} \qquad (5)$$

and E = Young's modulus, h = plate thickness, υ = Poisson's ratio. The potential energy of external conservative lateral and in-plate forces on a plate is given by

$$\Omega = \frac{1}{2} \iint \left[\theta\right]^T \left[N\right] \left[\theta\right] dx\, dy - \iint pw\, dx\, dy \tag{6}$$

where p = lateral load and

$$\theta = \begin{Bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{Bmatrix} \tag{7}$$

$$N = \begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix} \tag{8}$$

in which N_x, N_y, N_{xy} are the extant in-plane stress resultants.

The external non-conservative forces can only be expressed as virtual work expressions. The particular form of the virtual work of non-conservative forces will depend on the nature and location of load. In general

$$\int_{t_0}^{t_1} \delta W\, dt = \int_{t_0}^{t_1} \Big[- \int f\,(w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial t})\, \delta w\, dx\, dy - \oint g\,(w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial t})\, \delta w\, ds \Big] dt \tag{9}$$

For example, in the case of supersonic panel flutter with flow parallel to the plate (24),

$$\int_{t_0}^{t_1} \delta W\, dt = \int_{t_0}^{t_1} - \int (\lambda \frac{\partial w}{\partial \xi} + \mu \frac{\partial w}{\partial t})\, \delta w\, dx\, dy \tag{10}$$

where $\lambda = \frac{2\bar{q}}{\sqrt{M_\infty^2 - 1}}$ and $\mu = \frac{\lambda}{V} \frac{M_\infty^2 - 2}{M_\infty^2 - 1}$

$\bar{q}$ = dynamic pressure, M_∞ = Mach number, V = flow velocity, and ξ = direction of flow.

In the case of a follower load applied to an edge of the plate

$$\delta W = - \left(\int N_\ell \frac{\partial \omega}{\partial \eta}\, \delta\omega\, d\ell \right) \tag{11}$$

where N_ℓ = intensity of applied load per unit length of edge, ℓ = coordinate along loaded edge, η is direction of load application.

3. Finite Element Analog

The derivation of an isoparametric triangular plate element as presented in Ref. (25) is adopted herein. Based on cubic interpolation functions within an area coordinate system, the transverse deflection at an arbitrary point within an element is given by

$$\omega = \left[L\right] \left\{U\right\} = \left[L_1\ L_2\ L_3 \right] \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} \tag{12}$$

where

$$\left\{U_i\right\}^T = \left\{w_i, \theta_{xi}, \theta_{yi}\right\}^T \tag{13}$$

w_i, θ_{xi}, θ_{yi} being deflection and slopes at the i^{th} node and $\left[L\right]$ is given in Ref. (25) as a function of nodal coordinates of the element.

Substituting Eqs. (12,13) into Eqs. (1, 2, 3, 6,9), one obtains for each element the following matrix equation:

$$M \ddot{U} + K U + S U - \alpha E^T \dot{U} - \beta G^T U - F = 0 \tag{14}$$

where a dot (.) denotes time rate, and M is the mass matrix

$$M = \int L^T L \, dx \, dy \tag{15}$$

K is the stiffness matrix

$$K = \int C^T D C \, dx \, dy \tag{16a}$$

$$C^T = \left[\frac{\partial^2 L^T}{\partial x^2} , \frac{\partial^2 L^T}{\partial y^2} , \frac{\partial^2 L^T}{\partial x \partial y} \right]^T \tag{16b}$$

S is the initial stress matrix

$$S = \int \left[\frac{\partial L^T}{\partial x} \quad \frac{\partial L^T}{\partial y} \right] N \begin{Bmatrix} \frac{\partial L}{\partial x} \\ \frac{\partial L}{\partial y} \end{Bmatrix} dx \, dy \tag{17}$$

F is the lateral load vector

$$F = \int \rho \, (x,y) \, L \, dx \, dy \tag{18}$$

and $\alpha E^T U + \beta G^T \dot{U}$ are derived from the virtual non-conservative work

$$\int_{t_1}^{t_2} \delta W \, dt = \int_{t_1}^{t_w} (\alpha \, u^T E + \beta \, \dot{u}^T G) \, \delta u dt \tag{19}$$

α, β, E, G depending on the parameters of non-conservative loading.

In the case of supersonic panel flutter, Eqs. (10, 12, 13, 19) yield

$$\delta W = - \left[\lambda \int U^T \frac{\partial L^T}{\partial \xi} L \, \delta U \, dx \, dy + \mu \int \dot{U}^T L^T L \, \delta U \, dx \, dy \right] \tag{20a}$$

and therefore

$$\alpha = -\lambda \qquad \beta = -\mu \tag{20b}$$

$$E = \int \frac{\partial L^T}{\partial \xi} L \, dx \, dy \tag{20c}$$

$$G = \int L^T L \, dx \, dy \tag{20d}$$

for use in Eqs. (14,19).

In the case of follower loads distributed along one edge of an element

$$\delta W = -N_\ell \, U^T \int \frac{\partial L^T}{\partial \eta} L \, d\ell \, \delta U \tag{21a}$$

and therefore

$$\alpha = N_\ell \qquad \beta = 0 \tag{21b}$$

$$E = \int \frac{\partial L^T}{\partial \eta} L \, d\ell \tag{21c}$$

$$G = 0 \tag{21d}$$

for use in Eqs. (14,19)

4. Eigenvalue and Stability Analysis

If the lateral load term in Eq. (14) is deleted, Eq. (14) can be converted to an eigenvalue problem by assuming

$$U(t) = e^{\omega t} V \tag{22}$$

and then, Eq. (14) becomes

$$(M \omega^2 + K + S - \alpha E^T - \beta \omega G^T) V = 0 \tag{23}$$

the nontrivial solution for which, for given mass and stiffness properties, exists only for certain values of the parameters α, β, ω, S. Thus Eq. (23) constitutes an eigenvalue problem to determine the values ω_n for which nontrivial solutions V_n exists.

Since the matrix of coefficients in Eq. (23) may in general be non-symmetric, the QR method (26) was employed to determine the real and/or complex eigenvalues of Eq. (23). First the matrix is balanced by reducing the norm of the matrix through diagonal similarity transformations. Next, the matrix is reduced to upper Hessenberg form by unitary similarity transformation. After balancing and reducing the matrix, the QR method is employed: the matrix is deformed into a multiple of an orthogonal (Q) matrix and an upper-triangular (R) matrix from which the eigenvalues are obtained by deleting rows and columns. Origin shifts are used to speed convergence.

Once the eigenvalues ω_n have been determined for specified values of the parameters α, β, S, the system needs to be investigated to determine whether it is stable or not. Furthermore if the plate is unstable, one should determine the mode of instability - flutter or divergence. If the eigenvalue ω_n are real and negative, the system is stable. If the eigenvalues are complex, the instability will occur by flutter. If the eigenvalues are real and positive the instability will occur by divergence.

To obtain the critical values of the parameters α, β, S for which instability occurs, those parameters are increased incrementally while testing the value of the frequency parameters ω_n to determine the stability of the system. First, the load is increased using large increments. Then a binary search is instituted between the two values of the parameter corresponding to a stable regime and an unstable regime, respectively. Thus, one converges on the critical value of the parameter on the boundary of the stable and unstable regimes.

5. Example Problem:

Several problems were considered to verify the programming aspects of the previously described solution techniques. In most cases it was found that the use of a lumped mass model rather than a consistent mass model led to more accurate and efficient representations when an equal number of degrees of freedom were included in the model. Condensation of rotational degrees of freedom was used (27).

Here we will present results obtained for cantilevered trapezoidal and triangular plates subjected to a follower force on one edge of the plate. See Figure 1 for the layout of typical meshes of elements considered. In the trapezoidal problem, the effect of geometry on the critical load was studied. The span L and the length a of the fixed edge were held constant (a/L = 1.2). The length b of the loaded edge was varied from b=0 to b=a=1.2L. Results are shown in Fig 2 where is plotted the total non-dimensional edge load. Note that the form of instability is a function of the plate geometry. For a rectangle and those trapezoidal shapes with larger b/L ratios, the plate undergoes flutter instability. As the plate geometry approaches that of a triangle, the instability occurs by divergence. However, in the limiting case of a triangle, the instability again occurs by flutter.

The anomaly of the above noted shift back to flutter instability for a trianglar shape prompted a parametric study of apex loaded triangular plates. Various ratios of height L to base a were considered. As can be seen in Fig. (3), the type of instability is a function of the ratio. In other words, for a trapezoidal plate, all three dimensions L, a, b have an effect on the form of instability and a nonlinear effect on the value of critical load at which instability occurs.

REFERENCES

(1) Nikolai, E.L., "On the Stability of the Rectilinear Form of Equilibrium of a Bar in Compression and Torsion," Izv. Leningr. Politekhn. in-ta., 31, 1928.

(2) Reut, V.I., "On the Theory of Elastic Stability," Trans. odessk. in-ta inzh. grazhd. ikomm. Str-va, No. 1, 1939.

(3) Nikolai, B.L., "On the Criterion of Stability of Elastic Systems," Trans. odessk. in-ta inzh. grazhd. ikomm. Str-va, No. 1, 1939.

(4) Bolotin, V.V., Non Conservative Problems of the Theory of Elastic Stability, MacMillan Co., New York, N.Y., 1963.

(5) Bolotin, V.V., Dynamic Stability of Elastic Systems, Pergamon Press, 1956.

(6) Feodos'ev, V.I., "On Stability Problem," Physic of Metals and Metallography, (translated from the Russian Journal), Vol. 29, No. 2, 1965.

(7) Nemat-Nasser, S., and Hermann, G., "Torsional Instability of Cantilevered Bars Subject to Non-Conservative Loading," J. Appl. Mech., Vol. 33, 1966.

(8) Przemieniecki, J.S., Theory of Matrix Structural Analysis, McGraw-Hill, 1968.

(9) Turner, M.J., Dill, E.H., Martin, H.C., and Melosh, R.J., "Large Deflections of Structures Subjected to Heating and External Loads," J. Aerospace Sciences, Vol. 27, Feb., 1960.

(10) Argyris, J.H., "Matrix Analysis of Three-Dimensional Elastic Media, Small and Large Displacements," AIAA J., Vol. 3, No. 1, Jan., 1965.

(11) Gallaghar, R.H., and Padlog, J., "Discrete Element Approach to Structural Instability Analysis," AIAA J., Vol. 1, June, 1963.

(12) Martin, H.C., "On the Derivation of Stiffness Matrices for the Analysis of Large Deflection Stability Problems," First Conference on Matrix Methods in Structural Mechanics, Wright-Patterson AFB, Dayton, Ohio, Oct., 1965.

(13) Oden, J.T., "Calculation of Geometric Stiffness Matrices for Complex Structures," AIAA J., Vol. 4, No. 6, Aug., 1965.

(14) Kapur, K.K., and Hartz, B.J., "Stability of Thin Plate Using the Finite Element Method," J. Eng. Mech. Div., ASCE, No. EM2, 1966.

(15) Carson, W.G., and Newton, R.E.,"Plate Buckling Analysis Using a Fully Compatible Finite Element," AIAA J., 8, 1969, pp. 527-529.

(16) Anderson, R.G., Irons, B.M., and Zienkiewicz, O.C., "Vibration and Stability of Plates Using Finite Element Method," Int. J. Solids Struct., 4, 1968, pp. 1031-1055.

(17) Hutt, J.M., "Dynamic Stability of Plates by Finite Element," Ph.D. Dissertation, Oklahoma State University, 1968.

(18) Vance, J.M., and Sitchin,A., "Numerican Solution of Dynamical Systems by Direct Application of Hamilton's Principle," Int. J. Num. Meth. Eng., 4, 207-216.

(19) Barsoum, R.S., "Finite Element Method Applied to the Problems of Stability of a Non-Conservative System," Int. J. Num. Meth. Eng., Vol. 3, 1971.

(20) Mote, C.D. Jr., "Non-Conservative Stability by Finite Element," J. Eng. Mech. Div., ASCE, No. EM3, 1971.

(21) Beck, M., "Die Knicklast des Einsei tig Eingespannten Tangential Gedruckten Stubes," Zietschrift fur Angewandte Mathematik und Physik, Vol, 3, Berlin, West Germany, 1952, pp. 225-228, 476-477.

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(22) Mote, C.D. Jr., and Matsumoto, G.Y., "Coupled, Non-Conservative Stability Finite Element," J. Eng. Mech. Div., ASCE, No. EM3, 1972.

(23) Langhaar, H.L., Energy Methods in Applied Mechanics, John Wiley & Sons, Inc., N.Y. 1962.

(24) Sander, G., Bon, C., and Geradin, M., "Finite Element Analysis of Supersonic Panel Flutter," Int. Journal for Numerical Methods in Engng., Vol. 7, pp. 379-394, 1973.

(25) Zienkiewicz, O.C., The Finite Element Method in Engineering Science, McGraw-Hill, 1971.

(26) Wilkinson, J.H., and Reinsch, C., Handbook for Automatic Computation Volume II: Linear Algebra, Springer-Verlag, New York, 1971.

(27) Desai, C.S., and Abel, J.F., Introduction to the Finite Element Method, Van Nostrand Reinhold Co., New York, 1972.

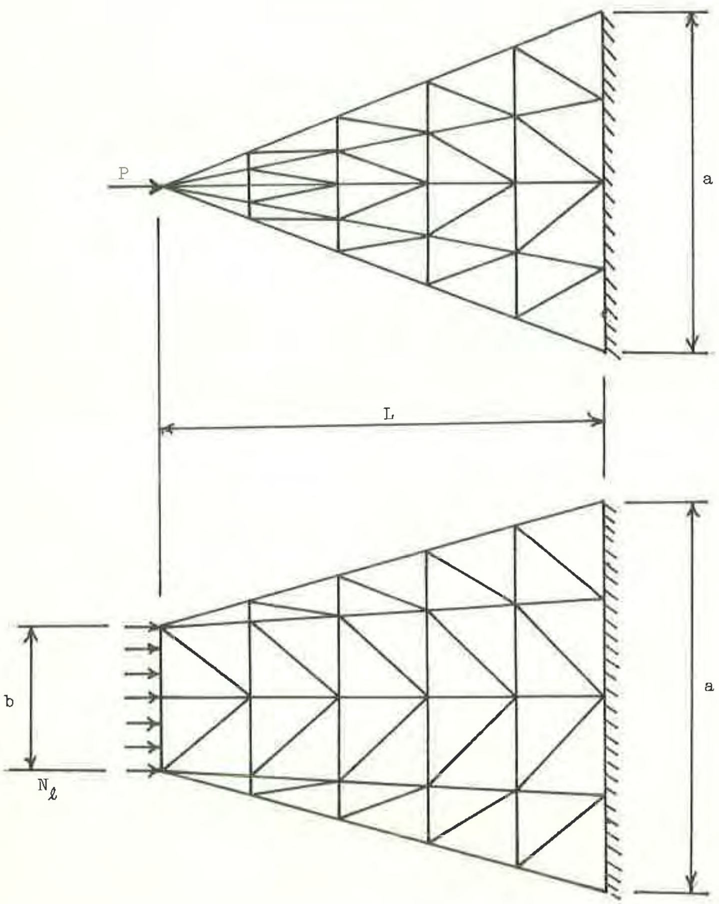

Fig. 1 Typical Trapezoidal and Triangular Meshes

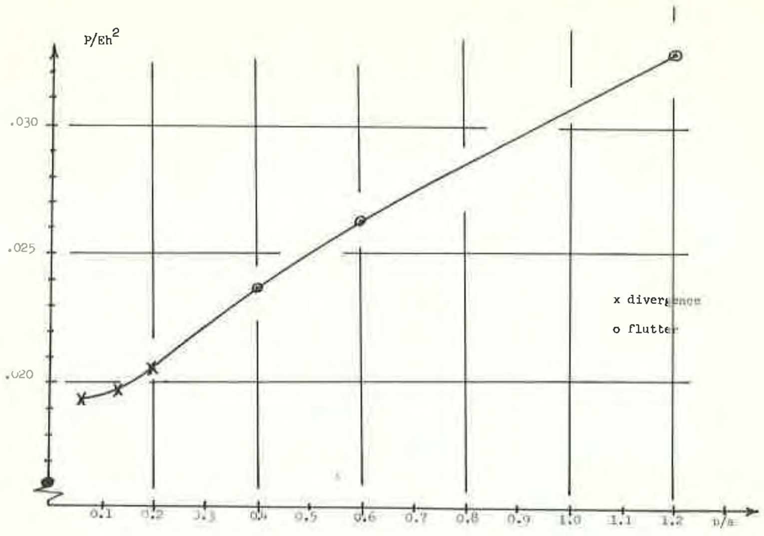

Fig. 2 Critical Load for Trapezoids

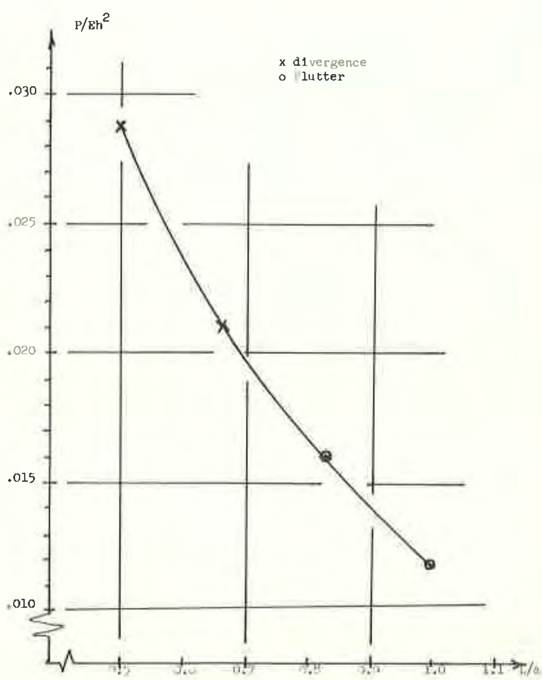

Fig. 3 Critical Load for Triangles